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# METRICAL MUSINGS ON LITTLEWOOD AND FRIENDS

A. HAYNES, J. L. JENSEN, AND S. KRISTENSEN

Abstract. We prove a metrical result on a family of conjectures related to the Littlewood conjecture, namely the original Littlewood conjecture, the mixed Littlewood conjecture of de Mathan and Teulié and a hybrid between a conjecture of Cassels and the Littlewood conjecture. It is shown that the set of numbers satisfying a strong version of all of these conjectures is large in the sense of Hausdorff dimension restricted to the set of badly approximable numbers.

## 1. Introduction

The Littlewood conjecture in Diophantine approximation is concerned with the simultaneous approximation of two real numbers by rationals with the same denominator. It states that for any pair $\alpha, \beta \in \mathbb{R}$,

$$\liminf_{q \in \mathbb{N}} q\|q\alpha\|\|q\beta\| = 0, \tag{1}$$

where $\|\cdot\|$ denotes the distance to the nearest integer. It follows from Dirichlet's theorem that for any single real number $\alpha$,

$$\liminf_{q \in \mathbb{N}} q\|q\alpha\| \leq 1. \tag{2}$$

This is best possible, apart from improvements in the constant on the right hand side. Indeed if $\alpha \notin \mathbb{Q}$ has bounded partial quotients in its simple continued fraction expansion, then the left hand side of (2) is positive. Such a number $\alpha$ is called *badly approximable*, and we denote the collection of badly approximable numbers by $\mathrm{Bad} \subseteq \mathbb{R}$.

By the Borel-Bernstein theorem (i.e. almost all real numbers have unbounded partial quotients) the Lebesgue measure of Bad is 0, but by a theorem of Jarnik [8] its Hausdorff dimension is 1. Furthermore since (1) is satisfied whenever $\alpha$ or $\beta$ lies in $\mathbb{R} \setminus \mathrm{Bad}$, it follows immediately that the set of exceptions to the Littlewood conjecture has Lebesgue measure 0.

The Littlewood conjecture has a long history. One of the first results, by Cassels and Swinnerton-Dyer [3], states that the conjecture is satisfied if $\alpha$ and $\beta$ lie in the same cubic extension of $\mathbb{Q}$. Although there have been several further advances toward this conjecture, the most widely quoted recent result is that of Einsiedler, Katok, and Lindenstrauss [5], who used powerful measure rigidity theorems in the space of unimodular lattices to show that the set of $(\alpha, \beta) \in \mathbb{R}^2$ for which (1) fails has Hausdorff dimension 0.

However one should not forget that this breakthrough was preceded by an equally significant theorem of Pollington and Velani [11], which states that for any $\alpha \in \mathrm{Bad}$, there is a set $G \subseteq \mathrm{Bad}$ of Hausdorff dimension 1 such that, for all $\beta \in G$,

$$\liminf_{q \to \infty} q(\log q)\|q\alpha\|\|q\beta\| \leq 1.$$

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The conclusion here is stronger than (1) and it is worth pointing out that, although it does not say as much about the exceptional set in the Littlewood conjecture, this result does not follow from the theorem of Einsiedler, Katok, and Lindenstrauss. More explicit constructions in the spirit of Pollington and Velani's approach have been obtained by Adamczewski and Bugeaud [1].

A problem related to the Littlewood conjecture is the so-called mixed Littlewood conjecture of de Mathan and Teulié [4]. In its most basic form, the conjecture states that for any $\alpha \in \mathbb{R}$ and any prime $p$,

$$\liminf_{q\to\infty} q|q|_p\|q\alpha\| = 0, \tag{3}$$

where $|q|_p$ denotes the $p$-adic absolute value of $q$. As above, (3) is trivially satisfied unless $\alpha \in \mathrm{Bad}$.

The mixed Littlewood conjecture has also attracted considerable interest in recent years. Like the Littlewood conjecture, it has a nice interpretation in terms of homogeneous dynamics. In additional there are new techniques from $p$-adic analysis and one dimensional dynamics which are available for studying this problem. The original statement of the conjecture is actually slightly more general then what we have mentioned so far. It involves the notion of a *pseudo-absolute value*, which we attend to presently.

