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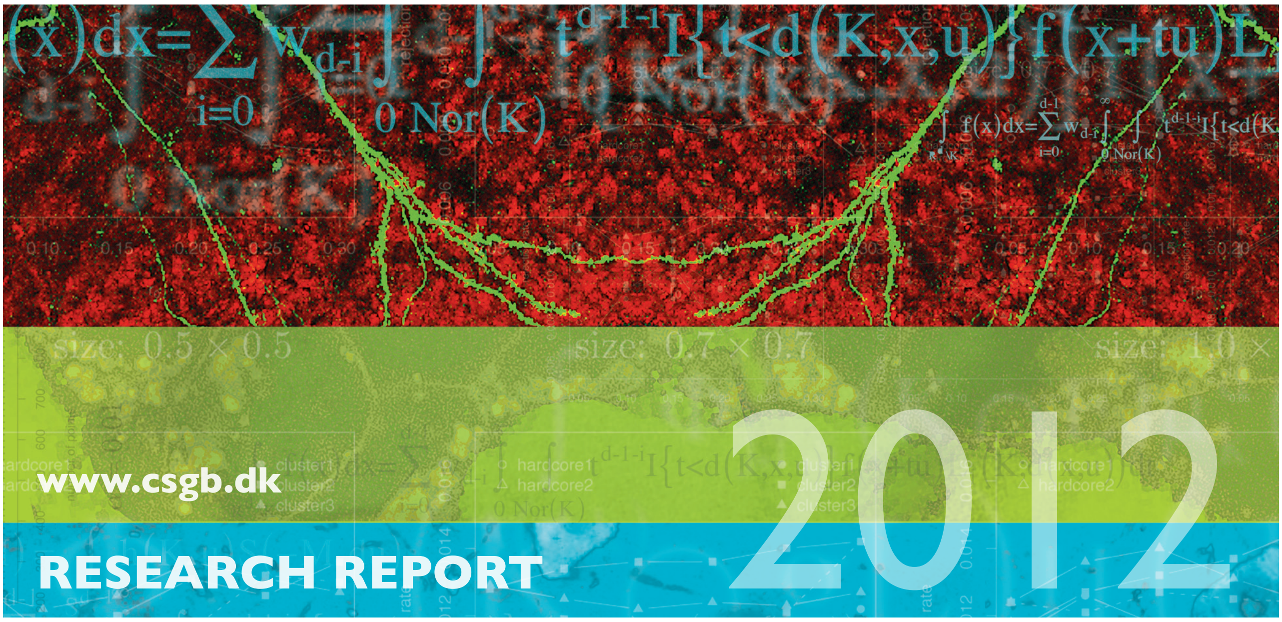

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Abstract

Wicksell's classical corpuscle problem deals with the retrieval of the size distribution of spherical particles from planar sections. We discuss the problem in a local stereology framework. Each particle is assumed to contain a reference point and the individual particle is sampled with an isotropic random plane through this reference point. Both the size of the section profile and the position of the reference point inside the profile are recorded and used to recover the distribution of the corresponding particle parameters. Theoretical results concerning the relationship between the profile and particle parameters, unfolding of the arising integral equations, uniqueness issues and domain of attraction relations are discussed. We illustrate the approach by reconstructing from simulated data using numerical unfolding algorithms.

Keywords: Wicksell's corpuscle problem; local stereology; inverse problems; numerical unfolding, stereology of extremes

1 Introduction and account of main results

This work discusses Wicksell's corpuscle problem in a local stereology framework, where the size distribution of spherical particles is recovered from plane sections through reference points. In order to describe similarities and differences of the local and the classical Wicksell problem, we start with a short outline of the latter.

Wicksell's classical corpuscle problem, described very figuratively as "tomato salad problem" by Günter Bach, asks how to recover the size distribution of random balls in $\mathbb{R}^3$ from the observed size distribution of two-dimensional section profiles. Although the stereological literature often refers to ball-shaped particles as "spheres", we decided to adopt terminology from pure mathematics, calling the solid particles "balls", and reserving the word "sphere" for the boundary of a ball. Assuming that the "density of the centers and the distribution of the sizes being the same in all parts of the body [where the observation is taken]", Wicksell showed Wicksell (1925) that the density f_r of the profile radii r is given by

$$f_r(x) = \frac{x}{\mathbb{E}R} \int_x^\infty \frac{1}{\sqrt{y^2 - x^2}} dF_R(y), \tag{1.1}$$

where F_R is the cumulative distribution function of the radius R of the balls in $\mathbb{R}^3$, and $\mathbb{E}R$ is the usual expectation of R. Stoyan & Mecke Mecke and Stoyan (1980) made Wicksell's arguments rigorous showing that (1.1) actually holds for all stationary particle processes of non-overlapping balls, when R has a finite mean. The right hand side of (1.1) is essentially an *Abel integral transform* of F_R. It can be inverted explicitly, and this shows in particular that F_R is determined by F_r.

Relation (1.1) is the result of two mutually counteracting sampling effects: As the probability that a ball is hit by the plane is proportional to its radius, the radius distribution of the intersected balls is size weighted, preferring large balls. On the other hand, the profile radius is always smaller than the radius of the intersected ball, as the section plane almost surely misses the ball's center. Already Wicksell was aware of the fact that these two effects can annihilate each other. When R follows a Rayleigh distribution (the distribution of the length of a centered normally distributed two-dimensional vector), r also follows this distribution, with the same parameter. The Rayleigh distribution is the only *reproducing distribution* in this sense; see Drees and Reiss (1992). If R_w denotes the radius-weighted radii distribution with cumulative distribution function

$$F_{R_w}(x) = \frac{1}{\mathbb{E}R}\left(xF_R(x) - \int_0^x F_R(s)\,ds\right),$$

$x \geq 0$, the two sampling effects can also be expressed by the relation

$$r = \Lambda R_w, \tag{1.2}$$

where Λ is a stochastic variable with Lebesgue-density $s \mapsto \mathbf{1}_{[0,1]}(s)s/\sqrt{1-s^2}$ that is independent of R_w; see Baddeley and Vedel Jensen (2005). This also gives the well-known *moment relations*

$$\tilde{m}_k = c_{k+1}\frac{\tilde{M}_{k+1}}{\tilde{M}_1}, \tag{1.3}$$

$k = -1, 0, 1, 2, \ldots$, where $\tilde{m}_k$ and $\tilde{M}_k$ are the kth moments of r and R, respectively, and

$$c_k = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left((k+1)/2\right)}{\Gamma\left(k/2+1\right)}. \tag{1.4}$$

The inversion of the integral equation (1.1) is ill-posed, which can informally be described as 'small deviations of the data can lead to arbitrarily large deviations of the solution'. There exist several methods, both distribution-free (non-parametric) and parametric ones, for numerically solving Wicksell's classical problem. Examples of distribution-free methods are finite difference methods, spectral differentiation and product integration methods and kernel methods. Parametric methods can be divided into maximum likelihood methods and methods that only use the moment relations. All methods have their advantages and disadvantages and none appears to be generally best. References for the respective methods and an overview of existing ones are given in for example (Stoyan et al., 1995, Section 11.4.1), Cruz-Orive (1983), Blödner et al. (1984), Anderssen and Jakeman (1975a) and (Anderssen and De Hoog, 1990, p. 373-410).

Stereology of extremes has received much interest due to applications in material science; see Takahashi and Sibuya (2002) for an application to metallurgy. The maximum size of balls in Wicksell's classical problem is studied in Drees and Reiss (1992), Takahashi and Sibuya (1996), Takahashi and Sibuya (1998) and Takahashi and Sibuya (2001). There it is shown that F_R and F_r belong to the same type of extreme value distribution. Extremes of the size and shape parameters of spheroidal particles are studied in Hlubinka (2003a), Hlubinka (2003b), Hlubinka (2006) and Beneš et al. (2003).

More detailed reviews of the classical Wicksell problem can be found in (Stoyan et al., 1995, Section 11.4), Cruz-Orive (1983), (Ohser and Mücklich, 2000, Chapter 6) and earlier contributions, that are listed in these sources.

In the above description of Wicksell's problem, we adopted the common model-based approach, where the particle system is random, and the probe can be taken with an arbitrary (deterministic) plane due to stationarity. Jensen Jensen (1984) proved that (1.1) also holds in a design-based setting, where the particle system is deterministic – possibly inhomogeneous – but the plane is randomized. In order to obtain a representative sample, it is enough to choose an FUR (*f*ixed orientation *u*niform *r*andom) plane. Inspired by local stereology, we discuss here a design-based sampling scheme, where each particle contains a reference point and the individual particle is sampled with an isotropic plane through that reference point. This design is tailor-made for applications e.g. in biology, where cells often are sampled using a confocal microscope by focusing on the plane through the nucleus or the nucleolus of a cell; see the monograph Jensen (1998) on local stereology. To our knowledge, the first explicit mention of Wicksell's problem in a local setup is Jensen (1991), where it is remarked that the problem is trivial whenever the reference point coincides with the ball's center, as the size distributions of balls and section profiles are then identical. This being obvious, we want to show here that the local Wicksell problem is far from trivial if this condition is violated.

Although biological application suggests to restrict considerations to $\mathbb{R}^3$, we will consider particles in n-dimensional space intersected by hyperplanes, as the general theory does not pose any essential extra difficulties. Like in the classical setting, we assume that the particles are (approximate) balls, and in order to incorporate the natural fluctuation, we assume that both the radius of the ball and the position of the reference point in the ball are random. This way, a ball-shaped particle is described by two quantities: firstly its random radius R, the *size of the ball*, and the distance of the reference point from the center of the ball. It turns out to be favorable to work with the *relative distance* $Q \in [0, 1]$ instead, meaning that QR is the distance of the reference point to the ball's center. The variable Q can be considered as *shape* descriptor. We do not take into account the direction of the reference point relative to the ball's center, as this direction appears to be of minor interest. In addition it cannot be determined from isotropic sections, unless we would also register the orientation of the section plane (which we will not do here). If a ball with a given reference point is intersected by an isotropic hyperplane through the reference point (independent of the ball), an $(n-1)$-dimensional ball is obtained. We let r be its radius and q be the relative distance of the reference point to its center.

