Conformally invariant differential operators on Heisenberg groups and minimal representations

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Abstract

For a simple real Lie group G with Heisenberg parabolic subgroup P, we study the corresponding degenerate principal series representations. For a certain induction parameter the kernel of the conformally invariant system of second order differential operators constructed by Barchini, Kable and Zierau is a subrepresentation which turns out to be the minimal representation. To study this subrepresentation, we take the Heisenberg Fourier transform in the non-compact picture and show that it yields a new realization of the minimal representation on a space of L^2-functions. The Lie algebra action is given by differential operators of order ≤ 3 and we find explicit formulas for the lowest K-type.

These L^2-models were previously known for the groups $\mathrm{SO}(n,n)$, $E_{6(6)}$, $E_{7(7)}$ and $E_{8(8)}$ by Kazhdan and Savin, for the group $G_{2(2)}$ by Gelfand, for the group $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ by Torasso, and for $\widetilde{\mathrm{SO}}(4,3)$ by Sabourin. Our new approach provides a uniform and systematic treatment of these cases and also gives new L^2-models for $E_{6(2)}$, $E_{7(-5)}$ and $E_{8(-24)}$ for which the minimal representation is a continuation of the quaternionic discrete series, and for the groups $\widetilde{\mathrm{SO}}(p,q)$ with either $p \geq q = 3$ or $p,q \geq 4$ and $p+q$ even.

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Contents

Introduction

The classification of all irreducible unitary representations of a semisimple Lie group is a natural problem in representation theory which is still unsolved for most groups. A guiding principle for the classification is the orbit philosophy which proposes a tight relation between the unitary dual of a semisimple group G and the set of coadjoint orbits in the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G by *quantization*. For elliptic and hyperbolic coadjoint orbits, cohomological and parabolic induction provide explicit constructions of the corresponding unitary representations and the resulting representations make up a large part of the unitary dual. For nilpotent orbits, however, it is not clear how to apply the orbit philosophy to construct unitary representations. Among the (finitely many) nilpotent coadjoint orbits, there are one or two of minimal dimension (depending on whether the group is of Hermitian type or not). Irreducible unitary representations corresponding to a minimal nilpotent coadjoint orbit are called *minimal representations*.

Minimal representations are not only of importance in representation theory, but are for instance related to Fourier coefficients of automorphic forms [2, 14, 15, 31], and their realizations often establish connections to families of special functions [24, 35] as well as interesting analytic and geometric questions [37, 38, 39]. It is therefore desirable to have convenient models for these representations.

For reductive groups G possessing a parabolic subgroup $P = MAN$ with abelian nilradical N, minimal representations can often be constructed as subrepresentations of the corresponding degenerate principal series $\operatorname{Ind}_P^G(\chi)$ for a certain character χ. The subrepresentations are given as the kernel of a system of second order differential operators. Realizing the degenerate principal series in the non-compact picture on a space of functions on the opposite nilradical $\overline{N}$ and identifying $\overline{N}$ with its Lie algebra $\overline{\mathfrak{n}}$, these differential operators become constant coefficient differential operators on $\overline{\mathfrak{n}}$. Taking the Euclidean Fourier transform $\mathcal{S}'(\overline{\mathfrak{n}}) \to \mathcal{S}'(\mathfrak{n})$ turns the systems of differential operators into a system of multiplication operators and hence the differential equations turn into a support condition on $\mathfrak{n}$. In this way, a realization of the minimal representation on a space of functions on the smallest non-trivial $\operatorname{Ad}(MA)$-orbit $\mathcal{O} \subseteq \mathfrak{n}$ is obtained. The relevant Hilbert space turns out to be $L^2(\mathcal{O})$ with respect to a certain $\operatorname{Ad}(MA)$-equivariant measure on $\mathcal{O}$. This was first observed by Vergne and Rossi [54] in the case of Hermitian groups and later generalized by Dvorsky and Sahi [6], Kobayashi and Ørsted [38] and Möllers and Schwarz [44] to cover all cases (see also Goncharov [13] for the underlying Lie algebra representation).

In the case of parabolic subgroups $P = MAN$ with Heisenberg nilradical N, analogous *conformally invariant systems of differential operators* on $\overline{\mathfrak{n}}$ have been constructed by Barchini, Kable and Zierau [1], but so far their kernels have only been studied in a few examples, mostly algebraically (see e.g. [27, 30, 29, 42]). In particular, an analysis of the corresponding representations is missing due to the fact that the Euclidean Fourier transform has to be replaced by the

Heisenberg group Fourier transform which is considerably more complicated.

In this work we systematically study the kernels of conformally invariant systems of differential operators and their Fourier transforms in the case of non-Hermitian groups. This leads to L^2-models for the minimal representations of the split groups $E_{6(6)}$, $E_{7(7)}$ and $E_{8(8)}$, the quaternionic groups $G_{2(2)}$, $E_{6(2)}$, $E_{7(-5)}$ and $E_{8(-24)}$, the groups $\mathrm{SL}(n,\mathbb{R})$ and $\widetilde{SL}(3,\mathbb{R})$ as well as the groups $\widetilde{\mathrm{SO}}(p,q)$ with either $p \geq q = 3$ or $p,q \geq 4$ with $p+q$ even. For some of these groups, the relevant L^2-models have been found earlier by different methods, and our approach gives a uniform construction for all these cases and provides an explanation for the corresponding formulas that occur in the literature. Moreover, our models seem to be new for the quaternionic groups $E_{6(2)}$, $E_{7(-5)}$ and $E_{8(-24)}$ and for the indefinite orthogonal groups $\widetilde{\mathrm{SO}}(p,q)$, except for $\widetilde{\mathrm{SO}}(4,3)$ where there seems to be a relation to the realization constructed by Sabourin [47].

We now describe our results in detail.

Conformally invariant systems

Let G be a connected non-compact simple real Lie group with finite center and denote by $\mathfrak{g}$ its Lie algebra. We assume that G has a parabolic subgroup $P = MAN$ whose nilradical N is a Heisenberg group and write $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{n}$ for the Lie algebras of M, A and N. There exists a unique element $H \in \mathfrak{a}$ such that $\mathrm{ad}(H)$ has eigenvalues $+1$ and $+2$ on $\mathfrak{n}$ and -1 and -2 on the opposite nilradical $\overline{\mathfrak{n}}$. We decompose

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

into eigenspaces for $\mathrm{ad}(H)$ where $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$. In all cases but $\mathfrak{g} \simeq \mathfrak{sl}(n,\mathbb{R})$ the parabolic subgroup P is maximal and hence $\mathfrak{a} = \mathbb{R}H$. To simplify notation we therefore put

$$\mathfrak{a} = \mathbb{R}H \qquad \text{and} \qquad \mathfrak{m} = \{X \in \mathfrak{g}_0 : \mathrm{ad}(X)|_{\mathfrak{g}_2} = 0\}$$

also in the case $\mathfrak{g} \simeq \mathfrak{sl}(n,\mathbb{R})$. Further, we write $\rho = \frac{1}{2}\,\mathrm{ad}\,|_{\mathfrak{n}} \in \mathfrak{a}^*$ as usual.

For an irreducible smooth admissible representation (ζ, V_ζ) of M and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ we form the degenerate principal series (smooth normalized parabolic induction)

$$\pi_{\zeta,\nu} = \mathrm{Ind}_P^G(\zeta \otimes e^\nu \otimes \mathbf{1})$$

and realize it on a subspace $I(\zeta,\nu) \subseteq C^\infty(\overline{\mathfrak{n}}) \mathbin{\widehat{\otimes}} V_\zeta$ of functions on the opposite nilradical $\overline{\mathfrak{n}} \simeq \overline{N}$ which is a Heisenberg Lie algebra (the *non-compact picture*). The representation $\pi_{\zeta,\nu}$ is irreducible for generic ν, but may contain irreducible subrepresentations for singular parameters.

Extending H to an $\mathfrak{sl}_2$-triple by $E \in \mathfrak{g}_2$ and $F \in \mathfrak{g}_{-2}$, we can endow $V = \mathfrak{g}_{-1}$ with a symplectic form ω characterized by

$$[x,y] = \omega(x,y)F \qquad \text{for } x,y \in V.$$

The identity component M_0 of M acts symplectically on (V,ω) and the 5-grading of $\mathfrak{g}$ gives rise to three additional symplectic invariants:

$$\begin{aligned} &\mu : V \to \mathfrak{m}, \quad \mu(x) = \frac{1}{2!}\,\mathrm{ad}(x)^2 E && \text{(the moment map)}\\ &\Psi : V \to V, \quad \Psi(x) = \frac{1}{3!}\,\mathrm{ad}(x)^3 E && \text{(the cubic map)}\\ &Q : V \to \mathbb{R}, \quad Q(x) = \frac{1}{4!}\,\mathrm{ad}(x)^4 E && \text{(the quartic)} \end{aligned}$$

which are all M-equivariant polynomials. In [1] Barchini, Kable and Zierau constructed for each of the invariants ω, μ, Ψ and Q a system of differential operators on $\overline{\mathfrak{n}}$ which is *conformally invariant*. These systems can be seen as quantizations of the symplectic invariants. The joint kernel of each system gives rise to a subrepresentation of a degenerate principal series representation. For instance, for the conformally invariant system $\Omega_\mu(T)$ $(T \in \mathfrak{m})$ corresponding to the moment map, the conformal invariance implies that for every simple or one-dimensional abelian ideal $\mathfrak{m}' \subseteq \mathfrak{m}$ there exists a parameter $\nu = \nu(\mathfrak{m}') \in \mathfrak{a}^*$ such that the joint kernel

$$I(\zeta,\nu)^{\Omega_\mu(\mathfrak{m}')} = \{u \in I(\zeta,\nu) : \Omega_\mu(T)u = 0 \,\forall\, T \in \mathfrak{m}'\}$$

is a subrepresentation of $I(\zeta,\nu)$ whenever ζ is trivial on the connected component M_0 of M. Note that $I(\zeta,\nu)^{\Omega_\mu(\mathfrak{m}')}$ could be trivial.

The Heisenberg group Fourier transform

The infinite-dimensional irreducible unitary representations $(\sigma_\lambda, \mathcal{H}_\lambda)$ of $\overline{N}$ are parameterized by their central character $\lambda \in \mathbb{R}^\times$ in the sense that $\sigma_\lambda(e^{tF}) = e^{i\lambda t}\,\mathrm{id}$. They give rise to operator-valued maps

$$\sigma_\lambda : L^1(\overline{N}) \to \mathrm{End}(\mathcal{H}_\lambda), \quad \sigma_\lambda(u) = \int_{\overline{N}} u(\overline{n})\sigma_\lambda(\overline{n})\,d\overline{n}.$$

The Heisenberg group Fourier transform is the collection of all σ_λ and it extends to a unitary isomorphism

$$\mathcal{F} : L^2(\overline{N}) \to L^2(\mathbb{R}^\times, \mathrm{HS}(\mathcal{H}); |\lambda|^{\frac{\dim V}{2}}\,d\lambda) \simeq L^2(\mathbb{R}^\times; |\lambda|^{\frac{\dim V}{2}}\,d\lambda)\,\widehat{\otimes}\,\mathrm{HS}(\mathcal{H}), \quad \mathcal{F}u(\lambda) = \sigma_\lambda(u),$$

where $\mathcal{H} = \mathcal{H}_\lambda$ is a Hilbert space which realizes all representations σ_λ and $\mathrm{HS}(\mathcal{H})$ denotes the Hilbert space of all Hilbert–Schmidt operators on $\mathcal{H}$. It is a non-trivial problem to extend $\mathcal{F}$ to tempered distributions (see e.g. [4, 8] on this issue). For our purpose it is enough to show that $\mathcal{F}$ extends for $\mathrm{Re}\,\nu > -\rho$ to an injective linear map (see Corollary 2.5.3)

$$\mathcal{F} : I(\zeta,\nu) \to \mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathrm{Hom}(\mathcal{H}^\infty, \mathcal{H}^{-\infty}),$$

where $\mathcal{H}^\infty$ denotes the space of smooth vectors in $\mathcal{H} = \mathcal{H}_\lambda$ and $\mathcal{H}^{-\infty} = (\mathcal{H}^\infty)'$ its dual space, the space of distribution vectors (both spaces are independent of λ).

For $\mathrm{Re}\,\nu > -\rho$ we call the realization

$$\widehat{\pi}_{\zeta,\nu}(g) = \mathcal{F} \circ \pi_{\zeta,\nu}(g) \circ \mathcal{F}^{-1} \qquad (g \in G)$$

on the subspace $\widehat{I}(\zeta,\nu) := \mathcal{F}(I(\zeta,\nu)) \subseteq \mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathrm{Hom}(\mathcal{H}^\infty, \mathcal{H}^{-\infty})$ the *Fourier transformed picture*. In order to understand the Fourier transformed picture of the subrepresentations $I(\zeta,\nu)^{\Omega_\mu(\mathfrak{m}')}$ we study the Fourier transform of the conformally invariant system Ω_μ. For this denote by $d\omega_{\mathrm{met},\lambda}$ the metaplectic representation of $\mathfrak{sp}(V,\omega)$ which is uniquely defined by

$$d\sigma_\lambda([T,X]) = [d\omega_{\mathrm{met},\lambda}(T), d\sigma_\lambda(X)] \qquad (T \in \mathfrak{sp}(V,\omega), X \in \overline{\mathfrak{n}}).$$

Note that the adjoint representation $\mathrm{ad} : \mathfrak{m} \to \mathfrak{gl}(V)$, $T \mapsto \mathrm{ad}(T)|_V$ identifies $\mathfrak{m}$ with a subalgebra of $\mathfrak{sp}(V,\omega)$ so that we can restrict $d\omega_{\mathrm{met},\lambda}$ to $\mathfrak{m}$.

Theorem A. *For* $\lambda\in\mathbb{R}^\times$, $T\in\mathfrak{m}$ *and* $u\in I(\zeta,\nu)$ *we have*

$$\sigma_\lambda(\Omega_\mu(T)u)=2i\lambda\,\sigma_\lambda(u)\circ d\omega_{\mathrm{met},\lambda}(T).$$

This implies that the Fourier transform of $u\in I(\zeta,\nu)^{\Omega_\mu(\mathfrak{m}')}$ satisfies

$$\mathcal{F}u(\lambda)\circ d\omega_{\mathrm{met},\lambda}(T)=0\qquad\text{for all } T\in\mathfrak{m}'.$$

We also obtain formulas for the Fourier transform of the conformally invariant systems Ω_ω, Ω_Ψ and Ω_Q associated to ω, Ψ and Q in Sections 3.3, 3.5 and 3.6.

The Fourier transformed picture of the minimal representation

In this work we restrict our attention to the case of non-Hermitian G. More details about the differences between Hermitian and non-Hermitian G can be found in Section 1.9. We hope to return to the Hermitian case in a future work.

Since G is non-Hermitian, one can use the structure theory developed in [49] to obtain a bigrading on $\mathfrak{g}$ (see Section 4.2 for details). This results in a particular choice of a Lagrangian subspace $\Lambda\subseteq V$ with decomposition $\Lambda=\mathbb{R}A\oplus\mathcal{J}$, where in most cases $\mathcal{J}$ is a semisimple Jordan algebra of degree 3 with norm function $n(z)$ defined by $\Psi(z)=n(z)A$. (In fact, all semisimple Jordan algebras of rank three arise in this way, cf. Table D.1). The two exceptions are $\mathfrak{g}\simeq\mathfrak{g}_{2(2)}$ where $\mathcal{J}\simeq\mathbb{R}$ and $n(z)=z^3$ and $\mathfrak{g}\simeq\mathfrak{sl}(n,\mathbb{R})$ where $n(z)=0$ for all $z\in\mathcal{J}$. We write (a,z) for $aA+z\in\mathbb{R}A\oplus\mathcal{J}=\Lambda$.

We realize the representations σ_λ on the common Hilbert space $\mathcal{H}=L^2(\Lambda)$, the Schrödinger model of σ_λ. In this realization we have $\mathcal{H}^\infty=\mathcal{S}(\Lambda)$, the space of Schwartz functions on Λ, and $\mathcal{H}^{-\infty}=\mathcal{S}'(\Lambda)$, the space of tempered distributions, so that the Schwartz Kernel Theorem implies

$$\mathrm{Hom}(\mathcal{H}^\infty,\mathcal{H}^{-\infty})\simeq\mathcal{S}'(\Lambda\times\Lambda)\simeq\mathcal{S}'(\Lambda)\mathbin{\widehat{\otimes}}\mathcal{S}'(\Lambda).$$

From here on we assume that the parameter $\nu=\nu(\mathfrak{m}')\in\mathfrak{a}^*$, for which the joint kernel of $\Omega_\mu(\mathfrak{m}')$ is a subrepresentation of $I(\zeta,\nu)$, is the same for all factors of $\mathfrak{m}$. This is in particular the case when $\mathfrak{m}$ is simple, but also for $\mathfrak{g}\simeq\mathfrak{sl}(3,\mathbb{R})$ where $\mathfrak{m}\simeq\mathfrak{gl}(1,\mathbb{R})$ and for $\mathfrak{g}\simeq\mathfrak{so}(4,4)$ where $\mathfrak{m}\simeq\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$. The reason for this assumption is that we need invariance under the full Lie algebra $\mathfrak{m}$ in the following result:

Theorem B. *For every* $\lambda\in\mathbb{R}^\times$ *the space* $L^2(\Lambda)^{-\infty,\mathfrak{m}}$ *of* $\mathfrak{m}$*-invariant distribution vectors in* $d\omega_{\mathrm{met},\lambda}$ *is two-dimensional and spanned by* $\xi_{\lambda,\varepsilon}$ ($\varepsilon\in\mathbb{Z}/2\mathbb{Z}$), *where*

$$\xi_{\lambda,\varepsilon}(a,z)=\mathrm{sgn}(a)^\varepsilon|a|^{s_{\min}}e^{-i\lambda\frac{n(z)}{a}},\qquad(a,z)\in\mathbb{R}\times\mathcal{J},$$

with $s_{\min}=-\frac{1}{6}(\dim\Lambda+2)$.

We remark that these distributions also occur in the classification [7] of certain generalized functions whose Euclidean Fourier transform is of the same type.

Theorem B implies that the Fourier transform $\mathcal{F}u \in \mathcal{D}'(\mathbb{R}^\times)\mathbin{\widehat{\otimes}}\mathcal{S}'(\Lambda)\mathbin{\widehat{\otimes}}\mathcal{S}'(\Lambda)$ of a function $u \in I(\zeta,\nu)^{\Omega_\mu(\mathfrak{m})}$ in the kernel of $\Omega_\mu(\mathfrak{m})$ can be written as

$$\mathcal{F}u(\lambda,x,y) = \xi_{-\lambda,\varepsilon}(x)\widetilde{u}(\lambda,y)$$

for some $\widetilde{u}(\lambda,\cdot) \in \mathcal{S}'(\Lambda)$, where $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ is determined in Corollary 6.2.8. The map

$$I(\zeta,\nu)^{\Omega_\mu(\mathfrak{m})} \to \mathcal{D}'(\mathbb{R}^\times)\mathbin{\widehat{\otimes}}\mathcal{S}'(\Lambda), \quad u \mapsto \widetilde{u}$$

is injective and provides a new realization $\rho_{\min}$ of the subrepresentation $I(\zeta,\nu)^{\Omega_\mu(\mathfrak{m})}$ on $J_{\min} \subseteq \mathcal{D}'(\mathbb{R}^\times)\mathbin{\widehat{\otimes}}\mathcal{S}'(\Lambda)$. Note that still $J_{\min}$ could be trivial. To show that there exists a representation ζ of M such that $J_{\min} \neq \{0\}$, we compute the Lie algebra action in the new realization and find K-finite vectors.

We remark that this construction only excludes the non-Hermitian Lie algebras $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ ($n > 3$) and $\mathfrak{g} = \mathfrak{so}(p,q)$ ($p,q \geq 3$, $(p,q) \neq (4,4)$). In Section 4.6 we explain how for $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ a generalization of the first order system Ω_ω (a quantization of the symplectic form ω) to the case of vector-valued principal series induced from characters $\zeta = \zeta_r$ of M ($r \in \mathbb{C}$) yields a one-parameter family $d\rho_{\min,r}$ of subrepresentations of $I(\zeta_r,\nu)$ on $\mathcal{D}'(\mathbb{R}^\times)\mathbin{\widehat{\otimes}}\mathcal{S}'(\Lambda)$. And for $\mathfrak{g} = \mathfrak{so}(p,q)$ we combine in Section 4.7 generalizations of both Ω_ω and Ω_μ to the vector-valued degenerate principal series induced from representations of $\mathrm{SL}(2,\mathbb{R}) \subseteq M$ to find the analogous representation $d\rho_{\min}$ of $\mathfrak{g}$ on $\mathcal{D}'(\mathbb{R}^\times)\mathbin{\widehat{\otimes}}\mathcal{S}'(\Lambda)$.

The Lie algebra action and lowest K-types

Heuristic arguments involving the standard Knapp–Stein intertwining operators (see Remark 4.5.5) show that $\widetilde{\pi}_{\min}$ should be unitary on $L^2(\mathbb{R}^\times \times \Lambda, |\lambda|^{\dim\Lambda - 2s_{\min}}d\lambda\,dy)$. To obtain a unitary representation on $L^2(\mathbb{R}^\times \times \Lambda)$, we twist the representation with the isomorphism

$$\Phi_\delta : \mathcal{D}'(\mathbb{R}^\times)\mathbin{\widehat{\otimes}}\mathcal{S}'(\Lambda) \to \mathcal{D}'(\mathbb{R}^\times)\mathbin{\widehat{\otimes}}\mathcal{S}'(\Lambda), \quad \Phi_\delta u(\lambda,x) = \operatorname{sgn}(\lambda)^\delta|\lambda|^{-s_{\min}}u(\lambda,\tfrac{x}{\lambda})$$

which restricts to an isometry $L^2(\mathbb{R}^\times \times \Lambda, |\lambda|^{\dim\Lambda - 2s_{\min}}d\lambda\,dy) \to L^2(\mathbb{R}^\times \times \Lambda)$ (see Corollary 6.2.8 for the choice of $\delta \in \mathbb{Z}/2\mathbb{Z}$). Let

$$I_{\min} := \Phi_\delta(J_{\min}), \qquad \pi_{\min}(g) := \Phi_\delta \circ \rho_{\min}(g) \circ \Phi_\delta^{-1}.$$

Theorem C. *The Lie algebra action $d\pi_{\min}$ is by algebraic differential operators of degree ≤ 3 on $\mathbb{R}^\times \times \Lambda$ and is explicitly computed in Proposition 4.5.6. It extends naturally to $\mathcal{D}'(\mathbb{R}^\times)\mathbin{\widehat{\otimes}}\mathcal{S}'(\Lambda)$ and is infinitesimally unitary on $L^2(\mathbb{R}^\times \times \Lambda)$.*

Comparing this action with formulas in the literature shows that our representation agrees with the one for $\mathfrak{g} = \mathfrak{so}(n,n), \mathfrak{e}_{6(6)}, \mathfrak{e}_{7(7)}, \mathfrak{e}_{8(8)}$ in [31, 32], for $\mathfrak{g} = \mathfrak{g}_{2(2)}$ in [12, 48], and for $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$ in [53] (see Section 4.8 for details). There also seems to be a relation to the formulas for $\mathfrak{g} = \mathfrak{so}(4,3)$ in Sabourin [47]. Similar formulas also appear in [19, 20] but without addressing the question of unitarizability. In this sense, our computations give a new explanation of the formulas in the literature and provide an explicit (degenerate) principal series embedding of the representations as well as generalize them to a larger class of groups.

In order to show that the Lie algebra representation $d\pi_{\min}$ on $I_{\min} \subseteq \mathcal{D}'(\mathbb{R}^\times)\mathbin{\widehat{\otimes}}\mathcal{S}'(\Lambda)$ integrates to an irreducible unitary representation on $L^2(\mathbb{R}^\times \times \Lambda)$ for some representation ζ, we find the lowest K-type in the representation.

Theorem D. *(1) For* $\mathfrak{g}=\mathfrak{e}_{6(2)},\mathfrak{e}_{7(-5)},\mathfrak{e}_{8(-24)}$ *there exists a* $\mathfrak{k}$*-subrepresentation* $W\subseteq\mathcal{D}'(\mathbb{R}^\times)\widehat{\otimes}\mathcal{S}'(\Lambda)$*, explicitly given in Theorem 5.2.1, which is isomorphic to the representation* $S^{2,4,8}(\mathbb{C}^2)\boxtimes\mathbb{C}$ *of* $\mathfrak{k}\simeq\mathfrak{su}(2)\oplus\mathfrak{k}''$*.*

(2) For $\mathfrak{g}=\mathfrak{e}_{6(6)},\mathfrak{e}_{7(7)},\mathfrak{e}_{8(8)}$ *there exists a* $\mathfrak{k}$*-subrepresentation* $W\subseteq\mathcal{D}'(\mathbb{R}^\times)\widehat{\otimes}\mathcal{S}'(\Lambda)$*, explicitly given in Theorem 5.3.1, which is isomorphic to the trivial representation.*

(3) For $\mathfrak{g}=\mathfrak{g}_{2(2)}$ *there exists a* $\mathfrak{k}$*-subrepresentation* $W\subseteq\mathcal{D}'(\mathbb{R}^\times)\widehat{\otimes}\mathcal{S}'(\Lambda)$*, explicitly given in Theorem 5.4.1, which is isomorphic to the representation* $S^2(\mathbb{C}^2)\boxtimes\mathbb{C}$ *of* $\mathfrak{k}\simeq\mathfrak{k}'\oplus\mathfrak{su}(2)$*.*

(4) For $\mathfrak{g}=\mathfrak{sl}(n,\mathbb{R})$ *there exist for every* $r\in\mathbb{C}$ *two* $\mathfrak{k}$*-subrepresentations* $W_{\varepsilon,r}\subseteq\mathcal{D}'(\mathbb{R}^\times)\widehat{\otimes}\mathcal{S}'(\Lambda)$ *(*$\varepsilon\in\mathbb{Z}/2\mathbb{Z}$*) of the representation* $d\pi_{\min,r}$ *of* $\mathfrak{g}$*, explicitly given in Theorem 5.5.1. The representation* $W_{0,r}$ *is isomorphic to the trivial representation of* $\mathfrak{k}$ *and* $W_{1,r}$ *is isomorphic to the standard representation* $\mathbb{C}^n$ *of* $\mathfrak{k}\simeq\mathfrak{so}(n)$*.*

(5) For $\mathfrak{g}=\mathfrak{sl}(3,\mathbb{R})$ *and* $r=0$ *there exists a third* $\mathfrak{k}$*-subrepresentation* $W_{\frac{1}{2}}\subseteq\mathcal{D}'(\mathbb{R}^\times)\widehat{\otimes}\mathcal{S}'(\Lambda)$ *of* $d\pi_{\min,r}$*, explicitly given in Theorem 5.6.1, which is isomorphic to the representation* $\mathbb{C}^2$ *of* $\mathfrak{k}\simeq\mathfrak{su}(2)$*.*

(6) For $\mathfrak{g}=\mathfrak{so}(p,q)$*,* $p\geq q\geq 4$ *with* $p+q$ *even, there exists a* $\mathfrak{k}$*-subrepresentation* $W\subseteq\mathcal{D}'(\mathbb{R}^\times)\widehat{\otimes}\mathcal{S}'(\Lambda)$*, explicitly given in Theorem 5.7.6, which is isomorphic to the representation* $\mathbb{C}\boxtimes\mathcal{H}^{\frac{p-q}{2}}(\mathbb{R}^q)$ *of* $\mathfrak{k}\simeq\mathfrak{so}(p)\oplus\mathfrak{so}(q)$*.*

(7) For $\mathfrak{g}=\mathfrak{so}(p,3)$*,* $p\geq 3$*, there exists a* $\mathfrak{k}$*-subrepresentation* $W\subseteq\mathcal{D}'(\mathbb{R}^\times)\widehat{\otimes}\mathcal{S}'(\Lambda)$*, explicitly given in Theorem 5.8.3, which is isomorphic to the representation* $\mathbb{C}\boxtimes S^{p-3}(\mathbb{C}^2)$ *of* $\mathfrak{k}\simeq\mathfrak{so}(p)\oplus\mathfrak{su}(2)$*.*

The explicit form for the spherical vector for $\mathfrak{g}=\mathfrak{so}(n,n),\mathfrak{e}_{6(6)},\mathfrak{e}_{7(7)},\mathfrak{e}_{8(8)}$ has previously been found by Kazhdan, Pioline and Waldron [31] and a similar formula for a K-finite vector in the case $\mathfrak{g}=\mathfrak{g}_{2(2)}$ can be found in [18]. Further, for $\mathfrak{sl}(3,\mathbb{R})$ the lowest K-types $W_{0,r}$, $W_{1,r}$ and $W_{\frac{1}{2}}$ were obtained by Torasso [53]. We believe that the other formulas are new.

The minimal representation

A careful study of the action of $\mathfrak{g}$ on the lowest K-type W shows:

Theorem E. *The* K*-type* W *generates an irreducible* $(\mathfrak{g},K)$*-module* $\overline{W}=d\pi_{\min}(U(\mathfrak{g}))W$ *with lowest* K*-type* W*. This* $(\mathfrak{g},K)$*-module integrates to an irreducible unitary representation of* G *on* $L^2(\mathbb{R}^\times\times\Lambda)$ *which is minimal in the sense that its annihilator is a completely prime ideal with associated variety equal to the minimal nilpotent coadjoint orbit. For* $\mathfrak{g}$ *not of type* A*, the annihilator is the Joseph ideal.*

For the split groups $G=\mathrm{SO}(n,n),E_{6(6)},E_{7(7)},E_{8(8)}$ the same realization has been constructed by Kazhdan and Savin [32] and for $G=G_{2(2)}$ by Gelfand [12] (see also Savin [48]). For $G=\widetilde{\mathrm{SL}}(3,\mathbb{R})$ the representations were studied in detail by Torasso [53], and for $G=\widetilde{\mathrm{SO}}(4,3)$ the realization is similar to the one constructed by Sabourin [47]. We believe that the realization is new for the quaternionic groups $G=E_{6(2)},E_{7(-5)},E_{8(-24)}$ in which case the representations can also be obtained as continuation of the quaternionic discrete series by algebraic methods (see Gross and Wallach [16, 17]) and also for the groups $G = \widetilde{\mathrm{SO}}(p,q)$.

In view of the classification of minimal representations in [52], the construction in Theorem E together with the constructions in [25], [44] and [48] yield L^2-models for all minimal representations except for the ones of $F_{4(4)}$ and the complex groups $E_8(\mathbb{C})$ and $F_4(\mathbb{C})$. We believe that the case of $F_{4(4)}$ can be treated with a slight generalization of our methods to the vector-valued case (see Remark 5.2.2). It is further feasible that a construction similar to the one in [48, Section 7] may construct an L^2-model for the complex groups $E_8(\mathbb{C})$ and $F_4(\mathbb{C})$ from the one for $E_{8(8)}$ and $F_{4(4)}$. This would give L^2-models for all minimal representations of simple Lie groups.

The action of a non-trivial Weyl group element

The group G is generated by its maximal parabolic subgroup $\overline{P}$ and a non-trivial Weyl group element $w_1 \in K$. While the action of $\overline{P}$ in $\pi_{\min}$ is relatively simple (see Lemma 6.1.3), it is non-trivial to find explicit formulas for other group elements. In [32, 48] the authors were able to construct the above L^2-models for split groups by extending the action of $\overline{P}$ to G in terms of $\pi_{\min}(w_1)$ (see also [46] for the case of p-adic groups). Here, one has to verify that the definition for the operator $\pi_{\min}(w_1)$ satisfies several relations, and the methods does not seem to generalize in a straightforward way.

Having constructed the representations $\pi_{\min}$ by different methods, we are able to find the action $\pi_{\min}(w_1)$ of the Weyl group element w_1 in all cases explicitly. The action depends on the eigenvalues of a certain Lie algebra element on the lowest K-type. Those can be integers of half-integers, and we refer to these two cases as the *integer case* and the *half-integer case* (see Section 6.2 for details).

Theorem F (see Theorem 6.2.1)**.** *The element w_1 acts in the L^2-model of the minimal representation by*

$$\pi_{\min}(w_1)f(\lambda,a,x) = e^{-i\frac{n(x)}{\lambda a}} f(\sqrt{2}a, -\tfrac{\lambda}{\sqrt{2}}, x) \times \begin{cases} 1 & \textit{in the integer case,} \\ \varepsilon(a\lambda) & \textit{in the half-integer case,} \end{cases}$$

where

$$\varepsilon(x) = \begin{cases} 1 & \textit{for } x > 0, \\ i & \textit{for } x < 0. \end{cases}$$

This gives a complete description of $\pi_{\min}$ on the generators $\overline{P}$ and w_1 and generalizes the formulas for $\pi_{\min}(w_1)$ in [32, 48] . We remark that this viewpoint was also advocated in [36] where the action of a non-trivial Weyl group element was obtained in a different L^2-model for the minimal representation of $\mathrm{O}(p,q)$.

Outlook

Minimal representations have shown to be of importance in the theory of automorphic representations, for instance in the construction of exceptional theta series (see e.g. [31]), the study of

Fourier coefficients of automorphic forms (see e.g. [14, 15]) or the study of local components of global automorphic representations (see e.g. [2, 40]). Some of these works use L^2-realizations of minimal representations. We hope that our new L^2-models might help to generalize some of their results.

Another possible application concerns branching laws for unitary representations, i.e. the restriction of representations to subgroups. L^2-models have proven to be useful in the decomposition of restricted representations since here classical spectral theory of differential operators can be applied (see e.g. [5, 38, 43]). We expect our new L^2-models to be useful for the decomposition of restrictions of minimal representations.

It has further been observed that explicit realizations of small representations have fruitful connections to geometry, analysis and special functions (see e.g. [10, 24, 26, 35, 36, 37, 39, 41]). In our L^2-models the explicit K-finite vectors exhibited in Chapter 5 are for instance expressed in terms of K-Bessel functions. We believe that there are many additional connections between our new realizations and other branches of mathematics. For instance, it would be interesting to relate the L^2-model for the minimal representation of $G = \mathrm{SO}(p, q)$ constructed in this paper to the one obtained in [36, 39] by an explicit integral transformation.

A more direct further line of research is the investigation of the missing case $G = F_{4(4)}$ for which we expect a similar, possibly vector-valued, L^2-model. Also the case of Hermitian groups, which is missing in this work, is a possible further research question (see Section 1.9 for some structural results in this situation).

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Chapter 1

Structure theory for Heisenberg parabolic subgroups

In this preliminary section we study the structure of Heisenberg graded real Lie algebras. For much of this we follow [49], the statements in Sections 1.7, 1.8 and 1.9 are new.

1.1 Heisenberg parabolic subgroups

Let G be a connected non-compact simple real Lie group with finite center. We assume that G has a parabolic subgroup P whose nilradical is a Heisenberg group, i.e. two-step nilpotent with one-dimensional center. Then P is maximal parabolic except in the case where G is locally isomorphic to $\mathrm{SL}(n,\mathbb{R})$. Let $P = MAN$ be the Langlands decomposition of P and denote by $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{n}$ the corresponding Lie algebras of G, P, M, A and N. See Table D.1 for a classification due to Cheng [3].

There is a unique grading element $H \in \mathfrak{a}$ such that $\mathrm{ad}(H)$ has eigenvalues 1 and 2 on $\mathfrak{n}$. Write

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$$

for the decomposition of $\mathfrak{g}$ into eigenspaces of $\mathrm{ad}(H)$, so that $\mathfrak{m} \oplus \mathfrak{a} = \mathfrak{g}_0$ and $\mathfrak{n} = \mathfrak{g}_1 + \mathfrak{g}_2$. Denote by $\overline{\mathfrak{p}} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0$ the opposite parabolic subalgebra with nilradical $\overline{\mathfrak{n}} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}$ and let $\overline{P}$ and $\overline{N}$ be the corresponding groups. In all cases but $\mathfrak{g} \simeq \mathfrak{sl}(n,\mathbb{R})$ we then have

$$\mathfrak{a} = \mathbb{R}H, \qquad \mathfrak{m} = \{X \in \mathfrak{g}_0 : \mathrm{ad}(X)|_{\mathfrak{g}_2} = 0\}.$$

To simplify notation, we use this as a definition for $\mathfrak{a}$ and $\mathfrak{m}$ in the case $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$.