A pseudo-absolute value is defined in terms of a sequence $\mathcal{D} = (n_k)_{k=0}^{\infty}$, satisfying the conditions $n_0 = 1$ and $n_{k-1}|n_k$ for all $k$. The associated pseudo-absolute value of an integer $q$ is then defined as

$$|q|_{\mathcal{D}} = \min\{n_k^{-1} : q \in n_k\mathbb{Z}\}. \tag{4}$$

For an integer $a$ if we set $n_k = a^k$, then associated pseudo-absolute value becomes the usual $a$-adic absolute value. The more general version of the mixed Littlewood conjecture is the assertion that

$$\liminf_{q\in\mathbb{N}} q|q|_{\mathcal{D}}\|q\alpha\| = 0, \tag{5}$$

for any $\alpha \in \mathbb{R}$ and any pseudo-absolute value sequence $\mathcal{D}$.

We remark that with an additional $p$-adic absolute value multiplied onto the left hand side of (5), and with a mild growth condition on $\mathcal{D}$, it is proved in [7] that

$$\liminf_{q\in\mathbb{N}} q|q|_p|q|_{\mathcal{D}}\|q\alpha\| = 0,$$

for all $\alpha \in \mathbb{R}$. This shows that a slightly weaker statement than the mixed Littlewood conjecture is certainly true.

A final problem related to the Littlewood conjecture is the an old conjecture of Cassels, recently resolved by Shapira [12]. Shapira's theorem states that for Lebesgue-almost every $\alpha, \beta$,

$$\liminf_{q\to\infty} q\|q\alpha - \gamma_1\|\|q\beta - \gamma_2\| = 0,$$

for all $\gamma_1, \gamma_2 \in \mathbb{R}$. In our main result below we address the case when $\gamma_1 = 0$ and consider more closely the collection of $\alpha, \beta$ for which

$$\liminf_{q\to\infty} q\|q\alpha\|\|q\beta - \gamma\| = 0, \tag{6}$$

for every value of $\gamma$. Of course as before the problem is trivial unless $\alpha \in \mathrm{Bad}$. We will call (6) the hybrid Cassels–Littlewood equation.

Now we present our main result, in which we consider a simultaneous version of the three above mentioned problems.

**Theorem 1.** *Fix $\epsilon > 0$ and let $\{\alpha_i\} \subseteq \mathrm{Bad}$ be a countable set of badly approximable numbers, and $\{\mathcal{D}_j\}$ a countable set of pseudo-absolute value sequences. Then there is set of $G \subseteq \mathrm{Bad}$ of Hausdorff dimension 1 such that for any $\beta \in G$,*

(i) *For any $i \in \mathbb{N}$ and $\gamma \in \mathbb{R}$ the inequality*

$$q\|q\alpha_i\|\|q\beta - \gamma\| < \frac{1}{(\log q)^{1/2-\epsilon}}, \tag{7}$$

*has infinitely many solutions $q \in \mathbb{N}$, and*

(ii) *For any $j \in \mathbb{N}$ and $\delta \in \mathbb{R}$ we have that*

$$\liminf_{q\to\infty} q|q|_{\mathcal{D}_j}\|q\beta - \delta\| = 0. \tag{8}$$

*Furthermore, for each $j$ such that $\mathcal{D}_j = (n_k)$ satisfies the inequality $n_k \leq C^k$ for some $C > 1$, we may replace (8) by the stronger statement that for any $\delta \in \mathbb{R}$ the inequality*

$$q|q|_{\mathcal{D}_j}\|q\beta - \delta\| < \frac{1}{(\log q)^{1/2-\epsilon}}, \tag{9}$$

*has infinitely many solutions $q \in \mathbb{N}$.*

We should stress that while the Littlewood conjecture and the mixed Littlewood conjecture are trivial whenever $\beta \notin \mathrm{Bad}$, this is no longer the case for the inhomogeneous versions in the theorem. Hence, the content of our result is of interest not only for badly approximable numbers. Luckily, applying an argument analogous to our proof below with the discrepancy estimate of R. C. Baker [2] in place of our Corollary 8, we could get Theorem 1 for Lebesgue-almost every $\beta$. For comparison with Shapira's result [12] this remains worth stating.

A weaker version of our theorem is the following corollary.

**Corollary 2.** *Let $\{\alpha_i\} \subseteq \mathrm{Bad}$ be a countable set of badly approximable numbers and $\{\mathcal{D}_j\}$ a countable set of pseudo-absolute value sequences. The set of $\beta \in \mathrm{Bad}$ for which the $\mathcal{D}_j$-mixed Littlewood conjectures are all satisfied and for which all pairs $(\alpha_i, \beta)$ satisfy the hybrid Cassels–Littlewood equation is of Hausdorff dimension 1.*

Considering only the cases where $\gamma = 0$, we get the following.