Our first result, Theorem 3.1, shows that the joint distributions of (r, q) and

(R, Q) are connected by an explicit integral transform. Here and in the following we assume that all balls have a positive radius, and that the reference point does almost surely not coincide with the center of a ball, that is, we agree on $\mathbb{P}(Q = 0) = 0$. The last assumption is not essential, and most of our results can be extended to the case $\mathbb{P}(Q = 0) > 0$, as outlined in Remark 3.2. Like in the classical Wicksell problem we show that the marginals of r and q always have probability densities f_r and f_q, respectively, and we determine their explicit forms in Corollary 3.4. However, the joint distribution of (r, q) need not have a density. Corollary 3.4 also shows that f_q only depends on the distribution of Q and not on R, which explains why we are working with *relative* distances. In Proposition 3.5 we show that

$$r = \Gamma R \tag{1.5}$$

with a stochastic variable Γ, whose density can be given explicitly. This is in analogy to (1.2) in the classical case. However, there is no size-weighting in our local stereological design, and the variable Γ is now depending on the distribution of Q. Thus, when R and Q are independent, so are R and Γ, and moment relations in analogy to (1.3) are readily obtained: if m_k and M_k are the kth moments of r and R, respectively, then

$$m_k = c_k(Q) M_k, \tag{1.6}$$

$k = 0, 1, 2, \ldots$. The constants $c_k(Q)$ depend on the distribution of Q and are given in Remark 3.6. However, (1.6) cannot be applied directly to obtain (estimates of the) moments M_k, as $c_k(Q)$ depends on the shape of the full particle. Corollary 5.2 shows that $c_k(Q)$ can be expressed by the distribution of q, making it possible to estimate both m_k and $c_k(Q)$ from the section profiles and thus to access M_k. Simulation studies showed that this estimation procedure is quite stable, as described after Corollary 5.2.

We then turn to uniqueness in Section 4. That the distribution of Q is uniquely determined by the distribution of q follows from the fact that the two distributions are connected by an Abel type integral equation. It can be inverted explicitly; see Proposition 5.1 for $n = 3$. Theorem 4.1 shows that even the joint distribution of (R, Q) is uniquely determined by the distribution of the profile quantities (r, q), but only under the assumption that R and Q are stochastically independent. The two marginals of (r, q) do not uniquely determine (R, Q) without this extra assumption, as shown by an example after Theorem 4.1. It is an open problem whether the *joint distribution of* (r, q) determines the joint distribution of (R, Q) without the independence assumption.

To reconstruct F_R and F_Q from simulated data, when R and Q are independent, we chose to use distribution-free methods. Maximum likelihood methods and method of moments can though also be used as for the classical Wicksell problem. In Section 6 we describe the implementation of a Scheil-Schwartz-Saltykov type method Saltykov (1974). Following Cruz-Orive (1983), the method can be classified as a finite difference method, more specifically a 'successive subtraction algorithm'. The data is grouped and the distributions F_Q and F_R discretized. Then F_q written in terms of F_Q, F_r written in terms of $F_{R,Q}$, respectively, become systems of linear equations, which can be solved. We carried out a number of stochastic simulation studies which illustrate the feasibility of the approach. A few reconstructions are reported in Figures 2 and 3.

In Section 7 we discuss practical examples and then turn to stereology of extremes in Section 8. Similar results as in the classical case are obtained. Proposition 8.1 shows that if the particle parameters, R and Q, are independent, F_R and F_r belong to the same type of extreme value distribution. An analogous result holds for the shape parameters. To our knowledge, stereology of extremes in a local setting has only been treated in Pawlas (2012). There the shape and size parameters of spheroids are studied but the isotropic section plane is always taken through the center of the spheroid.

2 Preliminaries

Throughout we let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and O its origin. The Euclidean scalar product is denoted by $\langle \cdot, \cdot \rangle$ and the Euclidean norm by $\|\cdot\|$. We let e_i be the vector in $\mathbb{R}^n$ with 1 in the ith place and zeros elsewhere. For a set $Y \subseteq \mathbb{R}^n$, we define

$$Y + x = \{y + x \mid y \in Y\},\ x \in \mathbb{R}^n, \qquad \alpha Y = \{\alpha y \mid y \in Y\},\ \alpha > 0. \tag{2.1}$$

We use ∂Y for the boundary and $\mathbf{1}_Y$ for the indicator function of Y. The unit ball in $\mathbb{R}^n$ is $B_n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and the boundary of it is the unit sphere (in $\mathbb{R}^n$) S^{n-1}, $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. A ball in $\mathbb{R}^n$ of radius R centred at O is denoted by RB_n, in accordance to (2.1). We write σ_n for the surface area of the unit ball in $\mathbb{R}^n$, i.e. $\sigma_n = \mathcal{H}_n^{n-1}(S^{n-1})$, where $\mathcal{H}_n^d$ is the d-dimensional *Hausdorff measure* in $\mathbb{R}^n$. When n is clear from the context, we abbreviate $\mathcal{H}_n^d(\mathrm{d}u)$ by $\mathrm{d}u^d$. For $p = 0, 1, \ldots, n$ let

$$\mathcal{L}_{p[O]}^n = \{L_{p[O]}^n \subseteq \mathbb{R}^n : L_{p[O]}^n \text{ is a } p\text{-dim. linear subspace}\},$$
$$\mathcal{L}_p^n = \{L_p^n \subseteq \mathbb{R}^n : L_p^n \text{ is a } p\text{-dim. affine subspace}\}.$$

be the family of all p-dimensional linear, respectively affine, subspaces of $\mathbb{R}^n$. These spaces are equipped with their standard topologies (Schneider and Weil (2008)) and we denote their Borel σ-algebras by $\mathcal{B}(\mathcal{L}_{p[O]}^n)$, $\mathcal{B}(\mathcal{L}_p^n)$, respectively. The spaces are furthermore endowed with their natural invariant measures, and we write $\mathrm{d}L_{p[O]}^n$ and $\mathrm{d}L_p^n$, respectively, when integrating with respect to these measures. We use the same normalization as in Schneider and Weil (2008)

$$\int_{\mathcal{L}_{p[O]}^n} \mathrm{d}L_{p[O]}^n = 1.$$

A random subspace $L_{p[O]}^n$ is called *isotropic random* (IR) if and only if its distribution is given by

$$\mathbb{P}_{L_{p[O]}^n}(A) = \int_{\mathcal{L}_{p[O]}^n} \mathbf{1}_A \, \mathrm{d}L_{p[O]}^n, \qquad A \in \mathcal{B}(\mathcal{L}_{p[O]}^n).$$

Similarly, a random flat $L_p^n \in \mathcal{L}_p^n$ is called *isotropic uniform random* (IUR) *hitting a compact object* Y if and only if its distribution is given by

$$\mathbb{P}_{L_p^n}(A) = c \int_{\mathcal{L}_p^n} \mathbf{1}_{A \cap \{L_p^n \in \mathcal{L}_p^n : L_p^n \cap Y \neq \emptyset\}} \, \mathrm{d}L_p^n, \qquad A \in \mathcal{B}(\mathcal{L}_p^n), \tag{2.2}$$

where c is a normalizing constant. We let $x|L^n_{p[O]}$ be the orthogonal projection of $x \in \mathbb{R}^n$ onto $L^n_{p[O]}$. We furthermore adopt the convention of writing $v^\perp$ for the hyperplane with unit normal $v \in S^{n-1}$. We use $B(z;a,b)$ to denote the *incomplete Beta function*, given by

$$B(z;a,b) = \int_0^z t^{a-1}(1-t)^{b-1}\mathrm{d}t, \qquad 0 < z < 1, \quad a,b > 0.$$

When $z = 1$ we write $B(a,b)$. Note in particular that $B(1/2,(n-1)/2) = \sigma_n/\sigma_{n-1}$. For an arbitrary function f we let $f^+(x) = \max\{f(x),0\}$ be its positive part. Given a random variable X, its *characteristic function* (or Fourier transform) is defined by

$$\varphi_X(t) = \mathbb{E}e^{itX}, \qquad t \in \mathbb{R}.$$

The reader is referred to Haan (1970) for the following important results from stereology of extremes.

Let $X_1,\dots,X_n$ be independent and identically distributed random variables with distribution function F. Denote the maximum of $X_1,\dots,X_n$ by $X_{(n)}$. Then we have $F_{X_{(n)}}(x) = F^n(x)$. If there exist sequences of constants $\{a_n\}$ ($a_n > 0$, $n = 1,2,\dots$) and $\{b_n\}$ such that

$$\lim_{n\to\infty} F^n(a_n x + b_n) = L(x), \qquad x \in \mathbb{R},$$

F is said to belong to the *domain of attraction* of the distribution function L. We refer to $\{a_n\}$ and $\{b_n\}$ as *normalizing constants*. We follow the notation in Haan (1970) and write $F \in \mathcal{D}(L)$ if F is in the domain of attraction of L. Up to affine transformation, L must be one of the extreme value distributions: Fréchet, Weibull and Gumbel. The following notations are used for these distributions:

- $L_{1,\gamma}(x) = \exp(-x^{-\gamma})$, $x,\gamma > 0$ (Fréchet),
- $L_{2,\gamma}(x) = \exp(-(-x)^{\gamma})$, $x \le 0$, $\gamma > 0$ (Weibull),
- $L_3(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$ (Gumbel).

Let $\omega_F = \sup\{x : F(x) < 1\}$ be the right endpoint of F. There are sufficient and necessary conditions for $F \in \mathcal{D}(L)$. These conditions are ($\gamma > 0$)

- $F \in \mathcal{D}(L_{1,\gamma}) \Leftrightarrow \omega_F = \infty$, $\lim_{x\to\infty} \frac{1-F(yx)}{1-F(x)} = y^{-\gamma}$ for all $y > 0$.
- $F \in \mathcal{D}(L_{2,\gamma}) \Leftrightarrow \omega_F < \infty$, $\lim_{x\to 0+} \frac{1-F(\omega_F - yx)}{1-F(\omega_F - x)} = y^{\gamma}$ for all $y > 0$.
- $F \in \mathcal{D}(L_3) \Leftrightarrow \lim_{x\to\omega_F-} \frac{1-F(x+yb(x))}{1-F(x)} = e^{-y}$ for all $y \in \mathbb{R}$. The function b is some auxiliary function (it can be chosen such that it is differentiable for $x < \omega_F$ with $\lim_{x\to\omega_F-} b'(x) = 0$ and $\lim_{x\to\infty} b(x)/x = 0$ if $\omega_F = \infty$, or $\lim_{x\to\omega_F-} b(x)/(\omega_F - x) = 0$ if $\omega_F < \infty$).

The following lemma is a slight generalisation of (Haan, 1970, Lemma 1.2.1) and the proof is analogous to the one given there. A similar result can be found in (Kötzer and Molchanov, 2006, Lemma 2.4).

Lemma 2.1. *Let f and g be positive functions on $\mathbb{R}^2$ and $a \in \mathbb{R}$, such that*

$$\int_a^\omega f(s,t)\,dt < \infty, \qquad \int_a^\omega g(s,t)\,dt < \infty$$

for some $\omega \in (a, \infty]$ and all $s \geq a$. If

$$\lim_{\substack{(s,t)\to(\omega,\omega)\\ s\leq t<\omega}} \frac{f(s,t)}{g(s,t)} = c, \qquad \text{where } c \in [0,\infty]. \tag{2.3}$$

then

$$\lim_{s\to\omega-} \frac{\int_s^\omega f(s,t)\,dt}{\int_s^\omega g(s,t)\,dt} = c.$$

An analogous result holds when s approaches ω from the right.