Since M commutes with $A = \exp(\mathbb{R}H)$, M preserves the 5-grading of $\mathfrak{g}$. In particular, M acts on the subspaces $\mathfrak{g}_{\pm 2}$ by the adjoint representation. These subspaces are one-dimensional since they are the center of the Heisenberg algebra $\mathfrak{n}$ resp. $\overline{\mathfrak{n}}$, so there exists a character $\chi : M \to \{\pm 1\}$ such that

$$\mathrm{Ad}(m)|_{\mathfrak{g}_{\pm 2}} = \chi(m) \cdot \mathrm{id}_{\mathfrak{g}_{\pm 2}}.$$

1.2 The $\mathfrak{sl}_2$-triple and the Weyl group element w_0

We can choose $E \in \mathfrak{g}_2$ and $F \in \mathfrak{g}_{-2}$ such that $[E, F] = H$. Then $\{E, H, F\}$ forms an $\mathfrak{sl}_2$-triple. Put

$$w_0 := \exp\left(\frac{\pi}{2}(E - F)\right),$$

then $\mathrm{Ad}(w_0) : \mathfrak{g} \to \mathfrak{g}$ is of order four. On the $\mathfrak{sl}_2$-triple it is given by

$$\mathrm{Ad}(w_0)E = -F, \qquad \mathrm{Ad}(w_0)H = -H, \qquad \mathrm{Ad}(w_0)F = -E,$$

and it acts trivially on $\mathfrak{m}$. Further, $\mathrm{Ad}(w_0)$ restricts to isomorphisms $\mathfrak{g}_1 \to \mathfrak{g}_{-1}$ and $\mathfrak{g}_{-1} \to \mathfrak{g}_1$ which compose to -1 times the identity. We put

$$\begin{aligned} \overline{X} &:= \mathrm{ad}(X)F = \mathrm{Ad}(w_0)X = -\mathrm{Ad}(w_0^{-1})X, && X \in \mathfrak{g}_1, \\ \overline{Y} &:= \mathrm{ad}(Y)E = -\mathrm{Ad}(w_0)Y = \mathrm{Ad}(w_0^{-1})Y, && Y \in \mathfrak{g}_{-1}. \end{aligned}$$

Note that these maps are mutually inverse and

$$\overline{\mathrm{Ad}(m)X} = \chi(m)\,\mathrm{Ad}(m)\overline{X} \qquad \forall\, m \in M, X \in \mathfrak{g}_{\pm 1}.$$

We further remark that P and $\overline{P}$ are conjugate via w_0:

$$\overline{P} = w_0 P w_0^{-1} = w_0^{-1} P w_0.$$

1.3 Minimal adjoint orbits

The adjoint orbit

$$\mathcal{O}_{\min} = \mathrm{Ad}(G)E \subseteq \mathfrak{g}$$

is a minimal nilpotent orbit. If G is non-Hermitian then $\mathcal{O}_{\min} = -\mathcal{O}_{\min}$ is the unique minimal nilpotent orbit. If G is Hermitian then $\mathcal{O}_{\min}$ and $-\mathcal{O}_{\min}$ are the two distinct minimal nilpotent orbits.

Lemma 1.3.1. *The stabilizer of E in G is given by M_1N where $M_1 = \{m \in M : \mathrm{Ad}(m)E = E\} = \{m \in M : \chi(m) = 1\} \subseteq M$ is a subgroup of index at most 2.*

Proof. Let $g \in G$ such that $\mathrm{Ad}(g)E = E$. We claim that $g \in N_G(\mathfrak{p}) = P$. In fact, for $x \in \mathfrak{g}_1$ and $T \in \mathfrak{m}$ we have

$$[\mathrm{Ad}(g)x, E] = \mathrm{Ad}(g)[x, E] = 0 \qquad \text{and} \qquad [\mathrm{Ad}(g)T, E] = \mathrm{Ad}(g)[T, E] = 0$$

and hence $\mathrm{Ad}(g)x, \mathrm{Ad}(g)T \in Z_{\mathfrak{g}}(E) = \mathfrak{m} + \mathfrak{g}_1 + \mathfrak{g}_2$. Further,

$$[\mathrm{Ad}(g)H, E] = \mathrm{Ad}(g)[H, E] = 2\,\mathrm{Ad}(g)E = 2E$$

and hence $\mathrm{Ad}(g)H \in H + \mathfrak{m} + \mathfrak{g}_1 + \mathfrak{g}_2$. This shows that $\mathrm{Ad}(g)\mathfrak{p} \subseteq \mathfrak{p}$ and therefore $g \in N_G(\mathfrak{p}) = P = MAN$. Write $g = man$ with $m \in M$, $a = \exp(rH) \in A$ and $n \in N$, then $\mathrm{Ad}(g)E = \chi(m)e^{2r}E$. This shows that $r = 0$ and hence $a = 1$, and $m \in M_1 = \{g \in M : \chi(g) = 1\}$. ☐

1.4 The symplectic invariants

Put $V := \mathfrak{g}_{-1}$. The bilinear form $\omega : V \times V \to \mathbb{R}$ given by

$$[X,Y] = \omega(X,Y)F, \qquad X,Y \in V,$$

turns V into a symplectic vector space. We note that for $X, Y \in \mathfrak{g}_1$ we have

$$[X,Y] = -\omega(\overline{X},\overline{Y})E.$$

We now define three maps on V related to the symplectic structure.

The moment map

Put

$$\mu : V \to \mathfrak{g}_0, \quad x \mapsto \frac{1}{2!}\,\mathrm{ad}(x)^2 E.$$

Then μ actually maps into $\mathfrak{m}$ and is $\mathrm{Ad}(M)$-equivariant, i.e.

$$\mu(\mathrm{Ad}(m)x) = \chi(m)\,\mathrm{Ad}(m)\mu(x), \qquad m \in M.$$

The cubic map

Put

$$\Psi : V \to V, \quad x \mapsto \frac{1}{3!}\,\mathrm{ad}(x)^3 E.$$

Then Ψ is $\mathrm{Ad}(M)$-equivariant, i.e.

$$\Psi(\mathrm{Ad}(m)x) = \chi(m)\,\mathrm{Ad}(m)\Psi(x), \qquad m \in M.$$

The quartic

Define $Q : V \to \mathbb{R}$ by

$$Q(x)F = \frac{1}{4!}\,\mathrm{ad}(x)^4 E.$$

Then Q transforms under the action $\mathrm{Ad}(M)$ by the character χ, i.e.

$$Q(\mathrm{Ad}(m)x) = \chi(m)Q(x), \qquad m \in M.$$

Symplectic formulas

We state some formulas for the symplectic covariants μ, Ψ and Q proved in [49, 51]. (Note that μ, Ψ and Q_n in their notation are -2μ, 6Ψ and $36Q$ in our notation.)

Lemma 1.4.1 ([49, Proposition 3.36]). *For $x \in V$ the following identities hold:*

(1) $\mu(ax + b\Psi(x)) = (a^2 - b^2 Q(x))\mu(x)$,

(2) $\Psi(ax+b\Psi(x)) = (a^2-b^2Q(x))(bQ(x)x+a\Psi(x))$,

(3) $Q(ax+b\Psi(x)) = (a^2-b^2Q(x))^2Q(x)$,

(4) $\mu(x)\Psi(x) = -3Q(x)x$.

Let B_μ, B_Ψ and B_Q denote the symmetrizations of μ, Ψ and Q, respectively. Further, let

$$\tau(x)y = \omega(x,y)x \qquad \text{and} \qquad B_\tau(x,y)z = \tfrac{1}{2}(\omega(x,z)y+\omega(y,z)x).$$

Lemma 1.4.2 ([51, Proposition 2.2]). *Let $x,y,z,w \in V$.*

(1) $B_\mu(x,y) = \frac{1}{4}([x,[y,E]]+[y,[x,E]])$,

(2) $B_\Psi(x,y,z) = -\frac{1}{3}[B_\mu(x,y),z] - \frac{1}{6}B_\tau(x,y)z$,

(3) $B_Q(x,y,z,w) = \frac{1}{4}\omega(x,B_\Psi(y,z,w))$.

Note that this implies in particular that

$$\omega(B_\mu(x,y)z,w) = \omega(B_\mu(z,w)x,y) \qquad \text{for all } x,y,z,w \in V.$$

Lemma 1.4.3 ([49, Proposition 3.9]). *For $x,y,z \in V$*

$$4[B_\mu(x,y),z] - 4[B_\mu(x,z),y] = \omega(x,y)z - \omega(x,z)y - 2\omega(y,z)x.$$

The Lie bracket

We express the Lie bracket $[\mathfrak{g}_1,\mathfrak{g}_{-1}]$ in terms of the moment map and the symplectic form.

Lemma 1.4.4. *For $v \in \mathfrak{g}_1$ and $w \in \mathfrak{g}_{-1}$ the decomposition of $[v,w] \in \mathfrak{g}_0$ in terms of the decomposition $\mathfrak{g}_0 = \mathfrak{m}+\mathfrak{a}$ is given by*

$$[v,w] = -2B_\mu(\overline{v},w) - \tfrac{1}{2}\omega(\overline{v},w)H.$$

Proof. We have

$$\begin{aligned}
[v,w] &= [\overline{\overline{v}},w] = [[\overline{v},E],w] = -[w,[\overline{v},E]] \\
&= -\tfrac{1}{2}\left([\overline{v},[w,E]]+[w,[\overline{v},E]]\right) + \tfrac{1}{2}[\operatorname{ad}(\overline{v}),\operatorname{ad}(w)]E \\
&= -2B_\mu(\overline{v},w) + \tfrac{1}{2}\operatorname{ad}([\overline{v},w])E = -2B_\mu(\overline{v},w) + \tfrac{1}{2}\omega(\overline{v},w)\operatorname{ad}(F)E \\
&= -2B_\mu(\overline{v},w) - \tfrac{1}{2}\omega(\overline{v},w)H.
\end{aligned}$$

□

Simple factors of $\mathfrak{m}$

We note that $\mathfrak{m}$ is reductive with at most one-dimensional center. The proof of the following result was communicated to us by R. Stanton:

Lemma 1.4.5. *Let (e_α) be a basis of V and $(\widehat{e}_\alpha)$ its dual basis with respect to the symplectic form ω, i.e. $\omega(e_\alpha, \widehat{e}_\beta) = \delta_{\alpha\beta}$. Then the map*

$$\mathfrak{m} \to \mathfrak{m}, \quad T \mapsto \sum_\alpha B_\mu(Te_\alpha, \widehat{e}_\alpha)$$

is a scalar multiple $\mathcal{C}(\mathfrak{m}') \cdot \mathrm{id}_{\mathfrak{m}'}$ of the identity on the center and on each simple factor $\mathfrak{m}'$ of $\mathfrak{m}$. In particular, if $\mathfrak{m}$ is simple the map is a scalar multiple of the identity.

Proof. The expression is obviously independent of the chosen basis, so that we may assume $\{e_\alpha\}$ to be a symplectic basis of the form $\{e_i, f_j\}$ with $\omega(e_i, f_j) = \delta_{ij}$ and $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$. Then $\{\widehat{e}_\alpha\} = \{f_i, -e_j\}$ and the sum becomes

$$\sum_i \Big(B_\mu(Te_i, f_i) + B_\mu(Tf_i, -e_i) \Big) = \sum_i \Big(B_\mu(Te_i, f_i) - B_\mu(e_i, Tf_i) \Big).$$

For $x, y \in V$ we define

$$\mathrm{Bz}(x, y) : \mathfrak{m} \to \mathfrak{m}, \quad \mathrm{Bz}(x, y)T = B_\mu(Tx, y) - B_\mu(x, Ty).$$

Bz is called the *Bezoutian* and it is easy to see that $\mathrm{Bz} : \Lambda^2 V \to \mathrm{End}(\mathfrak{m})$ is $\mathfrak{m}$-equivariant. Now, the trivial representation of $\mathfrak{m}$ is contained in $\Lambda^2 V$ as the symplectic form, and therefore its image in $\mathrm{End}(\mathfrak{m})$ under Bz has to be a copy of the trivial representation. Since $\mathfrak{m}$ is reductive with at most one-dimensional center, the copies of the trivial representation in $\mathrm{End}(\mathfrak{m})$ are given by the identity on the center and on each simple factor of $\mathfrak{m}$. This shows the claim. □

In [1] the numbers $\mathcal{C}(\mathfrak{m}')$ are computed for all simple factors $\mathfrak{m}'$ of $\mathfrak{m}$ and we have included them in Table D.1.

Corollary 1.4.6. *For every factor $\mathfrak{m}'$ of $\mathfrak{m}$ we have*

$$\mathrm{tr}(T\mu(x)) = \mathcal{C}(\mathfrak{m}')\omega(Tx, x) \qquad \forall\, x \in V, T \in \mathfrak{m}'.$$

Proof. Let (e_α) be a basis of V and $\widehat{e}_\alpha$ the dual basis with respect to the symplectic form, i.e. $\omega(e_\alpha, \widehat{e}_\beta) = \delta_{\alpha\beta}$. Then

$$\begin{aligned}
\mathrm{tr}(T\mu(x)) &= \sum_\alpha \omega(T\mu(x)e_\alpha, \widehat{e}_\alpha) = \sum_\alpha \omega(T\widehat{e}_\alpha, \mu(x)e_\alpha) \\
&= \sum_\alpha \omega(T\widehat{e}_\alpha, -3B_\Psi(x, x, e_\alpha) - \frac{1}{2}\tau(x)e_\alpha) \\
&= -12 \sum_\alpha B_Q(x, x, e_\alpha, T\widehat{e}_\alpha) + \frac{1}{2}\omega(Tx, x) \\
&= -3 \sum_\alpha \omega(x, B_\Psi(e_\alpha, T\widehat{e}_\alpha, x)) + \frac{1}{2}\omega(Tx, x) \\
&= \sum_\alpha \omega(x, B_\mu(e_\alpha, T\widehat{e}_\alpha)x) + \frac{1}{2}\omega(x, B_\tau(e_\alpha, T\widehat{e}_\alpha)x) + \frac{1}{2}\omega(Tx, x) \\
&= \mathcal{C}(\mathfrak{m}')\omega(Tx, x).
\end{aligned}$$

□

1.5 The Killing form

We compute the Killing form $\kappa(X, Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ on $\mathfrak{g}$. For this let

$$p := \dim \mathfrak{g}_1 + 4.$$

Lemma 1.5.1. *Let $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$, then $\kappa(X, Y) = 0$ unless $i + j = 0$. Further,*

$$\begin{aligned} \kappa(E, F) &= p, & \\ \kappa(v, w) &= -p\omega(\overline{v}, w), & v \in \mathfrak{g}_1, w \in \mathfrak{g}_{-1}, \\ \kappa(S + aH, T + bH) &= \kappa_{\mathfrak{m}}(S, T) + 2\,\mathrm{tr}(\mathrm{ad}(S) \circ \mathrm{ad}(T))|_{\mathfrak{g}_1} + 2pab, & S, T \in \mathfrak{m}, a, b \in \mathbb{R}. \end{aligned}$$

Proof. It is clear that $\kappa(\mathfrak{g}_i, \mathfrak{g}_j) = \{0\}$ unless $i + j = 0$. The formulas for $\kappa(E, F)$ and $\kappa(H, H)$ are proven in [49, Proposition 2.2] and the formula for $\kappa(S, T)$ is clear since $\mathfrak{m}$ acts trivially on E and F. It therefore remains to show the formula for $\kappa(v, w)$, $v \in \mathfrak{g}_1$ and $w \in \mathfrak{g}_{-1}$. Using ad-invariance of the Killing form we have

$$\kappa(v, w) = \kappa([\overline{v}, E], w) = -\kappa(E, [\overline{v}, w]) = -\kappa(E, \omega(\overline{v}, w)F) = -p\omega(\overline{v}, w). \qquad \square$$

1.6 The Heisenberg nilradical

The unipotent subgroup $\overline{N}$ is a Heisenberg group and hence diffeomorphic to its Lie algebra $\overline{\mathfrak{n}}$. We identify $\overline{N} \simeq \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \simeq V \times \mathbb{R}$ via

$$V \times \mathbb{R} \xrightarrow{\sim} \overline{N}, \quad (x, s) \mapsto \overline{n}_{(x,s)} := \exp(x + sF).$$

The group multiplication in $\overline{N}$ is given by

$$\overline{n}_{(x,s)} \cdot \overline{n}_{(y,t)} = \overline{n}_{(x+y,s+t+\frac{1}{2}\omega(x,y))}, \qquad x, y \in V, s, t \in \mathbb{R}.$$

Hence, the map $V \times \mathbb{R} \to \overline{N}$, $(x, s) \mapsto \overline{n}_{(x,s)}$ turns into a group isomorphism if we equip $V \times \mathbb{R}$ with the product

$$(x, s) \cdot (y, t) := (x + y, s + t + \tfrac{1}{2}\omega(x, y)).$$

1.7 Bruhat decomposition

The natural multiplication map

$$\overline{N} \times M \times A \times N \to G$$

is a diffeomorphism onto an open dense subset of G, the open dense Bruhat cell. Hence, every $g \in \overline{N}MAN \subseteq G$ decomposes uniquely into

$$g = \overline{n}(g)m(g)a(g)n.$$

We identify $\mathfrak{a}_{\mathbb{C}}^* \simeq \mathbb{C}$ by $\nu \mapsto \nu(H)$. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we write $a^\lambda = e^{\lambda(X)}$ where $a = e^X \in A$ with $X \in \mathfrak{a}$.

Lemma 1.7.1. *For* $(x,s) \in V \times \mathbb{R}$ *we have* $w_0^{-1}\overline{n}_{(x,s)} \in \overline{N}MAN$ *if and only if* $s^2 - Q(x) \neq 0$. *In this case*

$$a(w_0^{-1}\overline{n}_{(x,s)})^\lambda = |s^2 - Q(x)|^{\lambda/2} \qquad \textit{and} \qquad \log \overline{n}(w_0^{-1}\overline{n}_{(x,s)}) = \frac{1}{s^2 - Q(x)}(\Psi(x) - sx, -s).$$

Moreover, the quartic Q *is non-positive if and only if the character* $\chi : M \to \{\pm 1\}$ *is trivial.*

Proof. Assume $w_0^{-1}\overline{n}_{(x,s)} = \overline{n}_{(y,t)} m \exp(rH) n \in \overline{N}MAN$. We let both sides act on E by the adjoint representation and then compare the results. Let us first compute $\mathrm{Ad}(w_0^{-1}\overline{n}_{(x,s)})E$. We have

$$\begin{aligned} \mathrm{ad}(x + sF)E &= \overline{x} - sH, \\ \mathrm{ad}(x + sF)^2 E &= 2(\mu(x) - sx - s^2 F), \\ \mathrm{ad}(x + sF)^3 E &= 6\Psi(x), \\ \mathrm{ad}(x + sF)^4 E &= 24 Q(x) F, \\ \mathrm{ad}(x + sF)^5 E &= 0, \end{aligned}$$

and hence

$$\mathrm{Ad}(\overline{n}_{(x,s)})E = E + \overline{x} + \mu(x) - sH + \Psi(x) - sx + (Q(x) - s^2)F. \tag{1.7.1}$$

Applying $\mathrm{Ad}(w_0^{-1}) = \mathrm{Ad}(w_0)^{-1}$ yields

$$\mathrm{Ad}(w_0^{-1}\overline{n}_{(x,s)})E = (s^2 - Q(x))E + \overline{\Psi(x) - sx} + \mu(x) + sH - x - F. \tag{1.7.2}$$

Now let us compute $\mathrm{Ad}(\overline{n}_{(y,t)} m \exp(rH) n)E$. First, N acts trivially on E. Further, M acts on E by the character $\chi : M \to \{\pm 1\}$. Therefore,

$$\begin{aligned} \mathrm{Ad}(\overline{n}_{(y,t)} m \exp(rH) n)E &= \chi(m) e^{2r} \,\mathrm{Ad}(\overline{n}_{(y,t)})E \\ &= \chi(m) e^{2r} \left(E + \overline{y} + \mu(y) - tH + \Psi(y) - ty + (Q(y) - t^2)F\right). \end{aligned} \tag{1.7.3}$$

Comparing with (1.7.2) shows that

$$s^2 - Q(x) = \chi(m)e^{2r}, \quad \Psi(x) - sx = \chi(m)e^{2r} y, \quad \mu(x) = \chi(m)e^{2r}\mu(y)$$

$$s = -\chi(m)e^{2r} t, \qquad x = -\chi(m)e^{2r}(\Psi(y) - ty), \qquad 1 = \chi(m)e^{2r}(t^2 - Q(y)).$$

The first identity shows that if $\{x \in \overline{V} : Q(x) > 0\} \neq \emptyset$ then there exists $m \in M$ such that $\chi(m) = -1$, because otherwise the non-empty open set of all $w_0^{-1}\overline{n}_{(x,t)} man$ with $t^2 - Q(x) < 0$ and $m \in M$, $a \in A$, $n \in N$, would have trivial intersection with the open dense Bruhat cell $\overline{N}MAN$. Further, the first, second and fourth identities show that

$$s^2 - Q(x) = \pm e^{2r} \neq 0 \qquad \text{and} \qquad (y,t) = \frac{1}{s^2 - Q(x)}(\Psi(x) - sx, -s).$$

Conversely, if $s^2 - Q(x) \neq 0$ then let

$$r := \frac{1}{2}\log|s^2 - Q(x)| \qquad \text{and} \qquad (y,t) = \frac{1}{s^2 - Q(x)}(\Psi(x) - sx, -s),$$

and choose $m \in M$ such that $\chi(m) = \operatorname{sgn}(s^2 - Q(x))$. Using the above computation as well as Lemma 1.4.1 one can show that

$$\operatorname{Ad}(w_0^{-1}\overline{n}_{(x,s)})E = \operatorname{Ad}(\overline{n}_{(y,t)}me^{rH})E.$$

Since the stabilizer of E in G is equal to M_1N there exist $m' \in M_1$ and $n \in N$ such that $w_0^{-1}\overline{n}_{(x,s)} = \overline{n}_{(y,t)}mm'e^{rH}n \in \overline{N}MAN$. This show the claim. ∎

Lemma 1.7.2. *For $(x,t) \in V \times \mathbb{R}$ and $s \in \mathbb{R}$ sufficiently close to 0 we have*

$$\log \overline{n}(e^{-sE}\overline{n}_{(x,t)}) = \left(\frac{x + s(\Psi(x) - tx)}{1 - 2st - s^2(Q(x) - t^2)}, \frac{t + s(Q(x) - t^2)}{1 - 2st - s^2(Q(x) - t^2)}\right),$$
$$\left.\frac{d}{ds}\right|_{s=0} m(e^{-sE}\overline{n}_{(x,t)}) = -\mu(x),$$
$$a(e^{-sE}\overline{n}_{(x,t)})^\lambda = (1 - 2st - s^2(Q(x) - t^2))^{\frac{\lambda}{2}}.$$

Proof. Write

$$e^{-sE}\overline{n}_{(x,t)} = \exp(y + uF)m_s e^{rH}\exp(z + vE) \tag{1.7.4}$$

for $y, z \in V$, $r, u, v \in \mathbb{R}$ and $m_s \in M$. We first act with both sides of (1.7.4) on E by the adjoint action. By (1.7.1) we have

$$\operatorname{Ad}(\overline{n}_{(x,t)})E = E + \overline{x} + \mu(x) - tH + (\Psi(x) - tx) + (Q(x) - t^2)F.$$

Now, $\operatorname{Ad}(e^{-sE}) = e^{-s\operatorname{ad}(E)}$ and

$$\begin{aligned}
\operatorname{ad}(E)\operatorname{Ad}(\overline{n}_{(x,t)})E &= 2tE + \overline{tx - \Psi(x)} + (Q(x) - t^2)H,\\
\operatorname{ad}(E)^2\operatorname{Ad}(\overline{n}_{(x,t)})E &= -2(Q(x) - t^2)E,\\
\operatorname{ad}(E)^3\operatorname{Ad}(\overline{n}_{(x,t)})E &= 0,
\end{aligned}$$

hence

$$\begin{aligned}
\operatorname{Ad}(e^{-sE}\overline{n}_{(x,t)})E = (1 - 2st - s^2(Q(x) - t^2))E + \overline{x + s(\Psi(x) - tx)} + \mu(x)\\
- (t + s(Q(x) - t^2))H + (\Psi(x) - tx) + (Q(x) - t^2)F.
\end{aligned}$$

On the other hand, by (1.7.3):

$$\begin{aligned}
&\operatorname{Ad}\left(\exp(y + uF)m_s e^{rH}\exp(z + vE)\right)E\\
&\qquad = \chi(m_s)e^{2r}\left(E + \overline{y} + \mu(y) - uH + \Psi(y) - uy + (Q(y) - u^2)F\right).
\end{aligned}$$

Comparing the two expressions shows the formulas for $\overline{n}(e^{-sE}\overline{n}_{(x,t)})$ and $a(e^{-sE}\overline{n}_{(x,t)})$. To find $\left.\frac{d}{ds}\right|_{s=0} m_s$ we let both sides of (1.7.4) act on $\overline{a} \in \mathfrak{g}_1$. By similar computations we arrive at

$$\begin{aligned}
\operatorname{Ad}\left(e^{-sE}\overline{n}_{(x,t)}\right)\overline{a} = \big(-s\omega(x,a) + s^2\omega(tx + \Psi(x), a)\big)E + \overline{(a - s(ta + \mu(x)a + \omega(x,a)x))}\\
+ 2B_\mu(x,a) + \big(-\tfrac{1}{2}\omega(x,a) + s\omega(tx + \Psi(x), a)\big)H\\
- \big(ta + \mu(x)a + \omega(x,a)x\big) - \omega(tx + \Psi(x), a)F
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Ad}\big(\exp(y+uF)m_se^{rH}\exp(z+vE)\big)\overline{a}\\
&= e^r\Big(\overline{m_sa}+2B_\mu(y,m_s)-\tfrac{1}{2}\omega(y,m_sa)H-\big(um_sa+\mu(y)m_sa+\omega(y,m_sa)y\big)-\omega(uy+\Psi(y),m_sa)F\Big)\\
&\qquad\qquad +\omega(a,z)e^{2r}\Big(E+\overline{y}+\mu(y)-uH+\big(\Psi(y)-uy\big)+\big(Q(y)-u^2\big)F\Big).
\end{aligned}$$

Comparing the coefficients of E shows

$$z=\frac{sx-s^2(tx+\Psi(x))}{1-2st-s^2(Q(x)-t^2)}.$$

Next, comparing the terms in $\mathfrak{g}_1$ and using the previously obtained formula for y yields

$$e^rm_sa=a-s(ta+\mu(x)a+\omega(x,a)x)+\omega(z,a)(x-s(tx-\Psi(x))).$$

Note that $\frac{d}{ds}\big|_{s=0}e^r=-t$ and $\frac{d}{ds}\big|_{s=0}z=x$, hence differentiating the above identity gives

$$-ta+\frac{d}{ds}\bigg|_{s=0}m_sa=-(ta+\mu(x)a+\omega(x,a)x)+\omega(x,a)x=-ta-\mu(x)a$$

and the claim follows. □

Lemma 1.7.3. *Assume that $\mathfrak{g}_\mathbb{C}$ is not of type A. Then the elements*

$$2\cdot F\cdot T-\sum_{\alpha,\beta}\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\,e_\alpha\cdot e_\beta \qquad T\in\mathfrak{m},$$

$$6(F\overline{v}+\overline{v}F)+3(H\overline{v}+\overline{v}H)+4\sum_\alpha\big(e_\alpha B_\mu(v,\widehat{e}_\alpha)+B_\mu(v,\widehat{e}_\alpha)e_\alpha\big), \qquad v\in V,$$

$$4(EF+FE)+2H^2+\sum_\alpha\big(e_\alpha\overline{\widehat{e}_\alpha}+\overline{\widehat{e}_\alpha}e_\alpha\big)$$

generate an ideal in $U(\mathfrak{g}_\mathbb{C})$ whose associated variety is the minimal nilpotent coadjoint orbit in $\mathfrak{g}_\mathbb{C}$.

Proof. The minimal nilpotent adjoint orbit is given by $\mathcal{O}_{\min}=\mathrm{Ad}(G)E$. Since $\overline{N}MAN$ is open dense in G, the set $\mathrm{Ad}(\overline{N}MAN)E\subseteq$ is dense in $\mathcal{O}_{\min}$. We have $\mathrm{Ad}(MAN)E=\pm\mathbb{R}_+E$, the sign depending on whether M acts trivially on E or not. It follows that $\mathrm{Ad}(\overline{N}MAN)E=\pm\mathbb{R}_+\,\mathrm{Ad}(\overline{N})E$ and hence

$$\overline{\mathcal{O}_{\min,\mathbb{C}}}=\mathbb{C}\,\mathrm{Ad}(\overline{N}_\mathbb{C})E.$$

From (1.7.1) it is easy to see that this affine variety is the zero set of the second order (possibly vector-valued) polynomials

$$aS-\mu(x),\quad 3ay+Sx-3tx,\quad ab-\frac{1}{4}\omega(x,y)+t^2$$

in the variable $X=aE+\overline{x}+S+tH+y+bF$. Carefully applying the identification $\mathfrak{g}_\mathbb{C}\simeq\mathfrak{g}_\mathbb{C}^*$ given by the Killing form and the natural isomorphism $\mathbb{C}[\mathfrak{g}_\mathbb{C}^*]\simeq S(\mathfrak{g}_\mathbb{C})\simeq\operatorname{gr}U(\mathfrak{g}_\mathbb{C})$ produces the claimed generators. □

1.8 Maximal compact subgroups

Let θ be a Cartan involution of G and denote by θ also the corresponding involution on $\mathfrak{g}$. We may choose θ such that $\theta\mathfrak{g}_i = \mathfrak{g}_{-i}$, $i \in \{-2,-1,0,1,2\}$ and also $\theta\mathfrak{m} = \mathfrak{m}$ and $\theta H = -H$. Possibly rescaling E and F we may further assume that $\theta E = -F$ and $\theta F = -E$. Define $J \in \operatorname{End}(V)$ by

$$Jv := \overline{\theta x} = -\theta\overline{x}, \qquad x \in V.$$

Then $J^2 = -\mathbf{1}$ and the bilinear form on V given by.

$$(x|y) = \tfrac{1}{4}\omega(Jx, y), \qquad x, y \in V,$$

is positive definite. Write $|x|^2 = (x|x) = \frac{1}{4}\omega(Jx, x)$ for the corresponding norm on V. Further note that

$$\operatorname{Ad}(\theta(T))|_V = J \circ \operatorname{Ad}(T)|_V \circ J^{-1} \qquad \forall T \in \mathfrak{m}.$$

It is easy to see that

$$\omega(Jx, Jy) = \omega(x, y) \qquad \forall x, y \in V,$$

so that $J \in \operatorname{Sp}(V, \omega)$. For the symplectic covariants we further have for $x \in V$:

$$\mu(Jx) = J \circ \mu(x) \circ J^{-1}, \qquad \Psi(Jx) = J\Psi(x), \qquad Q(Jx) = Q(x). \tag{1.8.1}$$

The following result is a converse to the construction of J from θ:

Lemma 1.8.1. *Let $J \in \operatorname{End}(V)$ and define $\theta : \mathfrak{g} \to \mathfrak{g}$ by*

$$\theta E = -F, \qquad \theta H = -H, \qquad \theta F = -E,$$

and for $x \in \mathfrak{g}_1$, $y \in \mathfrak{g}_{-1}$ and $T \in \mathfrak{m}$ by

$$\theta x = -J\overline{x}, \qquad \theta T = JTJ^{-1}, \qquad \theta y = \overline{Jy}.$$

Then θ is a Cartan involution of $\mathfrak{g}$ if and only if the following conditions are satisfied:

(1) $J^2 = -\mathbf{1}$,

(2) $\omega(Jx, x) \geq 0$ for all $x \in V$,

(3) $\omega(Jx, Jy) = \omega(x, y)$ for all $x, y \in V$,

(4) $\mu(Jx) = J\mu(x)J^{-1}$ for all $x \in V$.

Proof. It is easy to see that conditions (3) and (4) imply (1.8.1). Then a short computation shows that θ is indeed a Lie algebra automorphism. That $\kappa(\theta X, X) \geq 0$ for all $X \in \mathfrak{g}$ now follows from (2) and Lemma 1.5.1, and that $\theta^2 = \mathbf{1}$ is a consequence of (1). □

Let $K = G^\theta$ be the subgroup of θ-fixed points in G. Then K is maximal compact in G and its Lie algebra $\mathfrak{k}$ has the form

$$\mathfrak{k} = \mathbb{R}(E - F) \oplus \{x + \theta(x) : x \in \mathfrak{g}_1\} \oplus \mathfrak{k}_\mathfrak{m},$$

where $\mathfrak{k}_\mathfrak{m} = \mathfrak{k} \cap \mathfrak{m} \subseteq \mathfrak{m}$ is maximal compact in $\mathfrak{m}$. Write $K_M = K \cap M \subseteq M$ for the corresponding group. Using the decomposition $G = KMAN$ we define a map $H : G \to \mathfrak{a}$ by

$$g \in KMe^{H(g)}N.$$

We now compute H on $\overline{N}$.

Lemma 1.8.2. *For $(x,t)\in V\times\mathbb{R}$ we have*

$$e^{\lambda(H(\overline{n}_{(x,t)}))}=\Big(1+4|x|^2-\frac{1}{2}\omega(\mu(Jx)x,x)+2t^2+4|tx-\Psi(x)|^2+(t^2-Q(x))^2\Big)^{\lambda/4}.$$

We remark that for some special cases a similar formula was obtained in [21, equation (5.5)] and [22, equation (5.4)].

Remark 1.8.3. Since

$$\begin{aligned}\kappa(\mu(x),T)&=\frac{1}{2}\kappa(\mathrm{ad}(x)^2E,T)=\frac{1}{2}\kappa(E,\mathrm{ad}(x)^2T)\\&=\frac{1}{2}\kappa(E,[[T,x],x])=\frac{1}{2}\omega(Tx,x)\kappa(E,F)=\frac{p}{2}\omega(Tx,x)\end{aligned}$$

we have

$$\omega(\mu(Jx)x,x)=\frac{2}{p}\kappa(\mu(x),\mu(Jx))=\frac{2}{p}\kappa(\mu(x),\theta\mu(x))\leq 0.$$

Proof. Let $(x,t)\in V\times\mathbb{R}$ and write $\overline{n}_{(x,t)}=kman$, so that $a=e^{H(\overline{n}_{(x,t)})}$. Then

$$\theta(\overline{n}_{(x,t)})^{-1}\overline{n}_{(x,t)}=\theta(n)^{-1}(\theta(m)^{-1}m)a^2n\in\overline{N}MAN.$$

Hence $a=a(\theta(\overline{n}_{(x,t)})^{-1}\overline{n}_{(x,t)})^{1/2}$ which we compute by letting $\theta(\overline{n}_{(x,t)})^{-1}\overline{n}_{(x,t)}$ act on E via the adjoint action. First, by (1.7.1) we have

$$\mathrm{Ad}(\overline{n}_{(x,t)})E=E+\overline{x}+\mu(x)-tH+\Psi(x)-tx+(Q(x)-t^2)F. \tag{1.8.2}$$

Next, $\theta(\overline{n}_{(x,t)})^{-1}=\exp(-\overline{Jx}+tE)$. We let this act on each of the summands of (1.8.2). First, it acts trivially on E:

$$\mathrm{Ad}(\exp(-\overline{Jx}+tE))E=E.$$

The action on the $\mathfrak{g}_1$-part $\overline{x}$ is given by

$$\mathrm{Ad}(\exp(-\overline{Jx}+tE))\overline{x}=\overline{x}+\mathrm{ad}(-\overline{Jx}+tE)\overline{x}=\overline{x}+\omega(Jx,x)E=\overline{x}+4|x|^2E.$$

Next, on the $\mathfrak{m}$-part $\mu(x)$ we have

$$\begin{aligned}&\mathrm{Ad}(\exp(-\overline{Jx}+tE))\mu(x)\\&=\mu(x)+\mathrm{ad}(-\overline{Jx}+tE)\mu(x)+\tfrac{1}{2}\,\mathrm{ad}(-\overline{Jx}+tE)^2\mu(x)\\&=\mu(x)+[\mu(x),\overline{Jx}]+\tfrac{1}{2}\omega(Jx,[\mu(x),Jx])E\\&=\mu(x)+[\mu(x),\overline{Jx}]-\Big(\tfrac{3}{2}\omega(Jx,B_\Psi(x,x,Jx))+\tfrac{1}{4}\omega(Jx,\omega(x,Jx)x\Big)E\\&=\mu(x)+[\mu(x),\overline{Jx}]-(6B_Q(x,x,Jx,Jx)-4|x|^4)E.\end{aligned}$$

For the action on H we find

$$\mathrm{Ad}(\exp(-\overline{Jx}+tE))H=H+\mathrm{ad}(-\overline{Jx}+tE)H=H+\overline{Jx}-2tE.$$

Next, the action on the $\mathfrak{g}_{-1}$-part $\Psi(x) - tx$ is given by

$$\begin{aligned}
&\operatorname{Ad}(\exp(-\overline{Jx}+tE))(\Psi(x)-tx)\\
&= (\Psi(x)-tx) + \operatorname{ad}(-\overline{Jx}+tE)(\Psi(x)-tx) + \tfrac{1}{2}\operatorname{ad}(-\overline{Jx}+tE)^2(\Psi(x)-tx)\\
&\quad + \tfrac{1}{6}\operatorname{ad}(-\overline{Jx}+tE)^3(\Psi(x)-tx)\\
&= (\Psi(x)-tx) + \Big(2B_\mu(Jx,\Psi(x)-tx) + \tfrac{1}{2}\omega(Jx,\Psi(x)-tx)H - t\overline{(\Psi(x)-tx)}\Big)\\
&\quad + \tfrac{1}{2}\Big(2[B_\mu(Jx,\Psi(x)-tx),\overline{Jx}] + \tfrac{1}{2}\omega(Jx,\Psi(x)-tx)\overline{Jx} - 2t\omega(Jx,\Psi(x)-tx)E\Big)\\
&\quad + \tfrac{1}{6}\Big(2\omega(Jx,[B_\mu(Jx,\Psi(x)-tx),Jx])E\Big)\\
&= \Big(-t\omega(Jx,\Psi(x)-tx) + \tfrac{1}{3}\omega(Jx,[B_\mu(Jx,\Psi(x)-tx),Jx])\Big)E + \cdots
\end{aligned}$$

And finally, for the action on F we have

$$\begin{aligned}
\operatorname{Ad}(\exp(-\overline{Jx}+tE))F &= F + \operatorname{ad}(-\overline{Jx}+tE)F + \tfrac{1}{2}\operatorname{ad}(-\overline{Jx}+tE)^2F + \tfrac{1}{6}\operatorname{ad}(-\overline{Jx}+tE)^3F\\
&\quad + \tfrac{1}{24}\operatorname{ad}(-\overline{Jx}+tE)^4F\\
&= F + \Big(-Jx+tH\Big) + \tfrac{1}{2}\Big(-2\mu(Jx) + 2t\overline{Jx} - 2t^2E\Big)\\
&\quad + \tfrac{1}{6}\Big(6\overline{\Psi(Jx)}\Big) + \tfrac{1}{24}\Big(24Q(Jx)E\Big).
\end{aligned}$$

Altogether we obtain

$$\begin{aligned}
\operatorname{Ad}(\theta(\overline{n}_{(x,t)})^{-1}\overline{n}_{(x,t)})E &= \Big(1 + 4|x|^2 - 6B_Q(x,x,Jx,Jx) + 4|x|^4 + 2t^2 - t\omega(Jx,\Psi(x)-tx)\\
&\quad + \tfrac{1}{3}\omega(Jx,[B_\mu(Jx,\Psi(x)-tx),Jx]) + (Q(Jx)-t^2)(Q(x)-t^2)\Big)E + \cdots
\end{aligned}$$

This expression can be simplified. In fact,

$$\begin{aligned}
\omega(Jx,\Psi(x)) &= 4B_Q(Jx,x,x,x),\\
\omega(Jx,[B_\mu(Jx,x),Jx]) &= -12B_Q(Jx,Jx,Jx,x),\\
\omega(Jx,[B_\mu(Jx,\Psi(x)),Jx]) &= -12B_Q(Jx,Jx,Jx,\Psi(x)).
\end{aligned}$$

Hence

$$\begin{aligned}
&1 + 4|x|^2 - 6B_Q(Jx,Jx,x,x) + 4|x|^4 + 2t^2 - 4tB_Q(Jx,x,x,x) + 4t^2|x|^2\\
&\qquad - 4B_Q(Jx,Jx,Jx,\Psi(x)) + 4tB_Q(Jx,Jx,Jx,x) + (Q(Jx)-t^2)(Q(x)-t^2).
\end{aligned}$$

Finally, using (1.8.1) we find that

$$\begin{aligned}
B_Q(Jx,x,x,x) &= (x|\Psi(x)), & B_Q(Jx,Jx,Jx,\Psi(x)) &= -\|\Psi(x)\|^2,\\
B_Q(Jx,Jx,Jx,x) &= -(x|\Psi(x)), & B_Q(Jx,Jx,x,x) &= \frac{1}{12}\omega(\mu(Jx)x,x) + \frac{2}{3}|x|^4
\end{aligned}$$

and $Q(Jx) = Q(x)$ and the claimed formula follows. □

1.9 Hermitian vs. non-Hermitian

We derive several equivalent properties characterizing the Hermitian Lie algebras among all Heisenberg graded Lie algebras.