**Corollary 3.** *Let $\{\alpha_i\} \subseteq \mathrm{Bad}$ be a countable set of badly approximable numbers and $\{\mathcal{D}_j\}$ a countable set of pseudo-absolute value sequences. The set of $\beta \in \mathrm{Bad}$ for which the $\mathcal{D}_j$-mixed Littlewood conjectures are all satisfied and for which all pairs $(\alpha_i, \beta)$ satisfy the Littlewood conjecture is of Hausdorff dimension 1.*

We point out that while these corollaries can be deduced using currently available techniques concerning measure rigidity in homogeneous spaces, our main theorem can not.

Our main result generalizes the result of Pollington and Velani [11] to cover both a countable number of $\alpha$'s and a countable number of pseudo-absolute values. We also believe that, although we use some of the same techniques, our proof is substantially simpler.

We will deduce Theorem 1 from a result on the discrepancy of certain sequences defined in terms of a generic point with respect to a very special measure called Kaufman's measure. In the next section, we will describe the properties of this measure and state the result on uniform distribution. In Section 2, we will prove Theorem 1. Finally, we give a few concluding remarks.

Throughout we will use the Vinogradov notation and write $f \ll g$ for two real quantities $f$ and $g$ if there exists a constant $C > 0$ such that $f \leq Cg$. If $f \ll g$ and $g \ll f$, we will write $f \asymp g$. We will also as usual define the function $e(x) = \exp(2\pi i x)$.

## 2. Kaufman's measure and discrepancy

The key tool in proving our main theorem is a result on the discrepancy of certain sequences, which holds true for almost all $\alpha$ with respect to a certain measure introduced by Kaufman [9].

Kaufman's measure $\mu_M$ is a measure supported on the set of real numbers with partial quotients bounded above by $M$. To be explicit, for each real number $\alpha \in [0, 1)$, let

$$\alpha = [a_1, a_2, \dots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots}}}$$

be the simple continued fraction expansion of $\alpha$. For $M \geq 3$, let

$$F_M = \{\alpha \in [0, 1) : a_i(\alpha) \leq M \text{ for all } i \in \mathbb{N}\}. \tag{10}$$

Kaufman [9] proved that the set $F_M$ supports a measure $\mu_M$ satisfying a number of nice properties. For our purposes, we need the following two properties.

(i) For any $s < \dim(F_M)$, there are positive constants $c, l > 0$ such that for any interval $I \subseteq [0, 1)$ of length $|I| \leq l$,

$$\mu_M(I) \leq c|I|^s.$$

(ii) For any $M$, there are positive constants $c, \eta > 0$ such that the Fourier transform $\hat{\mu}_M$ of the Kaufman measure $\mu_M$ satisfies

$$\hat{\mu}_M(u) \leq c|u|^{-\eta}.$$

The first property allows us to connect the Kaufman measure with the Hausdorff dimension of the set $F_M$ via the Mass Distribution Principle. The second property provides a positive lower bound on the Fourier dimension of the set $F_M$, but for our purposes the property is used only in computations.

The second key tool is the notion of discrepancy from the theory of uniform distribution. The discrepancy of a sequence in $[0, 1)$ measures how uniformly distributed a sequence is in the interval. Specifically, the discrepancy of the sequence $(x_n)$ is defined as

$$D_N(x_n) = \sup_{I \subseteq [0,1]} \left| \sum_{n=1}^{N} \chi_I(x_n) - N|I| \right|,$$

where $I$ is an interval and $\chi_I$ is the corresponding characteristic function. A sequence $(x_n)$ is uniformly distributed if $D_N(x_n) = o(N)$.

Our key result is the following discrepancy estimate.

**Theorem 4.** *Let $\mu_M$ be a Kaufman measure and assume that for positive integers $u < v$ we have*

$$\sum_{n,m=u}^{v} |a_n - a_m|^{-\eta} \ll \frac{1}{\log v} \sum_{n=u}^{v} \psi_n$$

*where $(\psi_n)$ is a sequence of non-negative numbers and $\eta > 0$ is the constant from property* (ii) *of the Kaufman measure. Then for $\mu_M$-almost every $x \in [0, 1]$ we have*

$$D_N(a_n x) \ll (N \log(N)^2 + \Psi_N)^{1/2} \log(N \log(N)^2 + \Psi_N)^{3/2+\varepsilon} + \max_{n \leq N} \psi_n$$

*where $\Psi_N = \psi_1 + \cdots + \psi_N$.*

## 3. Proofs

Initially, we prove Theorem 4 and then proceed to deduce Theorem 1. For the proof of Theorem 4, we will need the following result found in [6].