3 The direct problem

Consider a random ball in $\mathbb{R}^n$ with positive radius, centered at O' and containing the origin. Let R and Q denote the stochastic variables giving the radius of the ball, and the relative distance of the center of the ball from O, respectively. Intersect the ball with an IR hyperplane, $L^n_{n-1[O]}$, independent of the ball. Then an $(n-1)$-dimensional ball is obtained. Let r be its radius and $q = \frac{1}{r}\|O'|L^n_{n-1[O]}\|$ the relative distance of its center from O, see Figure 1. Note that r is almost surely positive.

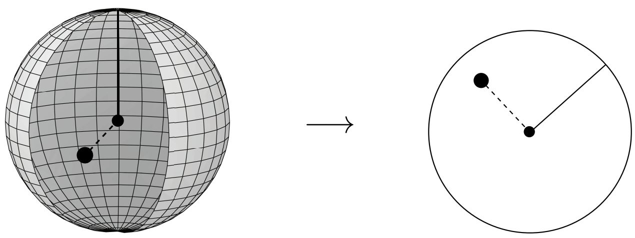

Figure 1: To the left: $RB_3 + O'$ with reference point (bold). The full line segment has length R and the broken line segment has length RQ. To the right: Section plane with profile. The full line segment and the broken line segment have length r and rq, respectively.

When $Q = 0$ the ball is centered at the origin and all hyperplanes give equivalent $(n-1)$-dimensional balls of radius R. We exclude this throughout, i.e. we assume that

$$\mathbb{P}(Q > 0) = 1. \tag{3.1}$$

This assumption can easily be relaxed, see Remark 3.2. The cumulative distribution function $F_{(r,q)}$ of (r, q) is given in the following theorem.

Theorem 3.1. *Let $RB_n + O'$ be a random ball in $\mathbb{R}^n$ containing O with $\|O'\| = RQ$, and let*

$$Z = Z(R,Q,x,y) = \frac{1}{Q^2}\max\left\{\frac{(R^2-x^2)^+}{R^2}, \frac{(Q^2-y^2)^+}{1-y^2}\right\} \tag{3.2}$$

for $x \in [0, \infty)$, $y \in [0, 1)$. If $L^n_{n-1[O]}$ is an IR hyperplane, independent of $RB_n + O'$, we have

$$F_{(r,q)}(x, y) = 1 - \frac{\sigma_{n-1}}{\sigma_n}\mathbb{E}\Big[B\Big(1 - \Big(1 - Z\Big)^+; \frac{1}{2}, \frac{n-1}{2}\Big)\Big], \tag{3.3}$$

for $x \geq 0$ and $0 \leq y < 1$. For $y = 1$, we obtain the marginal distribution function of r by

$$F_r(x) = F_{(r,q)}(x, 1) = 1 - \frac{\sigma_{n-1}}{\sigma_n}\mathbb{E}\Big[B\Big(1 - \Big(1 - \frac{1}{Q^2R^2}(R^2 - x^2)^+\Big)^+; \frac{1}{2}, \frac{n-1}{2}\Big)\Big], \tag{3.4}$$

where $x \geq 0$.

Proof. To avoid confusion we adopt the notation $\mathbb{E}_X$, $\mathbb{E}_{X,Y}$ for the expectation with respect to the random variable X and the pair of random variables (X, Y), respectively. Assume without loss of generality that $O' = RQe_n$. (Otherwise both $RB_n + O'$ and the section plane can be appropriately rotated. As the rotation is independent of the section plane, the rotated plane is still IR.) Let v be an isotropic vector on S^{n-1} representing the unit normal direction of $L^n_{n-1[O]}$. Applying Pythagoras' Theorem, we obtain

$$r = R\sqrt{1 - Q^2 \langle e_n, v\rangle^2} \quad \text{and} \quad q = Q\sqrt{\frac{1 - \langle e_n, v\rangle^2}{1 - Q^2 \langle e_n, v\rangle^2}}. \tag{3.5}$$

Using conditional expectation we have for $x \in [0, \infty)$ and $0 \leq y \leq 1$, that

$$\begin{aligned} F_{(r,q)}(x, y) &= \mathbb{E}_{R,Q}\mathbb{E}_v\Big[\mathbf{1}\Big\{R\sqrt{1 - Q^2 \langle e_n, v\rangle^2} \leq x,\ Q\sqrt{\frac{1 - \langle e_n, v\rangle^2}{1 - Q^2 \langle e_n, v\rangle^2}} \leq y\Big\}\ \Big|\ R, Q\Big] \\ &= \mathbb{E}_{R,Q}\frac{1}{\sigma_n}\int_{S^{n-1}} \mathbf{1}\Big\{R\sqrt{1 - Q^2 \langle e_n, v\rangle^2} \leq x,\ \frac{1 - \langle e_n, v\rangle^2}{1 - Q^2 \langle e_n, v\rangle^2} \leq \frac{y^2}{Q^2}\Big\}\mathrm{d}v^{n-1}. \end{aligned}$$

We use cylindrical coordinates (Müller, 1966, p.1), writing $v = te_n + \sqrt{1 - t^2}\omega$, with $\omega \in S^{n-1} \cap e_n^\perp$ and $t \in [-1, 1]$. Using $\langle e_n, v\rangle = t$ and $\mathcal{H}^{n-2}_{n-1}(S^{n-1} \cap e_n^\perp) = \sigma_{n-1}$, the cumulative distribution function becomes

$$F_{(r,q)}(x, y) = \mathbb{E}_{R,Q}\frac{\sigma_{n-1}}{\sigma_n}\int_{-1}^{1} \mathbf{1}\Big\{R\sqrt{1 - Q^2t^2} \leq x,\ \frac{1 - t^2}{1 - Q^2t^2} \leq \frac{y^2}{Q^2}\Big\}\left(1 - t^2\right)^{\frac{n-3}{2}} \mathrm{d}t.$$

The integrand is an even function of t. This and rearranging the indicator functions gives

$$\begin{aligned} F_{(r,q)}(x, y) = \frac{2\sigma_{n-1}}{\sigma_n}\mathbb{E}_{R,Q}\int_0^1 \Big(&\mathbf{1}\Big\{t \geq \frac{1}{QR}\sqrt{(R^2 - x^2)^+}, y = 1\Big\} \\ &+ \mathbf{1}\{t \geq \sqrt{Z}, y < 1\}\Big)\left(1 - t^2\right)^{\frac{n-3}{2}} \mathrm{d}t, \end{aligned}$$

where Z is given by (3.2). Using the substitution $s = t^2$ the cumulative distribution function of (r, q), under the assumption $0 \leq y < 1$, becomes

$$\begin{aligned} F_{(r,q)}(x, y) &= \frac{\sigma_{n-1}}{\sigma_n}\mathbb{E}_{R,Q}\int_{\min\{1,Z\}}^1 s^{\frac{1}{2}-1}(1 - s)^{\frac{n-1}{2}-1}\mathrm{d}s \\ &= 1 - \frac{\sigma_{n-1}}{\sigma_n}\mathbb{E}_{R,Q}\Big[B\Big(1 - (1 - Z)^+; \frac{1}{2}, \frac{n-1}{2}\Big)\Big]. \end{aligned}$$

Using similar calculations, we obtain (3.4) for the marginal distribution of r.

$\square$

Remark 3.2. If $\mathbb{P}(Q > 0) < 1$, the result of Theorem 3.1 and, similarly, results in the subsequent sections can be generalized by conditioning on the event $Q > 0$. When $Q = 0$, we have $r = R$, and hence

$$F_{(r,q)}(x, y) = \mathbb{P}(Q > 0)\Big(1 - \frac{\sigma_{n-1}}{\sigma_n}\mathbb{E}\Big[B\Big(1 - (1 - Z)^+; \frac{1}{2}, \frac{n-1}{2}\Big)\Big| Q > 0\Big]\Big) \\ + \mathbb{P}(R \le x, Q = 0),$$

when $x \ge 0,\ 0 \le y < 1$. A similar modification allows to generalize (3.4).

The distribution functions given by (3.3) and (3.4) simplify considerably when $n = 3$.

Corollary 3.3. *When $n = 3$,*

$$F_{(r,q)}(x, y) = \mathbb{E}\big(1 - \sqrt{Z}\big)^+, \qquad 0 \le x,\ 0 \le y < 1, \tag{3.6}$$

and

$$F_r(x) = \mathbb{E}\Big[\Big(1 - \frac{1}{RQ}\sqrt{(R^2 - x^2)^+}\Big)^+\Big], \qquad x \ge 0. \tag{3.7}$$

From (3.3) we immediately infer that the marginal distribution of q is given by

$$F_q(y) = 1 - \frac{\sigma_{n-1}}{\sigma_n}\mathbb{E}\Big[B\Big(1 - \Big(1 - \frac{(Q^2 - y^2)^+}{Q^2(1 - y^2)}\Big)^+; \frac{1}{2}, \frac{n-1}{2}\Big)\Big], \qquad 0 \le y < 1, \tag{3.8}$$

which does not depend on the distribution of R. This is an important fact, which we will use later on. It follows from (3.5) that one of the variables r and q determines the other whenever Q and R are given. This implies that a joint probability density function of (r, q) need not exist. However, it is elementary to show that the marginal probability density functions exist. Let $\phi(\cdot)$ be any smooth function having a compact support in $(0, \infty)$. Then $\mathbb{E}\phi(X) = \int_0^\infty \phi'(x)(1 - F_X(x))\mathrm{d}x$, where $X = r$, respectively q. Using Fubini and Tonelli arguments, integration by parts and Leibnitz's rule we obtain the following corollary.

Corollary 3.4. *Adopt the set-up in Theorem 3.1. The probability density functions of r and q exist. The function*

$$f_r(x) = \frac{2\sigma_{n-1}}{\sigma_n}\mathbb{E}\Big[\frac{\mathbf{1}\{R\sqrt{1 - Q^2} \le x < R\}x}{QR\sqrt{R^2 - x^2}}\Big(1 - \frac{R^2 - x^2}{Q^2R^2}\Big)^{(n-3)/2}\Big], \tag{3.9}$$

$x \ge 0$, is a density function of r and

$$f_q(y) = \frac{2\sigma_{n-1}}{\sigma_n}\mathbb{E}\Big[\mathbf{1}\{0 < y < Q\}\Big(\frac{y}{Q}\Big)^{n-2}\frac{(1 - Q^2)^{(n-1)/2}}{\sqrt{Q^2 - y^2}(1 - y^2)^{n/2}}\Big], \tag{3.10}$$

$0 \le y < 1$, a density function of q.