Theorem 1.9.1. *The following are equivalent:*

(1) The group G is of Hermitian type,

(2) There exists $J \in \mathfrak{m}$ such that $\mathrm{ad}(J)^2|_{\mathfrak{g}_{\pm 1}} = -1$ and $(X,Y) \mapsto \omega(\mathrm{ad}(J)X,Y)$ is positive definite on V,

(3) The minimal adjoint orbits $\mathcal{O}_{\min}$ and $-\mathcal{O}_{\min}$ are distinct,

(4) The quartic Q is non-positive,

(5) The character χ of M is trivial.

Proof. We first show (1)⇔(2). If G is of Hermitian type then the center of $\mathfrak{k}$ is non-trivial, i.e. there exists $0 \neq X \in \mathfrak{k}$ such that $[X,Y] = 0$ for any $Y \in \mathfrak{k}$. Write $X = s(E-F) + (x + \theta(x)) + T$ with $s \in \mathbb{R}$, $x \in \mathfrak{g}_1$ and $T \in \mathfrak{k}_{\mathfrak{m}}$. Then

$$0 = [X, E-F] = -\overline{x} + \overline{\theta(x)}$$

where $\overline{x} \in \mathfrak{g}_{-1}$ and $\overline{\theta(x)} \in \mathfrak{g}_1$. Hence $x = 0$. Further, for every $v \in \mathfrak{g}_1$ we have

$$0 = [X, v + \theta(v)] = s(\overline{\theta(v)} - \overline{v}) + [T, v] + [T, \theta(v)],$$

and therefore $[T,v] + s\overline{\theta(v)} = 0$ and $[T,\theta(v)] - s\overline{v} = 0$. This implies

$$\mathrm{ad}(T)v = -s\overline{\theta(v)} \qquad \text{and} \qquad \mathrm{ad}(T)w = s\overline{\theta(w)}$$

for $v \in \mathfrak{g}_1$ and $w \in \mathfrak{g}_{-1}$. Now, if $s = 0$ then $\mathrm{ad}(T) = 0$ on $\mathfrak{g}_{\pm 1}$, and trivially also on $\mathfrak{g}_{\pm 2}$, hence on $\mathfrak{g}$ which is only possible if $T = 0$, because $\mathfrak{g}$ is assumed to be simple. So $s \neq 0$ and therefore $\mathrm{ad}(T)^2|_{\mathfrak{g}_{\pm 1}} = -s^2$. Then $J := s^{-1}T \in \mathfrak{m}$ satisfies $\mathrm{ad}(J)^2|_{\mathfrak{g}_{\pm 1}} = -1$. Further, for $X, Y \in V = \mathfrak{g}_{-1}$ we have

$$0 \leq -B(X, \theta Y) = -B(\theta X, Y) = p \cdot \omega(\mathrm{ad}(J)X, Y)$$

and hence $\omega(\mathrm{ad}(J)X,Y)$ is positive definite.

Conversely, let $J \in \mathfrak{m}$ with $\mathrm{ad}(J)^2|_{\mathfrak{g}_{\pm 1}} = -1$ and $\omega(\mathrm{ad}(J)X,Y)$ positive definite on V. Note that $\widetilde{J} := \exp(\frac{\pi}{2}J) \in M$ satisfies $\mathrm{Ad}(\widetilde{J}^2) = -1$ and $\mathrm{Ad}(\widetilde{J})|_{\mathfrak{g}_{\pm 1}} = \mathrm{ad}(J)|_{\mathfrak{g}_{\pm 1}}$. Then one can define an involution θ on $\mathfrak{g}$ by

$$\theta(E) := -F, \qquad \theta(F) := -E, \qquad \theta(H) := -H,$$

and for $v \in \mathfrak{g}_1$, $w \in \mathfrak{g}_{-1}$ and $T \in \mathfrak{m}$ by

$$\theta(v) := -\overline{\mathrm{ad}(J)v}, \qquad \theta(w) := \overline{\mathrm{ad}(J)w}, \qquad \theta(T) = \mathrm{Ad}(\widetilde{J})T.$$

It is immediate that $\theta^2 = 1$ and hence θ is in fact an involution. We now show that θ is a Cartan involution. Then, by the same computations as above, the center of the corresponding maximal compact subalgebra $\mathfrak{k} = \mathfrak{g}^\theta$ is spanned by $X = E - F + J$ and hence G is of Hermitian type. To show that θ is a Cartan involution we compute $B(X, \theta(X))$ for $X \in \mathfrak{g}_i$, $i = -2, -1, 0, 1, 2$.

(1) For $X = E \in \mathfrak{g}_2$ we have $B(X, \theta(X)) = -p < 0$.

(2) For $X \in \mathfrak{g}_1$ we have $B(X, \theta(X)) = p \cdot \omega(\overline{X}, \overline{\operatorname{ad}(J)X}) = -p \cdot \omega(\operatorname{ad}(J)\overline{X}, \overline{X})$ which is strictly negative for $X \neq 0$.

(3) For $X \in \mathfrak{m}$ we have $\theta(X) = \widetilde{J}X\widetilde{J}^{-1}$. Since $\mathfrak{m} \subseteq \mathfrak{sp}(V, \omega)$ via $X \mapsto \operatorname{ad}(X)|_{\mathfrak{g}_{-1}}$ the Killing form $B_{\mathfrak{m}}$ has to be a scalar multiple of the Killing form of $\mathfrak{sp}(V, \omega)$ on each simple factor of $\mathfrak{m}$. It is well-known that the involution $\theta(X) = \widetilde{J}X\widetilde{J}^{-1}$ extends to a Cartan involution of $\mathfrak{sp}(V, \omega)$ and hence $B(X, \theta(X)) < 0$ for all $X \neq 0$.

(4) For $X = H \in \mathfrak{a}$ we have $B(X, \theta(X)) = -2p < 0$.

(5) For $X \in \mathfrak{g}_{-1}$ we have $B(X, \theta(X)) = -p \cdot \omega(\operatorname{ad}(J)X, X)$ and hence strictly negative for $X \neq 0$.

(6) For $X = F \in \mathfrak{g}_{-2}$ we have $B(X, \theta(X)) = -p < 0$.

Next, the equivalence (1)⇔(3) follows from [45, Theorem 1.4]. (Note that $\mathfrak{g}$ cannot have a complex structure, because in this case $\mathfrak{g}_2$ as the highest root space would have a complex structure and have real dimension ≥ 2.)

Let us show (3)⇔(4). If the orbits $\mathcal{O}_{\min}$ and $-\mathcal{O}_{\min}$ are distinct then $\operatorname{Ad}(M) \cdot E = \{E\}$. Hence, by Lemma 1.7.1 the quartic Q is non-positive. If conversely $Q \leq 0$ then by Lemma 1.7.1 we have $\operatorname{Ad}(m)E = E$ for all $m \in M$. We obtain $\operatorname{Ad}(MAN)E = \{e^{2r}E : r \in \mathbb{R}\}$ and further

$$\operatorname{Ad}(\overline{N}MAN)E = \{e^{2r}\left(E + \overline{x} + \mu(x) - sH + (\Psi(x) - sx) + (Q(x) - s^2)F\right) : \\ r \in \mathbb{R}, (x, s) \in V \times \mathbb{R}\}.$$

In particular, the coefficient of E of every element in $\operatorname{Ad}(\overline{N}MAN)E$ is positive. Since the set $\operatorname{Ad}(\overline{N}MAN)E$ is dense in $\mathcal{O}_{\min}$, the element $-E$ cannot be in $\mathcal{O}_{\min}$ and therefore, $\mathcal{O}_{\min}$ and $-\mathcal{O}_{\min}$ are distinct.

Finally, (4)⇔(5) follows from Lemma 1.7.1. □

Corollary 1.9.2. *If G is Hermitian, then the formula in Lemma 1.8.2 simplifies to*

$$e^{\lambda(H(\overline{n}_{(x,t)}))} = (1 + 2|x|^2 - Q(x) + t^2)^{\lambda/2}.$$

Proof. If G is Hermitian, $J \in \mathfrak{m}$ and hence

$$J\Psi(x) = 3B_\Psi(Jx, x, x) \qquad \text{and} \qquad B_Q(Jx, x, x, x) = 0,$$

so that

$$\begin{aligned}
\omega(\mu(Jx)x, x) &= 12B_Q(Jx, Jx, x, x) - 8|x|^4 = 3\omega(Jx, B_\Psi(Jx, x, x)) - 8|x|^4 \\
&= \omega(Jx, J\Psi(x)) - 8|x|^4 = 4Q(x) - 8|x|^4, \\
(x|\Psi(x)) &= \frac{1}{4}\omega(Jx, \Psi(x)) = B_Q(Jx, x, x, x) = 0.
\end{aligned}$$

Further, polarizing Lemma 1.4.1 (2) gives $B_\Psi(\Psi(x), x, x) = \frac{1}{3}Q(x)x$ and hence

$$\begin{aligned}
|\Psi(x)|^2 &= \frac{1}{4}\omega(J\Psi(x), \Psi(x)) = B_Q(J\Psi(x), x, x, x) = -3B_Q(\Psi(x), x, x, Jx) \\
&= -\frac{3}{4}\omega(Jx, B_\Psi(\Psi(x), x, x)) = -Q(x)|x|^2.
\end{aligned}$$

Inserting this into the formula in Lemma 1.8.2 and rearranging shows the claim. □

Remark 1.9.3. Note that if $\mathfrak{g} = \mathfrak{su}(n+1,1)$ then $V = \mathbb{C}^n$ and $Q(x) = -|x|^4$, so that the above expression becomes

$$e^{\lambda(H(\overline{n}_{(x,t)}))} = ((1+|x|^2)^2 + t^2).$$

This formula is well-known (see e.g. [23, Theorem IX.3.8]).

In the case where G is Hermitian we further show that the pair $(\mathfrak{sp}(V,\omega), \mathfrak{m})$ is of *holomorphic type* in the sense of Kobayashi [34, Definition 1.4]. Recall that for reductive Hermitian Lie algebras $\mathfrak{h} \subseteq \mathfrak{g}$, the pair $(\mathfrak{g}, \mathfrak{h})$ is said to be of holomorphic type if there exists a Cartan involution θ of $\mathfrak{g}$ which leaves $\mathfrak{h}$ invariant and an element $z \in \mathfrak{k}_{\mathfrak{h}} = \mathfrak{h}^\theta$ such that $\operatorname{ad}(z) = 0$ on $\mathfrak{k} = \mathfrak{g}^\theta$ and $\operatorname{ad}(z)^2 = -1$ on $\mathfrak{p} = \mathfrak{g}^{-\theta}$. In this case the natural embedding $H/(K \cap H) \subseteq G/K$ of Hermitian symmetric spaces is holomorphic.

Corollary 1.9.4. *If G is Hermitian then $\mathfrak{m}$ is also Hermitian and the pair $(\mathfrak{sp}(V,\omega), \mathfrak{m})$ is holomorphic.*

Proof. We can choose the element z to be $z = \frac{1}{2}J \in \mathfrak{m}$. A Cartan involution of $\mathfrak{sp}(V,\omega)$ is given by $\theta(T) = JTJ^{-1}$, hence $z \in \mathfrak{sp}(V,\omega)^\theta$. Further $\operatorname{ad}(z) = 0$ on $\mathfrak{sp}(V,\omega)^\theta = \{T \in \mathfrak{sp}(V,\omega) : TJ = JT\}$, and for $T \in \mathfrak{sp}(V,\omega)^{-\theta} = \{T \in \mathfrak{sp}(V,\omega) : TJ = -JT\}$ we have $\operatorname{ad}(z)T = JT$ and $\operatorname{ad}(z)^2T = \frac{1}{2}[J, JT] = J^2T = -T$. □

Chapter 2

Principal series representations and intertwining operators

We study the degenerate principal series representations induced from the parabolic subgroup P and intertwining operators between them.

2.1 Degenerate principal series representations

For a smooth admissible representation (ζ, V_ζ) of M and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ we let $(\widetilde{\pi}_{\zeta,\nu}, \widetilde{I}(\zeta,\nu))$ be the induced representation $\operatorname{Ind}_P^G(\zeta \otimes e^\nu \otimes \mathbf{1})$, acting by left-translation on

$$\widetilde{I}(\zeta,\nu) = \{f \in C^\infty(G, V_\zeta) : f(gman) = a^{-\nu-\rho}\zeta(m)^{-1}f(g)\, \forall\, man \in MAN\}.$$

Here $\rho \in \mathfrak{a}^*$ denotes as usual the half sum of all positive roots.

2.2 The non-compact picture

Since $\overline{N}MAN \subseteq G$ is open dense, functions in $\widetilde{I}(\zeta,\nu)$ are uniquely determined by their restriction to $\overline{N}$. Therefore, we define for any $f \in \widetilde{I}(\zeta,\nu)$ a function $f_{\overline{\mathfrak{n}}}$ on $V \times \mathbb{R}$ by

$$f_{\overline{\mathfrak{n}}}(x,s) := f(\overline{n}_{(x,s)}), \qquad (x,s) \in V \times \mathbb{R}$$

and let

$$I(\zeta,\nu) := \{f_{\overline{\mathfrak{n}}} : f \in \widetilde{I}(\zeta,\nu)\}.$$

The representation $\widetilde{\pi}_{\zeta,\nu}$ on $\widetilde{I}(\zeta,\nu)$ defines an equivalent representation $\pi_{\zeta,\nu}$ on $I(\zeta,\nu)$ by

$$\pi_{\zeta,\nu}(g)f_{\overline{\mathfrak{n}}} = (\widetilde{\pi}_{\zeta,\nu}(g)f)_{\overline{\mathfrak{n}}}, \qquad g \in G, f \in \widetilde{I}(\zeta,\nu).$$

The realization $(I(\zeta,\nu), \pi_{\zeta,\nu})$ is called the *non-compact picture* of the degenerate principal series. Note that

$$\mathcal{S}(V \times \mathbb{R}) \widehat{\otimes} V_\zeta \subseteq I(\zeta,\nu) \subseteq C^\infty_{\mathrm{temp}}(V \times \mathbb{R}) \widehat{\otimes} V_\zeta.$$

We compute the action of $MA\overline{N}$ and w_0 in this realization:

Proposition 2.2.1. *For $f \in I(\zeta,\nu)$ and $(x,s) \in V \times \mathbb{R}$ we have*

$$\begin{aligned} \pi_{\zeta,\nu}(\overline{n}_{(y,t)})f(x,s) &= f(x-y, s-t+\tfrac{1}{2}\omega(x,y)), && \overline{n}_{(y,t)} \in \overline{N},\\ \pi_{\zeta,\nu}(m)f(x,s) &= \zeta(m)f(m^{-1}x, \chi(m)^{-1}s), && m \in M,\\ \pi_{\zeta,\nu}(e^{rH})f(x,s) &= e^{(\nu+\rho)r}f(e^r x, e^{2r}s), && e^{rH} \in A. \end{aligned}$$

Moreover, for $\zeta = \mathbf{1}$ the trivial representation of M we have

$$\pi_{\mathbf{1},\nu}(w_0^{\pm 1})f(x,s) = |s^2 - Q(x)|^{-\frac{\nu+\rho}{2}} f\left(\pm\frac{\Psi(x) - sx}{s^2 - Q(x)}, -\frac{s}{s^2-Q(x)}\right).$$

Proof. The formulas for M, A and $\overline{N}$ are obvious. For $\pi_{\mathbf{1},\nu}(w_0)$ we use Lemma 1.7.1. □

This yields the differentiated action $d\pi_{\zeta,\nu}(X) = \frac{d}{dt}\big|_{t=0}\pi_{\zeta,\nu}(\exp(tX))$ of the Lie algebra $\mathfrak{g}$ on $I(\zeta,\nu)$. To simplify the formulas we $\mathbb{E}$ denote the weighted Euler operator on $V \times \mathbb{R}$, i.e.

$$\mathbb{E} = \sum_\alpha x_\alpha \frac{\partial}{\partial x_\alpha} + 2s\frac{\partial}{\partial s},$$

where $x = \sum_\alpha x_\alpha e_\alpha$ for any basis (e_α) of V.

Corollary 2.2.2. *The Lie algebra representation $d\pi_{\zeta,\nu}$ of $\mathfrak{g}$ is given by*

$$\begin{aligned} d\pi_{\zeta,\nu}(F) &= -\partial_s,\\ d\pi_{\zeta,\nu}(v) &= -\partial_v + \tfrac{1}{2}\omega(x,v)\partial_s, && v \in \mathfrak{g}_{-1}\\ d\pi_{\zeta,\nu}(T) &= -\partial_{Tx} + d\zeta(T), && T \in \mathfrak{m},\\ d\pi_{\zeta,\nu}(H) &= \mathbb{E} + (\nu+\rho),\\ d\pi_{\zeta,\nu}(w) &= \partial_{\mu(x)\overline{w}+\omega(x,\overline{w})x-s\overline{w}} + \tfrac{1}{2}\omega(sx+\psi(x),\overline{w})\partial_s + \tfrac{\nu+\rho}{2}\omega(x,\overline{w}) - 2d\zeta(B_\mu(x,\overline{w})), && w \in \mathfrak{g}_1,\\ d\pi_{\zeta,\nu}(E) &= \partial_{sx+\Psi(x)} + (s^2+Q(x))\partial_s + (\nu+\rho)s + d\zeta(\mu(x)). \end{aligned}$$

Proof. The formulas for $d\pi_\nu(F)$, $d\pi_\nu(v)$, $d\pi_\nu(T)$ and $d\pi_\nu(H)$ follow easily by differentiating the corresponding group actions in Proposition 2.2.1. For $d\pi_\nu(E)$ we use Lemma 1.7.2 and $d\pi_\nu(w)$ can be obtained from $d\pi_\nu(E)$ and $d\pi_{\zeta,\nu}(\overline{w})$ by $d\pi_{\zeta,\nu}(w) = [d\pi_{\zeta,\nu}(\overline{w}), d\pi_{\zeta,\nu}(E)]$. □

2.3 Intertwining operators

For $\operatorname{Re}\nu \gg 0$ the standard Knapp–Stein intertwining operator $\widetilde{A}(\zeta,\nu) : \widetilde{I}(\zeta,\nu) \to \widetilde{I}(w_0\zeta,-\nu)$ (with $w_0\zeta(m) = \zeta(w_0^{-1}mw_0)$) is given by the convergent integral

$$\widetilde{A}(\zeta,\nu)f(g) = \int_{\overline{N}} f(gw_0\overline{n})\,d\overline{n}.$$

It is well-known that $\widetilde{A}(\zeta,\nu)$ extends meromorphically in $\nu \in \mathbb{C}$. We consider the Knapp–Stein operator $A(\zeta,\nu)$ in the non-compact picture:

$$A(\zeta,\nu)f_{\overline{\mathfrak{n}}} := (\widetilde{A}(\zeta,\nu)f)_{\overline{\mathfrak{n}}} \qquad (f \in \widetilde{I}(\zeta,\nu)).$$

Proposition 2.3.1. *Assume $\zeta = \mathbf{1}$ is the trivial representation, then the operator $A(\zeta,\nu)$ is the convolution operator*

$$A(\mathbf{1},\nu)f(x,s) = \int_{V\times\mathbb{R}} |t^2 - Q(y)|^{\frac{\nu-\rho}{2}} f((x,s)\cdot(y,t))\,d(y,t).$$

Proof. In [33, Chapter VII, §7] it is shown that

$$\widetilde{A}(\zeta,\nu)f(g) = \int_{\overline{N}} a(w_0^{-1}\overline{n})^{\nu-\rho}\zeta(m(w_0^{-1}\overline{n}))f(g\overline{n})\,d\overline{n}.$$

Then Lemma 1.7.1 immediately yields the claim. □

Remark 2.3.2. Of course one can also write down a formula for $A(\zeta,\nu)$ for general ζ, but we will not need this in what follows.

2.4 The Fourier transform on the Heisenberg group

The infinite-dimensional irreducible unitary representations of the Heisenberg group $\overline{N}$ are parameterized by their central character $i\lambda \in i\mathbb{R}^\times$. More precisely, for each $\lambda \in \mathbb{R}^\times$ there exists a unique (up to equivalence) infinite-dimensional unitary representation $(\sigma_\lambda, \mathcal{H}_\lambda)$ of $\overline{N}$ such that $d\sigma_\lambda(0,t) = i\lambda t$ for $(0,t) \in \overline{\mathfrak{n}} \simeq V \times \mathbb{R}$. There are two standard realizations of σ_λ, the Schrödinger model and the Fock model. The Schrödinger model is realized on the space $\mathcal{H}_\lambda = L^2(\Lambda)$ for a Lagrangian subspace $\Lambda \subseteq V$, whereas the Fock model is realized on the Fock space $\mathcal{H}_\lambda = \mathcal{F}(V)$ consisting of holomorphic functions on V (with respect to a certain complex structure) which are square-integrable with respect to a Gaussian measure on V.

Since M acts on $\overline{N}$ by automorphisms, the map $\overline{n} \mapsto \sigma_\lambda(\mathrm{Ad}(m)\overline{n})$ defines an irreducible unitary representation of $\overline{N}$ with central character $i\chi(m)\lambda$ and hence there exists a projective unitary representation $\omega_{\mathrm{met},\lambda}$ of M on the same representation space such that

$$\sigma_\lambda(\mathrm{Ad}(m)\overline{n}) = \omega_{\mathrm{met},\lambda}(m) \circ \sigma_{\chi(m)\lambda}(\overline{n}) \circ \omega_{\mathrm{met},\lambda}(m)^{-1} \qquad \forall\, m \in M, \overline{n} \in \overline{N}. \tag{2.4.1}$$

Since $M_1 = \{m \in M : \chi(m) = 1\}$ acts symplectically on V, the representation $\omega_{\mathrm{met},\lambda}|_{M_1}$ is simply the restriction of the metaplectic representation of $\mathrm{Sp}(V,\omega)$ to M_1.

For $f \in L^1(\overline{N})$ we form

$$\sigma_\lambda(f) = \int_{\overline{N}} f(\overline{n})\sigma_\lambda(\overline{n})\,d\overline{n},$$

where $d\overline{n}$ is a fixed Haar measure on $\overline{N}$. If we identify $X \in \overline{\mathfrak{n}}$ with the corresponding left-invariant vector field on $\overline{N}$, then

$$\sigma_\lambda(Xf) = -\sigma_\lambda(f) \circ d\sigma_\lambda(X).$$

Further, if $f * g$ denotes the convolution of $f, g : \overline{N} \to \mathbb{C}$ given by

$$f * g(x) = \int_{\overline{N}} f(y)g(xy^{-1})\,dy,$$

then

$$\sigma_\lambda(f * g) = \sigma_\lambda(g) \circ \sigma_\lambda(f).$$

We have the following Plancherel Formula (after appropriate normalization of the measures involved):

$$\|f\|^2_{L^2(\overline{N})} = \int_{\mathbb{R}^\times} \|\sigma_\lambda(f)\|^2_{\mathrm{HS}} |\lambda|^{\frac{\dim \mathfrak{g}_1}{2}} \, d\lambda.$$

Here $\|T\|^2_{\mathrm{HS}} = \mathrm{tr}(TT^*)$ denotes the Hilbert–Schmidt norm of a Hilbert–Schmidt operator on a Hilbert space. Realizing all infinite-dimensional irreducible unitary representations of $\overline{N}$ on the same Hilbert space $\mathcal{H}_\lambda = \mathcal{H}$ and writing $\mathrm{HS}(\mathcal{H})$ for the Hilbert space of Hilbert–Schmidt operators on $\mathcal{H}$, the Fourier transform can be viewed as an isometric isomorphism

$$\mathcal{F} : L^2(\overline{N}) \to L^2(\mathbb{R}^\times, \mathrm{HS}(\mathcal{H}); |\lambda|^{\frac{\dim \mathfrak{g}_1}{2}} \, d\lambda), \quad \mathcal{F}f(\lambda) = \sigma_\lambda(f).$$

2.5 The Schrödinger model and the Fourier transform of distributions

For the Schrödinger model of the infinite-dimensional irreducible unitary representations of $\overline{N} \simeq V \times \mathbb{R}$ one has to choose a Lagrangian subspace $\Lambda \subseteq V$ and a Lagrangian complement $\Lambda^* \subseteq V$. Then the Schrödinger model is a realization of σ_λ on $\mathcal{H} = L^2(\Lambda)$ given by

$$\sigma_\lambda(z,t)\varphi(x) = e^{i\lambda t} e^{i\lambda(\omega(z'',x) + \frac{1}{2}\omega(z',z''))} \varphi(x - z'), \qquad (x \in \Lambda) \tag{2.5.1}$$

for $z = (z', z'') \in \Lambda \oplus \Lambda^* = V$ and $t \in \mathbb{R}$. The corresponding differentiated representation of $\overline{\mathfrak{n}} \simeq V \times \mathbb{R}$ is given by

$$d\sigma_\lambda(z,t) = -\partial_{z'} + i\lambda\omega(z'', x) + i\lambda t, \qquad z = (z', z'') \in \Lambda \oplus \Lambda^* = V, t \in \mathbb{R}.$$

For $\varphi \in \mathcal{S}(\overline{N})$ we have

$$\begin{aligned}
\sigma_\lambda(u)\varphi(y) &= \int_{\overline{N}} u(z,t)\sigma_\lambda(z,t)\varphi(y) \, dz \, dt \\
&= \int_\Lambda \int_{\Lambda^*} \int_{\mathbb{R}} u(z', z'', t) e^{i\lambda t} e^{i\lambda(\omega(z'',y) + \frac{1}{2}\omega(z',z''))} \varphi(y - z') \, dt \, dz'' \, dz' \\
&= \int_\Lambda \left(\int_{\Lambda^*} \int_{\mathbb{R}} u(y - x, z'', t) e^{i\lambda t} e^{-\frac{i\lambda}{2}(\omega(x+y,z''))} \, dt \, dz'' \right) \varphi(x) \, dx \\
&= \int_\Lambda \widehat{u}(\lambda, x, y)\varphi(x) \, dx,
\end{aligned}$$

where

$$\widehat{u}(\lambda, x, y) = \int_{\Lambda^*} \int_{\mathbb{R}} u(y - x, z'', t) e^{i\lambda t} e^{-\frac{i\lambda}{2}\omega(x+y,z'')} \, dt \, dz'' = \mathcal{F}_2\mathcal{F}_3 u(y - x, \tfrac{\lambda}{2}(x + y), -\lambda).$$

Here $\mathcal{F}_2$ denotes the symplectic Fourier transform with respect to ω in the second variable, and $\mathcal{F}_3$ denotes the Euclidean Fourier transform with respect to the third variable.

Proposition 2.5.1. *The linear map*

$$\mathcal{S}'(\overline{N}) \to \mathcal{D}'(\mathbb{R}^\times) \mathbin{\widehat{\otimes}} \mathcal{S}'(\Lambda \times \Lambda) \simeq \mathcal{D}'(\mathbb{R}^\times) \mathbin{\widehat{\otimes}} \mathrm{Hom}(\mathcal{S}(\Lambda), \mathcal{S}'(\Lambda)), \quad u \mapsto \widehat{u},$$

is defined and continuous. Its kernel is given by those distributions which are polynomial in t.

Proof. Clearly $\mathcal{F}_2\mathcal{F}_3$ is a topological isomorphism $\mathcal{S}'(\overline{N}) \to \mathcal{S}'(\Lambda\times\Lambda\times\mathbb{R})$. Restricting the last coordinate to $\mathbb{R}^\times$ defines a continuous linear map $\mathcal{S}'(\Lambda\times\Lambda\times\mathbb{R}) \simeq \mathcal{S}'(\Lambda\times\Lambda)\widehat{\otimes}\mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\Lambda\times\Lambda)\widehat{\otimes}\mathcal{D}'(\mathbb{R}^\times)$. Finally, the change of coordinates $(x,y,\lambda)\mapsto(y-x,\frac{\lambda}{2}(x+y),-\lambda)$ induces a continuous linear isomorphism on $\mathcal{S}'(\Lambda\times\Lambda)\widehat{\otimes}\mathcal{D}'(\mathbb{R}^\times)$. Composing these three maps shows continuity of the map $u\mapsto\widehat{u}$. To determine its kernel we observe that the only non-bijective map in this three-fold composition is the restriction to $\mathbb{R}^\times$. Its kernel is given by all distributions $v\in\mathcal{S}'(\Lambda\times\Lambda\times\mathbb{R})$ with $\operatorname{supp} v\subseteq\Lambda\times\Lambda\times\{0\}$. Such distributions are necessarily of the form

$$v(x,y,\lambda)=\sum_{k=0}^{m}v_k(x,y)\delta^{(k)}(\lambda)$$

for some distributions $v_k\in\mathcal{S}'(\Lambda\times\Lambda)$. Taking the inverse Fourier transforms $\mathcal{F}_2^{-1}\circ\mathcal{F}_3^{-1}$ shows the claim. □

Remark 2.5.2. The map $u\mapsto\widehat{u}$ is essentially the group Fourier transform of $\overline{N}$, but only evaluated at the infinite-dimensional unitary representations σ_λ, $\lambda\in\mathbb{R}^\times$. Therefore, it has a kernel which can be treated using the finite-dimensional unitary representations of $\overline{N}$.

Corollary 2.5.3. *For* $\operatorname{Re}\nu>-\rho$ *the Fourier transform*

$$\mathcal{F}:I(\zeta,\nu)\subseteq\mathcal{S}'(V\times\mathbb{R})\to\mathcal{D}'(\mathbb{R}^\times)\widehat{\otimes}\mathcal{S}'(\Lambda\times\Lambda)$$

is injective.

Proof. Assume $u=f_{\overline{\mathfrak{n}}}\in I(\zeta,\nu)$ is in the kernel of the Fourier transform. By Proposition 2.5.1 u is polynomial in t. On the other hand, we can write $\overline{n}_{(z,t)}=kme^{H(\overline{n}_{(z,t)})}n\in KMAN$ and hence

$$u(z,t)=f(k)\zeta(m)^{-1}e^{-(\nu+\rho)H(\overline{n}_{(z,t)})}.$$

Since K is compact, the first factor $f(k)$ is bounded. Further, since ζ is a unitary character, we have $|\zeta(m)|=1$. By Lemma 1.8.2 the third factor $e^{-(\nu+\rho)H(\overline{n}_{(z,t)})}$ is a polynomial in t lifted to the power $-\frac{1}{4}(\nu+\rho)$ which decays as $|t|\to\infty$ since $\operatorname{Re}(\nu+\rho)>0$. The only polynomial $u(z,t)$ in t with this property is the zero polynomial, so the kernel of the Fourier transform restricted to $I(\zeta,\nu)$ is trivial. □

Remark 2.5.4. We note that

$$\mathcal{S}'(\Lambda\times\Lambda)\simeq\operatorname{Hom}(\mathcal{S}(\Lambda),\mathcal{S}'(\Lambda))=\operatorname{Hom}(\mathcal{H}^\infty,\mathcal{H}^{-\infty})$$

since the space $\mathcal{H}^\infty$ of smooth vectors in $\mathcal{H}=L^2(\Lambda)$ is given by the space $\mathcal{S}(\Lambda)$ of Schwarz functions.

The previous observation allows us to define a representation $\widehat{\pi}_{\zeta,\nu}$ of G on $\widehat{I}(\zeta,\nu)=\mathcal{F}(I(\zeta,\nu))\subseteq\mathcal{D}'(\mathbb{R}^\times)\widehat{\otimes}\mathcal{S}'(\Lambda\times\Lambda)$ by

$$\widehat{\pi}_{\zeta,\nu}(g)=\mathcal{F}\circ\pi_{\zeta,\nu}(g)\circ\mathcal{F}^{-1},\qquad g\in G.$$

We call this realization the *Fourier transformed picture*. In this picture, the action of the opposite parabolic subgroup $\overline{P}$ is easily expressed in terms of the representation σ_λ and the metaplectic representation $\omega_{\mathrm{met},\lambda}$.

Proposition 2.5.5. *For $f \in \widehat{I}(\zeta,\nu) \subseteq \mathcal{D}'(\mathbb{R}^\times) \mathbin{\widehat{\otimes}} \operatorname{Hom}(\mathcal{H}^\infty, \mathcal{H}^{-\infty})$ we have*

$$\begin{aligned} \widehat{\pi}_{\zeta,\nu}(\overline{n}_{(z,t)})f(\lambda) &= \sigma_\lambda(z,t) \circ f(\lambda), & \overline{n}_{(z,t)} &\in \overline{N},\\ \widehat{\pi}_{\zeta,\nu}(m)f(\lambda) &= \zeta(m) \cdot \omega_{\mathrm{met},\lambda}(m) \circ f(\chi(m)\lambda) \circ \omega_{\mathrm{met},\lambda}(m)^{-1}, & m &\in M,\\ \widehat{\pi}_{\zeta,\nu}(e^{tH})f(\lambda) &= e^{(\nu-\rho)t}\delta_{e^t} \circ f(e^{-2t}\lambda) \circ \delta_{e^{-t}}, & e^{tH} &\in A, \end{aligned}$$

where $\delta_s\varphi(x) = \varphi(sx)$ $(s>0)$. Alternatively, viewing f as a distribution in $(\lambda,x,y) \in \mathbb{R}^\times \times \Lambda \times \Lambda$, we have

$$\begin{aligned} \widehat{\pi}_{\zeta,\nu}(\overline{n}_{(z,t)})f(\lambda,x,y) &= e^{i\lambda t}e^{i\lambda(\omega(z'',y)+\frac{1}{2}\omega(z',z''))}f(\lambda,x,y-z'), & \overline{n}_{(z,t)} &\in \overline{N},\\ \widehat{\pi}_{\zeta,\nu}(m)f(\lambda,x,y) &= \zeta(m)(\mathrm{id}^*_{\mathbb{R}^\times} \otimes \omega_{\mathrm{met},-\lambda}(m) \otimes \omega_{\mathrm{met},\lambda}(m))f(\chi(m)\lambda,x,y), & m &\in M,\\ \widehat{\pi}_{\zeta,\nu}(e^{tH})f(\lambda,x,y) &= e^{(\nu-1)t}f(e^{-2t}\lambda, e^t x, e^t y), & e^{tH} &\in A, \end{aligned}$$

Proof. Use Proposition 2.2.1 and note that

$$\sigma_\lambda(\overline{n})^\top = \sigma_{-\lambda}(\overline{n}^{-1}) \qquad \text{and} \qquad \omega_{\mathrm{met},\lambda}(m)^\top = \omega_{\mathrm{met},-\lambda}(m^{-1})$$

as operators $\mathcal{S}(\Lambda) \to \mathcal{S}(\Lambda)$, $\mathcal{S}(\Lambda) \to \mathcal{S}'(\Lambda)$ or $\mathcal{S}'(\Lambda) \to \mathcal{S}'(\Lambda)$. □

It seems difficult to express the action of N or w_0 in the Fourier transformed picture. More accessible is the action of the Lie algebra $\mathfrak{g}$ in the differentiated representation $d\widehat{\pi}_{\zeta,\nu}$ which can be obtained using Corollary 2.2.2 and the formulas in the following lemma. We will not carry out the computation of the Lie algebra action on the whole principal series representation, but rather restrict to a certain subrepresentation in Section 4.5.