**Lemma 5.** *Let $(X,\mu)$ be a measure space with $\mu(X)<\infty$. Let $F(n,m,x),\ n,m\geq 0$ be $\mu$-measurable functions and let $\phi_n$ be a sequence of real numbers such that $|F(n-1,n,x)|\leq\phi_n$ for $n\in\mathbb{N}$. Let $\Phi_N=\phi_1+\cdots+\phi_N$ and assume that $\Phi_N\to\infty$. Suppose that for $0\leq u<v$ we have*

$$\int_X |F(u,v,x)|^2 d\mu \ll \sum_{n=u}^{v}\phi_n.$$

*Then for $\mu$-almost all $x$, we have*

$$F(0,N,x)\ll \Phi_N^{1/2}\log(\Phi_N)^{3/2+\varepsilon}+\max_{n\leq N}\phi_n.$$

We will also need the Erdős–Turán inequality (see e.g. [10]).

**Theorem 6.** *For any positive integer $K$ and any sequence $(x_n)\subseteq[0,1)$,*

$$D_N(x_n)\leq \frac{N}{K+1}+3\sum_{k=1}^{K}\frac{1}{k}\Big|\sum_{n=1}^{N}e(kx_n)\Big|.$$

*Proof of Theorem 4.* Suppose $M\geq 3$ and for integers $0\leq u<v$ let

$$F(u,v,x)=\sum_{h=1}^{v}\frac{1}{h}\Big|\sum_{n=u}^{v}e(ha_nx)\Big|.$$

Theorem 6 with $K=N$ tells us that

$$D_N(x_n)\ll F(0,N,x).$$

Integrating with respect to $d\mu_M(x)$ and applying Cauchy-Schwartz gives

$$\begin{aligned}\int|F(u,v,x)|^2d\mu_M &\leq \sum_{h,k=1}^{v}\frac{1}{hk}\int\Big|\sum_{n=u}^{v}e(ha_nx)\Big|^2d\mu_M\\ &=\sum_{h,k=1}^{v}\frac{1}{hk}\Big(v-u+1+\sum_{\substack{n,m=u\\ n\neq m}}^{v}\hat{\mu}_M(h(a_n-a_m))\Big).\end{aligned}$$

Finally using property (ii) of the Kaufman measure we have

$$\begin{aligned}\int|F(u,v,x)|^2d\mu_M &\ll \sum_{h,k=1}^{v}\frac{1}{hk}\Big(v-u+1+h^{-\eta}\sum_{\substack{n,m=u\\ n\neq m}}^{v}|a_n-a_m|^{-\eta}\Big)\\ &\ll\sum_{n=u}^{v}[\log(n)^2+\psi_n].\end{aligned}$$

Since $F(n-1,n,x)\ll\log(n)^2+\psi_n$ for all $n\geq 1$, the theorem then follows from Lemma 5. □

We now state a corollary, which is of interest in its own right. Assuming nothing about the sequence $(a_n)$ we get the following corollary.

**Corollary 7.** *Let $\mu$ be a Kaufman measure. For $\mu$-almost every $x \in [0,1]$ we have $D_N(a_n x) \ll N^{1-\nu}$ for some $\nu > 0$. In particular $(a_n x)$ is uniformly distributed modulo 1.*

*Proof.* Without loss of generality we may assume that $a_n < a_{n+1}$, and we have that

$$\sum_{n,m=u}^{v} |a_n - a_m|^{-\eta} \leq 2 \sum_{m=u}^{v-1} \sum_{n=m+1}^{v} |n-m|^{-\eta} \ll v^{2-\eta} - u^{2-\eta} \ll \frac{1}{\log v} \sum_{n=u}^{v} n^{1-2\nu'}$$

for some $\nu' > 0$, so for $\mu$-a.e. $x$ we have

$$D_N(a_n x) \ll (N \log(N)^2 + N^{2-2\nu'})^{1/2} (\log(N \log(N)^2 + N^{2-2\nu'}))^{3/2+\varepsilon} + N^{1-2\nu'} \ll N^{1-\nu}$$

for any $\nu > 0$ with $\nu < \nu'$. □

Corollary 7 is sufficient to prove Corollary 3 directly. However, we are aiming for a proof of the more general Theorem 1. For the purposes of this result, we can specialize to lacunary sequences $a_n$.