We remark that when R and Q are independent and $n = 3$, (3.9) simplifies

$$f_r(x) = x\mathbb{E}_R\Big[\frac{\mathbf{1}\{x < R\}}{R\sqrt{R^2 - x^2}}\Big(\mathbb{E}_Q\frac{\mathbf{1}\{R\sqrt{1-Q^2} \le x\}}{Q}\Big)\Big], \qquad x \ge 0. \tag{3.11}$$

As mentioned in (1.2) the radius of the section profile can be written as a multiple of the size-weighted radius R_w of the intersected ball in the classical Wicksell problem. A similar result holds in the local Wicksell problem but here the radii of the intersected balls are not size-weighted.

Proposition 3.5. *If the assumptions of Theorem 3.1 hold, then*

$$r = \Gamma R,$$

where the random variable Γ *has density*

$$f_\Gamma(z) = \frac{2\sigma_{n-1}}{\sigma_n}\mathbb{E}\Big[\mathbf{1}\{\sqrt{1-Q^2} \le z < 1\}\frac{z}{Q^{n-2}\sqrt{1-z^2}}(Q^2 - 1 + z^2)^{\frac{n-3}{2}}\Big], \tag{3.12}$$

$z \ge 0$. *If* R *and* Q *are independent, then* R *and* Γ *are independent.*

Proof. Let $h(x|R = R_0),\ x \ge 0$, be the conditional density of r given that a ball of radius R_0 is cut by the section plane. By (3.9) we have that

$$h(x|R = R_0) = \frac{2\sigma_{n-1}}{\sigma_n}\mathbb{E}\Big[\frac{\mathbf{1}\{R_0\sqrt{1-Q^2} \le x < R_0\}x}{QR_0\sqrt{R_0^2 - x^2}}\Big(1 - \frac{R_0^2 - x^2}{Q^2R_0^2}\Big)^{\frac{n-3}{2}}\Big|R = R_0\Big],$$

$x \ge 0$, is a version of this density. We note that h satisfies the scaling property

$$h(x|R = R_0) = \frac{1}{R_0}h\Big(\frac{x}{R_0}\Big|R = 1\Big).$$

Let Γ be the stochastic variable defined by $\Gamma = r/R$. Using conditional expectation, conditioning on $R = R_0$, and then the substitution $s = r/R_0$, we obtain

$$\mathbb{P}(\Gamma \le z) = \mathbb{E}\mathbb{P}(r \le zR_0|R = R_0) = \mathbb{E}[\int_0^z R_0h(sR_0|R = R_0)\mathrm{d}s],$$

for $z \ge 0$. Applying the scaling property and conditional expectation we see that a density of Γ is given by (3.12). When Q and R are independent, R and Γ are independent by construction. $\square$

The cumulative distribution function of the random variable Γ in Proposition 3.5 is immediately obtained using (3.4) with $R = 1$,

$$F_\Gamma(z) = 1 - \frac{\sigma_{n-1}}{\sigma_n}\mathbb{E}\Big[B\Big(1 - \Big(1 - \frac{(1-z^2)}{Q^2}\Big)^+; \frac{1}{2}, \frac{n-1}{2}\Big)\Big], \quad 0 \le z < 1. \tag{3.13}$$

Remark 3.6. As Γ in Proposition 3.5 only depends on (R,Q) through Q we may write $c_k(Q)=\mathbb{E}\Gamma^k$, $k=0,1,2,\ldots$, for the kth moment of Γ. These moments are given by

$$c_k(Q)=\frac{2\sigma_{n-1}}{\sigma_n}\mathbb{E}\Big[\frac{1}{Q^{n-2}}\int_{\sqrt{1-Q^2}}^{1}\frac{z^{k+1}}{\sqrt{1-z^2}}(Q^2-1+z^2)^{(n-3)/2}\mathrm{d}z\Big], \tag{3.14}$$

for $k=0,1,2,\ldots$. In particular for $n=3$, the substitution $x=z^2$, yields

$$c_k(Q)=\tfrac{1}{2}\mathbb{E}\Big[\frac{1}{Q}\Big(\frac{\sigma_{k+3}}{\sigma_{k+2}}-B\Big(1-Q^2,\frac{k+2}{2},\frac{1}{2}\Big)\Big)\Big]. \tag{3.15}$$

Denote the kth moment of R by M_k, and that of r by m_k. When R and Q are independent, Proposition 3.5 gives us the following moment relation

$$m_k=c_k(Q)M_k, \tag{3.16}$$

where $c_k(Q)$ is given by (3.14).

4 Uniqueness

The distribution F_q uniquely determines F_Q. This can be seen from (3.10), which can be rewritten as

$$f_q(y)=\frac{2\sigma_{n-1}y^{n-2}}{\sigma_n(1-y^2)^{n/2}}\mathbf{1}\{y>0\}\int_y^1\frac{(1-s^2)^{\frac{n-1}{2}}}{s^{n-2}\sqrt{s^2-y^2}}\mathrm{d}F_Q(s).$$

This is essentially an Abel transform of the positive measure $(1-s^2)^{\frac{n-1}{2}}/s^{n-2}\mathrm{d}F_Q(s)$. The Abel transform has a unique solution, see e.g. (Gorenflo and Vessella, 1991, Section 1.2.), and hence F_q uniquely determines F_Q. An explicit solution is given in Proposition 5.1 for $n=3$.

When the spatial parameters R and Q are stochastically independent, we also can show that F_r determines F_R uniquely.

Theorem 4.1. *Adopt the set-up in Theorem 3.1. If R and Q are independent, $F_{(r,q)}$ uniquely determines $F_{(R,Q)}$.*

Proof. From Proposition 3.5 we have that $r=\Gamma R$ and thus $\log r=\log\Gamma+\log R$. When R and Q are independent, Γ and R are independent and hence the characteristic functions obey

$$\varphi_{-\log r}(t)=\varphi_{-\log\Gamma}(t)\varphi_{-\log R}(t),\quad t\in\mathbb{R}. \tag{4.1}$$

We want to show that the characteristic function $\varphi_{-\log\Gamma}$ is an analytic function. We note that $F_{-\log\Gamma}(z)=1-F_\Gamma(e^{-z})$. Using the dominated convergence theorem, we get $\lim_{z\to 0}f_\Gamma(z)=0$. Hence, applying l'Hopital's rule, we obtain

$$\lim_{z\to\infty}\frac{1-F_{-\log\Gamma}(z)}{e^{-z}}=\lim_{z\to 0}\frac{F_\Gamma(z)}{z}=\lim_{z\to 0}f_\Gamma(z)=0,$$

that is $1 - F_{-\log\Gamma}(z) = o(e^{-z})$ as $z \to \infty$. As $F_{-\log\Gamma}(-z) = 0$ when $z \geq 0$, we have

$$1 - F_{-\log\Gamma}(z) + F_{-\log\Gamma}(-z) = o(e^{-z}).$$

Therefore $\varphi_{-\log\Gamma}$ is an analytic function (Lukacs, 1960, p.137). This implies that $\varphi_{-\log\Gamma}$ has only countably many zeros. Therefore, in view of (4.1), $\varphi_{-\log r}$ uniquely determines the continuous function $\varphi_{-\log R}$. It then follows from the Fourier uniqueness theorem that F_r determines F_R uniquely. As F_q uniquely determines F_Q, this finishes the proof. $\square$

It is of interest to ask if the same holds without the independence assumption: Does the joint distribution $F_{(r,q)}$ uniquely determine $F_{(R,Q)}$ when the independence assumption is dropped? This is an open question, but the following example shows that the answer is negative if only the marginals F_q and F_r are given.

Example 4.2. Assume $n = 3$ and let the joint density of (R, Q) be given by

$$f_{(R,Q)}(t, s) = 3s\,\mathbf{1}\{0 < t < s < 1\}. \tag{4.2}$$

Then the two marginals are

$$f_Q(s) = 3s^2\,\mathbf{1}\{0 < s < 1\}, \quad f_R(t) = \tfrac{3}{2}(1 - t^2)\,\mathbf{1}\{0 < t < 1\}.$$

Inserting (4.2) into (3.9) and using elementary but tedious calculations, we obtain that f_r is given by

$$\begin{aligned} f_r(x) &= 3\,\mathbf{1}\{x < 1\} \tan^{-1}\left(\sqrt{1 - x^2}/x\right) \\ &- \mathbf{1}\{1/2 < x < 1\} 3x(\log(1 + \sqrt{1 - x^2}) - \log x) \\ &- \mathbf{1}\{x < 1/2\} 3x \left[\log\left(\sqrt{(1 - \sqrt{1 - 4x^2})/2} + \sqrt{(1 - \sqrt{1 - 4x^2})/2 - x^2}\right)\right. \\ &+ \log(1 + \sqrt{1 - x^2}) - \log\left(\sqrt{(1 + \sqrt{1 - 4x^2})/2} + \sqrt{(1 + \sqrt{1 - 4x^2})/2 - x^2}\right) \\ &\left. - \log x - ((1 + \sqrt{1 - 4x^2})/2)^{-1/2} + ((1 - \sqrt{1 - 4x^2})/2)^{-1/2}\right]. \end{aligned} \tag{4.3}$$

We now show that there is another pair (R', Q') of size and shape variables which are stochastically independent, but lead to the same section marginals F_q and F_r as does the pair (R, Q) with density (4.2). As $F_{Q'}$ is uniquely determined by F_q, we necessarily have $F_{Q'} = F_Q$. If we assume that R' has a density $f_{R'}$, this and (3.9) imply that this density must satisfy

$$f_r(x) = \tfrac{3}{2} x^3 \int_x^\infty \frac{f_{R'}(s)}{s^3 \sqrt{s^2 - x^2}} \mathrm{d}s. \tag{4.4}$$

Define

$$h(x) = \frac{2}{3x^3} f_r(x) \quad \text{and} \quad g(s) = \frac{f_{R'}(s)}{s^3}.$$

Equation (4.4) is an Abel transform of $g(s)$ with solution given by

$$f_{R'}(s) = -\frac{2s^4}{\pi} \int_s^\infty \frac{h'(x)}{\sqrt{x^2 - s^2}} \mathrm{d}x, \tag{4.5}$$

see e.g. (Gorenflo and Vessella, 1991, p.35). Hence, we would define $f_{R'}$ by (4.5) if we can guarantee that this function is a density. Inserting f_r given by (4.3) it can be shown that the function $h'(x)$ is negative for all x. Furthermore, using Tonelli's theorem, we obtain

$$\begin{aligned}\int_0^\infty f_{R'}(s)\mathrm{d}s &= -\frac{2}{\pi}\int_0^\infty h'(x)\int_0^x \frac{s^4}{\sqrt{x^2-s^2}}\mathrm{d}s\mathrm{d}x\\ &= -\frac{3}{8}\int_0^\infty h'(x)x^4\mathrm{d}x.\end{aligned}$$

Using partial integration and inserting for h, we find that $f_{R'}$ integrates to one. Hence $f_{R'}$ is a density. We thus have shown that populations of balls with size-shape parameters (R, Q) and (R', Q'), respectively, lead to the same size-shape distributions of their profiles, although $F_{(R,Q)} \neq F_{(R',Q')}$.