Lemma 2.5.6. *Let $u \in \mathcal{S}'(V \times \mathbb{R})$.*

(1) For $v \in \Lambda$ we have

$$\begin{aligned} \widehat{\omega(v,z)u}(\lambda,x,y) &= -\frac{1}{i\lambda}(\partial_{v,x} + \partial_{v,y})\widehat{u}(\lambda,x,y),\\ \widehat{\partial_v u}(\lambda,x,y) &= -\frac{1}{2}(\partial_{v,x} - \partial_{v,y})\widehat{u}(\lambda,x,y). \end{aligned}$$

(2) For $w \in \Lambda^*$ we have

$$\begin{aligned} \widehat{\omega(z,w)u}(\lambda,x,y) &= \omega(y-x,w)\widehat{u}(\lambda,x,y),\\ \widehat{\partial_w u}(\lambda,x,y) &= \frac{i\lambda}{2}\omega(x+y,w)\widehat{u}(\lambda,x,y). \end{aligned}$$

(3) For differentiation and multiplication with respect to the central variable we have

$$\begin{aligned} \widehat{\partial_t u}(\lambda,x,y) &= -i\lambda\widehat{u}(\lambda,x,y),\\ \widehat{tu}(\lambda,x,y) &= -i\partial_\lambda\widehat{u}(\lambda,x,y) - \frac{1}{2i\lambda}(\partial_{x+y,x} + \partial_{x+y,y})\widehat{u}(\lambda,x,y). \end{aligned}$$

Proof. We only show the last formula, the rest is standard. For this let $(e_\alpha) \subseteq \Lambda$ be a basis of Λ with dual basis $(\widehat{e}_\alpha) \subseteq \Lambda^*$. Then using (1) we find

$$\begin{aligned}
\widehat{tu}(\lambda, x, y) &= -i \int_{\Lambda^*} \int_{\mathbb{R}} u(y - x, z'', t) \partial_\lambda \left[e^{i\lambda t} \right] e^{-\frac{i\lambda}{2} \omega(x+y, z'')} \, dt \, dz'' \\
&= -i\partial_\lambda \widehat{u}(x, y, \lambda) + \frac{1}{2} \int_{\Lambda^*} \int_{\mathbb{R}} \omega(x + y, z'') u(y - x, z'', t) e^{i\lambda t} e^{-\frac{i\lambda}{2} \omega(x+y, z'')} \, dt \, dz'' \\
&= -i\partial_\lambda \widehat{u}(\lambda, x, y) + \frac{1}{2} \sum_\alpha \omega(x + y, \widehat{e}_\alpha) \omega(\widehat{e_\alpha, z''}) u(\lambda, x, y) \\
&= -i\partial_\lambda \widehat{u}(\lambda, x, y) - \frac{1}{2i\lambda} \sum_\alpha \omega(x + y, \widehat{e}_\alpha)(\partial_{e_\alpha, x} + \partial_{e_\alpha, y}) \widehat{u}(\lambda, x, y)
\end{aligned}$$

and the claimed formula follows. □

Chapter 3

Conformally invariant systems and their Fourier transform

We recall the construction of conformally invariant systems on Heisenberg nilradicals due to [1] and compute their action in the Fourier transformed picture.

3.1 Quantization of the symplectic covariants

In [1] four conformally invariant systems of differential operators on $\overline{N}$ are constructed. We briefly recall their construction and properties. For this let $(e_\alpha) \subseteq V$ be a basis and $\widehat{e}_\alpha$ be the dual basis with respect to the symplectic form, i.e. $\omega(e_\alpha, \widehat{e}_\beta) = \delta_{\alpha\beta}$. Denote by X_α the left-invariant differential operator on $\overline{N} \simeq V \times \mathbb{R}$ corresponding to e_α, i.e.

$$X_\alpha f(\overline{n}) = \frac{d}{dt}\Big|_{t=0} f(\overline{n}e^{te_\alpha}).$$

In the coordinates $(x,t) \in V \times \mathbb{R}$ on $\overline{N}$ this operator takes the form

$$X_\alpha = \partial_\alpha + \tfrac{1}{2}\omega(x, e_\alpha)\partial_t, \tag{3.1.1}$$

where $\partial_\alpha = \partial_{e_\alpha}$.

Quantization of ω

For $v \in V$ we let

$$\Omega_\omega(v) := \sum_\alpha \omega(v, \widehat{e}_\alpha)X_\alpha = \partial_v + \frac{1}{2}\omega(x,v)\partial_t.$$

Quantization of μ

For $T \in \mathfrak{m}$ we let

$$\Omega_\mu(T) = \sum_{\alpha,\beta} \omega(T\widehat{e}_\alpha, \widehat{e}_\beta)X_\alpha X_\beta.$$

Using the explicit expression (3.1.1) of X_α in the coordinates $(x,t)\in V\times\mathbb{R}$ we find

$$\begin{aligned}\Omega_\mu(T) &= \tfrac{1}{2}\sum_{\alpha,\beta}\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\Big[(\partial_\alpha+\tfrac{1}{2}\omega(x,e_\alpha)\partial_t)(\partial_\beta+\tfrac{1}{2}\omega(x,e_\beta)\partial_t)\\ &\qquad\qquad\qquad\qquad\qquad+(\partial_\beta+\tfrac{1}{2}\omega(x,e_\beta)\partial_t)(\partial_\alpha+\tfrac{1}{2}\omega(x,e_\alpha)\partial_t)\Big]\\ &= \sum_{\alpha,\beta}\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\Big[\partial_\alpha\partial_\beta+\tfrac{1}{2}\omega(x,e_\alpha)\partial_\beta\partial_t+\tfrac{1}{2}\omega(x,e_\beta)\partial_\alpha\partial_t+\tfrac{1}{4}\omega(x,e_\alpha)\omega(x,e_\beta)\partial_t^2\Big]\\ &= \sum_{\alpha,\beta}\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\partial_\alpha\partial_\beta-\partial_{Tx}\partial_t+\tfrac{1}{4}\omega(Tx,x)\partial_t^2. \qquad (3.1.2)\end{aligned}$$

Quantization of Ψ and Q

For $X\in V$ we let

$$\Omega_\Psi(X) := \sum_{\alpha,\beta,\gamma}\omega(X,B_\Psi(\widehat{e}_\alpha,\widehat{e}_\beta,\widehat{e}_\gamma))X_\alpha X_\beta X_\gamma,$$

$$\Omega_Q := \sum_{\alpha,\beta,\gamma,\delta}B_Q(\widehat{e}_\alpha,\widehat{e}_\beta,\widehat{e}_\gamma,\widehat{e}_\delta)X_\alpha X_\beta X_\gamma X_\delta.$$

3.2 Conformal invariance

In [1] it is shown that all four systems are conformally invariant for $d\zeta=0$ and certain special parameters ν. Since we also need to involve non-trivial representations $d\zeta$ in Sections 4.6 and 4.7, we give a self-contained proof of conformal invariance for the systems Ω_ω and Ω_μ to have all relevant formulas available.

Theorem 3.2.1. *For every $v\in V$ we have*

$$\begin{aligned}[]
[\Omega_\omega(v),d\pi_{\zeta,\nu}(X)] &= 0 && (X\in\overline{\mathfrak{n}}),\\
[\Omega_\omega(v),d\pi_{\zeta,\nu}(H)] &= \Omega_\omega(v),\\
[\Omega_\omega(v),d\pi_{\zeta,\nu}(S)] &= -\Omega_\omega(Sv) && (S\in\mathfrak{m}).
\end{aligned}$$

Moreover,

$$[\Omega_\omega(v),d\pi_{\zeta,\nu}(E)] = t\Omega_\omega(v)-\Omega_\omega(\mu(x)v)+\frac{\nu+\rho}{2}\omega(x,v)+2d\zeta(B_\mu(x,v)).$$

In particular, for $d\zeta=0$ and $\nu=-\rho$ the space

$$I(\zeta,\nu)^{\Omega_\omega(V)} = \{f\in I(\zeta,\nu):\Omega_\omega(v)f=0\,\forall\,v\in V\}$$

is a subrepresentation of $(\pi_{\zeta,\nu},I(\zeta,\nu))$.

Proof. This is a straightforward verification using the formulas in Corollary 2.2.2. □

Remark 3.2.2. It is easy to see that $I(\zeta,\nu)^{\Omega_\omega(V)}$ only consists of constant functions and is therefore the trivial representation. To obtain a non-trivial representation we try to reduce the conformally invariant system $\Omega_\omega(V)$ to $\Omega_\omega(W)$ for a subspace $W\subseteq V$. However, in all cases except $\mathfrak{g}\simeq\mathfrak{sl}(n,\mathbb{R})$ the Lie algebra $\mathfrak{m}$ acts irreducibly on V so that $[\Omega_\omega(w),d\pi_{\zeta,\nu}(S)]f=0$ for some $w\in V$ implies $\Omega_\omega(v)f=0$ for all $v\in V$. For $\mathfrak{g}=\mathfrak{sl}(n,\mathbb{R})$ the adjoint representation of $\mathfrak{m}$ on V splits into two irreducible subspaces. In Section 4.6 we show that restricting Ω_ω to one of those subspaces yields a conformally invariant subsystem for certain parameters (ζ,ν).

Theorem 3.2.3 ([1, Theorem 5.2])**.** *For every $T\in\mathfrak{m}$ we have*

$$\begin{aligned}
[\Omega_\mu(T),d\pi_{\zeta,\nu}(X)] &= 0 && (X\in\overline{\mathfrak{n}}),\\
[\Omega_\mu(T),d\pi_{\zeta,\nu}(H)] &= 2\Omega_\mu(T),\\
[\Omega_\mu(T),d\pi_{\zeta,\nu}(S)] &= \Omega_\mu([T,S]) && (S\in\mathfrak{m}).
\end{aligned}$$

Further, if $T\in\mathfrak{m}'$ where $\mathfrak{m}'$ is any simple or abelian factor of $\mathfrak{m}$, then

$$\begin{aligned}
[\Omega_\mu(T),d\pi_{\zeta,\nu}(E)] &= 2s\Omega_\mu(T)+\Omega_\mu([T,\mu(x)])+(2\,\mathcal{C}(\mathfrak{m}')-2-(\nu+\rho))\Omega_\omega(Tx)\\
&\qquad +4\sum_\alpha d\zeta(B_\mu(x,e_\alpha))\Omega_\omega(T\widehat{e}_\alpha)-2\,\mathcal{C}(\mathfrak{m}')d\zeta(T).
\end{aligned}$$

In particular, for any simple or abelian factor $\mathfrak{m}'$ of $\mathfrak{m}$, $d\zeta=0$ and $\nu=2\,\mathcal{C}(\mathfrak{m}')-\rho-2$ the space

$$I(\zeta,\nu)^{\Omega_\mu(\mathfrak{m}')}=\{f\in I(\zeta,\nu):\Omega_\mu(T)f=0\,\forall\,T\in\mathfrak{m}'\}$$

is a subrepresentation of $(\pi_{\zeta,\nu},I(\zeta,\nu))$.

Proof. The first three identities are easy to verify. To calculate $[\Omega_\mu(T),d\pi_{\zeta,\nu}(E)]$ we compute all commutators between the different summands of

$$\Omega_\mu(T)=\sum_{\alpha,\beta}\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\partial_\alpha\partial_\beta-\partial_{Tx}\partial_s+\tfrac14\omega(Tx,x)\partial_s^2$$

and

$$d\pi_{\zeta,\nu}(E)=\partial_{sx}+\partial_{\Psi(x)}+(s^2+Q(x))\partial_s+(\nu+\rho)s+d\zeta(\mu(x))$$

separately. First,

$$\Big[\sum_{\alpha,\beta}\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\partial_\alpha\partial_\beta,\partial_{sx}\Big]=2s\sum_{\alpha,\beta}\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\partial_\alpha\partial_\beta.$$

Next, using Lemma 1.4.2 (2) we have

$$\begin{aligned}
\Big[\sum_{\alpha,\beta}\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\partial_\alpha\partial_\beta,\partial_{\Psi(x)}\Big] &= 6\sum_{\alpha,\beta}\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\partial_{B_\Psi(e_\alpha,x,x)}\partial_\beta+6\sum_{\alpha,\beta}\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\partial_{B_\Psi(e_\alpha,e_\beta,x)}\\
&= -\sum_{\alpha,\beta}\Big(2\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\partial_{\mu(x)e_\alpha}\partial_\beta+\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\omega(x,e_\alpha)\partial_x\partial_\beta\Big)\\
&\quad -\sum_{\alpha,\beta}\Big(2\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\partial_{B_\mu(e_\alpha,e_\beta)x}+\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\partial_{\omega(e_\alpha,x)e_\beta}\Big)\\
&= \sum_{\alpha,\beta}\omega([T,\mu(x)]\widehat{e}_\alpha,\widehat{e}_\beta)\partial_\alpha\partial_\beta+\sum_{\alpha,\beta}\omega(x,\widehat{e}_\alpha)\omega(Tx,\widehat{e}_\beta)\partial_\alpha\partial_\beta\\
&\quad -\sum_{\alpha,\beta}2\partial_{B_\mu(e_\alpha,T\widehat{e}_\alpha)x}-\partial_{Tx}.
\end{aligned}$$

By Lemma 1.4.5 the sum in the third summand evaluates to $\sum_\alpha B_\mu(e_\alpha, T\widehat{e}_\alpha) = -\mathcal{C}(\mathfrak{m}')T$ and together we obtain

$$\begin{aligned} &= \sum_{\alpha,\beta} \omega([T,\mu(x)]\widehat{e}_\alpha, \widehat{e}_\beta)\partial_\alpha\partial_\beta + \sum_{\alpha,\beta} \omega(x,\widehat{e}_\alpha)\omega(Tx,\widehat{e}_\beta)\partial_\alpha\partial_\beta \\ &\quad + (2\,\mathcal{C}(\mathfrak{m}') - 1)\partial_{Tx}. \end{aligned}$$

Next, by Lemma 1.4.2 (2) and (3)

$$\begin{aligned} &\Big[\sum_{\alpha,\beta} \omega(T\widehat{e}_\alpha, \widehat{e}_\beta)\partial_\alpha\partial_\beta, (s^2 + Q(x))\partial_s\Big] \\ &= 8\sum_{\alpha,\beta} \omega(T\widehat{e}_\alpha, \widehat{e}_\beta) B_Q(e_\alpha, x, x, x)\partial_\beta\partial_s + 12\sum_{\alpha,\beta} \omega(T\widehat{e}_\alpha, \widehat{e}_\beta) B_Q(e_\alpha, e_\beta, x, x)\partial_s \\ &= 2\partial_{T\Psi(x)}\partial_s - \sum_{\alpha,\beta} \omega(T\widehat{e}_\alpha, \widehat{e}_\beta)\Big(\omega(x, B_\mu(e_\alpha, e_\beta)x) + \frac{1}{2}\omega(x, e_\beta)\omega(e_\alpha, x)\Big)\partial_s \\ &= -\frac{2}{3}\partial_{T\mu(x)x}\partial_s - \sum_\alpha \omega(x, B_\mu(e_\alpha, T\widehat{e}_\alpha)x)\partial_s + \frac{1}{2}\omega(Tx, x)\partial_s \end{aligned}$$

which is, again by Lemma 1.4.5, equal to

$$= -\frac{2}{3}\partial_{T\mu(x)x}\partial_s + (\frac{1}{2} - \mathcal{C}(\mathfrak{m}'))\omega(Tx, x)\partial_s.$$

Finally, the last commutator of this type is

$$\Big[\sum_{\alpha,\beta} \omega(T\widehat{e}_\alpha, \widehat{e}_\beta)\partial_\alpha\partial_\beta, (\nu + \rho)s\Big] = 0.$$

Next, we have

$$[\partial_{Tx}\partial_s, \partial_{sx}] = \partial_{Tx} + \sum_{\alpha,\beta} \omega(x, \widehat{e}_\alpha)\omega(Tx, \widehat{e}_\beta)\partial_\alpha\partial_\beta.$$

Further, again by Lemma 1.4.2

$$\begin{aligned} [\partial_{Tx}\partial_s, \partial_{\Psi(x)}] &= \sum_\alpha \Big(3\omega(B_\Psi(Tx, x, x), \widehat{e}_\alpha) - \omega(T\Psi(x), \widehat{e}_\alpha)\Big)\partial_\alpha\partial_s \\ &= -\partial_{\mu(x)Tx}\partial_s + \frac{1}{2}\omega(Tx, x)\partial_x\partial_s + \frac{1}{3}\partial_{T\mu(x)x}\partial_s. \end{aligned}$$

Next,

$$\begin{aligned} [\partial_{Tx}\partial_s, (s^2 + Q(x))\partial_s] &= 2s\partial_{Tx}\partial_s + 4B_Q(Tx, x, x, x)\partial_s^2 \\ &= 2s\partial_{Tx}\partial_s + \omega(Tx, \Psi(x))\partial_s^2 \\ &= 2s\partial_{Tx}\partial_s - \frac{1}{6}\omega([T, \mu(x)]x, x)\partial_s^2. \end{aligned}$$

And the last commutator of this type is

$$[\partial_{Tx}\partial_s, (\nu + \rho)s] = (\nu + \rho)\partial_{Tx}.$$

Next, we have

$$[\omega(Tx,x)\partial_s^2, \partial_{sx}] = 2\omega(Tx,x)\partial_x\partial_s - 2s\omega(Tx,x)\partial_s^2.$$

Further,

$$[\omega(Tx,x)\partial_s^2, \partial_{\Psi(x)}] = -\Big(\omega(T\Psi(x),x) + \omega(Tx,\Psi(x))\Big)\partial_s^2 = \frac{1}{3}\omega([T,\mu(x)]x,x)\partial_s^2.$$

Next

$$[\omega(Tx,x)\partial_s^2, (s^2+Q(x))\partial_s] = 2\omega(Tx,x)\partial_s + 4s\omega(Tx,x)\partial_s^2$$

and

$$[\omega(Tx,x)\partial_s^2, (\nu+\rho)s] = 2(\nu+\rho)\omega(Tx,x)\partial_s.$$

Finally,

$$\begin{aligned}[\Omega_\mu(T), d\zeta(\mu(x))] &= 2\sum_\alpha d\zeta(B_\mu(e_\alpha, T\widehat{e}_\alpha)) + 4\sum_\alpha d\zeta(B_\mu(x,e_\alpha))\partial_{T\widehat{e}_\alpha} - 2d\zeta(B_\mu(Tx,x))\partial_t \\ &= -2\,\mathcal{C}(\mathfrak{m}')d\zeta(T) + 4\sum_\alpha d\zeta(B_\mu(x,e_\alpha))\Omega_\omega(T\widehat{e}_\alpha)\end{aligned}$$

Collecting all terms shows the claimed formula. □

3.3 The Fourier transform of Ω_ω

Since $\Omega_\omega(v)$ is acting by the left-invariant vector field corresponding to $v \in V$, it follows immediately that

$$\sigma_\lambda(\Omega_\omega(v)u) = -\sigma_\lambda(u)d\sigma_\lambda(v). \tag{3.3.1}$$

3.4 The Fourier transform of Ω_μ

We show that in the Fourier transformed picture the conformally invariant system $\Omega_\mu(T)$, $T\in\mathfrak{m}$, is nothing else but the metaplectic representation of $\mathfrak{sp}(V,\omega)$ restricted to $\mathfrak{m}$ as defined in Section 2.4. Differentiating (2.4.1) we obtain that $d\omega_{\mathrm{met},\lambda}$ is the unique representation of $\mathfrak{m}$ on $\mathcal{S}(\Lambda)$ by differential operators such that

$$d\sigma_\lambda([T,X]) = [d\omega_{\mathrm{met},\lambda}(T), d\sigma_\lambda(X)] \qquad \forall\, T\in\mathfrak{m}, X\in\overline{\mathfrak{n}}.$$

Theorem 3.4.1. *For every $\lambda\in\mathbb{R}^\times$ and $T\in\mathfrak{m}$ we have*

$$d\sigma_\lambda(\Omega_\mu(T)) = 2i\lambda d\omega_{\mathrm{met},\lambda}(T).$$

Proof. It suffices to show that

$$[d\sigma_\lambda(\Omega_\mu(T)), d\sigma_\lambda(X)] = 2i\lambda d\sigma_\lambda([T, X]) \qquad \forall\, X \in V.$$

We compute

$$\begin{aligned}
[d\sigma_\lambda(\Omega_\mu(T)), d\sigma_\lambda(X)] &= \sum_{\alpha,\beta} \omega([T, \widehat{e}_\alpha], \widehat{e}_\beta)[d\sigma_\lambda(X_\alpha)\sigma_\lambda(X_\beta), d\sigma_\lambda(X)] \\
&= \sum_{\alpha,\beta} \omega([T, \widehat{e}_\alpha], \widehat{e}_\beta)\Big(d\sigma_\lambda(X_\alpha)[d\sigma_\lambda(X_\beta), d\sigma_\lambda(X)] + [d\sigma_\lambda(X_\alpha), d\sigma_\lambda(X)]d\sigma_\lambda(X_\beta)\Big) \\
&= \sum_{\alpha,\beta} \omega([T, \widehat{e}_\alpha], \widehat{e}_\beta)\Big(d\sigma_\lambda(X_\alpha)d\sigma_\lambda([X_\beta, X]) + d\sigma_\lambda([X_\alpha, X])d\sigma_\lambda(X_\beta)\Big) \\
&= i\lambda \sum_{\alpha,\beta} \omega([T, \widehat{e}_\alpha], \widehat{e}_\beta)\left(\omega(X_\beta, X)d\sigma_\lambda(X_\alpha) + \omega(X_\alpha, X)d\sigma_\lambda(X_\beta)\right) \\
&= 2i\lambda \sum_{\beta} \omega([T, X], \widehat{e}_\beta)d\sigma_\lambda(X_\beta) = 2i\lambda d\sigma_\lambda([T, X]).
\end{aligned}$$

□

3.5 The Fourier transform of Ω_Ψ

Corollary 3.5.1. *For every $\lambda \in \mathbb{R}^\times$ and $X \in V$ we have*

$$\begin{aligned}
d\sigma_\lambda(\Omega_\Psi(X)) &= \frac{2i\lambda}{3}\sum_\alpha \sigma_\lambda(e_\alpha)d\omega_{\mathrm{met},\lambda}(B_\mu(X, \widehat{e}_\alpha)) + \frac{i\lambda}{12}(\dim V + 1)\sigma_\lambda(X) \\
&= \frac{i\lambda p}{3}\sum_i \sigma_\lambda(T_i' X)d\omega_{\mathrm{met},\lambda}(T_i) + \frac{i\lambda}{12}(\dim V + 1)\sigma_\lambda(X),
\end{aligned}$$

where (e_α) is a basis of V, $(\widehat{e}_\alpha)$ the dual basis with respect the symplectic form ω, (T_i) a basis of $\mathfrak{m}$ and (T_i') the dual basis with respect to the Killing form κ of $\mathfrak{g}$.

Proof. By Lemma 1.4.2 and Theorem 3.4.1 we have

$$\begin{aligned}
\sigma_\lambda(\Omega_\Psi(X)) &= \sum_{\alpha,\beta,\gamma} \omega(\widehat{e}_\beta, B_\Psi(X, \widehat{e}_\alpha, \widehat{e}_\gamma))\sigma_\lambda(e_\alpha)\sigma_\lambda(e_\beta)\sigma_\lambda(e_\gamma) \\
&= -\frac{1}{3}\sum_{\alpha,\beta,\gamma} \omega(\widehat{e}_\beta, B_\mu(X, \widehat{e}_\alpha)\widehat{e}_\gamma)\sigma_\lambda(e_\alpha)\sigma_\lambda(e_\beta)\sigma_\lambda(e_\gamma) \\
&\qquad - \frac{1}{12}\sum_{\alpha,\beta,\gamma} \omega(\widehat{e}_\beta, \omega(X, \widehat{e}_\gamma)\widehat{e}_\alpha + \omega(\widehat{e}_\alpha, \widehat{e}_\gamma)X)\sigma_\lambda(e_\alpha)\sigma_\lambda(e_\beta)\sigma_\lambda(e_\gamma) \\
&= \frac{1}{3}\sum_\alpha \sigma_\lambda(e_\alpha)\sigma_\lambda(\Omega_\mu(B_\mu(X, \widehat{e}_\alpha)) \\
&\qquad + \frac{1}{12}\sum_\alpha \Big(\sigma_\lambda(e_\alpha)\sigma_\lambda(\widehat{e}_\alpha)\sigma_\lambda(X) + \sigma_\lambda(e_\alpha)\sigma_\lambda(X)\sigma_\lambda(\widehat{e}_\alpha)\Big) \\
&= \frac{2i\lambda}{3}\sum_\alpha \sigma_\lambda(e_\alpha)d\omega_{\mathrm{met},\lambda}(B_\mu(X, \widehat{e}_\alpha)) \\
&\qquad + \frac{1}{6}\sum_\alpha \sigma_\lambda(e_\alpha)\sigma_\lambda(\widehat{e}_\alpha)\sigma_\lambda(X) + \frac{1}{12}\sum_\alpha \sigma_\lambda(e_\alpha)\sigma_\lambda([X, \widehat{e}_\alpha]).
\end{aligned}$$

Using the independence of the chosen basis we find

$$\sum_\alpha \sigma_\lambda(e_\alpha)\sigma_\lambda(\widehat{e}_\alpha) = \frac{1}{2}\sum_\alpha \big(\sigma_\lambda(e_\alpha)\sigma_\lambda(\widehat{e}_\alpha) - \sigma_\lambda(\widehat{e}_\alpha)\sigma_\lambda(e_\alpha)\big) = \frac{1}{2}\sum_\alpha \sigma_\lambda([e_\alpha,\widehat{e}_\alpha]) = \frac{i\lambda}{2}\dim V \tag{3.5.1}$$

and $\sigma_\lambda([X,\widehat{e}_\alpha]) = i\lambda\omega(X,\widehat{e}_\alpha)$. This shows

$$\sigma_\lambda(\Omega_\Psi(X)) = \frac{2i\lambda}{3}\sum_\alpha \sigma_\lambda(e_\alpha)d\omega_{\mathrm{met},\lambda}(B_\mu(X,\widehat{e}_\alpha)) + \frac{i\lambda}{12}(\dim V+1)\sigma_\lambda(X).$$

The second identity follows by expanding $B_\mu(X,\widehat{e}_\alpha) = \sum_i \kappa(B_\mu(X,\widehat{e}_\alpha),T_i')T_i$ and using $\kappa(B_\mu(x,y),T) = \frac{p}{2}\omega(Tx,y)$ (see Remark 1.8.3). □

3.6 The Fourier transform of Ω_Q

The Fourier transform of the conformally invariant differential operator Ω_Q is essentially the Casimir element in the restriction of the metaplectic representation of $\mathfrak{sp}(V,\omega)$ to $\mathfrak{m}$.

Corollary 3.6.1. *For every $\lambda\in\mathbb{R}^\times$ we have*

$$\sigma_\lambda(\Omega_Q) = \frac{p}{3}\lambda^2 d\omega_{\mathrm{met},\lambda}(\mathrm{Cas}_{\mathfrak{m}}) - \frac{(\dim V)^2}{96}\lambda^2,$$

where $\mathrm{Cas}_{\mathfrak{m}}\in U(\mathfrak{m})$ denotes the Casimir element of $\mathfrak{m}$ with respect to the Killing form κ.

Proof. Using Lemma 1.4.2 and we find

$$\begin{aligned}\sigma_\lambda(\Omega_Q) &= -\frac{1}{12}\sum_{\alpha,\beta,\gamma,\delta}\omega(\widehat{e}_\gamma,B_\mu(\widehat{e}_\alpha,\widehat{e}_\beta)\widehat{e}_\delta)\sigma_\lambda(X_\alpha)\sigma_\lambda(X_\beta)\sigma_\lambda(X_\gamma)\sigma_\lambda(X_\delta)\\ &\qquad -\frac{1}{24}\sum_{\alpha,\beta,\gamma,\delta}\omega(\widehat{e}_\gamma,B_\tau(\widehat{e}_\alpha,\widehat{e}_\beta)\widehat{e}_\delta)\sigma_\lambda(X_\alpha)\sigma_\lambda(X_\beta)\sigma_\lambda(X_\gamma)\sigma_\lambda(X_\delta)\\ &= \frac{1}{12}\sum_{\alpha,\beta}\sigma_\lambda(X_\alpha)\sigma_\lambda(X_\beta)\sigma_\lambda(\Omega_\mu(B_\mu(\widehat{e}_\alpha,\widehat{e}_\beta))\\ &\qquad +\frac{1}{24}\sum_{\alpha,\beta}\sigma_\lambda(e_\alpha)\sigma_\lambda(e_\beta)\sigma_\lambda(\widehat{e}_\beta)\sigma_\lambda(\widehat{e}_\alpha).\end{aligned}$$

With (3.5.1) the latter sum becomes $-\frac{1}{4}\lambda^2(\dim V)^2$. For the first sum let (T_i) be a basis of $\mathfrak{m}$ and (T_i') its dual basis with respect to the Killing form κ of $\mathfrak{g}$. Then

$$\begin{aligned}\sum_{\alpha,\beta}\sigma_\lambda(X_\alpha)\sigma_\lambda(X_\beta)\sigma_\lambda(\Omega_\mu(B_\mu(\widehat{e}_\alpha,\widehat{e}_\beta)) &= \sum_i\sum_{\alpha,\beta}\kappa(B_\mu(\widehat{e}_\alpha,\widehat{e}_\beta),T_i')\sigma_\lambda(X_\alpha)\sigma_\lambda(X_\beta)\sigma_\lambda(\Omega_\mu(T_i))\\ &= -p\sum_i \sigma_\lambda(\Omega_\mu(T_i'))\sigma_\lambda(\Omega_\mu(T_i))\\ &= 4p\lambda^2\sum_i d\omega_{\mathrm{met},\lambda}(T_i')d\omega_{\mathrm{met},\lambda}(T_i)\\ &= 4p\lambda^2 d\omega_{\mathrm{met},\lambda}(\mathrm{Cas}_{\mathfrak{m}}),\end{aligned}$$

where $\mathrm{Cas}_{\mathfrak{m}} = \sum_i T_i'T_i\in U(\mathfrak{m})$ is the Casimir element of $\mathfrak{m}$. □

Chapter 4

Analysis of the Fourier transform of Ω_μ

We study the subrepresentation $I(\zeta,\nu)^{\Omega_\mu(\mathfrak{m})}$ and its image in $\mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda\times\Lambda)$ under the Fourier transform in the case where G is non-Hermitian. For this we first develop some more structure theory following [49].

4.1 The Lagrangian decomposition

By Theorem 1.9.1 the group G is non-Hermitian if and only if there exists $O\in V$ such that $Q(O)>0$. We renormalize O such that $Q(O)=1$. Any such $O\in V$ has by [49, Theorem 6.35] a Lagrangian decomposition

$$O = A + B$$

where $A,B\in Z=\mu^{-1}(0)$ and $\omega(A,B)=2$. This decomposition is unique and A and B are given by

$$A=\tfrac{1}{2}(O-\Psi(O)), \qquad B=\tfrac{1}{2}(O+\Psi(O)).$$

Further, the tangent spaces

$$\Lambda := T_A Z \qquad \text{and} \qquad \Lambda^* := T_B Z$$

are complementary Lagrangian subspaces. We use this particular Lagrangian decomposition for the Schrödinger model of the representation σ_λ of $\overline{N}$.

Note that we use the same letter A for the element $A\in V$ and the one-dimensional subgroup $A=\exp(\mathbb{R}H)\subseteq G$. It should be clear from the context which object is meant.

4.2 The bigrading

Let

$$H_\alpha=\tfrac{1}{2}(H+\mu(O))=\tfrac{1}{2}(H+2B_\mu(A,B)), \qquad H_\beta=\tfrac{1}{2}(H-\mu(O))=\tfrac{1}{2}(H-2B_\mu(A,B)).$$

Then $[H_\alpha, H_\beta] = 0$ and $\mathrm{ad}(H_\alpha)$ and $\mathrm{ad}(H_\beta)$ are simultaneously diagonalizable. Write

$$\mathfrak{g}_{(i,j)} := \{X \in \mathfrak{g} : \mathrm{ad}(H_\alpha)X = iX, \mathrm{ad}(H_\beta)X = jX\},$$

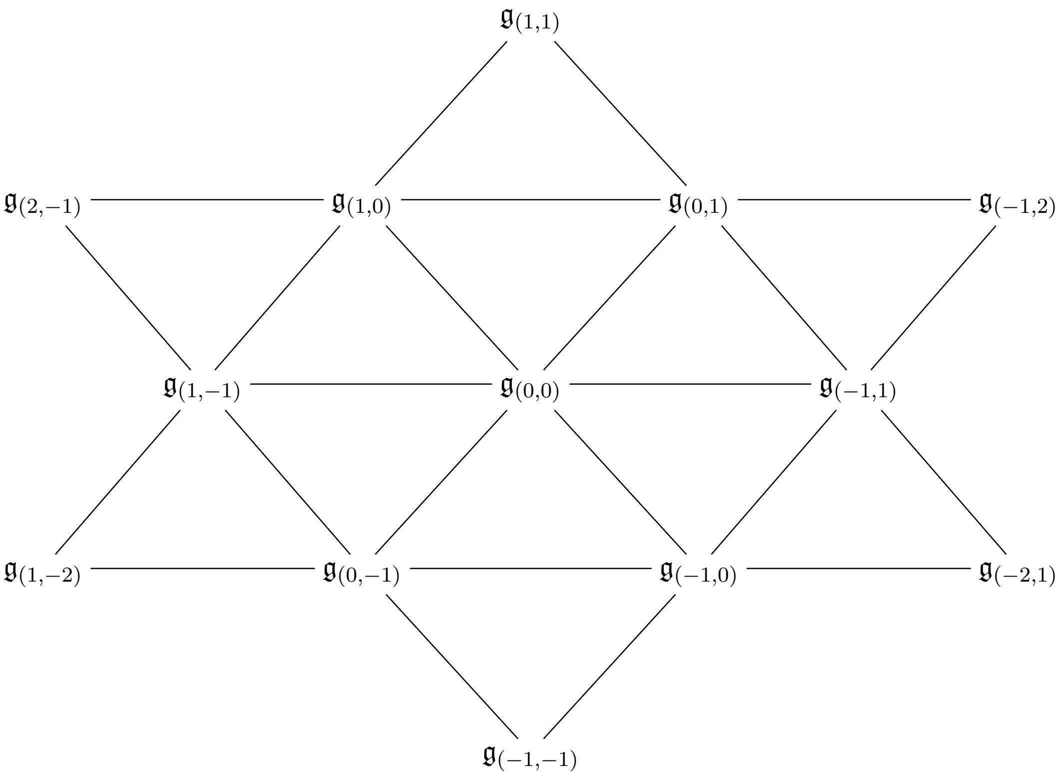
then we have the bigrading

$$\mathfrak{g} = \bigoplus_{i,j} \mathfrak{g}_{(i,j)}.$$

More precisely

$$\begin{aligned}
\mathfrak{g}_2 &= \mathfrak{g}_{(1,1)}, \\
\mathfrak{g}_1 &= \mathfrak{g}_{(2,-1)} + \mathfrak{g}_{(1,0)} + \mathfrak{g}_{(0,1)} + \mathfrak{g}_{(-1,2)}, \\
\mathfrak{g}_0 &= \mathfrak{g}_{(1,-1)} + \mathfrak{g}_{(0,0)} + \mathfrak{g}_{(-1,1)}, \\
\mathfrak{g}_{-1} &= \mathfrak{g}_{(1,-2)} + \mathfrak{g}_{(0,-1)} + \mathfrak{g}_{(-1,0)} + \mathfrak{g}_{(-2,1)}, \\
\mathfrak{g}_{-2} &= \mathfrak{g}_{(-1,-1)}.
\end{aligned}$$

Note that $\mathrm{Ad}(w_0)\mathfrak{g}_{(i,j)} = \mathfrak{g}_{(-j,-i)}$, i.e. w_0 flips the star diagram along the axis $i + j = 0$.