**Corollary 8.** *Let $\nu > 0$, let $\mu$ be a Kaufman measure and $(a_n)$ a lacunary sequence of integers. For $\mu$-almost every $x \in [0,1]$ we have $D_N(a_n x) \ll N^{1/2} (\log N)^{5/2+\nu}$.*

*Proof.* We apply again Theorem 4. Using lacunarity of the sequence $(a_n)$, we see that

$$\sum_{n,m=1}^{\infty} |a_n - a_m|^{-\eta} < \infty.$$

Consequently, we can absorb all occurrences of $\Psi_N$ as well as the final term $\max_{n \leq N} \psi_n$ in the discrepancy estimate of Theorem 4 into the implied constant. It follows that

$$D_N(a_n x) \ll (N \log(N)^2)^{1/2} \log(N \log(N)^2)^{3/2+\varepsilon} \ll N^{1/2} (\log N)^{5/2+\nu}$$

for $\mu$-almost every $x$, where $\nu$ can be made as small as desired by picking $\varepsilon$ small enough. □

We are now ready to prove the main result, Theorem 1.

*Proof of Theorem 1.* Let $G$ denote the set from the statement of the theorem and suppose, contrary to what we are to prove, that $\dim g < 1$. Pick an $M \geq 3$ such that $\dim F_M > \dim G$ (this can be done in light of Jarník's Theorem). Let $\mu = \mu_M$ denote the Kaufman measure on $F_M$.

Consider first one of the $\alpha_i$, and let $(q_k)$ denote the sequence of denominators of convergents in the simple continued fraction expansion of $\alpha_i$. The sequence $q_k$ is lacunary hence by Corollary 8, for $\mu-$almost every $x$,

$$D_N(q_n x) \ll N^{1/2} (\log N)^{5/2+\nu}.$$

Let $\psi(N) = N^{-1/2+\epsilon}$ for some $\epsilon > 0$ and consider the interval

$$I_N^{\gamma} = [\gamma - \psi(N), \gamma + \psi(N)].$$

By the definition of discrepancy, for every $\gamma \in [0,1]$ and $\mu$-almost every $\beta$

$$|\#\{k \leq N : \{q_k \beta\} \in I_N^{\gamma}\} - 2N\psi(N)| \ll N^{1/2} (\log N)^{5/2+\nu}.$$

Hence,

$$\begin{aligned} \#\{k \leq N : \{q_k \beta\} \in I_N^{\gamma}\} &\geq 2N\psi(N) - KN^{1/2} (\log N)^{5/2+\nu} \\ &= 2N^{1/2+\epsilon} - KN^{1/2} (\log N)^{5/2+\nu}, \end{aligned}$$

where $K > 0$ is the implied constant from Corollary 8. Next let $N_h^\gamma$ denote the increasing sequences defined by

$$N_h^\gamma = \min\big\{N \in \mathbb{N} : \#\{k \le N : \{q_k\beta\} \in I_N^\gamma\} = h\big\}.$$

We claim that the sequence $q_{N_h^\gamma}$ satisfies (7) for the given $\alpha_i$ and every $\gamma$ with $\mu$-almost every $\beta$. Indeed, since each $q_{N_h^\gamma}$ is a denominator of a convergent to $\alpha_i$,

$$q_{N_h^\gamma}\|q_{N_h^\gamma}\alpha_i\| \le 1.$$

Hence,

$$q_{N_h^\gamma}\|q_{N_h^\gamma}\alpha_i\|\|q_{N_h^\gamma}\beta - \gamma\| \le \|q_{N_h^\gamma}\beta - \gamma\| \le (N_h^\gamma)^{-1/2+\epsilon}.$$

Since $\alpha_i \in \mathrm{Bad}$, the sequence of denominators of convergents $q_n$ is bounded between the Fibonacci sequence and $(2M)^n$, where $M$ is the maximal partial quotient in the simple continued fraction expansion of $\alpha_i$. Hence, $n \asymp \log q_n$, and it follows that

$$q_{N_h^\gamma}\|q_{N_h^\gamma}\alpha_i\|\|q_{N_h^\gamma}\beta - \gamma\| \le (\log q_{N_h^\gamma})^{-1/2+\epsilon/2},$$

whenever $h$ is large enough. This establishes our claim and shows that the exceptional set $E_i \subseteq F_M$ for which (7) does not hold has $\mu(E_i) = 0$.