5 The unfolding problem

We mentioned in the introduction that there exists a reproducing distribution for the radii in the classical Wicksell problem (the Rayleigh distribution). In the local Wicksell problem a reproducing radii distribution does typically not exist. When the balls are not a.s. centered at O, the radii of the section profiles are smaller than the radii of the respective balls with positive probability. This implies that there does not exist a reproducing distribution in the local Wicksell problem under the assumption $\mathbb{P}(Q > 0) > 0$.

In the previous section we saw that under the assumption that R and Q are independent, $F_{(r,q)}$ uniquely determines $F_{(R,Q)}$. In this section we will present analytical unfolding formulae and moment relations. In order to avoid technicalities, we restrict attention to the three-dimensional case, which is most relevant for practical applications. The following proposition gives F_Q in terms of F_q.

Proposition 5.1. *If the assumptions of Theorem 3.1 hold, and $n = 3$, then*

$$1 - F_Q(y) = -\frac{2y^3}{\pi}\Big(\int_y^1 \frac{(s^2-2)(1-F_q(s))}{s^3\sqrt{1-s^2}\sqrt{s^2-y^2}}ds - \int_y^1 \frac{\sqrt{1-s^2}}{s^2\sqrt{s^2-y^2}}dF_q(s)\Big), \quad (5.1)$$

$y \in [0, 1)$.

Proof. Let $0 \leq y < 1$. From (3.8) we know that the distributions of Q and q are connected by

$$F_q(y) = 1 - \mathbb{E}\Big[\frac{\sqrt{(Q^2-y^2)^+}}{Q\sqrt{1-y^2}}\Big].$$

Using integration by parts and rearranging, we obtain

$$1 - F_q(y) = \frac{y^2}{\sqrt{1-y^2}}\int_y^1 \frac{1}{s^2\sqrt{s^2-y^2}}(1-F_Q(s))\mathrm{d}s. \quad (5.2)$$

Define

$$h(y) = \frac{\sqrt{1-y^2}}{y^2}(1-F_q(y)) \quad \text{and} \quad g(s) = \frac{1-F_Q(s)}{s^2}. \quad (5.3)$$

Equation (5.2) is an Abel transform of $g(s)$ with solution given by

$$g(y) = -\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}y}\int_y^1 \frac{sh(s)}{\sqrt{s^2-y^2}}\mathrm{d}s = -\frac{2y}{\pi}\int_y^1 \frac{h'(s)}{\sqrt{s^2-y^2}}\mathrm{d}s; \tag{5.4}$$

see for instance (Gorenflo and Vessella, 1991, p.35). Inserting for g and the derivative of h, (5.1) is obtained. □

Using similar arguments as in the proof of Proposition 5.1, we obtain that when Q has a density f_Q, it is given by

$$f_Q(s) = \frac{8s^2}{\pi}\frac{\mathrm{d}^2}{\mathrm{d}(s^2)^2}\int_s^1 \frac{t\sqrt{1-t^2}}{\sqrt{t^2-s^2}}(1-F_q(t))\mathrm{d}t, \qquad 0 \le s < 1. \tag{5.5}$$

We can use (5.4) to write the moments of Γ in Proposition 3.5 as functions of q only.

Corollary 5.2. *Adopt the set-up in Theorem 3.1. For $n = 3$ the moments of the random variable Γ in Proposition 3.5, can be written as*

$$\mathbb{E}\Gamma^k = 1 - \frac{k}{\pi}\mathbb{E}[\tilde{\Gamma}(q)], \qquad k = 0, 1, 2, \ldots, \tag{5.6}$$

where

$$\tilde{\Gamma}(y) = \int_0^y \frac{\sqrt{1-s^2}}{s}\int_0^{s^2} \frac{\sqrt{t}(1-t)^{\frac{k-2}{2}}}{\sqrt{s^2-t}}dt\,ds. \tag{5.7}$$

Proof. Using (3.13) with $n = 3$ we note that the distribution function of Γ can be written as

$$F_\Gamma(z) = \mathbb{E}\Big[\Big(1 - \frac{1}{Q}\sqrt{1-z^2}\Big)^+\Big] = \sqrt{1-z^2}\int_{\sqrt{1-z^2}}^1 g(s)\mathrm{d}s, \quad 0 \le z < 1,$$

where $g(s)$ is given by (5.3). Using the first equality in (5.4) and substituting $t = 1-z^2$, the moments of Γ are given by

$$\mathbb{E}\Gamma^k = \int_0^1 kz^{k-1}(1-F_\Gamma(z))\mathrm{d}z = 1 - \frac{k}{\pi}\int_0^1 (1-t)^{\frac{k-2}{2}}\sqrt{t}\int_{\sqrt{t}}^1 \frac{yh(y)}{\sqrt{y^2-t}}\mathrm{d}y\mathrm{d}t.$$

Inserting h and using Tonelli's theorem we arrive at

$$\begin{aligned}\mathbb{E}\Gamma^k &= 1 - \frac{k}{\pi}\int_0^1 \frac{\sqrt{1-y^2}}{y}(1-F_q(y))\int_0^{y^2} \frac{\sqrt{t}(1-t)^{\frac{k-2}{2}}}{\sqrt{y^2-t}}\mathrm{d}t\mathrm{d}y \\ &= 1 - \frac{k}{\pi}\mathbb{E}[\tilde{\Gamma}(q)],\end{aligned}$$

where $\tilde{\Gamma}$ is given by (5.7). □

If $k = 2m$, $m \geq 1$, equation (5.6) simplifies to

$$\mathbb{E}\Gamma^{2m} = 1 - \frac{m}{\pi} \sum_{j=0}^{m-1} \binom{m-1}{j} (-1)^j \frac{\sigma_{2j+4}}{\sigma_{2j+3}} \mathbb{E}\big[B\big(q^2; j+1, \tfrac{3}{2}\big)\big],$$

and in particular

$$\mathbb{E}\Gamma^2 = \tfrac{1}{3}(2 + \mathbb{E}[(1-q^2)^{3/2}]).$$

Thus the average surface area of balls in $\mathbb{R}^3$ can be estimated ratio unbiasedly by

$$\frac{\frac{12\pi}{N}\sum_{i=1}^N r_i^2}{2 + \frac{1}{N}\sum_{i=1}^N (1-q_i^2)^{3/2}},$$

where $(r_1, q_1), \ldots, (r_N, q_N)$ are N independent observations of profile parameters.

When Q has a density, similar arguments as applied in the proof of (5.5) can be used to show that the moments of Γ can be written as

$$\mathbb{E}\Gamma^k = \tfrac{1}{2\pi}\mathbb{E}[\tilde{\Gamma}_1(q^2)] + \tfrac{1}{\pi}\mathbb{E}[\tilde{\Gamma}_2(q^2)], \tag{5.8}$$

where

$$\tilde{\Gamma}_1(y) = \int_0^y \frac{(1-t)^{k/2}}{\sqrt{t}} B\Big(\frac{y-t}{1-t}, \frac{1}{2}, \frac{1}{2}\Big)\mathrm{d}t, \quad \tilde{\Gamma}_2(y) = \sqrt{1-y}\int_0^y \frac{(1-t)^{k/2}}{\sqrt{t}\sqrt{y-t}}\mathrm{d}t.$$

By (3.16) and Corollary 5.2, the moments of R can be estimated from the section profiles when R and Q are independent. A number of stochastic simulation studies were carried out, varying F_Q, F_R and the number of observations. The function `quad` in the language and interactive environment `Matlab` was used to determine $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ numerically. We used (5.8) instead of the more general expression given in Corollary 5.2 as it is more straight-forward to implement. Then M_k was estimated by dividing the crude Monte Carlo (CMC) estimate of m_k by the CMC estimate obtained for $\mathbb{E}\Gamma^k$ using (5.8). The simulations suggest that the estimation procedure is quite stable for moments up to 7th order. The difference between the coefficient of error of this estimator and of M_k estimated by CMC, using the simulated particle radii, is typically less than 10% for $k = 1, \ldots, 7$. This seems to be true irrespective of the choice of F_Q and even after adding moderate measurement errors to r and q.

6 Reconstruction

In this section we will discuss the estimation of F_R and F_Q from data. We assume throughout this section that R and Q are stochastically independent, and that $n = 3$. As mentioned in the introduction, there exist various methods for numerically solving Wicksell's classical problem but none of these seems to be superior to all the others. In Blödner et al. (1984) six distribution-free methods are compared using several error criteria as well as studying their numerical stability and sensitivity to underlying distributions. Considering all criteria, the Scheil-Schwartz-Saltykov method (S^3M) from Saltykov (1974) is favoured, in particular when the underlying distribution is smooth. The method can be classified as a finite difference method, which are relatively easy to implement. We therefore chose to implement a variation of S^{3}M. The method and its advantages and disadvantages are discussed in Blödner et al. (1984). Although this method is rather crude, we obtain satisfactory results. We have also studied the product integration method explained in Anderssen and Jakeman (1975b). The method has been claimed to yield accurate results for Wicksell's classical problem, but our reconstruction of F_Q based on Proposition 5.1 is not satisfactory. The reconstruction $\hat{F}_Q$ is not monotonic and fluctuates far too much around F_Q to be of any use. We have also implemented kernel density estimators for obtaining $\hat{F}_Q$ but without obtaining stable results.

In the following we describe the implementation of S^{3}M in our setup. We start using S^{3}M to obtain a discrete approximation of F_Q based on the Abel-type relation (5.2). By discretizing F_Q, the integral becomes a finite sum. This produces a system of linear equations that can be solved. After having obtained an estimate $\hat{F}_Q$ we can apply S^{3}M again, this time discretizing F_R to solve (3.7) numerically (where $\hat{F}_Q$ is substituted for the unknown F_Q). We used `Matlab` for all simulations and for generating the figures.