Here $\mathfrak{m} = \mathfrak{g}_{(1,-1)} \oplus (\mathfrak{m} \cap \mathfrak{g}_{(0,0)}) \oplus \mathfrak{g}_{(-1,1)}$ and $\mathfrak{g}_{(0,0)} = \mathbb{R}H \oplus (\mathfrak{m} \cap \mathfrak{g}_{(0,0)})$. Further, $\mathfrak{m} \cap \mathfrak{g}_{(0,0)} = \mathbb{R}B_\mu(A,B) \oplus \mathfrak{m}^O$, where

$$\mathfrak{m}^O = \{T \in \mathfrak{m} : [T,O] = 0\} = \{T \in \mathfrak{m} : [T,A] = [T,B] = 0\}.$$

Further,

$$\mathfrak{g}_{(1,-2)} = \mathbb{R}A \qquad \text{and} \qquad \mathfrak{g}_{(-2,1)} = \mathbb{R}B,$$

and the Lagrangians Λ and Λ^* are given by

$$\Lambda = \mathbb{R}A + \mathfrak{g}_{(0,-1)}, \qquad \text{and} \qquad \Lambda^* = \mathbb{R}B + \mathfrak{g}_{(-1,0)}.$$

Note that $\omega(A, \mathfrak{g}_{(-1,0)}) = 0$ and $\omega(B, \mathfrak{g}_{(0,-1)}) = 0$. Moreover, the maps

$$\mathfrak{g}_{(0,-1)} \to \mathfrak{g}_{(-1,1)},\ X \mapsto B_\mu(X, B) \qquad \text{and} \qquad \mathfrak{g}_{(-1,0)} \to \mathfrak{g}_{(1,-1)},\ X \mapsto B_\mu(A, X)$$

are $\mathfrak{g}_{(0,0)}$-equivariant isomorphisms. Note that $B_\mu(A, B)$ acts on $\mathfrak{g}_{-1}$ by

$$B_\mu(A, B) = \begin{cases} \frac{3}{2} & \text{on } \mathfrak{g}_{(1,-2)}, \\ \frac{1}{2} & \text{on } \mathfrak{g}_{(0,-1)}, \\ -\frac{1}{2} & \text{on } \mathfrak{g}_{(-1,0)}, \\ -\frac{3}{2} & \text{on } \mathfrak{g}_{(-2,1)}. \end{cases} \tag{4.2.1}$$

Further, note that

$$\mathfrak{g}_{(2,-1)} = \mathbb{R}\overline{A}, \qquad \mathfrak{g}_{(-1,2)} = \mathbb{R}\overline{B},$$

and

$$[\overline{A}, B] = -2H_\alpha, \qquad [\overline{B}, A] = 2H_\beta.$$

The choice of $O \in V$ further determines a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(3, \mathbb{R})$ given by

$$\mathfrak{g}_{(1,1)} + \mathfrak{g}_{(2,-1)} + \mathfrak{g}_{(-1,2)} + \mathbb{R}H_\alpha + \mathbb{R}H_\beta + \mathfrak{g}_{(-2,1)} + \mathfrak{g}_{(1,-2)} + \mathfrak{g}_{(-1,-1)}.$$

For $z \in \mathfrak{g}_{(0,-1)}$ we have $\mu(z) \in \mathfrak{g}_{(1,-1)}$ and hence $\Psi(z) \in \mathfrak{g}_{(1,-2)} = \mathbb{C}A$. Write

$$\Psi(z) = n(z)A, \qquad z \in \mathfrak{g}_{(0,-1)}.$$

Then the function $n(z)$ is a polynomial of degree 3 which vanishes identically if and only if $\mathfrak{g} \simeq \mathfrak{sl}(n, \mathbb{R})$ (see [49, Proposition 7.9]). In all other cases,

$$\mathcal{J} = \mathfrak{g}_{(0,-1)}$$

carries the structure of a rank 3 Jordan algebra with Jordan determinant $n(z)$. Note that $\Psi^{-1}(0) \cap \mathcal{J}$ resp. $\mu^{-1}(0) \cap \mathcal{J}$ is the subvariety of elements of rank ≤ 2 resp. ≤ 1. We write $\mathcal{J}^* = \mathfrak{g}_{(-1,0)}$ which can be identified with the dual of $\mathcal{J}$ using the symplectic form.

Remark 4.2.1. A slightly different and more natural point of view is to endow $(V^+, V^-) = (\mathcal{J}, \mathcal{J}^*)$ with the structure of a *Jordan pair*. This structure consists of trilinear maps

$$\{\cdot, \cdot, \cdot\}_\pm : V^\pm \times V^\mp \times V^\pm \to V^\pm,$$

such that

(1) $\{u, v, w\}_\pm = \{w, v, u\}_\pm$ for all $u, w \in V^\pm$ and $v \in V^\mp$,

(2) $\{x, y, \{u, v, w\}_\pm\}_\pm = \{\{x, y, u\}_\pm, v, w\}_\pm - \{u, \{v, x, y\}_\mp, w\}_\pm + \{u, v, \{x, y, w\}_\pm\}_\pm$ for all $x, u, w \in V^\pm$ and $y, v \in V^\mp$.

If we define

$$\{u,v,w\}_\pm := \pm B_\Psi(u,v,w),$$

property (1) follows immediately from the symmetry of B_Ψ and with Lemma 1.4.2 it is easy to see that property (2) is equivalent to the $\mathfrak{m}$-equivariance of B_μ.

In addition to w_0 we define the additional Weyl group elements

$$w_1 = \exp\left(\frac{\pi}{2\sqrt{2}}(A-\overline{B})\right), \qquad w_2 = \exp\left(\frac{\pi}{2\sqrt{2}}(B+\overline{A})\right).$$

Lemma 4.2.2. *(1) The elements w_1 and w_2 have the following mapping properties:*

$$\mathrm{Ad}(w_1)\mathfrak{g}_{(i,j)} = \mathfrak{g}_{(i+j,-j)}, \qquad \mathrm{Ad}(w_2)\mathfrak{g}_{(i,j)} = \mathfrak{g}_{(-i,i+j)}.$$

(2) For $v\in\mathcal{J}$, $w\in\mathcal{J}^*$ and $T\in\mathfrak{m}^O$ we have

$$\begin{aligned}
\mathrm{Ad}(w_1)F &= -\frac{1}{\sqrt{2}}B, & \mathrm{Ad}(w_1)E &= \frac{1}{\sqrt{2}}\overline{A},\\
\mathrm{Ad}(w_1)A &= -\overline{B}, & \mathrm{Ad}(w_1)\overline{A} &= -\sqrt{2}E,\\
\mathrm{Ad}(w_1)v &= \sqrt{2}B_\mu(v,B), & \mathrm{Ad}(w_1)\overline{v} &= \overline{v},\\
\mathrm{Ad}(w_1)w &= w, & \mathrm{Ad}(w_1)\overline{w} &= \sqrt{2}B_\mu(A,w),\\
\mathrm{Ad}(w_1)B &= \sqrt{2}F, & \mathrm{Ad}(w_1)\overline{B} &= -A,\\
\mathrm{Ad}(w_1)B_\mu(v,B) &= -\frac{1}{\sqrt{2}}v, & \mathrm{Ad}(w_1)B_\mu(A,w) &= -\frac{1}{\sqrt{2}}\overline{w},\\
\mathrm{Ad}(w_1)H &= B_\mu(A,B)+\frac{1}{2}H, & \mathrm{Ad}(w_1)B_\mu(A,B) &= -\frac{1}{2}B_\mu(A,B)+\frac{3}{4}H,\\
\mathrm{Ad}(w_1)T &= T,
\end{aligned}$$

and similar for w_2 by substituting $(A,B)\mapsto(B,-A)$.

(3) We have $w_1^2, w_2^2\in M$ with $\chi(w_1^2)=\chi(w_2^2)=-1$ and

$$\begin{aligned}
\mathrm{Ad}(w_1^2)(aA+v+w+bB) &= aA-v+w-bB\\
\mathrm{Ad}(w_2^2)(aA+v+w+bB) &= -aA+v-w+bB
\end{aligned} \qquad (a,b\in\mathbb{R}, v\in\mathcal{J}, w\in\mathcal{J}^*).$$

(4) The following relations hold:

$$\begin{aligned}
w_0w_1w_0^{-1} &= w_2^{-1}, & w_1w_0w_1^{-1} &= w_2, & w_2w_0w_2^{-1} &= w_1^{-1}\\
w_0w_2w_0^{-1} &= w_1, & w_1w_2w_1^{-1} &= w_0^{-1}, & w_2w_1w_2^{-1} &= w_0.
\end{aligned}$$

Proof. (2) is an easy computation using the definitions in Section 1.2 as well as Lemma 1.4.4. The formulas for $\mathrm{Ad}(w_1)$ and $\mathrm{Ad}(w_2)$ then imply (1) and (3). Finally, (4) follows with the identity

$$w\exp(X)w^{-1} = \exp(\mathrm{Ad}(w)X)$$

and the definitions in Section 1.2. □

4.3 Identities for the moment map μ in the bigrading

We show several identities for the moment map μ and its symmetrization B_μ acting on different parts of the decomposition

$$V = \mathbb{R}A \oplus \mathcal{J} \oplus \mathcal{J}^* \oplus \mathbb{R}B.$$

Lemma 4.3.1. *Assume $\mathfrak{g} \not\simeq \mathfrak{sl}(n,\mathbb{R}), \mathfrak{so}(p,q)$. For $z \in \mathcal{J}$ and $w \in \mathcal{J}^*$ we have*

$$\mathrm{tr}(B_\mu(A,w) \circ B_\mu(z,B)|_{\mathcal{J}}) = \left(\frac{1}{2} + \frac{1}{6}\dim \mathcal{J}\right)\omega(z,w).$$

We remark that the formula also holds for $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$ and $\mathfrak{so}(4,4)$, but we do not need this.

Proof. First note that since B_μ is $\mathfrak{m}$-equivariant

$$[B_\mu(A,w), B_\mu(z,B)] = B_\mu(B_\mu(A,w)z, B) + B_\mu(z, B_\mu(A,w)B).$$

By Lemma 1.4.3 we have $B_\mu(A,w)z = \frac{1}{2}\omega(z,w)A$ and $B_\mu(A,w)B = -w$ and hence

$$[B_\mu(A,w), B_\mu(z,B)] = \frac{1}{2}\omega(z,w)B_\mu(A,B) - B_\mu(z,w).$$

Choose a basis (e_α) of $\mathfrak{g}_{(0,-1)}$ and let $(\widehat{e}_\alpha)$ be the basis of $\mathfrak{g}_{(-1,0)}$ such that $\omega(e_\alpha, \widehat{e}_\beta) = \delta_{\alpha\beta}$. Then

$$\mathrm{tr}(B_\mu(A,w) \circ B_\mu(z,B)|_{\mathfrak{g}_{(0,-1)}}) = \sum_\alpha \omega(B_\mu(A,w)B_\mu(z,B)e_\alpha, \widehat{e}_\alpha)$$
$$= \sum_\alpha \Big(\omega(B_\mu(z,B)B_\mu(A,w)e_\alpha, \widehat{e}_\alpha) + \frac{1}{2}\omega(z,w)\omega(B_\mu(A,B)e_\alpha, \widehat{e}_\alpha) - \omega(B_\mu(z,w)e_\alpha, \widehat{e}_\alpha)\Big).$$

Again by Lemma 1.4.3 we find that $B_\mu(A,w)e_\alpha = \frac{1}{2}\omega(e_\alpha,w)A$ and $B_\mu(z,B)A = z$, and further $B_\mu(A,B)e_\alpha = \frac{1}{2}e_\alpha$ so that

$$\mathrm{tr}(B_\mu(A,w) \circ B_\mu(z,B)|_{\mathfrak{g}_{(0,-1)}}) = (\frac{1}{2} + \frac{1}{4}\dim \mathfrak{g}_{(0,-1)})\omega(z,w) - \mathrm{tr}(B_\mu(z,w)|_{\mathfrak{g}_{(0,-1)}}).$$

We have $B_\mu(z,w) \in \mathfrak{g}_{(0,0)} = \mathbb{R}H_\alpha \oplus \mathbb{R}H_\beta \oplus \mathfrak{m}^O$, where $\mathfrak{m}^O = \{T \in \mathfrak{m} : [T,O] = 0\}$. Write $B_\mu(z,w) = aH_\alpha + bH_\beta + T$ with $T \in \mathfrak{m}^O$. Then $\mathrm{tr}(T|_{\mathfrak{g}_{(0,-1)}}) = 0$ (since $\mathfrak{g} \not\simeq \mathfrak{sl}(n,\mathbb{R}), \mathfrak{so}(p,q)$ and hence $\mathfrak{m}^O$ is semisimple) and $\mathrm{tr}(H_\alpha|_{\mathfrak{g}_{(0,-1)}}) = 0$, $\mathrm{tr}(H_\beta|_{\mathfrak{g}_{(0,-1)}}) = -\dim \mathfrak{g}_{(0,-1)}$. We determine a and b which will complete the proof. Since $[T,A] = [T,B] = 0$ we have

$$B_\mu(z,w)A = (a-2b)A \qquad \text{and} \qquad B_\mu(z,w)B = (b-2a)B.$$

On the other hand, by Lemma 1.4.3 we find

$$B_\mu(z,w)A = \frac{1}{2}\omega(B_\mu(z,w)A,B)A = \frac{1}{2}\omega(B_\mu(A,B)z,w)A = \frac{1}{4}\omega(z,w)A,$$
$$B_\mu(z,w)B = \frac{1}{2}\omega(A,B_\mu(z,w)B)B = \frac{1}{2}\omega(z,B_\mu(A,B)w)B = -\frac{1}{4}\omega(z,w)B.$$

Thus, $a = -b = \frac{1}{12}\omega(z,w)$ and the claimed formula follows. ∎

Lemma 4.3.2. *For $z \in \mathcal{J}$ we have*

$$\mu(z)^2 B = -4n(z)z.$$

Proof. Using Lemma 1.4.3 and $\mu(z)z = -3\Psi(z) = -3n(z)A$ we find

$$\begin{aligned}\mu(z)^2 B = \mu(z)B_\mu(z,z)B = \mu(z)B_\mu(z,B)z &= [\mu(z), B_\mu(z,B)]z + B_\mu(z,B)\mu(z)z \\ &= [\mu(z), B_\mu(z,B)]z - 3n(z)B_\mu(z,B)A = [\mu(z), B_\mu(z,B)]z - 3n(z)z.\end{aligned}$$

Now by the $\mathfrak{m}$-equivariance of B_μ:

$$\begin{aligned}[\mu(z), B_\mu(z,B)]z &= B_\mu(\mu(z)z, B)z + B_\mu(z, \mu(z)B)z \\ &= -3n(z)B_\mu(A,B)z + B_\mu(z, B_\mu(z,B)z)z \\ &= -\frac{3}{2}n(z)z + \frac{1}{2}[B_\mu(z,B), \mu(z)]z\end{aligned}$$

and hence $[\mu(z), B_\mu(z,B)]z = -n(z)z$ and the claim follows. □

Lemma 4.3.3. *If $\mathfrak{g}$ is non-Hermitian and $\mathfrak{m}$ is simple then the number $\mathcal{C}(\mathfrak{m})$ in Lemma 1.4.5 is given by*

$$\mathcal{C}(\mathfrak{m}) = \frac{3}{2} + \frac{\dim \mathcal{J}}{6}.$$

Proof. Let (e_α) be a basis of $\mathfrak{g}_{-1}$ and $(\widehat{e}_\alpha)$ its dual basis with respect to ω, then by Lemma 1.4.5

$$\sum_\alpha B_\mu(Te_\alpha, \widehat{e}_\alpha) = \mathcal{C}(\mathfrak{m}) \cdot T \qquad \forall T \in \mathfrak{m}. \tag{4.3.1}$$

We may choose $e_\alpha \in \{A, B\} \cup \mathfrak{g}_{(0,-1)} \cup \mathfrak{g}_{(-1,0)}$, then $\widehat{e}_\alpha \in \{-\frac{1}{2}A, \frac{1}{2}B\} \cup \mathfrak{g}_{(0,-1)} \cup \mathfrak{g}_{(-1,0)}$. Now put $T = B_\mu(A,B) \in \mathfrak{m}$, then the left hand side of (4.3.1) becomes

$$\frac{1}{2}B_\mu(TA,B) - \frac{1}{2}B_\mu(TB,A) + \sum_{e_\alpha \in \mathfrak{g}_{(0,-1)}} B_\mu(Te_\alpha, \widehat{e}_\alpha) + \sum_{e_\alpha \in \mathfrak{g}_{(-1,0)}} B_\mu(Te_\alpha, \widehat{e}_\alpha)$$

and by (4.2.1) this is

$$\frac{3}{2}B_\mu(A,B) + \sum_{e_\alpha \in \mathfrak{g}_{(0,-1)}} B_\mu(e_\alpha, \widehat{e}_\alpha).$$

Hence

$$\sum_{e_\alpha \in \mathfrak{g}_{(0,-1)}} B_\mu(e_\alpha, \widehat{e}_\alpha) = \left(\mathcal{C}(\mathfrak{m}) - \frac{3}{2}\right) B_\mu(A,B).$$

We apply both sides to A and find

$$\sum_{e_\alpha \in \mathfrak{g}_{(0,-1)}} B_\mu(e_\alpha, \widehat{e}_\alpha)A = \left(\mathcal{C}(\mathfrak{m}) - \frac{3}{2}\right) B_\mu(A,B)A = \frac{3}{2}\left(\mathcal{C}(\mathfrak{m}) - \frac{3}{2}\right) A.$$

As in the proof of Lemma 4.3.1 we have $B_\mu(e_\alpha, \widehat{e}_\alpha)A = \frac{1}{4}\omega(e_\alpha, \widehat{e}_\alpha)A = \frac{1}{4}A$ and the claim follows. □

Lemma 4.3.4. *Let $x = aA + z + w + bB$ with $a, b \in \mathbb{R}$, $z \in \mathfrak{g}_{(0,-1)}$ and $w \in \mathfrak{g}_{(-1,0)}$. Then*

$$\mu(x) = \underbrace{(\mu(z) + 2aB_\mu(A,w))}_{\in\mathfrak{g}_{(1,-1)}} + \underbrace{(2abB_\mu(A,B) + 2B_\mu(z,w))}_{\in\mathfrak{g}_{(0,0)}} + \underbrace{(\mu(w) + 2bB_\mu(z,B))}_{\in\mathfrak{g}_{(-1,1)}},$$

$$\Psi(x) = \underbrace{(-a^2b - \tfrac{1}{2}a\omega(z,w) + n(z))A}_{\in\mathfrak{g}_{(1,-2)}} + \underbrace{\Big[(-ab - \tfrac{1}{2}\omega(z,w))z - a\mu(w)A - \mu(z)w\Big]}_{\in\mathfrak{g}_{(0,-1)}}$$
$$+ \underbrace{\Big[(ab + \tfrac{1}{2}\omega(z,w))w - b\mu(z)B - \mu(w)z\Big]}_{\in\mathfrak{g}_{(-1,0)}} + \underbrace{(ab^2 + \tfrac{1}{2}b\omega(z,w) + n(w))B}_{\in\mathfrak{g}_{(-2,1)}},$$

$$Q(x) = a^2b^2 + ab\omega(z,w) - 2bn(z) + 2an(w) + \frac{1}{4}\omega(z,w)^2 + \frac{1}{2}\omega(\mu(z)w,w),$$

where $\Psi(z) = n(z)A$ and $\Psi(w) = n(w)B$. Moreover, we have $\mu(z) = -B_\mu(A,\mu(z)B)$, $\mu(w) = B_\mu(\mu(w)A,B)$ and

$$B_\mu(z,w) \in \tfrac{1}{6}\omega(z,w)B_\mu(A,B) + \mathfrak{m}^O.$$

4.4 Invariant distribution vectors in $\mathcal{S}'(\Lambda)$

Using Theorem 3.4.1 we compute $d\omega_{\mathrm{met},\lambda}(T)$ explicitly for $T \in \mathfrak{m}$. In view of the decomposition $\Lambda = \mathbb{R}A + \mathcal{J}$ we write $x \in \Lambda$ as $x = aA + z$ with $a \in \mathbb{R}$ and $z \in \mathcal{J}$.

Proposition 4.4.1. *For every $\lambda \in \mathbb{R}^\times$ the representation $d\omega_{\mathrm{met},\lambda}$ of $\mathfrak{m}$ is given by*

$$d\omega_{\mathrm{met},\lambda}(T) = \begin{cases} \frac{1}{2i\lambda}\sum_{e_\alpha,e_\beta\in\mathfrak{g}_{(0,-1)}} \omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\partial_\alpha\partial_\beta - \frac{1}{2}\omega(TB,z)\partial_A & T \in \mathfrak{g}_{(1,-1)}, \\ -\frac{1}{2}\omega(TA,B)a\partial_A - \partial_{Tz} - \frac{1}{2}\operatorname{tr}(T|_\Lambda) & T \in \mathfrak{g}_{(0,0)} \cap \mathfrak{m}, \\ -a\partial_{TA} + \frac{1}{2}i\lambda\omega(Tz,z) & T \in \mathfrak{g}_{(-1,1)}. \end{cases}$$

Proof. By Theorem 3.4.1:

$$d\omega_{\mathrm{met},\lambda}(T) = \frac{1}{2i\lambda}\sum_{\alpha,\beta}\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)d\sigma_\lambda(X_\alpha)d\sigma_\lambda(X_\beta).$$

Since this expression is independent of the choice of the basis (e_α) we may choose $e_\alpha \in \{A,B\} \cup \mathfrak{g}_{(0,-1)} \cup \mathfrak{g}_{(-1,0)}$. Then $\widehat{e}_\alpha \in \{-\frac{1}{2}A, \frac{1}{2}B\} \cup \mathfrak{g}_{(0,-1)} \cup \mathfrak{g}_{(-1,0)}$. Further, the representation $d\sigma_\lambda$ is in the coordinates (a,z) given by

$$d\sigma_\lambda(A) = -\partial_A, \qquad d\sigma_\lambda(X) = -\partial_X, \quad X \in \mathfrak{g}_{(0,-1)}$$

and

$$d\sigma_\lambda(B) = -2i\lambda a, \qquad d\sigma_\lambda(X) = i\lambda\omega(X,z), \quad X \in \mathfrak{g}_{(-1,0)}.$$

First let $T \in \mathfrak{g}_{(1,-1)}$, then

$$\begin{aligned}
d\omega_{\mathrm{met},\lambda}(T) &= \frac{1}{2i\lambda}\sum_{\alpha,\beta}\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)d\sigma_\lambda(X_\alpha)d\sigma_\lambda(X_\beta)\\
&= \frac{1}{2i\lambda}\Big(-\frac{1}{2}i\lambda\sum_{e_\beta\in\mathfrak{g}_{(-1,0)}}\omega(TB,\widehat{e}_\beta)\omega(e_\beta,z)\partial_A + \sum_{e_\alpha,e_\beta\in\mathfrak{g}_{(0,-1)}}\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\partial_\alpha\partial_\beta\\
&\quad -\frac{1}{2}i\lambda\sum_{e_\alpha\in\mathfrak{g}_{(-1,0)}}\omega(T\widehat{e}_\alpha,B)\omega(e_\alpha,z)\partial_A\Big)\\
&= \frac{1}{2i\lambda}\sum_{e_\alpha,e_\beta\in\mathfrak{g}_{(0,-1)}}\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\partial_\alpha\partial_\beta - \frac{1}{2}\omega(TB,z)\partial_A.
\end{aligned}$$

Next let $T \in \mathfrak{g}_{(0,0)}$, then $\mathrm{ad}(T)$ preserves each $\mathfrak{g}_{(i,j)}$ and we find

$$\begin{aligned}
d\omega_{\mathrm{met},\lambda}(T) &= \frac{1}{2i\lambda}\sum_{\alpha,\beta}\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)d\sigma_\lambda(X_\alpha)d\sigma_\lambda(X_\beta)\\
&= \frac{1}{2i\lambda}\Big(-\frac{1}{2}i\lambda\omega(TB,A)\partial_A a - i\lambda\sum_{\substack{e_\alpha\in\mathfrak{g}_{(0,-1)}\\ e_\beta\in\mathfrak{g}_{(-1,0)}}}\omega([T,\widehat{e}_\alpha],\widehat{e}_\beta)\partial_\alpha\omega(e_\beta,z)\\
&\quad - i\lambda\sum_{\substack{e_\alpha\in\mathfrak{g}_{(-1,0)}\\ e_\beta\in\mathfrak{g}_{(0,-1)}}}\omega([T,\widehat{e}_\alpha],\widehat{e}_\beta)\omega(e_\alpha,z)\partial_\beta - \frac{1}{2}i\lambda\omega(TA,B)a\partial_A\Big)\\
&= -\frac{1}{2}\Big(\omega(TA,B)a\partial_A + 2\partial_{Tz} + \mathrm{tr}(T|_\Lambda)\Big).
\end{aligned}$$

Finally let $T \in \mathfrak{g}_{(-1,1)}$, then

$$\begin{aligned}
d\omega_{\mathrm{met},\lambda}(T) &= \frac{1}{2i\lambda}\sum_{\alpha,\beta}\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)d\sigma_\lambda(X_\alpha)d\sigma_\lambda(X_\beta)\\
&= \frac{1}{2i\lambda}\Big(-i\lambda\sum_{e_\alpha\in\mathfrak{g}_{(0,-1)}}\omega(T\widehat{e}_\alpha,A)a\partial_\alpha - \lambda^2\sum_{e_\alpha,e_\beta\in\mathfrak{g}_{(-1,0)}}\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\omega(e_\alpha,z)\omega(e_\beta,z)\\
&\quad - i\lambda\sum_{e_\beta\in\mathfrak{g}_{(0,-1)}}\omega(TA,\widehat{e}_\beta)a\partial_\beta\Big)\\
&= -a\partial_{TA} + \frac{1}{2}i\lambda\omega(Tz,z).
\end{aligned}$$

□

Recall that $\mathcal{J} = \mathfrak{g}_{(0,-1)}$ is a rank 3 Jordan algebra with norm function $n(z) = \frac{1}{2}\omega(\Psi(z),B)$, except in the case $\mathfrak{g} \simeq \mathfrak{sl}(n,\mathbb{R})$ where $n(z) = 0$. We note that for any $w \in \mathcal{J}$ we have by Lemma 1.4.2

$$\partial_w n(z) = \frac{1}{2}\omega(3B_\Psi(z,z,w),B) = -\frac{1}{2}\omega(\mu(z)w + \frac{1}{2}\tau(z)w,B) = -\frac{1}{2}\omega(\mu(z)w,B). \qquad (4.4.1)$$

Theorem 4.4.2. *Assume $\mathfrak{g} \not\simeq \mathfrak{sl}(n,\mathbb{R}), \mathfrak{so}(p,q)$. For every $\lambda \in \mathbb{R}^\times$ the space $L^2(\Lambda)^{-\infty,\mathfrak{m}} = \mathcal{S}'(\Lambda)^\mathfrak{m}$ of $\mathfrak{m}$-invariant distribution vectors in $\omega_{\mathrm{met},\lambda}$ is two-dimensional. More precisely, $\mathcal{S}'(\Lambda)^\mathfrak{m} = \mathbb{C}\xi_{\lambda,0} \oplus \mathbb{C}\xi_{\lambda,1}$, where*

$$\xi_{\lambda,\varepsilon}(a,z) = \mathrm{sgn}(a)^\varepsilon |a|^{s_{\min}} e^{-i\lambda\frac{n(z)}{a}}, \qquad s_{\min} = -\tfrac{1}{6}(\dim\Lambda + 2), \varepsilon \in \mathbb{Z}/2\mathbb{Z}.$$

Remark 4.4.3. For $s_{\min} \leq -1$ the definition of $\xi_{\lambda,\varepsilon}(a,z)$ does not make sense as a function on Λ, but has to be interpreted as a distribution. In Appendix C we show that $\xi_{\lambda,\varepsilon}$ is indeed the special value of a meromorphic family of distributions on Λ at a regular point.

Proof of Theorem 4.4.2. Let $\xi \in \mathcal{S}'(\Lambda)$ such that $d\omega_{\mathrm{met},\lambda}(T)\xi = 0$ for all $T \in \mathfrak{m}_0$. By definition of $\mathfrak{m}_0$ this is equivalent to $d\omega_{\mathrm{met},\lambda}(T)\xi = 0$ for $T \in \mathfrak{g}_{(1,-1)}$ and $T \in \mathfrak{g}_{(-1,1)}$. First, let $T = B_\mu(w,B) \in \mathfrak{g}_{(-1,1)}$, $w \in \mathfrak{g}_{(0,-1)}$, then by Lemma 1.4.3

$$TA = B_\mu(B,w)A = B_\mu(B,A)w + \frac{1}{4}\omega(B,w)A - \frac{1}{4}\omega(B,A)w - \frac{1}{2}\omega(w,A)B = w$$

and for $z \in \mathcal{J}$:

$$\omega(Tz,z) = \omega(B_\mu(w,B)z,z) = \omega(B_\mu(w,z)B,z) = \omega(B_\mu(z,w)z,B) = \omega(\mu(z)w,B).$$

Hence $d\omega_{\mathrm{met},\lambda}(T)\xi = 0$ implies

$$a\partial_w\xi = \frac{1}{2}i\lambda\omega(\mu(z)w,B)\xi = -i\lambda(\partial_w n(z))\xi$$

which is equivalent to

$$a\partial_w(\xi \cdot e^{i\lambda\frac{n(z)}{a}}) = 0.$$

Since $w \in \mathfrak{g}_{(0,-1)}$ was arbitrary we have

$$\xi(a,z) = \xi_0(a)e^{-i\lambda\frac{n(z)}{a}} + \xi_1(a,z),$$

where $\xi_0(a)$ is independent of z and $\xi_1(a,z)$ has support on $\{a=0\}$. Next let $T = B_\mu(A,w) \in \mathfrak{g}_{(1,-1)}$, $w \in \mathfrak{g}_{(-1,0)}$, then again by Lemma 1.4.3

$$TB = B_\mu(A,w)B = B_\mu(A,B)w + \frac{1}{4}\omega(A,w)B - \frac{1}{4}\omega(A,B)w - \frac{1}{2}\omega(w,B)A = -w.$$

Therefore, $d\omega_{\mathrm{met},\lambda}(T)\xi = 0$ implies

$$\sum_{e_\alpha,e_\beta \in \mathfrak{g}_{(0,-1)}} \omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\partial_\alpha\partial_\beta\xi = i\lambda\omega(z,w)\partial_A\xi.$$

Let us first assume $a \neq 0$, then $\xi(a,z) = \xi_0(a)e^{-i\lambda\frac{n(z)}{a}}$ and hence

$$\begin{aligned}
\partial_A\xi(a,z) &= \xi_0'(a)e^{i\lambda\frac{n(z)}{a}} + i\lambda a^{-2}n(z)\xi(a,z) \\
\partial_\alpha\xi(a,z) &= \frac{1}{2}i\lambda a^{-1}\omega(\mu(z)e_\alpha,B)\xi(a,z), \\
\partial_\alpha\partial_\beta\xi(a,z) &= i\lambda a^{-1}\omega(B_\mu(z,e_\beta)e_\alpha,B)\xi(a,z) - \frac{1}{4}\lambda^2a^{-2}\omega(\mu(z)e_\alpha,B)\omega(\mu(z)e_\beta,B)\xi(a,z).
\end{aligned}$$

We sum the two terms for $\partial_\alpha\partial_\beta\xi$ over α and β separately. For the first term we obtain, using Lemma 1.4.3 and 4.3.1:

$$\begin{aligned}
&\sum_{e_\alpha,e_\beta \in \mathfrak{g}_{(0,-1)}} \omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\omega(B_\mu(z,e_\beta)e_\alpha,B) = \sum_{e_\alpha,e_\beta \in \mathfrak{g}_{(0,-1)}} \omega(T\widehat{e}_\beta,\widehat{e}_\alpha)\omega(B_\mu(z,e_\beta)B,e_\alpha) \\
&= \sum_{e_\beta \in \mathfrak{g}_{(0,-1)}} \omega(B_\mu(z,e_\beta)B,T\widehat{e}_\beta) = -\sum_{e_\beta \in \mathfrak{g}_{(0,-1)}} \omega(TB_\mu(z,B)e_\beta,\widehat{e}_\beta) \\
&= -\operatorname{tr}(B_\mu(A,w) \circ B_\mu(z,B)|_{\mathfrak{g}_{(0,-1)}}) = -\left(\frac{1}{2} + \frac{1}{6}\dim\mathfrak{g}_{(0,-1)}\right)\omega(z,w).
\end{aligned}$$

For the second term we have

$$\sum_{e_\alpha,e_\beta\in\mathfrak{g}_{(0,-1)}} \omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\omega(\mu(z)e_\alpha,B)\omega(\mu(z)e_\beta,B) = -\omega(\mu(z)T\mu(z)B,B).$$

But $[T,\mu(z)] = 2B_\mu(Tz,z) = 0$ since $T\in\mathfrak{g}_{(1,-1)}$ implies $Tz\in\mathfrak{g}_{(1,-2)}$ and $B_\mu(Tz,z)\in\mathfrak{g}_{(2,-2)} = 0$. Therefore

$$\omega(\mu(z)T\mu(z)B,B) = -\omega(\mu(z)^2B,TB) = -4n(z)\omega(z,w).$$

by Lemma 4.3.2. This implies

$$\sum_{e_\alpha,e_\beta\in\mathfrak{g}_{(0,-1)}} \omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\partial_\alpha\partial_\beta\xi = -i\lambda a^{-1}\left(\frac{1}{2}+\frac{1}{6}\dim\mathfrak{g}_{(0,-1)}\right)\omega(z,w)\xi - \lambda^2a^{-2}n(z)\omega(z,w)\xi$$

and hence $d\omega_{\mathrm{met},\lambda}(T)\xi = 0$ becomes

$$a\xi_0'(a) = s_{\min}\xi_0(a).$$

Hence $\xi_0(a) = c_1|a|^{s_{\min}} + c_2\operatorname{sgn}(a)|a|^{s_{\min}}$. Now assume $\xi_1(a,z)$ has support in $\{a=0\}$ and solves $d\omega_{\mathrm{met},\lambda}(T)\xi_1 = 0$. Then there exists $m\in\mathbb{N}$ and $\xi_k\in\mathcal{S}'(\mathcal{J})$, $k=0,\ldots,m$, such that $\xi_1(a,z) = \sum_{k=0}^m \xi_k(z)\delta^{(k)}(a)$, where $\delta^{(k)}(a)$ denotes the k-th derivative of the Dirac distribution $\delta(a)$. The differential equation $d\omega_{\mathrm{met},\lambda}(T)\xi_1 = 0$ then reads

$$\sum_{k=0}^m\sum_{e_\alpha,e_\beta\in\mathfrak{g}_{(0,-1)}} \omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\partial_\alpha\partial_\beta\xi_k(z)\delta^{(k)}(a) = -i\lambda\omega(z,w)\sum_{k=0}^m\xi_k(z)\delta^{(k+1)}(a).$$

Comparing coefficients of $\delta_a^{(k)}$ we find inductively that $\xi_k = 0$ so that $\xi_1 = 0$. This finishes the proof. □

4.5 The Fourier transformed picture

Assume $\mathfrak{g}\not\simeq\mathfrak{sl}(n,\mathbb{R}),\mathfrak{so}(p,q)$. Then $\mathfrak{m}$ is simple and $\mathcal{C} := \mathcal{C}(\mathfrak{m}) = \frac{3}{2}+\frac{\dim\mathcal{J}}{6}$. It follows from Theorem 3.2.3 that for $\nu = 2\mathcal{C}-\rho-2 = -\frac{2}{3}\dim\mathcal{J}-1$ the space

$$I(\zeta,\nu)^{\Omega_\mu(\mathfrak{m})} = \{u\in I(\zeta,\nu):\Omega_\mu(T)u = 0\,\forall\,T\in\mathfrak{m}\}\subseteq I(\zeta,\nu)$$

is a subrepresentation of $(\pi_{\zeta,\nu},I(\zeta,\nu))$. We study this subrepresentation in the Fourier transformed picture. Let $u\in I(\zeta,\nu)^{\Omega_\mu(\mathfrak{m})}$, then for every $\lambda\in\mathbb{R}^\times$:

$$0 = \sigma_\lambda(\Omega_\mu(T)u) = \sigma_\lambda(u)\circ d\sigma_\lambda(\Omega_\mu(T)) \qquad \forall\,T\in\mathfrak{m}.$$

Corollary 4.5.1. *Let $A:\mathcal{S}(\Lambda)\to\mathcal{S}'(\Lambda)$ be a continuous linear operator such that $A\circ d\sigma_\lambda(\Omega_\mu(T)) = 0$ for all $T\in\mathfrak{m}$. Then there exist $u_0,u_1\in\mathcal{S}'(\Lambda)$ such that*

$$A\varphi = \langle\varphi,\xi_{-\lambda,0}\rangle u_0 + \langle\varphi,\xi_{-\lambda,1}\rangle u_1 \qquad \forall\,\varphi\in\mathcal{S}(\Lambda).$$

Proof. Since $d\sigma_\lambda(\Omega_\mu(T)) = 2i\lambda d\omega_{\mathrm{met},\lambda}(T)$ we have $A \circ d\omega_{\mathrm{met},\lambda}(T) = 0$ for all $T \in \mathfrak{m}$. Let $A^\top : \mathcal{S}(\Lambda) \to \mathcal{S}'(\Lambda)$ denote the transpose of A and note that $d\omega_{\mathrm{met},\lambda}(T)^\top = -d\omega_{\mathrm{met},-\lambda}(T)$ for $T \in \mathfrak{m}$. Hence $d\omega_{\mathrm{met},-\lambda}(T) \circ A^\top = 0$ for all $T \in \mathfrak{m}$. This implies that the image of $A^\top$ is contained in $\mathcal{S}'(\Lambda)^{\mathfrak{m}} = \mathbb{C}\xi_{-\lambda,0} \oplus \mathbb{C}\xi_{-\lambda,1}$, so there exist unique $u_0, u_1 \in \mathcal{S}'(\Lambda)$ such that $A^\top\varphi = \langle\varphi, u_0\rangle\xi_{-\lambda,0} + \langle\varphi, u_1\rangle\xi_{-\lambda,1}$ for $\varphi \in \mathcal{S}(\Lambda)$. Passing to the transposed operator once more shows the claimed formula for A. ☐

By Corollary 4.5.1 we can write

$$\sigma_\lambda(u)\varphi(y) = \langle\varphi, \xi_{-\lambda,0}\rangle u_0(\lambda, y) + \langle\varphi, \xi_{-\lambda,1}\rangle u_1(\lambda, y)$$

for unique $u_0, u_1 \in \mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda)$. In terms of the integral kernel $\widehat{u}(\lambda, x, y)$ of $\sigma_\lambda(u)$ this can be written as

$$\widehat{u}(x, y, \lambda) = \xi_{-\lambda,0}(x)u_0(\lambda, y) + \xi_{-\lambda,1}(x)u_1(\lambda, y). \tag{4.5.1}$$

The map

$$I(\zeta,\nu)^{\Omega_\mu(\mathfrak{m})} \to (\mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda)) \oplus (\mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda)), \quad u \mapsto (u_0, u_1)$$

is injective and we denote its image by $J_{\min}$. Let $\rho_{\min}$ denote the representation of G on $J_{\min}$ which turns the map $u \mapsto (u_0, u_1)$ into an isomorphism of G-representations.