Consider now one of the absolute value sequences, $\mathcal{D}_| = \{r_k\}$. If we do not have an upper geometric growth rate of the sequence, applying Corollary 7 implies that (8) holds for $\mu$-almost every $\beta$. Indeed, let $\beta$ be such that $\{r_k\beta\}$ is uniformly distributed and suppose to the contrary that for some $\delta$ and $\epsilon > 0$,

$$r_k|r_k|_{\mathcal{D}_j}\|r_k\beta - \delta\| > \epsilon,$$

for every $k$. Since $r_k|r_k|_{\mathcal{D}_j} = 1$, this would violate the uniform distribution of the sequence $\{r_k\beta\}$ and complete the proof of the statement.

Next we show that when $r_k$ has the right upper growth rate, the stronger statement (9) holds. By the divisibility condition, this sequence must again be lacunary. Hence,

$$D_N(r_nx) \ll N^{1/2}(\log N)^{5/2+\nu}.$$

Defining $I_N^\delta$ with $\delta$ in place of $\gamma$ and $\Psi(N)$ as before and repeating the argument from the Littlewood-case, we see that

$$\begin{aligned}\#\{k \le N : \{q_k\beta\} \in I_N^\delta\} &\ge 2N\psi(N) - KN^{1/2}(\log N)^{5/2+\nu}\\ &= 2N^{1/2+\epsilon} - KN^{1/2}(\log N)^{5/2+\nu}.\end{aligned}$$

As before, we take a sequence $N_h^\delta$ and note that since $r_{N_h^\delta}|r_{N_h^\delta}|_{\mathcal{D}_j} = 1$, we immediately have for $\mu$-almost every $\beta$ a sequence $r_{N_h^\delta}$ satisfying the inequalities (9). Indeed,

$$r_{N_h^\delta}|r_{N_h^\delta}|_{\mathcal{D}_j}\|r_{N_h^\delta}\beta - \delta\| \le \|r_{N_h^\delta}\beta - \delta\| \le \left(N_h^\delta\right)^{-1/2+\epsilon}.$$

Since the upper and lower growth assumptions imply that $k \asymp \log r_k$, this implies that

$$r_{N_h^\delta}|r_{N_h^\delta}|_{\mathcal{D}_j}\|r_{N_h^\delta}\beta - \delta\| \le \left(\log q_{N_h^\delta}\right)^{-1/2+\epsilon},$$

whenever $h$ is large enough. The upshot is that the exceptional set $E_j' \subseteq F_M$ for which (8) does not hold has $\mu(E_j') = 0$.

To conclude, let $E$ be the set of $\beta \in F_M$ for which there is an $i$ or a $j$ such that either (1) or (5) is not satisfied. Then,

$$E = \bigcup_i E_i \cup \bigcup_j E_j',$$

and therefore $\mu(E) = 0$.

Finally $\mu(G)$ is maximal, so consider the trace measure $\tilde{\mu}$ of $\mu$ on $G$, defined by $\tilde{\mu}(X) = \mu(X \cap G)$. It follows from property (i) of Kaufman's measures that $\mu$ is a mass distribution on $[0, 1)$, and since $G$ is full, $\tilde{\mu}$ inherits the decay property of (i) from $\mu$. By the Mass Distribution Principle it then follows that $\dim(G) = \dim(F_M) > \dim(G)$, which contradicts our original assumption. Therefore we conclude that $\dim(G) = 1$. □

## 4. Concluding remarks

We suspect that the rate of convergence in Theorem 1 can be replaced by $(\log q)^{-1}$, at least in the case $\gamma = \delta = 0$ and with the $\mathcal{D}_j$ growing at most geometrically. This is certainly the case with (7) for a single fixed $\alpha_i$ as proved in [11]. However it doesn't seem possible to prove this using the method we have given here, without some significant modification.

We suspect that it is possible to improve the discrepancy estimate in Corollary 8 to replace the exponent $5/2 + \nu$ by $3/2 + \nu$ for $\mu_M$-almost every $x$. This is a natural conjecture in view of the Lebesgue-almost sure discrepancy estimates of R. C. Baker [2]. However, although it is an interesting problem in its own right, for the Littlewood-type statements this would only give a marginal improvement.

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