Assume that we observe N pairs $(r_1, q_1), \dots, (r_N, q_N)$ of size and shape variables in independent sections of (independent) particles. Assume further that all observed r's, are less than or equal to a constant c. Divide the intervals $(0, 1]$ and $(0, c]$ into classes of constant width. Let k_1 denote the number of classes for q, k_2 the number of classes for Q, k_3 for r and k_4 for R. Different to Blödner et al. (1984) we allow k_1, k_3, to be greater than k_2, k_4, respectively. Define $\Delta_i = 1/k_i$, $i = 1, 2$, $\Delta_i = c/k_i$, $i = 3, 4$. Let $n_i = \mathbb{P}(q \in ((i-1)\Delta_1, i\Delta_1])$ be the probability that q is in class i, $i = 1, \dots k_1$, and $N_j = \mathbb{P}(Q \in ((j-1)\Delta_2, j\Delta_2])$ the probability that Q is in class j, $j = 1, \dots k_2$.

The S^{3}M method approximates $F_Q(u)$ by the step function

$$u \longmapsto \sum_{j:\Delta_2 j \leq u} N_j.$$

Note that we use the standard definition of a cumulative distribution function and hence the notation is slightly different from Blödner et al. (1984) where left continuous distribution functions are considered. Inserting the approximation for F_Q in (5.2) and simplifying, we obtain

$$1 - F_q(i\Delta_1) \approx \frac{1}{\sqrt{1 - i^2\Delta_1^2}\Delta_2} \sum_{j=1}^{k_2} N_j \, \mathbf{1}\{i\Delta_1 < j\Delta_2\} \frac{\sqrt{j^2\Delta_2^2 - i^2\Delta_1^2}}{j}.$$

Hence n_i can be approximated by

$$n_i = (1 - F_q((i-1)\Delta_1)) - (1 - F_q(i\Delta_1)) \approx \sum_{j=1}^{k_2} b_{ij} N_j, \tag{6.1}$$

for $i = 1, \dots, k_1$, where

$$b_{ij} = \frac{1}{j\Delta_2}\Big(\mathbf{1}\{(i-1)\Delta_1 < j\Delta_2\} \frac{\sqrt{j^2\Delta_2^2 - (i-1)^2\Delta_1^2}}{\sqrt{1-(i-1)^2\Delta_1^2}} - \mathbf{1}\{i\Delta_1 < j\Delta_2\} \frac{\sqrt{j^2\Delta_2^2 - i^2\Delta_1^2}}{\sqrt{1-i^2\Delta_1^2}}\Big),$$

for $j = 1, \dots, k_2$, $i = 1, \dots, k_1 - 1$, and

$$b_{k_1 j} = \frac{1}{j\Delta_2} \mathbf{1}\{(k_1-1)\Delta_1 < j\Delta_2\} \frac{\sqrt{j^2\Delta_2^2 - (k_1-1)^2\Delta_1^2}}{\sqrt{1-(k_1-1)^2\Delta_1^2}}, \qquad j = 1, \dots, k_2.$$

The relative frequencies $\hat{n}_i = \frac{1}{N}\sum_{j=1}^N \mathbf{1}\{q_j \in ((i-1)\Delta_1, i\Delta_1]\}$ are approximations of the left hand side of (6.1). Using these and solving the corresponding linear system yields therefore approximations $\hat{N}_j$ of the unknown probabilities N_j, $j = 1, \dots, k_2$. The function

$$\hat{F}_Q(u) = \sum_{j:\Delta_2 j \le u} \hat{N}_j \tag{6.2}$$

is then the estimator for F_Q.

In a second step (3.7) is inverted using S^{3}M. We note that (3.7) can be rewritten as

$$F_r(x) = (1 - F_R(x)) + \mathbb{E}\Big[\mathbf{1}\{x < R\} \frac{\sqrt{R^2 - x^2}}{R} \int_{\frac{1}{R}\sqrt{R^2-x^2}}^1 \frac{1}{s^2}(1 - F_Q(s))\mathrm{d}s \Big],$$

for $x \in [0, \infty)$. Approximating F_Q by $\hat{F}_Q$ given by (6.2) we find

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}\{x < R\} \frac{\sqrt{R^2 - x^2}}{R} \int_{\frac{1}{R}\sqrt{R^2-x^2}}^1 \frac{1}{s^2}(1 - \hat{F}_Q(s))\mathrm{d}s \Big] \\
&= 1 - \sum_{j=1}^{k_2-1} \hat{N}_j \frac{1}{\Delta_2 j} \int_x^{x/\sqrt{1-\Delta_2^2 j^2}} \frac{x^2}{t^2\sqrt{t^2-x^2}}(1 - F_R(t))\mathrm{d}t \\
&\quad - F_R(x) - \hat{N}_{k_2} \frac{1}{\Delta_2 k_2} \int_x^\infty \frac{x^2}{t^2\sqrt{t^2-x^2}}(1 - F_R(t))\mathrm{d}t.
\end{aligned}$$

Hence

$$\begin{aligned}
F_r(x) \approx \frac{1}{3}\Big(2 &- \sum_{j=1}^{k_2-1} \hat{N}_j \frac{1}{\Delta_2 j} \int_x^{x/\sqrt{1-\Delta_2^2 j^2}} \frac{x^2}{t^2\sqrt{t^2-x^2}}(1 - F_R(t))\mathrm{d}t \\
&- \hat{N}_{k_2} \frac{1}{\Delta_2 k_2} \int_x^\infty \frac{x^2}{t^2\sqrt{t^2-x^2}}(1 - F_R(t))\mathrm{d}t\Big).
\end{aligned} \tag{6.3}$$

Now, in analogy to the treatment of F_Q, we use the S^{3}M method to approximate F_R: Let $m_a = \mathbb{P}(r \in ((a-1)\Delta_3, a\Delta_3])$ be the probability that r is in class a, $a = 1, \dots, k_3$, and $M_b = \mathbb{P}(R \in ((b-1)\Delta_4, b\Delta_4])$ the probability that R is in class b, $b = 1, \dots, k_4$. The S^{3}M method approximates $F_R(u)$ by

$$u \longmapsto \sum_{b:\Delta_4 b \le u} M_b.$$

Using this in (6.3) we obtain, again using elementary calculations, that

$$
\begin{aligned}
&F_r(a\Delta_3)\\
&\approx -\frac{1}{3}\Bigg[\sum_{j=1}^{k_2-1}\hat{N}_j\frac{1}{\Delta_2 j}\sum_{b=1}^{k_4}M_b\,\mathbf{1}\{\Delta_4 b\geq a\Delta_3\}\sqrt{1-\left(\min\left\{\frac{\Delta_4^2 b^2}{a^2\Delta_3^2},\frac{1}{1-\Delta_2^2 j^2}\right\}\right)^{-1}}\\
&\qquad\qquad +\hat{N}_{k_2}\frac{1}{\Delta_2 k_2}\sum_{b=1}^{k_4}M_b\,\mathbf{1}\{\Delta_4 b\geq a\Delta_3\}\sqrt{1-\frac{a^2\Delta_3^2}{\Delta_4^2 b^2}}-2\Bigg].
\end{aligned}
$$

We are thus lead to the following linear system involving the relative frequencies $\hat{m}_a=\frac{1}{N}\sum_{j=1}^{N}\mathbf{1}\{r_j\in((a-1)\Delta_3,a\Delta_3]\}$

$$
\hat{m}_a=\hat{F}_r(a\Delta_3)-\hat{F}_r((a-1)\Delta_3)=\sum_{b=1}^{k_4}c_{ab}\hat{M}_b, \tag{6.4}
$$

for $a=1,\dots,k_3$, where

$$
\begin{aligned}
c_{ab}&=\frac{1}{3}\sum_{j=1}^{k_2-1}\hat{N}_j\frac{1}{\Delta_2 j}\\
&\quad\times\Bigg[\mathbf{1}\{\Delta_4 b\geq(a-1)\Delta_3\}\sqrt{1-\left(\min\left\{\frac{\Delta_4^2 b^2}{(a-1)^2\Delta_3^2},\frac{1}{1-\Delta_2^2 j^2}\right\}\right)^{-1}}\\
&\qquad-\mathbf{1}\{\Delta_4 b\geq a\Delta_3\}\sqrt{1-\left(\min\left\{\frac{\Delta_4^2 b^2}{a^2\Delta_3^2},\frac{1}{1-\Delta_2^2 j^2}\right\}\right)^{-1}}\Bigg]\\
&\quad+\frac{\hat{N}_{k_2}}{3\Delta_2 k_2}\Bigg[\mathbf{1}\{\Delta_4 b\geq(a-1)\Delta_3\}\sqrt{1-\frac{(a-1)^2\Delta_3^2}{\Delta_4^2 b^2}}\\
&\qquad\qquad-\mathbf{1}\{\Delta_4 b\geq a\Delta_3\}\sqrt{1-\frac{a^2\Delta_3^2}{\Delta_4^2 b^2}}\Bigg]
\end{aligned}
$$

for $a=1,\dots,k_3,\ b=1,\dots,k_4$. The linear system of equations can be solved for the unknown $\hat{M}_b$ approximating M_b, $b=1,\dots,k_4$.

In our simulations, we used N independent realizations of (r_i,q_i) to estimate $\hat{n}_i$, $i=1,\dots,k_1$ and $\hat{m}_a$, $a=1,\dots,k_3$. Then we solved (6.1) using constrained minimum least squares, `lsqlin` in `Matlab.` To ensure that the estimated distribution functions are non-decreasing, we required that $\hat{N}_j\geq 0$, $j=1,\dots,k_2$, $\hat{M}_b\geq 0$, $b=1,\dots,k_4$. Non-negativity constraints have also been suggested in Taylor (1983) for the classical S^{3}M. The distribution function F_Q is then estimated by (6.2). Using the estimate $\hat{F}_Q$, (6.4) can be solved for $\hat{M}_b$, $b=1,\dots,k_4$, in exactly the same way and F_R is then estimated by $\hat{F}_R(u)=\sum_{b:\Delta_4 b\leq u}\hat{M}_b$.

The simulations in Blödner et al. (1984) show that classes with overestimation of the distribution function are usually close to classes with underestimation and the authors refer to this phenomenon quite intuitively as *waves*. In order to decrease the occurrence of waves, we allowed k_1, k_3, to be greater than k_2, k_4, respectively. This does though not seem to be of great importance in our setting. We ran two types of simulations, one with the number of classes equal and another one with the number of classes unequal. Then the Kullback Leibler divergence between the true probability distribution and each of the estimated ones was calculated using `KLDiv` in `Matlab`.The difference was negligible (even after adding independent measurement errors to r_j and q_j). Therefore we chose the number of classes equal in the figures presented below.