We remark that $J_{\min}$ could be zero, which is equivalent to $I(\zeta,\nu)^{\Omega_\mu(\mathfrak{m})} = \{0\}$. In order to show that there exists some representation ζ of M such that $J_{\min} \neq \{0\}$ and to extend $\rho_{\min}$ to an irreducible unitary representation of G, we compute the Lie algebra action $d\rho_{\min}(\mathfrak{g})$. For this we first state the action of the identity component of $\overline{P}$ which is easily derived from Proposition 2.5.5.

Proposition 4.5.2. *The representation $\rho_{\min}$ is for $g \in \overline{P}_0$ given by*

$$\rho_{\min}(g)(u_0, u_1) = (\rho_{\min,0}(g)u_0, \rho_{\min,1}u_1),$$

where $\rho_{\min,\varepsilon}$ is the representation of $\overline{P}_0$ on $\mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda)$ given by

$$\begin{aligned}
\rho_{\min,\varepsilon}(\overline{n}_{(z,t)})f(\lambda, y) &= e^{i\lambda t}e^{i\lambda(\omega(z'',y)+\frac{1}{2}\omega(z',z''))}f(\lambda, y - z') && \overline{n}_{(z,t)} \in \overline{N},\\
\rho_{\min,\varepsilon}(m)f(\lambda, y) &= (\mathrm{id}^*_{\mathbb{R}^\times} \otimes \omega_{\mathrm{met},\lambda}(m))f(\lambda, y) && m \in M_0,\\
\rho_{\min,\varepsilon}(e^{tH})f(\lambda, y) &= e^{(\nu+s_{\min}-1)t}f(e^{-2t}\lambda, e^t y) && e^{tH} \in A.
\end{aligned}$$

Since $\rho_{\min,\varepsilon}$ is independent of $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, we abuse notation and write $\rho_{\min} = \rho_{\min,0} = \rho_{\min,1}$.

To state the Lie algebra action we write $y \in \Lambda$ as $y = aA + y'$.

Proposition 4.5.3. *The Lie algebra representation $d\rho_{\min}$ of $\mathfrak{g}$ is given by*

$$d\rho_{\min}(X)(u_0, u_1) = (d\rho_{\min,0}(X)u_0, d\rho_{\min,1}(X)u_1)),$$

where $d\rho_{\min,\varepsilon}$ is the representation of $\mathfrak{g}$ on $\mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda)$ given by

$$\begin{aligned}
d\rho_{\min,\varepsilon}(F) &= i\lambda \\
d\rho_{\min,\varepsilon}(v) &= -\partial_v && (v\in\Lambda)\\
d\rho_{\min,\varepsilon}(w) &= -i\lambda\omega(y,w) && (w\in\Lambda^*)\\
d\rho_{\min,\varepsilon}(T) &= d\omega_{\mathrm{met},\lambda}(T) && (T\in\mathfrak{m})\\
d\rho_{\min,\varepsilon}(H) &= \partial_y - 2\lambda\partial_\lambda + 2s_{\min} - \frac{\dim\Lambda}{2} - 1\\
d\rho_{\min,\varepsilon}(\overline{A}) &= i\partial_\lambda\partial_A + \frac{s_{\min}-\frac{\dim\Lambda}{2}}{i\lambda}\partial_A - \frac{2}{\lambda^2}n(\partial')\\
d\rho_{\min,\varepsilon}(\overline{v}) &= i\partial_\lambda\partial_v + \frac{3s_{\min}+1}{i\lambda}\partial_v + \frac{1}{2}\omega(\mu(y')v,B)\partial_A\\
&\quad - \frac{1}{i\lambda}\sum_{\alpha,\beta}\omega(B_\mu(y',v)\widehat{e}_\alpha,\widehat{e}_\beta)\partial_{e_\alpha}\partial_{e_\beta} && (v\in\mathfrak{g}_{(0,-1)})\\
d\rho_{\min,\varepsilon}(\overline{w}) &= -\omega(y,w)\lambda\partial_\lambda + \omega(y,w)\partial_y + 2s_{\min}\omega(y,w)\\
&\quad + \partial_{\mu(y')w} - \frac{1}{2i\lambda}\omega(y,B)\sum_{\alpha,\beta}\omega(B_\mu(A,w)\widehat{e}_\alpha,\widehat{e}_\beta)\partial_{e_\alpha}\partial_{e_\beta} && (w\in\mathfrak{g}_{(-1,0)})\\
d\rho_{\min,\varepsilon}(\overline{B}) &= -\omega(y,B)\lambda\partial_\lambda + \omega(y,B)\partial_y + s_{\min}\omega(y,B) + 2i\lambda n(y')\\
d\rho_{\min,\varepsilon}(E) &= i\lambda\partial_\lambda^2 - ia\partial_\lambda\partial_A - i\partial_\lambda\partial_{y'} - 5is_{\min}\partial_\lambda - \frac{4s_{\min}+1}{i\lambda}a\partial_A + n(y')\partial_A\\
&\quad + \frac{2}{\lambda^2}an(\partial') + \frac{s_{\min}}{i\lambda}\dim\Lambda - \frac{3s_{\min}+1}{i\lambda}\partial_{y'}\\
&\quad + \frac{1}{2i\lambda}\sum_{\alpha,\beta}\omega(\mu(y')\widehat{e}_\alpha,\widehat{e}_\beta)\partial_{e_\alpha}\partial_{e_\beta},
\end{aligned}$$

where (e_α) is a basis of $\mathcal{J}=\mathfrak{g}_{(0,-1)}$, $(\widehat{e}_\alpha)$ the corresponding dual basis of $\mathfrak{g}_{(-1,0)}$ with respect to the symplectic form, and

$$n(\partial') = \frac{1}{2}\sum_{\alpha,\beta,\gamma}\omega(A,B_\Psi(\widehat{e}_\alpha,\widehat{e}_\beta,\widehat{e}_\gamma))\partial_{e_\alpha}\partial_{e_\beta}\partial_{e_\gamma}.$$

Since $d\rho_{\min,\varepsilon}$ is independent of $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$, we abuse notation and write $d\rho_{\min}=d\rho_{\min,0}=d\rho_{\min,1}$.

Proof. The formulas for $\mathfrak{m}$, $\mathfrak{a}$ and $\overline{\mathfrak{n}}$ follow easily by differentiating the formulas in Proposition 4.5.2. We next compute $d\rho_{\min}(\overline{B})$. Writing $z=aA+v+w+bB$ with $a,b\in\mathbb{R}$ and $v\in\mathcal{J}$, $w\in\mathcal{J}^*$ we have by Lemma 4.3.4:

$$\begin{aligned}
d\pi_{\zeta,\nu}(\overline{B}) = 2a^2\partial_A + 2a\partial_v + \partial_{\mu(v)B} - (ab + \tfrac{1}{2}\omega(v,w)+t)\partial_B&\\
+ (at + n(v) - \tfrac{1}{2}a\omega(v,w) - a^2b)\partial_t + (\nu+\rho)a.&
\end{aligned}$$

A careful application of Lemma 2.5.6 yields

$$\begin{aligned}
d\widehat{\pi}_{\zeta,\nu}(\overline{B}) = &-\omega(y,B)\lambda\partial_\lambda + (\omega(y,B)-\tfrac{1}{2}\omega(x,B))\omega(x,B)\partial_{A,x} + \omega(y-x,B)\partial_{x',x}\\
&+\omega(x,B)\partial_{y',x} + \tfrac{1}{2}\omega(y,B)^2\partial_{A,y} + \omega(y,B)\partial_{y',y}\\
&+4i\lambda n(x') + i\lambda\omega(\mu(x')y',B) + 2i\lambda n(y') + \tfrac{\nu+\rho}{2}\omega(y-x,B)
\end{aligned} \tag{4.5.2}$$

where $\partial_{v,x}$ resp. $\partial_{w,y}$ means differentiation in the direction v resp. w with respect to the variable x resp. y of $\widehat{u}(\lambda,x,y)$, and $x\in\mathbb{R}A+x'$, $y\in\mathbb{R}A+y'$. In view of (4.5.1) we compute with $x=aA+x'$:

$$\begin{aligned}\lambda\partial_\lambda\xi_{-\lambda,\varepsilon}(x) &= \tfrac{i\lambda n(x')}{a}\xi_{-\lambda,\varepsilon}(a,x'),\\ \omega(x,B)\partial_{A,x}\xi_{-\lambda,\varepsilon}(x) &= 2a\partial_A\xi_{-\lambda,\varepsilon}(a,x') = 2s_{\min}\xi_{-\lambda,\varepsilon}(a,x') - \tfrac{2i\lambda n(x')}{a}\xi_{-\lambda,\varepsilon}(a,x'),\\ \partial_{x',x}\xi_{-\lambda,\varepsilon}(x) &= \tfrac{3i\lambda n(x')}{a}\xi_{-\lambda,\varepsilon}(a,x'),\\ \partial_{y',x}\xi_{-\lambda,\varepsilon}(x) &= -\frac{i\lambda}{2a}\omega(\mu(x')y',B)\xi_{-\lambda,\varepsilon}(a,x').\end{aligned}$$

A short computation using $\frac{\nu+\rho}{2}=-s_{\min}$ then shows

$$\begin{aligned}(d\widehat{\pi_{\zeta,\nu}(\overline{B})u})(\lambda,x,y) = \sum_{\varepsilon\in\mathbb{Z}/2\mathbb{Z}}\xi_{-\lambda,\varepsilon}(x)\Big[&-\omega(y,B)\lambda\partial_\lambda+\tfrac{1}{2}\omega(y,B)^2\partial_A+\omega(y,B)\partial_{y'}\\ &+s_{\min}\omega(y,B)+2i\lambda n(y')\Big]u_\varepsilon(\lambda,y).\end{aligned}$$

This shows the claimed formula for $d\rho_{\min,\varepsilon}(\overline{B})$. Since $\mathfrak{g}$ is generated by $\mathfrak{m}$, $\mathfrak{a}$, $\overline{\mathfrak{n}}$ and $\overline{B}$, the remaining formulas can be obtained as commutators, more precisely we use that for $v\in\mathcal{J}$, $w\in\mathcal{J}^*$ and $T\in\mathfrak{g}_{(1,-1)}$ (so that $Tw\in\mathcal{J}^*$):

$$[\overline{B},B_\mu(A,w)]=\overline{w},\qquad [T,\overline{w}]=\overline{Tw},\qquad [B_\mu(A,w),\overline{v}]=\tfrac{1}{2}\omega(v,w)\overline{A},\qquad [\overline{B},\overline{A}]=2E.\qquad\square$$

Remark 4.5.4. The representation $d\rho_{\min}$ obviously extends to a representation of $\mathfrak{g}$ on the space $\mathcal{D}'(\mathbb{R}^\times)\mathbin{\widehat{\otimes}}\mathcal{S}'(\Lambda)$, and we will use the same notation for this extension.

Remark 4.5.5. We give a formal heuristic argument why the representation $\rho_{\min}$ should extend to a unitary representation on $L^2(\mathbb{R}^\times\times\Lambda,|\lambda|^{\dim\Lambda-2s_{\min}}d\lambda)$. Note that several steps of the argument need a certain regularization to make sense. However, since we prove that $\rho_{\min}$ extends to a unitary representation by different means, we do not justify all steps.
By Proposition 2.3.1 the invariant Hermitian form on $I(\zeta,\nu)$ is given by a regularization of

$$\begin{aligned}\langle u,\overline{\Delta^{\frac{\nu-\rho}{2}}*v}\rangle &= \int_{\mathbb{R}^\times}\operatorname{tr}\left(\sigma_\lambda(\Delta^{\frac{\nu-\rho}{2}}*v)^*\sigma_\lambda(u)\right)|\lambda|^{\dim\Lambda}\,d\lambda\\ &= \int_{\mathbb{R}^\times}\operatorname{tr}\left(\sigma_\lambda(\Delta^{\frac{\nu-\rho}{2}})\sigma_\lambda(v)^*\sigma_\lambda(u)\right)|\lambda|^{\dim\Lambda}\,d\lambda,\end{aligned}$$

where $\Delta(z,t)=t^2-Q(z)$. Assume $u,v\in I(\zeta,\nu)^{\Omega_\mu(\mathfrak{m})}$ satisfy $\sigma_\lambda(u)\varphi=\langle\xi_{-\lambda,\varepsilon},\varphi\rangle u_\varepsilon(\lambda)$ and $\sigma_\lambda(v)^*\psi=\langle\overline{v_\varepsilon(\lambda)},\psi\rangle\overline{\xi_{-\lambda,\varepsilon}}$ and hence

$$\sigma_\lambda(v)^*\sigma_\lambda(u)\varphi=\langle\xi_{-\lambda,\varepsilon},\varphi\rangle\langle u_\varepsilon(\lambda),\overline{v_\varepsilon(\lambda)}\rangle\overline{\xi_{-\lambda,\varepsilon}},$$

so that

$$\operatorname{tr}\left(\sigma_\lambda(\Delta^{\frac{\nu-\rho}{2}})\sigma_\lambda(v)^*\sigma_\lambda(u)\right)=\langle u_\varepsilon(\lambda),\overline{v_\varepsilon(\lambda)}\rangle\langle\sigma_\lambda(\Delta^{\frac{\nu-\rho}{2}})\overline{\xi_{-\lambda,\varepsilon}},\xi_{-\lambda,\varepsilon}\rangle.$$

Now, $\Delta(-z,-t)=\Delta(z,t)$ and $\sigma_\lambda(z,t)^\top=\sigma_{-\lambda}(-z,-t)$ so that

$$\langle\sigma_\lambda(\Delta^{\frac{\nu-\rho}{2}})\overline{\xi_{-\lambda,\varepsilon}},\xi_{-\lambda,\varepsilon}\rangle=\langle\sigma_{-\lambda}(\Delta^{\frac{\nu-\rho}{2}})\xi_{-\lambda,\varepsilon},\overline{\xi_{-\lambda,\varepsilon}}\rangle.$$

A short computation shows that

$$\sigma_\lambda(\Delta^{\frac{\nu-\rho}{2}}) = |\lambda|^{-\nu}\delta_{|\lambda|^{\frac{1}{2}}} \circ \sigma_{\operatorname{sgn}\lambda}(\Delta^{\frac{\nu-\rho}{2}}) \circ \delta_{|\lambda|^{-\frac{1}{2}}} \qquad \text{and} \qquad \xi_{\lambda,\varepsilon} = |\lambda|^{-\frac{s_{\min}}{2}}\delta_{|\lambda|^{\frac{1}{2}}}\xi_{\operatorname{sgn}\lambda,\varepsilon},$$

where $\delta_s\varphi(x) = \varphi(sx)$ $(s > 0)$, and hence

$$\begin{aligned}\langle\sigma_{-\lambda}(\Delta^{\frac{\nu-\rho}{2}})\xi_{-\lambda,\varepsilon}, \overline{\xi_{-\lambda,\varepsilon}}\rangle &= |\lambda|^{-\nu-s_{\min}}\langle\delta_{|\lambda|^{\frac{1}{2}}} \circ \sigma_{-\operatorname{sgn}\lambda}(\Delta^{\frac{\nu-\rho}{2}})\xi_{-\operatorname{sgn}\lambda,\varepsilon}, \delta_{|\lambda|^{\frac{1}{2}}}\overline{\xi_{-\operatorname{sgn}\lambda,\varepsilon}}\rangle \\ &= |\lambda|^{-\nu-s_{\min}-\frac{1}{2}\dim\Lambda}\langle\sigma_{-\operatorname{sgn}\lambda}(\Delta^{\frac{\nu-\rho}{2}})\xi_{-\operatorname{sgn}\lambda,\varepsilon}, \overline{\xi_{-\operatorname{sgn}\lambda,\varepsilon}}\rangle.\end{aligned}$$

Since $\Delta(z,t)$ is M_0-invariant, the operator $\sigma_\lambda(\Delta^{\frac{\nu-\rho}{2}})$ is $\omega_{\mathrm{met},\lambda}(M_0)$-invariant. The subspace of M_0-invariant vectors in $\omega_{\mathrm{met},\lambda}$ is spanned $\xi_{\lambda,0}$ and $\xi_{\lambda,1}$, so $\sigma_\lambda(\Delta^{\frac{\nu-\rho}{2}})\xi_{\lambda,\varepsilon}$ is a linear combination of $\xi_{\lambda,0}$ and $\xi_{\lambda,1}$. This shows that

$$\begin{aligned}\langle u, \overline{\Delta^{\frac{\nu-\rho}{2}} * v}\rangle &= \mathrm{const}\cdot\int_{\mathbb{R}^\times}\langle u_\varepsilon(\lambda), \overline{v_\varepsilon(\lambda)}\rangle|\lambda|^{\frac{1}{2}\dim\Lambda-\nu-s_{\min}}\,d\lambda \\ &= \mathrm{const}\cdot\int_{\mathbb{R}^\times}\int_\Lambda u_\varepsilon(\lambda,x)\overline{v_\varepsilon(\lambda,x)}|\lambda|^{\dim\Lambda-2s_{\min}}\,dx\,d\lambda.\end{aligned}$$

Hence, the representation $\rho_{\min}$ is unitary on $L^2(\mathbb{R}^\times\times\Lambda, |\lambda|^{\dim\Lambda-2s_{\min}}\,d\lambda\,dx)$.

We renormalize $\rho_{\min}$ to obtain a unitary representation on $L^2(\mathbb{R}^\times\times\Lambda)$. For $\delta\in\mathbb{Z}/2\mathbb{Z}$ let

$$\Phi_\delta : \mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda) \to \mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda), \quad \Phi_\delta u(\lambda,x) = \operatorname{sgn}(\lambda)^\delta|\lambda|^{-s_{\min}}u(\lambda,\tfrac{x}{\lambda}), \tag{4.5.3}$$

then Φ_δ restricts to an isometric isomorphism

$$L^2(\mathbb{R}^\times\times\Lambda, |\lambda|^{\dim\Lambda-2s_{\min}}\,d\lambda\,dy) \to L^2(\mathbb{R}^\times\times\Lambda).$$

We define

$$\pi_{\min}(g) := \Phi_\delta\circ\rho_{\min}(g)\circ\Phi_\delta^{-1},$$

then

$$\begin{aligned}\Phi_\delta\circ\partial_v\circ\Phi_\delta^{-1} &= \lambda\partial_v, & \Phi_\delta\circ\partial_\lambda\circ\Phi_\delta^{-1} &= (\partial_\lambda+\lambda^{-1}\partial_x+s\lambda^{-1}), \\ \Phi_\delta\circ\omega(y,w)\circ\Phi_\delta^{-1} &= \lambda^{-1}\omega(x,w), & \Phi_\delta\circ\lambda\circ\Phi_\delta^{-1} &= \lambda,\end{aligned}$$

and hence:

Proposition 4.5.6. *The representation $d\pi_{\min}$ of $\mathfrak{g}$ on $\mathcal{D}'(\mathbb{R}^\times)\widehat{\otimes}\mathcal{S}'(\Lambda)$ is given by*

$$\begin{aligned}
d\pi_{\min}(F) &= i\lambda \\
d\pi_{\min}(v) &= -\lambda\partial_v && (v\in\Lambda)\\
d\pi_{\min}(w) &= -i\omega(x,w) && (w\in\Lambda^*)\\
d\pi_{\min}(T) &= -\frac{i\lambda}{2}\sum_{\alpha,\beta}\omega(T\widehat{e}_\alpha,\widehat{e}_\beta)\partial_\alpha\partial_\beta-\frac{1}{2}\omega(TB,x')\partial_A && (T\in\mathfrak{g}_{(1,-1)})\\
d\pi_{\min}(T) &= -\partial_{Tx}-\frac{1}{2}\operatorname{tr}(T|_\Lambda) && (T\in\mathfrak{g}_{(0,0)}\cap\mathfrak{m})\\
d\pi_{\min}(T) &= -a\partial_{TA}-\frac{1}{2i\lambda}\omega(Tx',x') && (T\in\mathfrak{g}_{(-1,1)})\\
d\pi_{\min}(H) &= -\partial_x-2\lambda\partial_\lambda-\frac{\dim\Lambda+2}{2}\\
d\pi_{\min}(\overline{A}) &= i\lambda\partial_\lambda\partial_A+i\partial_x\partial_A+i\frac{\dim\Lambda+2}{2}\partial_A-2\lambda n(\partial')\\
d\pi_{\min}(\overline{v}) &= i\lambda\partial_\lambda\partial_v+i\partial_x\partial_v-2is_{\min}\partial_v+\frac{1}{2}\lambda^{-1}\omega(\mu(x')v,B)\partial_A\\
&\quad-\frac{1}{i}\sum_{\alpha,\beta}\omega(B_\mu(x',v)\widehat{e}_\alpha,\widehat{e}_\beta)\partial_{e_\alpha}\partial_{e_\beta} && (v\in\mathfrak{g}_{(0,-1)})\\
d\pi_{\min}(\overline{w}) &= -\omega(x,w)\partial_\lambda+s_{\min}\lambda^{-1}\omega(x,w)\\
&\quad+\lambda^{-1}\partial_{\mu(x')w}-\frac{1}{2i}\omega(x,B)\sum_{\alpha,\beta}\omega(B_\mu(A,w)\widehat{e}_\alpha,\widehat{e}_\beta)\partial_{e_\alpha}\partial_{e_\beta} && (w\in\mathfrak{g}_{(-1,0)})\\
d\pi_{\min}(\overline{B}) &= -\omega(x,B)\partial_\lambda+2i\lambda^{-2}n(x')\\
d\pi_{\min}(E) &= i\lambda\partial_\lambda^2+i\partial_\lambda\partial_x-3is_{\min}\partial_\lambda+\lambda^{-2}n(x')\partial_A+2an(\partial')\\
&\quad-\frac{is_{\min}}{3\lambda}(\dim\Lambda-1)+\frac{s_{\min}}{i\lambda}\partial_{x'}+\frac{1}{2i\lambda}\sum_{\alpha,\beta}\omega(\mu(x')\widehat{e}_\alpha,\widehat{e}_\beta)\partial_{e_\alpha}\partial_{e_\beta}.
\end{aligned}$$

Proposition 4.5.7. *The annihilator of $d\pi_{\min}$ in $U(\mathfrak{g}_\mathbb{C})$ is a completely prime ideal whose associated variety is equal to the closure of the minimal nilpotent coadjoint orbit. In particular, for $\mathfrak{g}_\mathbb{C}$ not of type A the annihilator is the Joseph ideal.*

Proof. It is easily verified that $d\pi_{\min}$ maps the generators of Lemma 1.7.3 to zero. Therefore, the annihilator ideal of $d\pi_{\min}$ has associated variety contained in the closure of the minimal nilpotent coadjoint orbit. Since this orbit is minimal, the associated variety is actually equal to the orbit's closure. By the same argument as in [25, Theorem 2.18] the annihilator ideal is completely prime since it is given by the kernel of an algebra homomorphism into the ring of regular differential operators on the irreducible variety $\mathbb{R}^\times\times\Lambda$ which does not have zero divisors. The second claim now follows from the uniqueness result for the Joseph ideal [11, Theorem 3.1]. $\square$

4.6 The case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$

For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ the previous arguments cannot be applied in the same way since here $\mathfrak{m}$ is not simple and the value $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ for which $\Omega_\mu(\mathfrak{m}')$ is conformally invariant is different for the two factors $\mathfrak{m}'$ of $\mathfrak{m}$. We discuss how to use the first order system Ω_ω instead to obtain a subrepresentation of $I(\zeta, \nu)$ for some ν which has a Fourier transformed picture similar to the other cases.

The subalgebra $\mathfrak{m}$ decomposes into the direct sum of two ideals

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1$$

with $\mathfrak{m}_0 = \mathbb{R}T_0$ and $\mathfrak{m}_1 \simeq \mathfrak{sl}(n-2, \mathbb{R})$. We can normalize T_0 such that it has eigenvalues ± 1 on V. Then the eigenspaces $\Lambda = \ker(T_0 - \mathrm{id}_V)$ and $\Lambda^* = \ker(T_0 + \mathrm{id}_V)$ are dual Lagrangian subspaces with $V = \Lambda \oplus \Lambda^*$. Note that both Λ and Λ^* are invariant under $\mathfrak{m}_1$. The following lemma is easily verified using the explicit realization of $\mathfrak{g}$ given in Appendix B.1:

Lemma 4.6.1. *(1) For $v \in \Lambda$ and $w \in \Lambda^*$ we have $\mu(v) = \mu(w) = 0$.*

(2) $B_\mu(x, y) \equiv \frac{n}{4(n-2)} \omega(T_0 x, y) T_0 \mod \mathfrak{m}_1$ for all $x, y \in V$.

For $r \in \mathbb{C}$ let ζ_r denote a character of M for which $d\zeta_r(\mathfrak{m}_1) = 0$ and $d\zeta_r(T_0) = \frac{n-2}{2} + \frac{n-2}{n} r$.

Theorem 4.6.2. *For any $r \in \mathbb{C}$ and $\zeta = \zeta_r$, the system of differential operators $\Omega_\omega(v)$ ($v \in \Lambda$) is conformally invariant for $\pi_{\zeta,\nu}$ with $\nu + \rho = \frac{n}{2} + r$.*

Proof. Since $[\mathfrak{m}, \Lambda] \subseteq \Lambda$, Theorem 3.2.1 implies that $\Omega_\omega(v)$ ($v \in \Lambda$) is conformally invariant if and only if

$$\frac{\nu + \rho}{2} \omega(x, v) + 2d\zeta(B_\mu(x, v)) = 0$$

for all $v \in \Lambda$. This follows from Lemma 4.6.1 (2). □

Remark 4.6.3. $\Omega_\omega(v)u = 0$ for all $v \in \Lambda$ implies $\Omega_\mu(T)u = 0$ for all $T \in \mathfrak{m}_1$. In fact, since $T\Lambda \subseteq \Lambda$ and $T\Lambda^* \subseteq \Lambda^*$ we have

$$\begin{aligned} \Omega_\mu(T) &= \sum_{e_\alpha \in \Lambda, e_\beta \in \Lambda^*} \omega(T\widehat{e}_\alpha, \widehat{e}_\beta)(X_\alpha X_\beta + X_\beta X_\alpha) \\ &= \sum_{e_\alpha \in \Lambda, e_\beta \in \Lambda^*} \omega(T\widehat{e}_\alpha, \widehat{e}_\beta)([X_\alpha, X_\beta] + 2X_\beta X_\alpha) \end{aligned}$$

for a basis (e_α) of V with $e_\alpha \in \Lambda \cup \Lambda^*$. If $\Omega_\omega(v)u = 0$ for all $v \in \Lambda$, then $X_\alpha u = 0$ for $e_\alpha \in \Lambda$. Further, $[X_\alpha, X_\beta] = \omega(e_\alpha, e_\beta)\partial_t$ so that $\sum \omega(T\widehat{e}_\alpha, \widehat{e}_\beta)[X_\alpha, X_\beta] = \mathrm{tr}(T|_\Lambda)\partial_t$ which vanishes for $T \in \mathfrak{m}_1 \simeq \mathfrak{sl}(n-2, \mathbb{R})$.

By (3.3.1) the Fourier transform of $\Omega_\omega(v)$ is given by composition with $\sigma_\lambda(v)$. In terms of the distribution kernel $\widehat{u}(\lambda, x, y)$ of $\sigma_\lambda(u)$ this means

$$\widehat{\Omega_\omega(v)u}(\lambda, x, y) = \sigma_{-\lambda}(v)_x \widehat{u}(\lambda, x, y).$$

This implies that for every $u \in I(\zeta_r, \nu)^{\Omega_\omega(\Lambda)}$, the distribution $\widehat{u}(\lambda, x, y)$ is in the x-variable a distribution vector in $L^2(\Lambda)^{-\infty} = \mathcal{S}'(\Lambda)$ which is invariant under $\sigma_{-\lambda}(v) = -\partial_v$ for all $v \in \Lambda$. These are obviously only the constant functions:

Proposition 4.6.4. *For every $\lambda \in \mathbb{R}^\times$ the space $L^2(\Lambda)^{-\infty,\Lambda} = \mathcal{S}'(\Lambda)^\Lambda$ of Λ-invariant distribution vectors in σ_λ is one-dimensional and spanned by the constant function*

$$\xi_\lambda(x) = 1.$$

It follows that for $u \in I(\zeta_r, \nu)^{\Omega_\omega(\Lambda)}$ we can write

$$\widehat{u}(\lambda, x, y) = \xi_{-\lambda}(x) u_0(\lambda, y) = u_0(\lambda, y)$$

for some $u_0 \in \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda)$. Let $J_{\min,r} \subseteq \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda)$ denote the image of the map

$$I(\zeta_r, \nu)^{\Omega_\omega(\Lambda)} \to \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda), \quad u \mapsto u_0$$

and write $\rho_{\min,r}$ for the representation of G on $J_{\min,r}$ which makes this map G-equivariant.

Proposition 4.6.5. *The representation $d\rho_{\min,r}$ of $\mathfrak{g}$ on $J_{\min,r} \subseteq \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda)$ is given by*

$$\begin{aligned}
d\rho_{\min,r}(F) &= i\lambda \\
d\rho_{\min,r}(v) &= -\partial_v && (v \in \Lambda) \\
d\rho_{\min,r}(w) &= -i\lambda\omega(y, w) && (w \in \Lambda^*) \\
d\rho_{\min,r}(T) &= -\partial_{Ty} - \tfrac{n-2r}{2n} \operatorname{tr}(T|_\Lambda) && (T \in \mathfrak{m}) \\
d\rho_{\min,r}(H) &= \partial_y - 2\lambda\partial_\lambda - \tfrac{n-2r}{2} \\
d\rho_{\min,r}(\overline{v}) &= i(\partial_\lambda + \tfrac{n-2r-2}{2\lambda})\partial_v \\
d\rho_{\min,r}(\overline{w}) &= \omega(y, w)(\partial_y - \lambda\partial_\lambda), \\
d\rho_{\min,r}(E) &= i(\lambda\partial_\lambda^2 - \partial_\lambda\partial_y - \tfrac{n-2r-2}{2\lambda}\partial_y + \tfrac{n-2r}{2}\partial_\lambda).
\end{aligned}$$

Proof. We proceed as in the proof of Proposition 4.5.3. The formulas for $\mathfrak{m}$, $\mathfrak{a}$ and $\overline{\mathfrak{n}}$ follow easily from Proposition 2.5.5, and for $w \in \Lambda^*$ we find, using Lemma 2.5.6 and Lemma 4.6.1

$$d\widehat{\pi}_{\zeta,\nu}(\overline{w}) = \omega(x, w) \left[\partial_{y-x,x} - \frac{\nu+\rho}{2} + \frac{n}{2(n-2)} d\zeta(T_0)\right] \\ + \omega(y, w) \left[\partial_{x,x} + \partial_{y,y} - \lambda\partial_\lambda + \frac{\nu+\rho}{2} - \frac{n}{2(n-2)} d\zeta(T_0)\right].$$

Since $\nu + \rho = \frac{n}{n-2} d\zeta_r(T_0)$ and $\partial_{v,x}\xi_\lambda(x) = 0$ for all $v \in \Lambda$ it follows that for $u(\lambda, x, y) = \xi_{-\lambda}(x) u_0(\lambda, y)$:

$$d\widehat{\pi}_{\zeta,\nu}(\overline{w}) u(\lambda, x, y) = \xi_{-\lambda}(x) \cdot \omega(y, w)(\partial_y - \lambda\partial_\lambda) u_0(\lambda, y).$$

The formula for $d\rho_{\min,r}(\overline{v})$ is obtained by a similar computation, and for $d\rho_{\min,r}(E)$ we use that $[\overline{v}, \overline{w}] = -\omega(v, w)E$. □

The change of coordinates $x = \lambda y$ finally yields a representation $d\pi_{\min,r}$ of $\mathfrak{g}$ on $\mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda)$ given by

$$
\begin{aligned}
d\pi_{\min,r}(F) &= i\lambda \\
d\pi_{\min,r}(v) &= -\lambda\partial_v && (v\in\Lambda)\\
d\pi_{\min,r}(w) &= -i\omega(x,w) && (w\in\Lambda^*)\\
d\pi_{\min,r}(T) &= -\partial_{Tx}-\tfrac{n-2r}{2n}\operatorname{tr}(T|_\Lambda) && (T\in\mathfrak{m})\\
d\pi_{\min,r}(H) &= -\partial_x-2\lambda\partial_\lambda-\tfrac{n-2r}{2}\\
d\pi_{\min,r}(\overline{v}) &= i(\lambda\partial_\lambda+\partial_x+\tfrac{n-2r}{2})\partial_v\\
d\pi_{\min,r}(\overline{w}) &= -\omega(x,w)\partial_\lambda,\\
d\pi_{\min,r}(E) &= i(\lambda\partial_\lambda+\partial_x+\tfrac{n-2r}{2})\partial_\lambda.
\end{aligned}
$$

Remark 4.6.6. It can be shown that for $\zeta=\zeta_r$, $r\in\mathbb{C}$, and $\nu+\rho=n-2$ the second order differential operator

$$\Omega_\mu^\zeta(T_0)=\Omega_\mu(T_0)+\frac{n}{n-2}d\zeta(T_0)\partial_t$$

is conformally invariant for $\pi_{\zeta,\nu}$. For $n>3$ the single equation $\Omega_\mu^\zeta(T_0)u=0$ is not sufficient to give a small representation similar to the previous cases; only for $n=3$ this is the case, since here $\mathfrak{m}=\mathfrak{m}_0=\mathbb{R}T_0$. In fact, for $n=3$ the same arguments as before identify $I(\zeta,\nu)^{\Omega_\mu^\zeta(T_0)}$ with a subspace of $\mathcal{D}'(\mathbb{R}^\times)\widehat{\otimes}\mathcal{S}'(\Lambda)$ and the corresponding Lie algebra action agrees with the one obtained in Proposition 4.6.5. This is due to the fact that for $n=3$ the parameter families $(\nu,r)=(-\frac{1}{2}+r,r)$ and $(\nu,r)=(-1,r)$ are related by the Weyl group element

$$w(\operatorname{diag}(H_1,H_2,H_3))=\operatorname{diag}(H_1,H_3,H_2).$$

It is likely that the corresponding standard intertwining operator identifies the two subrepresentations $I(\zeta,\nu)^{\Omega_\omega(\Lambda)}$ and $I(\zeta,\nu)^{\Omega_\mu^\zeta(T_0)}$.

4.7 The case $\mathfrak{g}=\mathfrak{so}(p,q)$

We also treat the case $\mathfrak{g}=\mathfrak{so}(p,q)$ separately since here $\mathfrak{m}$ is not simple either and the value $\nu\in\mathfrak{a}_\mathbb{C}^*$ for which $\Omega_\mu(\mathfrak{m}')$ is conformally invariant is different for the two factors $\mathfrak{m}'$ of $\mathfrak{m}$. Instead, we combine variations of the first order system Ω_ω and the second order system Ω_μ to the case of vector-valued principal series in order to obtain a subrepresentation of $I(\zeta,\nu)$ which has a Fourier transformed picture similar to the other cases.

For $\mathfrak{g}=\mathfrak{so}(p,q)$ the lack of simplicity of $\mathfrak{m}$ stems from the fact that $\mathcal{J}=\mathfrak{g}_{(0,-1)}$ is not a simple Jordan algebra but the sum of two simple Jordan algebras, the one-dimensional Jordan algebra which is of rank one and a $(p+q-6)$-dimensional Jordan algebra of rank two. Write $\mathcal{J}=\mathcal{J}_0\oplus\overline{\mathcal{J}}$ with $\mathcal{J}_0=\mathbb{R}P$ and $\overline{\mathcal{J}}\simeq\mathbb{R}^{p-3,q-3}$ and similarly $\mathcal{J}^*=\mathcal{J}_0^*\oplus\overline{\mathcal{J}}^*$ with $\mathcal{J}_0^*=\mathbb{R}Q$ such that $\omega(P,Q)=1$ and $\omega(\mathcal{J}_0,\overline{\mathcal{J}}^*)=0=\omega(\overline{\mathcal{J}},\mathcal{J}_0^*)$. We decompose $v\in\mathcal{J}$ into $v=v_0+\overline{v}$ with $v_0\in\mathcal{J}_0$ and $\overline{v}\in\overline{\mathcal{J}}$ and similar for $w\in\mathcal{J}^*$.

Note that we use the same letter P for the element $P\in\mathcal{J}$ and the parabolic subgroup $P=MAN\subseteq G$. It should be clear from the context which object is meant.

The following statement is the analog of Lemma 4.3.1:

Lemma 4.7.1. *For $v \in \mathcal{J}$ and $w \in \mathcal{J}^*$ we have*

$$\operatorname{tr}(B_\mu(A, w) \circ B_\mu(v, B)|_{\mathcal{J}}) = \frac{p + q - 6}{2}\omega(v_0, w_0) + \omega(\overline{v}, \overline{w}).$$

Proof. This is a straightforward computation using Appendix B.2. □

According to the decomposition $\mathcal{J} = \mathcal{J}_0 \oplus \overline{\mathcal{J}}$ the Lie algebra $\mathfrak{m}$ splits into

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \overline{\mathfrak{m}} \qquad \text{with} \qquad \mathfrak{m}_0 \simeq \mathfrak{sl}(2, \mathbb{R}) \quad \text{and} \quad \overline{\mathfrak{m}} \simeq \mathfrak{so}(p - 2, q - 2).$$

We collect a few more identities related to the bigrading, all which can be verified by direct computations using the explicit realization of $\mathfrak{g}$ given in Appendix B.2.

Lemma 4.7.2. *(1) $\mathfrak{m}_0 \simeq \mathfrak{sl}(2, \mathbb{R})$ is spanned by the $\mathfrak{sl}(2)$-triple*

$$(e, f, h) = (\sqrt{2}B_\mu(A, Q), \sqrt{2}B_\mu(P, B), B_\mu(A, B) - 2B_\mu(P, Q)).$$

(2) $B_\mu(A, Q) : \overline{\mathcal{J}} \to \overline{\mathcal{J}}^*$ and $B_\mu(P, B) : \overline{\mathcal{J}}^* \to \overline{\mathcal{J}}$ are isomorphisms satisfying

$$B_\mu(A, Q) \circ B_\mu(P, B)|_{\overline{\mathcal{J}}^*} = \frac{1}{2}\operatorname{id}_{\overline{\mathcal{J}}^*} \qquad \text{and} \qquad B_\mu(P, B) \circ B_\mu(A, Q)|_{\overline{\mathcal{J}}} = \frac{1}{2}\operatorname{id}_{\overline{\mathcal{J}}}.$$

(3) $B_\mu(v, B) \in \overline{\mathfrak{m}}$ if and only if $v \in \overline{\mathcal{J}}$.

(4) $\mu(P) = \mu(Q) = 0$ and $\mu(\overline{\mathcal{J}}) \in \mathbb{R}B_\mu(A, Q)$

(5) For $v \in \mathcal{J}$:

$$\begin{aligned}
&\mu(v)Q = -\omega(v_0, Q)\overline{v},\\
&\mu(v)w = -\omega(\overline{v}, w)\omega(v, Q)P + \omega(\mu(v)P, B)B_\mu(A, Q)w \qquad\qquad (w \in \overline{\mathcal{J}}^*),\\
&\mu(v)B = -\omega(\mu(\overline{v})P, B)Q + 2\omega(v_0, Q)B_\mu(P, B)\overline{v}.
\end{aligned}$$

(6) $B_\mu(P, Q)$ acts on V as follows:

$$\begin{aligned}
&B_\mu(P, Q)A = \frac{1}{4}A, \qquad B_\mu(P, Q)P = \frac{3}{4}P, \qquad B_\mu(P, Q)v = -\frac{1}{4}v \qquad (v \in \overline{\mathcal{J}}),\\
&B_\mu(P, Q)B = -\frac{1}{4}B, \qquad B_\mu(P, Q)Q = -\frac{3}{4}Q, \qquad B_\mu(P, Q)w = \frac{1}{4}w \qquad (v \in \overline{\mathcal{J}}^*).
\end{aligned}$$

Let ζ be a representation of M such that $d\zeta$ is trivial on $\mathfrak{m}_1$.

Proposition 4.7.3. *The system of differential operators*

$$\Omega^\zeta_\omega(v) = \sum_\alpha \Big((\nu + \rho - 2)\omega(v, \widehat{e}_\alpha) + 4d\zeta(B_\mu(v, \widehat{e}_\alpha))\Big)\Omega_\omega(e_\alpha)$$

is conformally invariant for $\pi_{\zeta,\nu}$ if and only if $d\zeta(\operatorname{Cas}_{\mathfrak{m}_0}) = (\nu + \rho)(\nu + \rho - 2)$, where $\operatorname{Cas}_{\mathfrak{m}_0} = h^2 + 2ef + 2fe$. In this case, the joint kernel

$$I(\zeta, \nu)^{\Omega^\zeta_\omega(\Lambda)} = \{u \in I(\zeta, \nu) : \Omega^\zeta_\omega(v)u = 0 \,\forall\, v \in \Lambda\}$$

is a subrepresentation of $I(\zeta, \nu)$.

Proof. Using Theorem 3.2.3 we find

$$\begin{aligned}
[\Omega^\zeta_\omega(v), d\pi_{\zeta,\nu}(X)] &= 0, && (X \in \overline{\mathfrak{n}}),\\
[\Omega^\zeta_\omega(v), d\pi_{\zeta,\nu}(H)] &= \Omega^\zeta_\omega(v),\\
[\Omega^\zeta_\omega(v), d\pi_{\zeta,\nu}(S)] &= -\Omega^\zeta_\omega(Sv) && (S \in \mathfrak{m}).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&[\Omega^\zeta_\omega(v), d\pi_{\zeta,\nu}(E)]\\
&= \sum_\alpha \Big((\nu+\rho-2)\omega(v,\widehat{e}_\alpha) + 4d\zeta(B_\mu(v,\widehat{e}_\alpha))\Big)[\Omega_\omega(e_\alpha), d\pi_{\zeta,\nu}(E)]\\
&\qquad\qquad + 4\sum_\alpha d\zeta([B_\mu(v,\widehat{e}_\alpha),\mu(x)])\Omega_\omega(e_\alpha)\\
&= \sum_\alpha \Big((\nu+\rho-2)\omega(v,\widehat{e}_\alpha) + 4d\zeta(B_\mu(v,\widehat{e}_\alpha))\Big)\Big(t\Omega_\omega(e_\alpha) - \Omega_\omega(\mu(x)e_\alpha)\\
&\qquad\qquad + \tfrac{\nu+\rho}{2}\omega(x,e_\alpha) + 2d\zeta(B_\mu(x,e_\alpha))\Big)\\
&\qquad - 4\sum_\alpha \Big(d\zeta(B_\mu(\mu(x)v,\widehat{e}_\alpha) + d\zeta(B_\mu(v,\mu(x)\widehat{e}_\alpha)])\Big)\Omega_\omega(e_\alpha)\\
&= t\Omega^\zeta_\omega(v) - \Omega^\zeta_\omega(\mu(x)v) + \tfrac{(\nu+\rho)(\nu+\rho-2)}{2}\omega(x,v) - 4d\zeta(B_\mu(x,v))\\
&\qquad\qquad + 8\sum_\alpha d\zeta(B_\mu(v,\widehat{e}_\alpha))d\zeta(B_\mu(x,e_\alpha)).
\end{aligned}$$

The last term is evaluated in the next lemma and the claim follows. □

Lemma 4.7.4. *For $v, w \in V$ we have*

$$\sum_\alpha B_\mu(v,\widehat{e}_\alpha)B_\mu(e_\alpha,w) \equiv \frac{1}{16}\omega(v,w)(h^2+2ef+2fe) + \frac{1}{2}B_\mu(v,w) \mod \overline{\mathfrak{m}}\,\mathcal{U}(\mathfrak{m}_0)$$

Proof. We first note that

$$B_\mu(v,w) \equiv \frac{1}{4}\omega(hv,w)h + \frac{1}{2}\omega(fv,w)e + \frac{1}{2}\omega(ev,w)f \mod \overline{\mathfrak{m}}.$$

This can be shown by applying both sides to A, B, P and Q and pairing with another element in this list with respect to the symplectic form. Plugging this into the sum and using the following identities on V (which is a direct sum of $p+q-4$ copies of the standard representation $\mathbb{R}^2$ of $\mathfrak{m}_0 \simeq \mathfrak{sl}(2,\mathbb{R})$)

$$\begin{aligned}
\operatorname{ad}(h)^2 &= 1, & \operatorname{ad}(e)^2 &= \operatorname{ad}(f)^2 = 0,\\
\operatorname{ad}(e)\operatorname{ad}(f) &= \frac{1}{2}(1+\operatorname{ad}(h)), & \operatorname{ad}(f)\operatorname{ad}(e) &= \frac{1}{2}(1-\operatorname{ad}(h)),\\
\operatorname{ad}(h)\operatorname{ad}(e) &= -\operatorname{ad}(e)\operatorname{ad}(h) = \operatorname{ad}(e), & \operatorname{ad}(h)\operatorname{ad}(f) &= -\operatorname{ad}(f)\operatorname{ad}(h) = -\operatorname{ad}(f),
\end{aligned} \tag{4.7.1}$$

shows the desired formula. □

Now let G be a connected Lie group with Lie algebra $\mathfrak{g} = \mathfrak{so}(p,q)$ such that the analytic subgroup $\langle \exp \mathfrak{m}_0 \rangle$ of M corresponding to $\mathfrak{m}_0 \simeq \mathfrak{sl}(2,\mathbb{R})$ is the non-trivial double cover of $SL(2,\mathbb{R})$. For $k \in \mathbb{Z}/4\mathbb{Z}$ and $s \in \mathbb{C}$ there exists a principal series representation $(\zeta_{k,s}, V_{k,s})$ of $\langle \exp \mathfrak{m}_0 \rangle$ with K-types v_n, $n \in 2\mathbb{Z} + \frac{k}{2}$, on which the basis

$$\kappa = f - e, \qquad x_\pm = h \mp i(e+f)$$

of $\mathfrak{m}_0$ acts by

$$d\zeta_{k,s}(\kappa)v_n = inv_n, \qquad d\zeta_{k,s}(x_\pm)v_n = (s \pm n + 1)v_{n\pm 2}. \tag{4.7.2}$$

Note that for $k = 0, 2$ these representations factor through $SL(2,\mathbb{R})$ and become the usual even and odd principal series for $\mathrm{SL}(2,\mathbb{R})$ while for $k = 1, 3$ the representations are genuine. We also write $(\zeta, V_\zeta) = (\zeta_{k,s}, V_{k,s})$ for short and denote by ζ any extension to M which is trivial on $\mathfrak{m}_1$. Note that for $\zeta = \zeta_{k,s}$ we have $d\zeta(\mathrm{Cas}_{\mathfrak{m}_0}) = s^2 - 1$, so that Ω^ζ_ω is conformally invariant for $\pi_{\zeta,\nu}$ if and only if $s = \pm(\nu + \rho - 1)$. We therefore let $s = -(\nu + \rho - 1)$.

The Fourier transform of the equation $\Omega^\zeta_\omega(v)u = 0$ is by (3.3.1):

$$0 = \widehat{\Omega^\zeta_\omega(v)u}(\lambda, x, y) = \Big((\nu + \rho - 2)d\sigma_{-\lambda}(v) + 4\sum_\alpha d\zeta(B_\mu(v, \widehat{e}_\alpha))d\sigma_{-\lambda}(e_\alpha)\Big)_x \widehat{u}(\lambda, x, y)$$

This motivates the following:

Proposition 4.7.5. *For every $\lambda \in \mathbb{R}^\times$ the space of all $\xi \in \mathcal{S}'(\Lambda) \otimes V_\zeta$ satisfying*

$$\Big((\nu + \rho - 2)d\sigma_\lambda(v) + 4\sum_\alpha d\zeta(B_\mu(v, \widehat{e}_\alpha))d\sigma_\lambda(e_\alpha)\Big)\xi = 0$$

consists of all distributions of the form $\xi(a,z) = \xi_0(a,p)e^{-i\lambda\frac{n(z)}{a}}$ with

$$\xi_0 = \sum_{n \equiv \frac{k}{2} \mod 2} \xi_{0,n} \otimes v_n \in \mathcal{S}'(\mathbb{R}^2) \otimes V_\zeta$$

and each $\xi_{0,n} \in \mathcal{S}'(\mathbb{R}^2)$ homogeneous of degree -1 satisfying the recurrence relation

$$(s + n + 1)(p - i\sqrt{2}a)\xi_{0,n} = (s - n - 1)(p + i\sqrt{a})\xi_{0,n+2},$$

where $p = \omega(x, Q)$.

Proof. A short computation shows that the above equation is equivalent to

$$((s+1) - d\zeta(h))\partial_A\xi - 2\sqrt{2}d\zeta(e)\partial_P\xi = 0, \tag{4.7.3}$$

$$((s+1) + d\zeta(h))a\xi + \sqrt{2}d\zeta(f)p\xi = 0, \tag{4.7.4}$$

$$((s+1) + d\zeta(h))\partial_P\xi - \sqrt{2}d\zeta(f)\partial_A\xi = 0, \tag{4.7.5}$$

$$((s+1) - d\zeta(h))p\xi + 2\sqrt{2}d\zeta(e)a\xi = 0, \tag{4.7.6}$$

and for $v \in \overline{\mathcal{J}}$

$$((s+1) - d\zeta(h))\partial_v\xi + 2\sqrt{2}i\lambda d\zeta(e)\omega(B_\mu(x,v)P, B)\xi = 0, \tag{4.7.7}$$

$$i\lambda((s+1) + d\zeta(h))\omega(B_\mu(x,v)P, B)\xi + \sqrt{2}d\zeta(f)\partial_v\xi = 0. \tag{4.7.8}$$

Combining (4.7.6) and (4.7.7) resp. (4.7.4) and (4.7.8) we find

$$d\zeta(e)\big(a\partial_v - i\lambda p\omega(B_\mu(x,v)P,B)\big)\xi = 0 = d\zeta(f)\big(a\partial_v - i\lambda p\omega(B_\mu(x,v)P,B)\big)\xi,$$

hence $(a\partial_v - i\lambda p\omega(B_\mu(x,v)P,B))\xi = 0$. Note that

$$\partial_v n(x) = -\frac{1}{2}\omega(\mu(x)v,B) = -p\omega(B_\mu(x,v)P,B),$$

so that this equation is equivalent to $\partial_v(\xi \cdot e^{i\lambda\frac{n(x)}{a}}) = 0$. It follows that $\xi(a,x) = \xi_0(a,p)e^{-i\lambda\frac{n(x)}{a}}$. Combining (4.7.3) and (4.7.6) resp. (4.7.4) and (4.7.5) yields

$$d\zeta(e)\big(a\partial_A + p\partial_P + 1\big)\xi_0 = 0 = d\zeta(f)\big(a\partial_A + p\partial_P + 1\big)\xi_0,$$

hence $(a\partial_A + p\partial_P + 1)\xi_0 = 0$ and ξ_0 is homogeneous of degree -1. Write $\xi_0 = \sum_n \xi_{0,n} \otimes v_n$, then using (4.7.2) we find that (4.7.3) and (4.7.5) are equivalent to

$$(s+n+1)(\partial_A + i\sqrt{2}\partial_P)\xi_{0,n} = (s-n-1)(\partial_A - i\sqrt{2}\partial_P)\xi_{0,n+2}$$

and that (4.7.4) and (4.7.6) are equivalent to

$$(s+n+1)(p - i\sqrt{2}a)\xi_{0,n} = (s-n-1)(p + i\sqrt{2}a)\xi_{0,n+2}.$$

It is easy to see that the latter identity implies the first one whenever $\xi_{0,n}$ and $\xi_{0,n+2}$ are homogeneous of degree -1. □

In Proposition 4.7.5 the space of invariant distribution vectors ξ is still infinite-dimensional. This indicates that the kernel of the system $\Omega^\zeta_\omega(v)$, $v \in \Lambda$, is not small enough to yield a representation in the same way as in Section 4.5. We therefore also use construct a vector-valued version of the second order system Ω_μ:

Proposition 4.7.6. *For $\nu = -\frac{p+q-2}{2}$ the system of differential operators*

$$\Omega^\zeta_\mu(T) = \Omega_\mu(T) + 2d\zeta(T)\partial_t \qquad (T \in \mathfrak{m})$$

is conformally invariant on the kernel of the system Ω^ζ_ω, i.e. the joint kernel

$$I(\zeta,\nu)^{\Omega^\zeta_\omega(\Lambda),\Omega^\zeta_\mu(\mathfrak{m})} = \{u \in I(\zeta,\nu)^{\Omega^\zeta_\omega(\Lambda)} : \Omega^\zeta_\mu(T)u = 0\,\forall\, T \in \mathfrak{m}\}$$

is a subrepresentation of $I(\zeta,\nu)$.

Proof. Using Theorem 3.2.3 we find that

$$\begin{aligned}
[\Omega^\zeta_\mu(T), d\pi_{\zeta,\nu}(\bar{\mathfrak{n}})] &= 0,\\
[\Omega^\zeta_\mu(T), d\pi_{\zeta,\nu}(H)] &= 2\Omega^\zeta_\mu(T),\\
[\Omega^\zeta_\mu(T), d\pi_{\zeta,\nu}(S)] &= \Omega^\zeta_\mu([T,S]) && (S \in \mathfrak{m}).
\end{aligned}$$

We further show that $[\Omega^\zeta_\mu(T), d\pi_{\zeta,\nu}(E)]$ can be expressed as a $C^\infty(\bar{\mathfrak{n}})$-linear combination of operators in $\Omega^\zeta_\mu(\mathfrak{m})$ and $\Omega^\zeta_\omega(\Lambda)$. First note that

$$[d\zeta(T)\partial_t, d\pi_{\zeta,\nu}(E)] = d\zeta(T)(\partial_x + 2t\partial_t) + (\nu+\rho)d\zeta(T) + d\zeta([T,\mu(x)])\partial_t.$$

Together with the formula for $[\Omega_\mu(T), d\pi_{\zeta,\nu}(E)]$ in Theorem 3.2.3 this yields

$$[\Omega_\mu^\zeta(T), d\pi_{\zeta,\nu}(E)] = 2t\Omega_\mu^\zeta(T) + \Omega_\mu^\zeta([T,\mu(x)]) + (2\mathcal{C}(\mathfrak{m}') - 2 - (\nu+\rho))\Omega_\omega(Tx) + 4\sum_\alpha d\zeta(B_\mu(x, T\widehat{e}_\alpha))\Omega_\omega(e_\alpha) + 2d\zeta(T)\partial_x + 2(\nu+\rho-\mathcal{C}(\mathfrak{m}'))d\zeta(T).$$

Note that $\mathcal{C}(\mathfrak{m}_0) = \frac{p+q-4}{2}$ and $\mathcal{C}(\overline{\mathfrak{m}}) = 2$ and $\nu+\rho = \frac{p+q-4}{2}$. Let us first assume that $\mathfrak{m}' = \overline{\mathfrak{m}}$, then $d\zeta(T) = 0$. Further, since $[T, B_\mu(x,\widehat{e}_\alpha)] \in \overline{\mathfrak{m}}$ we have $d\zeta([T, B_\mu(x,\widehat{e}_\alpha)]) = 0$ and hence

$$(2\mathcal{C}(\mathfrak{m}') - 2 - (\nu+\rho))\Omega_\omega(Tx) + 4\sum_\alpha d\zeta(B_\mu(x,T\widehat{e}_\alpha))\Omega_\omega(e_\alpha) = -(\nu+\rho-2)\Omega_\omega(Tx) - 4\sum_\alpha d\zeta(B_\mu(Tx,\widehat{e}_\alpha))\Omega_\omega(e_\alpha) = -\Omega_\omega^\zeta(Tx).$$

Now let $\mathfrak{m}' = \mathfrak{m}_0$, then $\nu+\rho-\mathcal{C}(\mathfrak{m}') = 0$, so the last term vanishes. Using $\Omega_\omega^\zeta(Tx) = 0$ the first two terms combine to

$$4\sum_\alpha d\zeta\big(B_\mu(x,T\widehat{e}_\alpha) - B_\mu(Tx,\widehat{e}_\alpha)\big)\Omega_\omega(e_\alpha).$$

which, by the following lemma, equals

$$-2d\zeta(T)\Omega_\omega(x) = -2d\zeta(T)\partial_x.$$

□

Lemma 4.7.7. *For $T \in \mathfrak{m}_0$ and $x,y \in V$ we have*

$$B_\mu(Tx,y) - B_\mu(x,Ty) \equiv \frac{1}{2}\omega(x,y)T \mod \overline{\mathfrak{m}}.$$

Proof. Modulo $\overline{\mathfrak{m}}$ we have

$$\begin{aligned} B_\mu(Tx,y) - B_\mu(x,Ty) &\equiv \frac{1}{4}\big(\omega(hTx,y) - \omega(hx,Ty)\big)h + \frac{1}{2}\big(\omega(fTx,y) - \omega(fx,Ty)\big)e \\ &\qquad + \frac{1}{2}\big(\omega(eTx,y) - \omega(ex,Ty)\big)f \\ &= \frac{1}{4}\omega((hT+Th)x,y)h + \frac{1}{2}\omega((fT+Tf)x,y)e + \frac{1}{2}\omega((eT+Te)x,y)f. \end{aligned}$$

Using (4.7.1) one shows that for $T = T_h h + T_e e + T_f f$ we have

$$(hT+Th)x = 2T_h x, \qquad (eT+Te)x = T_f x, \qquad (fT+Tf)x = T_e x,$$

so the result follows. □

We finally fix $\nu = -\frac{p+q-2}{2}$, $k = p+q$ and $s = -(\nu+\rho-1) = -\frac{p+q-6}{2}$. Note that $\zeta = \zeta_{k,s}$ is reducible and has a unique irreducible subrepresentation. This subrepresentation is finite-dimensional for $p+q$ even and spanned by

$$v_{\frac{p+q-8}{2}}, v_{\frac{p+q-8}{2}-2}, \ldots, v_{-\frac{p+q-8}{2}},$$

and it is infinite-dimensional for $p+q$ odd and spanned by

$$v_{\frac{p+q-8}{2}}, v_{\frac{p+q-8}{2}-2}, \ldots$$

By Theorem 3.4.1 and Lemma 2.5.6 the Fourier transform of $\Omega_\mu^\zeta(T)$ takes the form

$$\widehat{\Omega_\mu^\zeta(T)u}(\lambda,x,y) = -2i\lambda\left(d\omega_{\mathrm{met},-\lambda}(T)_x + d\zeta(T)\right)\widehat{u}(\lambda,x,y).$$

We therefore study $\mathfrak{m}$-invariant distribution vectors in $L^2(\Lambda)^{-\infty}\otimes V_\zeta = \mathcal{S}'(\Lambda)\otimes V_\zeta$.

Proposition 4.7.8. *For every $\lambda\in\mathbb{R}^\times$ the space $(\mathcal{S}'(\Lambda)\otimes V_\zeta)^{\mathfrak{m}}$ of $\mathfrak{m}$-invariant distribution vectors in $\omega_{\mathrm{met},-\lambda}\otimes\zeta$ is two-dimensional and spanned by the distributions*

$$\xi_{\lambda,\varepsilon}(a,z) = \xi_{0,\varepsilon}(a,p)e^{-i\lambda\frac{n(z)}{a}} \qquad \text{with} \qquad \xi_{0,\varepsilon} = \sum_{n\equiv\frac{k}{2} \mod 2}\xi_{0,\varepsilon,n}\otimes v_n$$

and

$$\xi_{0,\varepsilon,n}(a,p) = c_n\operatorname{sgn}(a)^\varepsilon|a|^{-\frac{p+q-6}{2}}(\sqrt{2}|a|+i\operatorname{sgn}(a)p)^n(2a^2+p^2)^{\frac{p+q-2n-8}{4}}$$

with $(c_n)_n$ satisfying

$$(n+2+\tfrac{p+q-8}{2})c_{n+2} = (n-\tfrac{p+q-8}{2})c_n.$$

Proof. We first study invariance under $T\in\mathfrak{m}_1$ in the same way as in Theorem 4.4.2. For $v\in\mathcal{J}$ we have

$$d\omega_{\mathrm{met},\lambda}(B_\mu(v,B))\xi = -a\partial_v(\xi\cdot e^{i\lambda\frac{n(z)}{a}})\cdot e^{-i\lambda\frac{n(z)}{a}} \tag{4.7.9}$$

so that invariance under $B_\mu(\mathcal{J}_1,B)\subseteq\mathfrak{m}_1$ implies $\xi(a,z)=\xi_0(a,p)e^{-i\lambda\frac{n(z)}{a}}$ with $\xi_0\in\mathcal{S}'(\mathbb{R}^2)\otimes V_\zeta$ and $p=\omega(z,Q)$. For $w\in\mathcal{J}^*$ we further have

$$d\omega_{\mathrm{met},\lambda}(B_\mu(A,w))\xi = \frac{1}{2i\lambda}\sum_{\alpha,\beta}\omega(B_\mu(A,w)\widehat{e}_\alpha,\widehat{e}_\beta)\partial_\alpha\partial_\beta\xi - \frac{1}{2}\omega(z,w)\partial_A\xi.$$

For ξ as above we find

$$\begin{aligned}
\partial_A\xi &= \partial_A\xi_0\cdot e^{-i\lambda\frac{n(z)}{a}} + \frac{i\lambda n(z)}{a^2}\xi,\\
\partial_\alpha\xi &= \omega(e_\alpha,Q)\partial_P\xi_0\cdot e^{-i\lambda\frac{n(z)}{a}} + \frac{i\lambda}{2a}\omega(\mu(z)e_\alpha,B)\xi,\\
\partial_\alpha\partial_\beta\xi &= \omega(e_\alpha,Q)\omega(e_\beta,Q)\partial_P^2\xi_0\cdot e^{-i\lambda\frac{n(z)}{a}} + \frac{i\lambda}{a}\omega(e_\alpha,Q)\omega(\mu(z)e_\beta,B)\partial_P\xi_0\cdot e^{-i\lambda\frac{n(z)}{a}},\\
&\quad - \frac{\lambda^2}{4a^2}\omega(\mu(z)e_\alpha,B)\omega(\mu(z)e_\beta,B)\xi + \frac{i\lambda}{a}\omega(B_\mu(z,e_\alpha)e_\beta,B)\xi.
\end{aligned}$$

Combined with Lemma 4.7.1 and 4.7.2 this gives

$$\begin{aligned}
d\omega_{\mathrm{met},\lambda}(B_\mu(A,w))\xi = {}& -\frac{\omega(z_0,w_0)}{2a}\left[a\partial_A\xi_0 + \frac{p+q-6}{2}\xi_0\right]e^{-i\lambda\frac{n(z)}{a}}\\
& - \frac{\omega(z_1,w_1)}{2a}\left[a\partial_A\xi_0 + p\partial_P\xi_0 + \xi_0\right]e^{-i\lambda\frac{n(z)}{a}},
\end{aligned} \tag{4.7.10}$$

so that invariance under $B_\mu(A,\mathcal{J}_1^*)$ implies that ξ_0 is homogeneous of degree -1. Next we consider the action of $\mathfrak{m}_0$ on ξ of this form. By (4.7.9) and (4.7.10) a distribution $\xi(a,z)=\xi(a,p)e^{-i\lambda\frac{n(z)}{a}}$ is invariant under $\mathfrak{m}_0$ if and only if

$$-\frac{p}{a}\left(a\partial_A + \frac{p+q-6}{2}\right)\xi_0 + \sqrt{2}d\zeta(e)\xi_0 = 0 \qquad \text{and} \qquad -2a\partial_P\xi_0 + \sqrt{2}d\zeta(f)\xi_0 = 0.$$

Writing $\xi_0 = \sum_n \xi_{0,n} \otimes v_n$ and using (4.7.2) shows that this is equivalent to

$$\left(2a\partial_P - p\partial_A - \frac{p+q-6}{2}\frac{p}{a}\right)\xi_{0,n} = in\sqrt{2}\xi_{0,n},$$
$$\left(2a\partial_P + p\partial_A + \frac{p+q-6}{2}\frac{p}{a}\right)\xi_{0,n} = i\frac{\sqrt{2}}{2}\big((s+n-1)\xi_{0,n-2} - (s-n-1)\xi_{0,n+2}\big).$$

The first equation has the solutions

$$\xi_{0,n}(a,p) = c_n \operatorname{sgn}(a)^\varepsilon |a|^{-\frac{p+q-6}{2}}(\sqrt{2}|a| + i\operatorname{sgn}(a)p)^n(2a^2+p^2)^{\frac{p+q-2n-8}{4}} \qquad (\varepsilon \in \mathbb{Z}/2\mathbb{Z}),$$

and for this choice of $\xi_{0,n}$ the second equation is equivalent to

$$(n-s+1)c_{n+2} = (n+s+1)c_n.$$

It follows that $c_n = 0$ for $n > -(s+1) = \frac{p+q-8}{2}$ and for $n \leq \frac{p+q-8}{2}$ the sequence (c_n) is uniquely determined by $c_{\frac{p+q-8}{2}}$. The result follows. ☐

Remark 4.7.9. Comparing the invariant distribution vectors in Proposition 4.7.5 and Proposition 4.7.8 suggests that $\Omega^\zeta_\mu(\mathfrak{m})u = 0$ implies $\Omega^\zeta_\omega(V)u = 0$. However, we were not able to show this only using the differential operators $\Omega^\zeta_\mu(T)$ and $\Omega^\zeta_\omega(v)$.

By the same arguments as in the other cases, $u \in I(\zeta,\nu)^{\Omega^\zeta_\omega(V),\Omega^\zeta_\mu(\mathfrak{m})}$ implies

$$\widehat{u}(\lambda,x,y) = \xi_{-\lambda,0}(x)u_0(\lambda,y) + \xi_{-\lambda,1}(x)u_1(\lambda,y)$$

and we obtain a representation $\rho_{\min} = (\rho_{\min,0},\rho_{\min,1})$ of G on a subspace $J_{\min} \subseteq (\mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda)) \oplus (\mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda))$ which makes the map $u \mapsto (u_0,u_1)$ equivariant. Also here, $d\rho_{\min,\varepsilon}$ is independent of ε and we simply write $d\rho_{\min} = d\rho_{\min,0} = d\rho_{\min,1}$ and extend $d\rho_{\min}$ to $\mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda)$.

Proposition 4.7.10. *The representation $d\rho_{\min}$ of $\mathfrak{g}$ on $\mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda)$ is given by the same formulas as in Proposition 4.5.3 with $s_{\min} = -1$, except for the following:*

$$d\rho_{\min}(\overline{v}) = i\partial_\lambda\partial_v + \frac{1}{2}\omega(\mu(y')v,B)\partial_A - \frac{1}{i\lambda}\sum_{\alpha,\beta}\omega(B_\mu(y',v)\widehat{e}_\alpha,\widehat{e}_\beta)\partial_{e_\alpha}\partial_{e_\beta}$$
$$+\begin{cases}(s_{\min}-1)\frac{1}{i\lambda}\partial_v & (v\in\mathcal{J}_0)\\ (s_{\min}-\frac{\dim\Lambda}{2}+1)\frac{1}{i\lambda}\partial_v & (v\in\mathcal{J}_1)\end{cases}$$

$$d\rho_{\min}(\overline{w}) = -\omega(y,w)\lambda\partial_\lambda + \omega(y,w)\partial_y + \partial_{\mu(y')w} - \frac{1}{2i\lambda}\omega(y,B)\sum_{\alpha,\beta}\omega(B_\mu(A,w)\widehat{e}_\alpha,\widehat{e}_\beta)\partial_{e_\alpha}\partial_{e_\beta}$$
$$+\begin{cases}(s_{\min}-\frac{\dim\Lambda}{2}+1)\omega(y,w) & (w\in\mathcal{J}_0^*)\\ (s_{\min}-1)\omega(y,w) & (w\in\mathcal{J}_1^*)\end{cases}$$

$$d\rho_{\min}(E) = i\lambda\partial_\lambda^2 - ia\partial_\lambda\partial_A - i\partial_\lambda\partial_{y'} - i(2s-\tfrac{\dim\Lambda}{2}-1)\partial_\lambda - \frac{s-\frac{\dim\Lambda}{2}}{i\lambda}a\partial_A + n(y')\partial_A$$
$$+\frac{2}{\lambda^2}an(\partial') - \frac{(s-1)(s-\frac{\dim\Lambda}{2}+1)}{i\lambda} - \frac{s-1}{i\lambda}\partial_{y_0'} - \frac{s-\frac{\dim\Lambda}{2}+1}{i\lambda}\partial_{y_1'}$$
$$+\frac{1}{2i\lambda}\sum_{\alpha,\beta}\omega(\mu(y')\widehat{e}_\alpha,\widehat{e}_\beta)\partial_{e_\alpha}\partial_{e_\beta}.$$

Proof. We proceed as in the proof of Proposition 4.5.3 and first compute $d\rho_{\min}(\overline{B})$ by taking the Fourier transform of $d\pi_{\zeta,\nu}(\overline{B})u$. Since

$$d\pi_{\zeta,\nu}(\overline{B}) = d\pi_{\mathbf{1},\nu}(\overline{B}) - 2d\zeta(B_\mu(x,B))$$

we can consider the two terms $d\pi_{\mathbf{1},\nu}(\overline{B})$ and $d\zeta(B_\mu(x,B))$ separately. For the first term the Fourier transform was computed in (4.5.2) and for the second term we use

$$B_\mu(x,B) = \frac{1}{2}\omega(x,B)B_\mu(A,B) + \omega(x,Q)B_\mu(P,B),$$

so that

$$d\zeta(\widehat{B_\mu(x,B)})u = \frac{1}{2}\omega(y-x,B)d\zeta(B_\mu(A,B))\widehat{u} + \omega(y-x,Q)d\zeta(B_\mu(P,B)).$$

Applying the result to

$$\widehat{u}(\lambda,x,y) = \sum_\varepsilon \xi_{-\lambda,\varepsilon}(x)u_\varepsilon(\lambda,y)$$

using

$$\begin{aligned}
\lambda\partial_\lambda\xi_{-\lambda,\varepsilon}(x) &= \tfrac{i\lambda n(x')}{a}\xi_{-\lambda,\varepsilon}(a,x'),\\
(a\partial_A + p\partial_P)\xi_{-\lambda,\varepsilon}(x) &= -\xi_{-\lambda,\varepsilon}(x)\\
\partial_{\overline{x}}\xi_{-\lambda,\varepsilon}(x) &= \tfrac{2i\lambda n(x')}{a}\xi_{-\lambda,\varepsilon}(a,x'),\\
\partial_{\overline{y}}\xi_{-\lambda,\varepsilon}(x) &= -\frac{i\lambda}{2a}\omega(\mu(x')\overline{y},B)\xi_{-\lambda,\varepsilon}(a,x').
\end{aligned}$$

gives

$$\begin{aligned}
d\widehat{\pi}_{\zeta,\nu}(\overline{B})\widehat{u}(\lambda,x,y) &= \sum_\varepsilon \xi_{-\lambda,\varepsilon}(x)\Big[-\omega(y,B)\lambda\partial_\lambda + \omega(y,B)\partial_y - \omega(y,B) + 2i\lambda n(y')\Big]u_\varepsilon(\lambda,y)\\
&+\omega(y-x,B)\sum_\varepsilon u_\varepsilon(\lambda,y)\left[\frac{1}{2}(a\partial_A - p\partial_P + \partial_{\overline{x}}) + \frac{\nu+\rho-1}{2} - d\zeta(B_\mu(A,B))\right]\xi_{-\lambda,\varepsilon}(x)\\
&+\omega(y-x,Q)\sum_\varepsilon u_\varepsilon(\lambda,y)\left[2a\partial_P + i\lambda\omega(\mu(z)P,B) - 2d\zeta(B_\mu(P,B))\right]\xi_{-\lambda,\varepsilon}(x).
\end{aligned}$$

The $\mathfrak{m}$-invariance of $\xi_{-\lambda,\varepsilon}$ further implies that

$$\begin{aligned}
d\zeta(B_\mu(P,B))\xi_{-\lambda,\varepsilon} &= \left(a\partial_P + \frac{1}{2}i\lambda\omega(\mu(z)P,B)\right)\xi_{-\lambda,\varepsilon}\\
d\zeta(B_\mu(A,B))\xi_{-\lambda,\varepsilon} &= \frac{1}{2}\left(a\partial_A - p\partial_P + \partial_{\overline{x}} + \frac{p+q-6}{2}\right)\xi_{-\lambda,\varepsilon},
\end{aligned}$$

so that the claimed formula follows. ∎

As before, we change coordinates using the map Φ_δ in (4.5.3) with $s_{\min} = -1$ and obtain a representation $d\pi_{\min}$ of $\mathfrak{g}$ which is given by the same formulas as in Proposition 4.5.6 except for

$$d\rho_{\min}(\overline{v}) = i\lambda\partial_\lambda\partial_v + i\partial_x\partial_v + \frac{1}{2\lambda}\omega(\mu(x')v,B)\partial_A + i\sum_{\alpha,\beta}\omega(B_\mu(x',v)\widehat{e}_\alpha,\widehat{e}_\beta)\partial_{e_\alpha}\partial_{e_\beta} + \begin{cases} 2i\partial_v & (v\in\mathcal{J}_0) \\ i\frac{p+q-4}{2}\partial_v & (v\in\mathcal{J}_1)\end{cases}$$

$$d\rho_{\min}(\overline{w}) = -\omega(x,w)\partial_\lambda + \frac{1}{\lambda}\partial_{\mu(x')w} + \frac{1}{2}i\omega(x,B)\sum_{\alpha,\beta}\omega(B_\mu(A,w)\widehat{e}_\alpha,\widehat{e}_\beta)\partial_{e_\alpha}\partial_{e_\beta} - \begin{cases} \frac{p+q-6}{2\lambda}\omega(x,w) & (w\in\mathcal{J}_0^*) \\ \frac{1}{\lambda}\omega(x,w) & (w\in\mathcal{J}_1^*)\end{cases}$$

$$d\rho_{\min}(E) = i\lambda\partial_\lambda^2 + i\partial_\lambda\partial_x + i\frac{p+q-2}{2}\partial_\lambda + \frac{n(x')}{\lambda^2}\partial_A + 2an(\partial') - \frac{p+q-6}{2i\lambda} - \frac{p+q-6}{2i\lambda}\partial_{x_0'} - \frac{1}{i\lambda}\partial_{x_1'} + \frac{1}{2i\lambda}\sum_{\alpha,\beta}\omega(\mu(x')\widehat{e}_\alpha,\widehat{e}_\beta)\partial_{e_\alpha}\partial_{e_\beta}.$$

4.8 Matching the Lie algebra action with the literature

For some cases, the Lie algebra representation $d\pi_{\min}$ can be found in the existing literature.