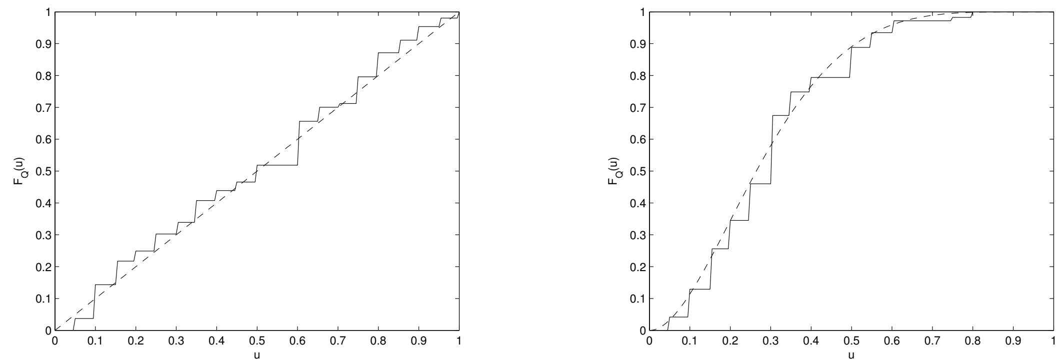

Figure 2: The step function is $\hat{F}_Q(u)$ and the dashed curve is $F_Q(u)$ when $k_1 = k_2 = 20$, $N = 100$. To the left $Q \sim \text{unf}(0, 1)$ whereas on the right $Q \sim \text{Beta}(2, 5)$.

Figure 2 shows the comparison between $\hat{F}_Q$ and F_Q. The values $k_1 = k_2 = 20$, $N = 100$ were chosen. To the left, Q is uniformly distributed on $[0, 1]$, and to the right, Q follows the Beta distribution with parameters 2 and 5. Figure 3 shows the comparison between $\hat{F}_R$ and F_R. The setup corresponds to the ones in Figure 2 with $c = \max\{r_1, \ldots, r_N\}$ and $k_3 = k_4 = 20$. To the left, R is exponentially distributed with mean one, and to the right R is uniformly distributed on $[0, 10]$.

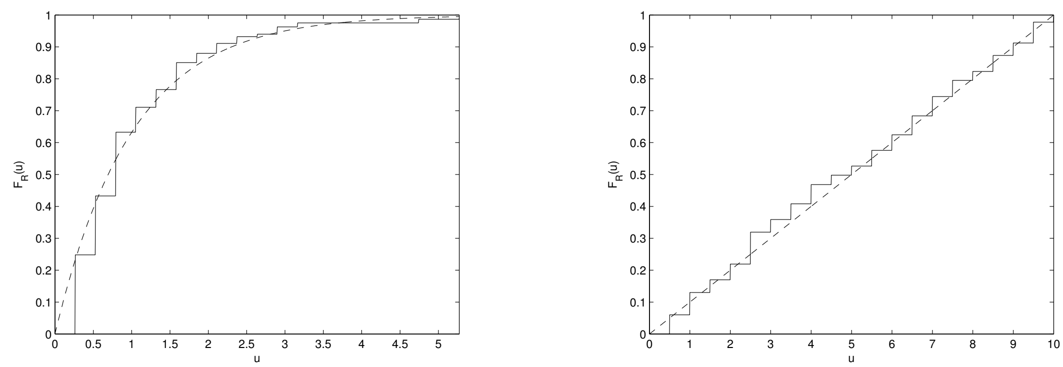

Figure 3: The stepwise function is $\hat{F}_R(u)$ and the dashed curve is $F_R(u)$ when $c = \max\{r_1, \ldots, r_N\}$, $N = 100$, $k_1 = k_2 = k_3 = k_4 = 20$. One the left side $Q \sim \text{unf}(0, 1)$ and $R \sim \text{e}(1)$ whereas on the right $Q \sim \text{Beta}(2, 5)$ and $R \sim \text{unf}(0, 10)$.

These results indicate that the S^{3}M method is satisfactory for reconstructing F_R and F_Q from simulated data. When the profile variables are measured with random errors a similar recommendation as in Blödner et al. (1984) applies: the sample size should be large (more than 400 profile sections) but the number of classes small (around 7).

7 Examples

For illustration we discuss some special cases and variants of the above general theory. We restrict attention to the three-dimensional case. Recall that we considered Q to be a variable describing the 'shape' of the random ball under consideration. We first show that the formulae simplify if Q is (almost surely) the same for all particles, meaning that all reference points have the same relative distance from their respective ball centers.

Example 7.1. Assume that $Q = Q_0$ a.s., $Q_0 > 0$. Then the marginal distribution function of q given by (5.2) becomes

$$F_q(y) = 1 - \mathbf{1}\{y < Q_0\} \frac{\sqrt{Q_0^2 - y^2}}{Q_0 \sqrt{1 - y^2}}.$$

Furthermore, the moments of Γ in (3.15) simplify to

$$c_k(Q) = \frac{1}{2Q_0} \Big(\frac{\sigma_{k+3}}{\sigma_{k+2}} - B\Big((1 - Q_0^2), \frac{k+2}{2}, \frac{1}{2} \Big) \Big).$$

For $k = 1, 2$, we obtain

$$c_1(Q) = \frac{\pi}{4Q_0} - \frac{\sin^{-1}(\sqrt{1 - Q_0^2})}{2Q_0} + \frac{\sqrt{1 - Q_0^2}}{2}, \quad c_2(Q) = 1 - \frac{Q_0^2}{3}.$$

The formulae simplify in particular if the reference point lies on the boundary of the object, that is, when $Q_0 = 1$.

Example 7.2. Let the reference point be located on the boundary of the ball $RB_3 + O'$. Then $Q = 1$ a.s. and we immediately obtain from Example 7.1 that $q = 1$ a.s. and $c_k(Q) = \sigma_{k+3}/(2\sigma_{k+2})$. Note that this constant is the same as c_{k+1} given by (1.4), which can be explained by the fact that the section plane is an IUR plane hitting the ball, as shown in Example 7.4. The moments of Γ can in fact be calculated explicitly in n-dimensional space and (3.16) becomes

$$m_k = \frac{\sigma_{n-1}\sigma_{k+n}}{\sigma_n \sigma_{k+n-1}} M_k.$$

Furthermore (3.11) becomes an Abel transform of the positive measure $\mathbb{P}_R(\mathrm{d}t)/t$. Hence, when R has a density function f_R, it is given by

$$\begin{aligned} f_R(t) &= -\frac{2t^2}{\pi} \int_t^\infty \frac{1}{\sqrt{x^2 - t^2}} \frac{\mathrm{d}}{\mathrm{d}x} \Big(\frac{f_r(x)}{x} \Big) \mathrm{d}x \\ &= -\frac{2t}{\pi} \frac{\mathrm{d}}{\mathrm{d}t} \int_t^\infty \frac{f_r(x)}{\sqrt{x^2 - t^2}} \mathrm{d}x; \end{aligned}$$

see for instance (Gorenflo and Vessella, 1991, p.35).

In practice a particle may have an easily identifiable kernel that cannot be treated as a mathematical point. Thus one has to work with a reference set of positive volume. In general, our methods do not apply in this case, but whenever the reference set is (approximately) ball-shaped, concentric with the whole particle, and has a radius proportional to the particle size, our method can be used. We suggest in the following examples two possible schemes for this situation.

Example 7.3. Begin by assuming that the particle $RB_3 + O'$ contains a reference set $Q_0 RB_3 + O'$, $Q_0 \in (0, 1]$. The first sampling design we suggest is the choice of an isotropic plane through a uniformly chosen boundary point of the reference set. Let $L^3_{2[O]}$ be an IR plane and z a uniformly distributed point on the boundary of the reference set, chosen independently of $L^3_{2[O]}$. Define $L = L^3_{2[O]} + z$ and adopt L as the sectioning plane, that is (r, q) refers to the parameters of the disk $(RB_3 + O') \cap L$. By construction (r, q) can thus be interpreted as section variables from a local IR plane through the reference point z, which has relative distance Q_0 from the ball's center. Hence, we can use the local Wicksell theory directly with $Q = Q_0$. The simplifications in Example 7.1 apply.

Example 7.4. As in Example 7.3 assume that the particle $RB_3 + O'$ contains a reference set $Q_0 RB_3 + O'$, $Q_0 \in (0, 1]$. In contrast to Example 7.3 we now use an IUR section plane L^3_2 hitting the reference set. By definition, the distribution of L^3_2 is

$$\mathbb{P}_{L^3_2}(A) = \frac{1}{2\sigma_3 Q_0} \int_{S^2} \int_{-Q_0}^{Q_0} \mathbf{1}_A(ru + u^\perp) \mathrm{d}r \mathrm{d}u^2, \qquad A \in \mathcal{B}(\mathcal{L}^3_2).$$

Using cylindrical coordinates, we have, equivalently

$$\mathbb{P}_{L^3_2}(A) = \frac{1}{\sigma_3^2} \int_{S^2} \int_{S^2} \mathbf{1}_A(Q_0 v + u^\perp) \mathrm{d}u^2 \mathrm{d}v^2,$$

so L^3_2 is an isotropic plane through the (independent) point $z = Q_0 v$, which is uniform on the boundary of the reference set. Concluding, we see that L^3_2 has the same distribution as L in Example 7.3. Hence, the two designs lead to the same sample distribution, and, again, Wicksell's local theory with $Q = Q_0$ applies.

The last two (coinciding) sampling schemes can in particular be applied when the reference set is taken to be the whole ball. This is equivalent to choosing $Q_0 = 1$ and the formulae in Example 7.2 can be applied in this case.

8 Stereology of extremes

In some practical applications, for instance when studying damage of materials Murakami and Beretta (1999), the distribution of the maximal size parameter is of more interest than the whole distribution. When extremal parameters are studied based on lower dimensional sections we speak of stereology of extremes. We will here discuss stereology of extremes in the context of the local Wicksell problem.

We assume that R and Q are independent and that $n = 3$. Given independent observations $(r_1, q_1), \ldots, (r_N, q_N)$ we are interested in the distribution of the extremal particle radius. Therefore relations between the domains of attraction of the distributions of the size parameters are of interest. These are given in the following theorem.

Proposition 8.1. *Let $n = 3$, $\gamma > 0$, and assume that R and Q are stochastically independent and have probability densities. The following statements hold*

- *if $F_R \in \mathcal{D}(L_{1,\gamma})$, then $F_r \in \mathcal{D}(L_{1,\gamma})$,*
- *if $F_R \in \mathcal{D}(L_{2,\gamma})$, then $F_r \in \mathcal{D}(L_{2,\gamma+1/2})$,*
- *if $F_R \in \mathcal{D}(L_3)$, then $F_r \in \mathcal{D}(L_3)$.*

Proof. Writing out (3.7) explicitly gives

$$1 - F_r(x) = \int_x^\infty \Big(\frac{\sqrt{t^2-x^2}}{t}\int_{\frac{1}{t}\sqrt{t^2-x^2}}^1 \frac{f_Q(s)}{s}\mathrm{d}s + F_Q\Big(\frac{1}{t}\sqrt{t^2-x^2}\Big)\Big) f_R(t)\mathrm{d}t.$$

Applying integration by parts on the outer integral, we find that

$$1 - F_r(x) = \int_x^\infty \frac{x^2}{t^2\sqrt{t^2-x^2}}(1-F_R(t))\int_{\frac{1}{t}\sqrt{t^2-x^2}}^1 \frac{f_Q(s)}{s}\mathrm{d}s\mathrm{d}t, \qquad x > 0. \tag{8.1}$$

We see from (3.5) that $\omega_{F_R} = \omega_{F_r}$ and we will call this common value ω. Let $y > 0$ and assume first that $F_R \in \mathcal{D}(L_{1,\gamma})$ and this implies $\omega = \infty$. Using (8.1) and then the substitution $t = yz$ in the numerator, we obtain after some simplification

$$\lim_{x\to\infty} \frac{1-F_r(yx)}{1-F_r(x)} = \lim_{x\to\infty} \frac{\int_x^\infty \frac{x^2}{t^2\sqrt{t^2-x^2}}(1-F_R(yt))\int_{\frac{1}{t}\sqrt{t^2-x^2}}^1 \frac{f_Q(s)}{s}\mathrm{d}s\mathrm{d}t}{\int_x^\infty \frac{x^2}{t^2\sqrt{t^2-x^2}}(1-F_R(t))\int_{\frac{1}{t}\sqrt{t^2-x^2}}^1 \frac{f_Q(s)}{s}\mathrm{d}s\mathrm{d}t}.$$

We note that

$$\lim_{\substack{(x,t)\to(\infty,\infty)\\ x\le t<\infty}} \frac{\frac{1}{t^2\sqrt{t^2-x^2}}(1-F_R(yt))\int_{\frac{1}{t}\sqrt{t^2-x^2}}^1 \frac{1}{s}f_Q(s)\mathrm{d}s}{\frac{1}{t^2\sqrt{t^2-x^2}}(1-F_R(t))\int_{\frac{1}{t}\sqrt{t^2-x^2}}^1 \frac{1}{s}f_Q(s)\mathrm{d}s} = \lim_{x\to\infty}\frac{1-F_R(yx)}{1-F_R(x)}.$$

Hence using Lemma 2.1, and $F_R \in \mathcal{D}(L_{1,\gamma})$, we find

$$\lim_{x\to\infty} \frac{1-F_r(yx)}{1-F_r(x)} = \lim_{x\to\infty}\frac{1-F_R(yx)}{1-F_R(x)} = y^{-\gamma},$$

that is $F_r \in \mathcal{D}(L_{1,\gamma})$.

Let us now assume that $F_R \in \mathcal{D}(L_{2,\gamma})$ implying in particular that $0 < \omega < \infty$. From (8.1) we have

$$\lim_{x\to 0+} \frac{1-F_r(\omega-yx)}{1-F_r(\omega-x)}$$
$$= \lim_{x\to 0+} \frac{y\int_0^x \frac{(\omega-yx)^2(1-F_R(\omega-zy))}{(\omega-zy)^2\sqrt{(\omega-zy)^2-(\omega-yx)^2}}\int_{\frac{1}{(\omega-zy)}\sqrt{(\omega-zy)^2-(\omega-yx)^2}}^1 \frac{f_Q(s)}{s}\mathrm{d}s\mathrm{d}z}{\int_0^x \frac{(\omega-x)^2(1-F_R(\omega-z))}{(\omega-z)^2\sqrt{(\omega-z)^2-(\omega-x)^2}}\int_{\frac{1}{(\omega-z)}\sqrt{(\omega-z)^2-(\omega-x)^2}}^1 \frac{f_Q(s)}{s}\mathrm{d}s\mathrm{d}z},$$

where we have substituted $t = \omega - zy$ in the numerator and $t = \omega - z$ in the denominator. As

$$\lim_{\substack{(x,z)\to(0,0)\\0<z\le x}} \frac{\frac{(\omega-yx)^2(1-F_R(\omega-zy))}{(\omega-zy)^2\sqrt{(\omega-zy)^2-(\omega-yx)^2}}\int_{\frac{1}{(\omega-zy)}\sqrt{(\omega-zy)^2-(\omega-yx)^2}}^1 \frac{f_Q(s)}{s}\mathrm{d}s}{\frac{(\omega-x)^2(1-F_R(\omega-z))}{(\omega-z)^2\sqrt{(\omega-z)^2-(\omega-x)^2}}\int_{\frac{1}{(\omega-z)}\sqrt{(\omega-z)^2-(\omega-x)^2}}^1 \frac{f_Q(s)}{s}\mathrm{d}s}$$
$$= \lim_{x\to 0+} \frac{1}{\sqrt{y}}\frac{(1-F_R(\omega-xy))}{(1-F_R(\omega-x))}$$

we have, using Lemma 2.1 and $F_R \in \mathcal{D}(L_{2,\gamma})$, that

$$\lim_{x\to 0+} \frac{1-F_r(\omega-yx)}{1-F_r(\omega-x)} = y \lim_{x\to 0+} \frac{1}{\sqrt{y}}\frac{(1-F_R(\omega-xy))}{(1-F_R(\omega-x))} = y^{\gamma+1/2}.$$

Thus, $F_r \in \mathcal{D}(L_{2,\gamma+1/2})$.

Assume next that $F_R \in \mathcal{D}(L_3)$ and let the auxiliary function b be differentiable for $x < \omega$ with $\lim_{x\to\omega-} b'(x) = 0$ and $\lim_{x\to\infty} b(x)/x = 0$ if $\omega = \infty$ or $\lim_{x\to\omega-} b(x)/(\omega - x) = 0$ if $\omega < \infty$. Using (8.1) we have

$$\lim_{x\to\omega-} \frac{1-F_r(x+yb(x))}{1-F_r(x)}$$
$$= \lim_{x\to\omega-} \frac{\int_{x+yb(x)}^{\omega} \frac{(x+yb(x))^2(1-F_R(t))}{t^2\sqrt{t^2-(x+yb(x))^2}} \int_{\sqrt{1-(x+yb(x))^2/t^2}}^1 \frac{1}{s} f_Q(s)\mathrm{d}s\mathrm{d}t}{\int_x^{\omega} \frac{x^2(1-F_R(t))}{t^2\sqrt{t^2-x^2}}\int_{\sqrt{1-(x/t)^2}}^1 \frac{1}{s}f_Q(s)\mathrm{d}s\mathrm{d}t}.$$

Due to the properties of b there exists an $x_0 \in (0,\omega)$ such that $g : x \mapsto x + yb(x)$ is a strictly increasing function on $[x_0,\omega)$; see for instance Drees and Reiss (1992). Using the substitution $t = z + yb(z)$ in the numerator and noting that from the properties of b we have $\lim_{x\to\omega-} g^{-1}(\omega) = \omega$, we obtain

$$\lim_{x\to\omega-} \frac{1-F_r(x+yb(x))}{1-F_r(x)}$$
$$= \lim_{x\to\omega-}\Big(1+y\frac{b(x)}{x}\Big)^2 \frac{\int_x^{\omega} \frac{(1-F_R(z+yb(z)))\int_{\sqrt{1-(x+yb(x))^2/(z+yb(z))^2}}^1 \frac{1}{s}f_Q(s)\mathrm{d}s}{(z+yb(z))^2\sqrt{(z+yb(z))^2-(x+yb(x))^2}}(1+yb'(z))\mathrm{d}z}{\int_x^{\omega}\frac{(1-F_R(z))}{z^2\sqrt{z^2-x^2}}\int_{\sqrt{1-(x/z)^2}}^1 \frac{1}{s}f_Q(s)\mathrm{d}s\mathrm{d}z}.$$

Considering the quotient of the integrands, we have

$$\lim_{\substack{(x,z)\to(\omega,\omega)\\x\le z<\omega}} \frac{\frac{(1-F_R(z+yb(z)))\int_{\sqrt{1-(x+yb(x))^2/(z+yb(z))^2}}^1 \frac{1}{s}f_Q(s)\mathrm{d}s}{(z+yb(z))^2\sqrt{(z+yb(z))^2-(x+yb(x))^2}}(1+yb'(z))}{\frac{(1-F_R(z))}{z^2\sqrt{z^2-x^2}}\int_{\sqrt{1-(x/z)^2}}^1\frac{1}{s}f_Q(s)\mathrm{d}s}$$
$$= \lim_{\substack{(x,z)\to(\omega,\omega)\\x\le z<\omega}} \Big(1-\Big(\frac{x}{z}\Big)^2\Big)^{\frac{1}{2}}\Big(1+y\frac{b(z)}{z}\Big)^{-3}\Big(1-\Big(\frac{x}{z}\Big)^2\Big(\frac{1+yb(x)/x}{1+yb(z)/z}\Big)^2\Big)^{-\frac{1}{2}}$$
$$\times \frac{1-F_R(z+yb(z))}{1-F_R(z)}\frac{\int_{\sqrt{1-(\frac{x}{z})^2(\frac{1+yb(x)/x}{1+yb(z)/z})^2}}^1 \frac{1}{s}f_Q(s)\mathrm{d}s}{\int_{\sqrt{1-(\frac{x}{z})^2}}^1\frac{1}{s}f_Q(s)\mathrm{d}s}(1+yb'(z)).$$

Using the properties of the function b, applying Lemma 2.1 and using $F_R \in \mathcal{D}(L_3)$, we find

$$\lim_{x \to \omega -} \frac{1 - F_r(x + yb(x))}{1 - F_r(x)} = \lim_{x \to \omega -} \frac{1 - F_R(x + yb(x))}{1 - F_R(x)} = e^{-y}.$$

Hence $F_r \in \mathcal{D}(L_3)$, which finishes the proof. $\square$

An analogous result holds for the shape parameters. Arguments similar to the proof of Proposition 8.1 and (5.2) show that $F_Q \in \mathcal{D}(L_{2,\gamma})$ implies $F_q \in \mathcal{D}(L_{2,\gamma+1/2})$ and $F_Q \in \mathcal{D}(L_3)$ implies $F_q \in \mathcal{D}(L_3)$.

In order to use these results in practical applications, the normalizing constants for both F_r and F_R are required. They can be estimated by a semi-parametric approach as in the classical Wicksell problem: First a parametric model for F_R is chosen. We know from Proposition 8.1 that F_R and F_r belong to the same domain of attraction. Hence normalizing constants based on $(r_1, q_1), \ldots, (r_N, q_N)$ can be found, for example using maximum likelihood estimators based on the k largest observations; cf. Weissman (1978). One then has to derive normalizing constants for F_R from the estimated normalizing constants for F_r. Methods regarding this are discussed in e.g. Hlubinka (2003b) and Takahashi (1987). When normalizing constants for F_R have been obtained, they can be used to approximate the distribution of the extremal particle radius.

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