The split cases $\mathfrak{g} = \mathfrak{so}(n,n), \mathfrak{e}_{6(6)}, \mathfrak{e}_{7(7)}, \mathfrak{e}_{8(8)}$

For the split cases $\mathfrak{g} = \mathfrak{so}(n,n)$, $\mathfrak{e}_{6(6)}$, $\mathfrak{e}_{7(7)}$ and $\mathfrak{e}_{8(8)}$ our formulas for the representation $d\pi_{\min}$ agree with the formulas in [31, Appendix].

The case $\mathfrak{g} = \mathfrak{g}_{2(2)}$

Let $\mathfrak{g} = \mathfrak{g}_{2(2)}$, the split real form of $\mathfrak{g}_2(\mathbb{C})$. Then the subspace $\mathfrak{h} = \mathbb{R}H_\alpha \oplus \mathbb{R}H_\beta$ is a Cartan subalgebra of $\mathfrak{g}$. The roots $\lambda = \alpha - 2\beta$ and $\mu = -\alpha + \beta$ form a system of simple roots with λ a long root and μ a short root. A Chevalley basis of $\mathfrak{g}$ is given by

$$\begin{aligned} &X_\mu = -2B_\mu(B,C), && X_{-\mu} = -2B_\mu(A,D), && X_{2\lambda+3\mu} = F, && X_{-2\lambda-3\mu} = E, \\ &X_\lambda = -\frac{1}{\sqrt{2}}A, && X_{\lambda+\mu} = -\sqrt{2}C, && X_{\lambda+2\mu} = -\sqrt{2}D, && X_{\lambda+3\mu} = -\frac{1}{\sqrt{2}}B, \\ &X_{-\lambda} = -\frac{1}{\sqrt{2}}\overline{B}, && X_{-\lambda-\mu} = -\sqrt{2}\overline{D}, && X_{-\lambda-2\mu} = \sqrt{2}\overline{C}, && X_{-\lambda-3\mu} = \frac{1}{\sqrt{2}}\overline{A}. \end{aligned}$$

Using the the coordinates

$$(\lambda,x) = \left(z, \frac{x}{\sqrt{2}}A + \sqrt{2}yC\right),$$

the Lie algebra action $d\pi_{\min}$ equals the one given in [48, pages 124–125] which is due to Gelfand [12]. (In [48] the simple roots are denoted by α and β instead of λ and μ. Further note that in [48] the term $-\frac{iz}{27}D_y^3$ in the formula for $T(X_{-\alpha-3\beta})$ has to be replaced by $-\frac{z}{27}D_y^3$, cf. [12].)

The case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$

For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ the Lie algebra action $d\pi_{\min,r}$ agrees with the action of $\mathfrak{g}$ on the Fourier transformed picture of a different degenerate principal series, namely one corresponding to a maximal parabolic subgroup. Let $Q = L_Q N_Q \subseteq G$ be a parabolic subgroup with $L_Q \simeq \mathrm{GL}(n-1, \mathbb{R})$. The characters $\chi_{r,\varepsilon}$ of L_Q are parameterized by $r \in \mathbb{C}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ and we form the degenerate principal series $\mathrm{Ind}_Q^G(\chi_{r,\varepsilon})$. In [44, Proposition 4.2] this representation is realized in the non-compact picture on $\overline{N}_Q \simeq \mathbb{R}^{n-1}$ and the Euclidean Fourier transform on $\mathbb{R}^{n-1}$ is applied. Surprisingly, this results in the same formulas as the ones obtained for $d\pi_{\min,r}$ in Section 4.6, if we identify the tuple $(\lambda, x) \in \mathbb{R}^\times \times \Lambda$ with a vector in $\mathbb{R}^{n-1} \simeq \overline{N}_Q$.

In Section 6.1 we integrate $d\pi_{\min,r}$ to irreducible unitary representations of $\mathrm{SL}(n, \mathbb{R})$ which are equivalent to the unitary degenerate principal series $\mathrm{Ind}_Q^G(\chi_{r,\varepsilon})$ with $r \in i\mathbb{R}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$.

The case $\mathfrak{g} = \mathfrak{so}(4, 3)$

For $\mathfrak{g} = \mathfrak{so}(4, 3)$ Sabourin [47] constructed an explicit L^2-realization of the minimal representation. His formulas have a lot in common with our realization, but the major difference is that in his model the Lie algebra acts by differential operators of order ≤ 2 while we need order 3 as well. We believe that Sabourin's realization can be obtained with our methods by choosing a different Lagrangian subspace $\Lambda \subseteq V$. More precisely, for $\mathfrak{g} = \mathfrak{so}(p, q)$ the Lie algebra $\mathfrak{m} \simeq \mathfrak{sl}(2, \mathbb{R}) \otimes \mathfrak{so}(p-2, q-2)$ acts on $V \simeq \mathbb{R}^2 \otimes \mathbb{R}^{p+q-4}$ by the tensor product of the two standard representations. We believe that choosing Λ and Λ^* to be $\mathfrak{so}(p-2, q-2)$-invariant one obtains Sabourin's formulas.

Chapter 5

Lowest K-types

To show that for some representation ζ of M, the subrepresentation $I(\zeta,\nu)^{\Omega_\mu(\mathfrak{m})}$ is non-trivial, we find the lowest K-type in the Fourier transformed picture explicitly. For this we first discuss maximal compact subalgebras of $\mathfrak{g}$.

5.1 Cartan involutions

We study Cartan involutions in the case where $\mathfrak{m}$ is simple. This excludes the cases $\mathfrak{g}=\mathfrak{sl}(n,\mathbb{R})$ and $\mathfrak{g}=\mathfrak{so}(p,q)$ for which we discuss Cartan involutions in Section 5.5 and 5.7, respectively.

To construct a Cartan involution on $\mathfrak{g}$, we first choose a unit in the Jordan algebra $\mathcal{J}$. Let $C\in\mathcal{J}=\mathfrak{g}_{(0,-1)}$ with $\Psi(C)\neq 0$ and put $D:=\mu(C)B\in\mathfrak{g}_{(-1,0)}$. Then by Lemma 4.3.4 we have $\mu(C)=-B_\mu(A,\mu(C)B)=-B_\mu(A,D)$ and $\mu(D)=B_\mu(\mu(D)A,B)$. On the other hand, by the $\mathfrak{m}$-equivariance of B_μ and Lemma 4.3.2:

$$\mu(D)=B_\mu(\mu(C)B,\mu(C)B)=[\mu(C),\underbrace{B_\mu(B,\mu(C)B)}_{\in\mathfrak{g}_{(-2,2)}=0}]-B_\mu(B,\mu(C)^2B)=4n(C)B_\mu(C,B).$$

It follows that $\mu(D)A=4n(C)C$ and hence $\mu(D)=4n(C)B_\mu(C,B)$. We therefore renormalize C such that $n(C)=\frac{1}{4}$, then

$$\mu(C)=-B_\mu(A,D)\qquad\text{and}\qquad\mu(D)=B_\mu(C,B).$$

as well as

$$\mu(C)D=-C\qquad\text{and}\qquad\mu(D)C=D.$$

It further follows that $\Psi(C)=\frac{1}{4}A$ and

$$\begin{aligned}\Psi(D)&=\frac{1}{2}\omega(A,\Psi(D))B=-\frac{1}{6}\omega(A,\mu(D)D)B=\frac{1}{6}\omega(\mu(D)A,D)B=\frac{1}{6}\omega(C,D)B\\&=\frac{1}{6}\omega(C,\mu(C)B)B=-\frac{1}{6}\omega(\mu(C)C,B)B=\frac{1}{2}\omega(\Psi(C),B)B=\frac{1}{4}B.\end{aligned}$$

Note that this computation also shows that $\omega(C,D)=\frac{3}{2}$.

Lemma 5.1.1. $B_\mu(C,D)=\frac{1}{4}B_\mu(A,B)$.

Proof. We have
$$[\mu(C),\mu(D)] = 2B_\mu(\mu(C)D,D) = -2B_\mu(C,D).$$
On the other hand, $\mu(D) = B_\mu(C,B)$, so that
$$[\mu(C),\mu(D)] = B_\mu(\mu(C)C,B) + B_\mu(C,\mu(C)B) = -\frac{3}{4}B_\mu(A,B) + B_\mu(C,D). \quad \square$$

Lemma 5.1.2. *The elements* $\{2\mu(C), 2B_\mu(A,B), -2\mu(D)\}$ *form an* $\mathfrak{sl}(2)$*-triple.*

Proof. By the proof of Lemma 5.1.1 we have $[\mu(C),\mu(D)] = -\frac{1}{2}B_\mu(A,B)$. Further,
$$[B_\mu(A,B),\mu(C)] = 2B_\mu(B_\mu(A,B)C,C) = \mu(C),$$
$$[B_\mu(A,B),\mu(D)] = 2B_\mu(B_\mu(A,B)D,D) = -\mu(D). \quad \square$$

Let $\mathcal{J}_0 = \{v \in \mathcal{J} : \omega(v,D) = 0\}$.

Lemma 5.1.3. $\mu(C)\mu(D)|_{\mathcal{J}_0} = -\frac{1}{4}\,\mathrm{id}_{\mathcal{J}_0}$

Proof. Let $v \in \mathcal{J}_0$, then
$$\mu(C)\mu(D)v = [\mu(C),\mu(D)]v + \mu(D)\mu(C)v = -\frac{1}{4}v + \mu(D)\mu(C)v.$$
Since
$$\mu(C)v = \frac{1}{2}\omega(\mu(C)v,B) = -\frac{1}{2}\omega(v,\mu(C)B) = -\frac{1}{2}\omega(v,D) = 0$$
the claim follows. $\square$

In [49, Theorem 7.32] it is shown that $\mathcal{J}$ can be endowed with a natural Jordan algebra structure with unit element C and norm function $N(v) = 4n(v)$. The corresponding trace form is given by
$$T(u,v) = \partial_u N(c)\partial_v N(c) - \partial_u\partial_v N(c) = 4\omega(u,D)\omega(v,D) + 4\omega(\mu(D)u,v), \qquad u,v \in \mathcal{J}.$$
Note that since $T(C,v) = 2\omega(v,D)$, the T-orthogonal complement of C in $\mathcal{J}$ equals $\mathcal{J}_0$. Write $u = u_0C + u'$ and $v = v_0C + v'$ with $u', v' \in \mathcal{J}_0$, then
$$T(u,v) = 3u_0v_0 + 4\omega(\mu(D)u',v'). \tag{5.1.1}$$

Definition 5.1.4. A *Cartan involution* of a Jordan algebra $\mathcal{J}$ with trace form T is an involutive algebra automorphism ϑ of $\mathcal{J}$ such that $T(\vartheta v,v) > 0$ for all $v \in \mathcal{J} \setminus \{0\}$.

Proposition 5.1.5. *For every Cartan involution* ϑ *of the Jordan algebra* $\mathcal{J}$ *the map* $J \in \mathrm{End}(V)$ *given by*
$$\begin{aligned} &JA = -B, && JC = -D, && Jv = 2\mu(D)\vartheta v && (x \in \mathcal{J}_0),\\ &JB = A, && JD = C, && Jw = 2\vartheta\mu(C)w && (w \in \mathfrak{g}_{(-1,0)}, \omega(C,w) = 0), \end{aligned}$$
satisfies the conditions of Lemma 1.8.1.

Proof. Condition (1) follows from $\vartheta^2 = \mathrm{id}$ and Lemma 5.1.3. Condition (2) follows from (5.1.1) and Lemma 5.1.3. The only non-trivial computation for condition (3) is

$$\begin{aligned}\omega(Jv, Jw) &= 4\omega(\mu(D)\vartheta v, \vartheta\mu(C)w) = T(\vartheta v, \vartheta\mu(C)w) = T(v, \mu(C)w)\\ &= 4\omega(\mu(D)v, \mu(C)w) = -4\omega(\mu(C)\mu(D)v, w) = \omega(v, w).\end{aligned}$$

by Lemma 5.1.3 and the fact that ϑ is a Jordan algebra automorphism and therefore leaves the trace form invariant. Condition (4) is equivalent to

$$JB_\mu(x, y)J^{-1} = B_\mu(Jx, Jy) \qquad \text{for all } x, y \in V,$$

which is checked by a lengthy case-by-case computation. □

We fix a Cartan involution ϑ of $\mathcal{J}$, let J be the map constructed in Proposition 5.1.5 and θ the corresponding Cartan involution of $\mathfrak{g}$ constructed in Lemma 1.8.1. Write $\mathfrak{k} = \mathfrak{g}^\theta$ for the corresponding maximal compact subalgebra. We let

$$T_0 = B_\mu(B, C) - B_\mu(A, D) = \mu(C) + \mu(D) \in \mathfrak{k} \cap \mathfrak{m}.$$

Proposition 5.1.6. *If $\mathcal{J}$ is a Euclidean Jordan algebra (i.e. T is positive definite and $\vartheta = \mathrm{id}_{\mathcal{J}}$), then the elements*

$$T_1 = 2T_0 - (E - F), \quad T_2 = A - 2D + \theta(A - 2D), \quad T_3 = B + 2C + \theta(B + 2C)$$

span an ideal $\mathfrak{k}_1 \subseteq \mathfrak{k}$ which is isomorphic to $\mathfrak{su}(2)$.

Proof. We first note the following commutator formulas in $\mathfrak{k}$:

$$\begin{aligned}
[E - F, x + \theta(x)] &= Jx + \theta(Jx) && \forall\, x \in V,\\
[S, x + \theta(x)] &= Sx + \theta(Sx) && \forall\, S \in \mathfrak{k} \cap \mathfrak{m}, x \in V,\\
[E - F, S] &= 0 && \forall\, S \in \mathfrak{k} \cap \mathfrak{m},\\
[x + \theta(x), y + \theta(y)] &= -\omega(x, y)(E - F) - 2(B_\mu(Jx, y) - B_\mu(x, Jy)) && \forall\, x, y \in V.
\end{aligned}$$

Further, we have

$$T_0A = C, \quad T_0B = D, \quad T_0C = -\frac{3}{4}A + D, \quad T_0D = -\frac{3}{4}B - C.$$

We first show that $\mathfrak{k}_1$ is a subalgebra. For this we compute

$$\begin{aligned}
[T_1, T_2] &= (2T_0 - J)(A - 2D) + \theta((2T_0 - J)(A - 2D)) = 4T_3,\\
[T_1, T_3] &= (2T_0 - J)(B + 2C) + \theta((2T_0 - J)(B + 2C)) = -4T_2,\\
[T_2, T_3] &= -\omega(A - 2D, B + 2C)(E - F)\\
&\quad - 2(B_\mu(J(A - 2D), B + 2C) - B_\mu(A - 2D, J(B + 2C)))\\
&= 8T_1.
\end{aligned}$$

It remains to show that $\mathfrak{k}_1$ is an ideal, i.e. $[\mathfrak{k}, \mathfrak{k}_1] \subseteq \mathfrak{k}_1$. First, by similar computations as above, T_1, T_2 and T_3 commute with

$$2T_0 + 3(E - F), \quad 3A + 2D + \theta(3A + 2D) \quad \text{and} \quad 3B - 2C + \theta(3B - 2C).$$

Finally, similar computations show that T_1, T_2 and T_3 commute with

- $v + \theta v$, $v \in \mathcal{J}_0$ or $v \in \overline{\mathcal{J}}_0$,
- $B_\mu(v, B) + B_\mu(A, Jv)$, $v \in \mathcal{J}_0$,
- $S \in \mathfrak{g}_{(0,0)} \cap \mathfrak{k}$. □

Remark 5.1.7. The renormalized generators

$$\widetilde{T}_1 = \frac{1}{2}T_1, \quad \widetilde{T}_2 = \frac{1}{2\sqrt{2}}T_2, \quad \widetilde{T}_3 = \frac{1}{2\sqrt{2}}T_3$$

satisfy the standard $\mathfrak{su}(2)$-relations

$$[\widetilde{T}_1, \widetilde{T}_2] = 2\widetilde{T}_3, \quad [\widetilde{T}_2, \widetilde{T}_3] = 2\widetilde{T}_1, \quad [\widetilde{T}_3, \widetilde{T}_1] = 2\widetilde{T}_2.$$

5.2 The quaternionic cases $\mathfrak{g} = \mathfrak{e}_{6(2)}, \mathfrak{e}_{7(-5)}, \mathfrak{e}_{8(-24)}$

Assume that the Jordan algebra $\mathcal{J}$ is simple and Euclidean, i.e. the identity $\vartheta = \mathrm{id}_{\mathcal{J}}$ is a Cartan involution. Then by Proposition 5.1.6 the group G is of quaternionic type. By the classification, we have $\mathfrak{g} \simeq \mathfrak{f}_{4(4)}, \mathfrak{e}_{6(2)}, \mathfrak{e}_{7(-5)}, \mathfrak{e}_{8(-24)}$ with $s_{\min} = -\frac{3}{2}, -2, -3, -5$ respectively. We decompose $\mathfrak{k}$ into simple ideals:

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$$

with $\mathfrak{k}_1 = \mathbb{R}T_1 \oplus \mathbb{R}T_2 \oplus \mathbb{R}T_3 \simeq \mathfrak{su}(2)$. We further abbreviate $n = -s_{\min} - 1 \in \{\frac{1}{2}, 1, 2, 4\}$.

Theorem 5.2.1. *For $\mathfrak{g} = \mathfrak{e}_{6(2)}, \mathfrak{e}_{7(-5)}, \mathfrak{e}_{8(-24)}$ the space $W = \bigoplus_{k=-n}^{n} \mathbb{C}f_k$ with*

$$f_k(\lambda, a, x) = (\lambda - i\sqrt{2}a)^k (\lambda^2 + 2a^2)^{\frac{s_{\min}-k}{2}} \exp\left(-\frac{2ian(x)}{\lambda(\lambda^2 + 2a^2)}\right) \sum_{m=-n}^{n} h_{k,m} K_m(r) e^{im\theta},$$

where

$$(r\cos\theta, r\sin\theta) = \left(\frac{2(\lambda^2 + 2a^2)I_1 - I_3}{\sqrt{2}(\lambda^2 + 2a^2)}, \frac{I_2 + \lambda^2 + 2a^2}{(\lambda^2 + 2a^2)^{\frac{1}{2}}}\right)$$

with

$$I_1 = \omega(x, D), \qquad I_2 = \omega(\mu(x)C, B), \qquad I_3 = \omega(\Psi(x), B)$$

and $(h_{k,m})_{m=-n,\dots,n}$ is given by (5.2.7), is a $\mathfrak{k}$-subrepresentation of $(d\pi_{\min}, \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda))$ isomorphic to the representation $S^{2n}(\mathbb{C}^2) \boxtimes \mathbb{C}$ of $\mathfrak{k} \simeq \mathfrak{su}(2) \oplus \mathfrak{k}_2$.

Remark 5.2.2. We excluded the case $\mathfrak{g} = \mathfrak{f}_{4(4)}$ in the theorem, because here $n = \frac{1}{2}$ and therefore the summation would have to be over $m = \pm\frac{1}{2}$. This is not immediately possible since it would require the use of $e^{\pm\frac{i\theta}{2}} = (\cos\theta + i\sin\theta)^{\frac{1}{2}}$ which cannot be defined as a smooth function on $\mathbb{R}_+ \times \Lambda$ or $\mathbb{R}_- \times \Lambda$, because the image of both $2(\lambda^2 + 2a^2)I_1 - I_3$ and $I_2 + \lambda^2 + 2a^2$ is $\mathbb{R}$. However, it might be possible to find the lowest K-type $S^1(\mathbb{C}^2) \boxtimes \mathbb{C}$ of the minimal representation in a space of vector-valued functions as a subrepresentation of a vector-valued degenerate principal series (cf. [17, Section 12]).

We prove this result in several steps. For this we decompose $\mathfrak{k}_2$ as follows:

$$\begin{aligned}\mathfrak{k}_2 = {} & \mathbb{R}(2T_0 + 3(E-F)) \oplus \mathbb{R}(3A+2D+\theta(3A+2D)) \oplus \mathbb{R}(3B-2C+\theta(3B-2C)) \\ & \oplus \{v+\theta v : v \in \mathcal{J}_0\} \oplus \{\overline{v}+\theta\overline{v} : v \in \mathcal{J}_0\} \\ & \oplus \{B_\mu(v,B)+B_\mu(A,Jv) : v \in \mathcal{J}_0\} \oplus (\mathfrak{g}_{(0,0)} \cap \mathfrak{k}).\end{aligned}$$

Lemma 5.2.3. *The Lie algebra $\mathfrak{k}_2$ is generated by*

$$\mathfrak{g}_{(0,0)} \cap \mathfrak{k} \oplus \{v+\theta v : v \in \mathcal{J}_0\} \oplus \{B_\mu(v,B)+B_\mu(A,Jv) : v \in \mathcal{J}_0\}.$$

Proof. Let $\mathfrak{h} \subseteq \mathfrak{k}$ denote the subalgebra generated by the above elements. For $v, w \in \mathcal{J}_0$ we have

$$\begin{aligned}[B_\mu(v,B)+B_\mu(A,Jv), w+\theta w] &= B_\mu(v,B)w + B_\mu(A,Jv)w + \theta(B_\mu(v,B)w + B_\mu(A,Jv)w) \\ &= -\frac{1}{6}\omega(Jv,w)(3A+2D+\theta(3A+2D)) + u + \theta(u)\end{aligned}$$

with $u = B_\mu(v,w)B + \frac{1}{3}\omega(Jv,w)D \in \overline{\mathcal{J}}_0$. Note that since the Jordan algebra $\mathcal{J}$ is simple, we can always find $v, w \in \mathcal{J}_0$ such that $u \neq 0$. If we further act by $S \in \mathfrak{g}_{(0,0)} \cap \mathfrak{k}$, using $SA = SD = 0$, we obtain

$$[S,[B_\mu(v,B)+B_\mu(A,Jv), w+\theta w]] = Su + \theta(Su).$$

Now $\mathfrak{g}_{(0,0)} \cap \mathfrak{k}$ is the Lie algebra of the automorphism group of the Jordan algebra $\mathcal{J}$ which acts irreducibly on $\mathcal{J}_0$. It follows that $\{u+\theta u : u \in \overline{\mathcal{J}}_0\} \subseteq \mathfrak{h}$. Further, by choosing $v, w \in \mathcal{J}_0$ above such that $\omega(Jv,w) \neq 0$ we obtain $3A+2D+\theta(3A+2D)$. A similar argument with

$$[B_\mu(v,B)+B_\mu(A,Jv), \overline{w}+\theta(\overline{w})]$$

shows $3B-2C+\theta(3B-2C) \in \mathfrak{h}$. Finally,

$$[3A+2D+\theta(3A+2D), 3B-2C+\theta(3B-2C)] = -8(2T_0+3(E-F)) \in \mathfrak{h}. \qquad \square$$

Lemma 5.2.4. *If $f \in \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda)$ is $(\mathfrak{g}_{(0,0)} \cap \mathfrak{k})$-invariant, it is of the form*

$$f(\lambda, a, x) = f_1(\lambda, a, I_1, I_2, I_3),$$

where

$$I_1 = \omega(x,D), \qquad I_2 = \omega(\mu(x)C, B), \qquad I_3 = \omega(\Psi(x), B).$$

Proof. The Lie algebra $\mathfrak{g}_{(0,0)}$ decomposes as $\mathbb{R}H \oplus (\mathfrak{m} \cap \mathfrak{g}_{(0,0)})$ and $\mathfrak{m} \cap \mathfrak{g}_{(0,0)}$ is the Lie algebra of the structure group of the Jordan algebra $\mathcal{J}$ (see [9] for details on Jordan algebras). Since $\mathcal{J}$ is Euclidean, the maximal compact subalgebra $\mathfrak{k} \cap \mathfrak{g}_{(0,0)}$ is therefore the Lie algebra of the automorphism group of $\mathcal{J}$. Its invariants are the coefficients $a_1(x), a_2(x), a_3(x)$ of the minimal polynomial $X^3 - a_1(x)X^2 + a_2(x)X - a_3(x)$ of a generic element $x \in \mathcal{J}$ (see . These are given by $a_1(x) = \mathrm{tr}(x) = T(x,C) = 2\omega(x,D)$, $a_3(x) = \det(x) = 4n(x) = 2\omega(\Psi(x),B)$ and

$$a_2(x) = \partial_C \det(x) = 6\omega(B_\Psi(x,x,C),B) = -2\omega(\mu(x)B,C). \qquad \square$$

Lemma 5.2.5. *If $f \in \mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda)$ is additionally an eigenfunction of $d\pi_{\min}(A-\overline{B})$ to the eigenvalue $ik\sqrt{2}$, it is of the form*

$$f(\lambda,a,x) = (\lambda - i\sqrt{2}a)^k \exp\left(-\frac{iaI_3}{\lambda R}\right) f_2(R,I_1,I_2,I_3),$$

where

$$R = \lambda^2 + 2a^2.$$

Proof. The method of characteristics applied to the first order equation

$$d\pi_{\min}(A-\overline{B})f = \left(-\lambda\partial_A + 2a\partial_\lambda - \frac{iI_3}{\lambda^2}\right) f = ik\sqrt{2}f$$

shows the claim. □

Lemma 5.2.6. *If $f \in \mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda)$ is additionally invariant under $\{\lambda\, d\pi_{\min}(v+\theta v) + 2a\, d\pi_{\min}(B_\mu(v,B) + B_\mu(A,Jv)) : v \in \mathcal{J}_0\}$, it is of the form*

$$f(\lambda,a,x) = (\lambda - i\sqrt{2}a)^k (\lambda^2+2a^2)^{\frac{s_{\min}-k}{2}} \exp\left(-\frac{iaI_3}{\lambda R}\right) f_3(S,T)$$

with

$$S = \frac{2RI_1 - I_3}{\sqrt{2}R}, \quad T = \frac{I_2 + R}{R^{\frac{1}{2}}}.$$

Proof. Applying

$$\begin{aligned}\lambda\, d\pi_{\min}(v+\theta v) + 2a\, d\pi_{\min}(B_\mu(v,B) + B_\mu(A,Jv)) &= -\omega(x,Jv)(\lambda\partial_\lambda + a\partial_A - s_{\min}) \\ &\quad - (2a^2+\lambda^2)\partial_v + \partial_{\mu(x)Jv} + i\frac{a}{\lambda}\omega(\mu(x)v,B)\end{aligned}$$

to $f(\lambda,a,x) = (\lambda - i\sqrt{2}a)^k \exp\left(-\frac{iaI_3}{\lambda R}\right) f_2(R,I_1,I_2,I_3)$ gives

$$\begin{aligned}(\lambda - i\sqrt{2}a)^k \exp\left(-\frac{iaI_3}{\lambda R}\right) \Bigg[\omega(x,Jv)\Big(-2R\partial_R + (R-I_2)\partial_2 - 2I_3\partial_3 + s_{\min} - k\Big) f_2 \\ + \omega(\mu(x)v,B)\Big(R\partial_3 + \frac{1}{2}\partial_1\Big) f_2\Bigg] = 0.\end{aligned}$$

Here we have used

$$\begin{aligned}\omega(B_\mu(x,v)C,B) &= -\frac{1}{2}\omega(x,Jv), & \omega(B_\mu(x,\mu(x)Jv)C,B) &= -\frac{1}{2}I_2\omega(x,Jv), \\ \omega(\mu(x)Jv,D) &= \frac{1}{2}\omega(\mu(x)v,B), & \omega(B_\Psi(x,x,\mu(x)Jv),B) &= -\frac{2}{3}I_3\omega(x,Jv).\end{aligned}$$

This yields two first order partial differential equations:

$$-2R\partial_R f_2 + (R-I_2)\partial_2 f_2 - 2I_3\partial_3 f_2 = (k - s_{\min}) f_2, \tag{5.2.1}$$

$$R\partial_3 f_2 + \frac{1}{2}\partial_1 f_2 = 0. \tag{5.2.2}$$

Solving (5.2.2) using the method of characteristics gives

$$f_2(R, I_1, I_2, I_3) = \widetilde{f}_2(R, I_2, U) \qquad \text{with } U = 2RI_1 - I_3.$$

Applying (5.2.1) to this expression and using again the method of characteristics yields

$$\widetilde{f}_2(\lambda, a, I_2, U) = R^{\frac{s_{\min}-k}{2}} f_3(S, T).$$ □

Lemma 5.2.7. *If $f \in \mathcal{D}'(\mathbb{R}^\times)\widehat{\otimes}\mathcal{S}'(\Lambda)$ is additionally invariant under $\{B_\mu(v, B) + B_\mu(A, Jv) : v \in \mathcal{J}_0\}$, it is of the form*

$$f(\lambda, a, x) = (\lambda - i\sqrt{2}a)^k(\lambda^2 + 2a^2)^{\frac{s_{\min}-k}{2}} \exp\left(-\frac{iaI_3}{\lambda R}\right) \sum_{m\in\mathbb{Z}} h_{k,m}K_m(r)e^{im\theta},$$

where $(S, T) = (r\cos\theta, r\sin\theta)$ and $(h_{k,m})_{m\in\mathbb{Z}}$ is a sequence satisfying

$$\frac{m + s_{\min}}{2} h_{k,m-1} - \frac{m - s_{\min}}{2} h_{k,m+1} + kh_{k,m} = 0. \tag{5.2.3}$$

Proof. Using the identities

$$\sum_{\alpha,\beta} \omega(e_\alpha, D)\omega(e_\beta, D)\omega(B_\mu(A, Jv)\widehat{e}_\alpha, \widehat{e}_\beta) = 0,$$

$$\sum_{\alpha,\beta} \omega(e_\alpha, D)\omega(B_\mu(x, e_\beta)C, B)\omega(B_\mu(A, Jv)\widehat{e}_\alpha, \widehat{e}_\beta) = -\frac{1}{4}\omega(x, Jv),$$

$$\sum_{\alpha,\beta} \omega(e_\alpha, D)\omega(\mu(x)e_\beta, B)\omega(B_\mu(A, Jv)\widehat{e}_\alpha, \widehat{e}_\beta) = \frac{1}{2}\omega(\mu(x)v, B),$$

$$\sum_{\alpha,\beta} \omega(B_\mu(e_\alpha, e_\beta)x, B)\omega(B_\mu(A, Jv)\widehat{e}_\alpha, \widehat{e}_\beta) = s_{\min}\omega(x, Jv),$$

$$\sum_{\alpha,\beta} \omega(\mu(x)e_\alpha, B)\omega(\mu(x)e_\beta, B)\omega(B_\mu(A, Jv)\widehat{e}_\alpha, \widehat{e}_\beta) = 2I_3\omega(x, Jv),$$

$$\sum_{\alpha,\beta} \omega(\mu(x)e_\alpha, B)\omega(B_\mu(x, e_\beta)C, B)\omega(B_\mu(A, Jv)\widehat{e}_\alpha, \widehat{e}_\beta) = -\frac{1}{2}I_2\omega(x, Jv),$$

$$\sum_{\alpha,\beta} \omega(B_\mu(x, e_\alpha)C, B)\omega(B_\mu(x, e_\beta)C, B)\omega(B_\mu(A, Jv)\widehat{e}_\alpha, \widehat{e}_\beta) = \frac{1}{2}I_1\omega(x, Jv) + \frac{1}{4}\omega(\mu(x)v, B)$$

the equation $d\pi_{\min}(B_\mu(v, B) + B_\mu(A, Jv))f = 0$ for f as in Lemma 5.2.6 becomes

$$\omega(x, Jv)\Big(\frac{I_3}{R}\partial_S^2 - \sqrt{2}T\partial_S\partial_T + 2I_1\partial_T^2 + \sqrt{2}s_{\min}\partial_S - \frac{I_3}{R} - k\sqrt{2}\Big)f_3 \\ + \omega(\mu(x)v, B)\Big(\partial_S^2 + \partial_T^2 - 1\Big)f_3 = 0.$$

Again, this gives rise to two partial differential equations, this time of second order:

$$\frac{I_3}{R}\partial_S^2 f_3 - \sqrt{2}T\partial_S\partial_T f_3 + 2I_1\partial_T^2 f_3 + \sqrt{2}s_{\min}\partial_S f_3 - \frac{I_3}{R}f_3 - k\sqrt{2}f_3 = 0, \tag{5.2.4}$$

$$\partial_S^2 f_3 + \partial_T^2 f_3 = f_3. \tag{5.2.5}$$

While (5.2.5) only contains the variables S and T, (5.2.4) also contains I_1 and I_3. We therefore subtract $2I_1$ times (5.2.5) from (5.2.4) to obtain the equivalent equation

$$(S\partial_S + T\partial_T - s_{\min})\partial_S f_3 - Sf_3 + kf_3 = 0. \tag{5.2.6}$$

Using polar coordinates $(S,T) = (r\cos\theta, r\sin\theta)$ we expand f_3 into a Fourier series

$$f_3(S,T) = \sum_{m\in\mathbb{Z}} g_m(r)e^{im\theta}.$$

Then (5.2.5) becomes

$$\partial_r^2 g_m + \frac{1}{r}\partial_r g_m - \Big(1 + \frac{m^2}{r^2}\Big)g_m = 0.$$

The two solutions to this ordinary differential equation are the Bessel functions $I_m(r)$ and $K_m(r)$. The I-Bessel function grows exponentially as $r\to\infty$ while the K-Bessel function decays exponentially. Since we are only interested in tempered distributions (in fact, only L^2-functions) we write

$$g_m(r) = h_{k,m}K_m(r)$$

for some scalars $h_{k,m}\in\mathbb{C}$. Applying (5.2.6) to the Fourier expansion finally yields the relation (5.2.3). □

Lemma 5.2.8. *For $-n\le k\le n$ there is a unique (up to scalar multiples) sequence $(h_{k,m})_{m\in\mathbb{Z}}$ satisfying (5.2.3). It satisfies $h_{k,m}=0$ for $|m|>n$ and, normalizing $h_{k,m}=1$ for $m=-n$, is given by*

$$h_{k,m} = \sum_{j=0}^{m+n}(-1)^{m+n-j}\binom{n+k}{j}\binom{n-k}{m+n-j} \tag{5.2.7}$$

for $m=-n,\dots,n$.

Proof. Let $(h_{k,m})_{m\in\mathbb{Z}}$ be a sequence satisfying (5.2.3) and form the generating function

$$h(t) = \sum_{m\in\mathbb{Z}} h_{k,m}t^m.$$

Then (5.2.3) becomes the differential equation

$$h'(t) = \frac{-nt^2+2kt-n}{t(1-t^2)}h(t).$$

Writing

$$\frac{-nt^2+2kt-n}{t(1-t^2)} = -\frac{n}{t} + \frac{k-n}{1-t} + \frac{k+n}{1+t}$$

reveals the solution

$$h(t) = t^{-n}(1-t)^{n-k}(1+t)^{n+k}.$$

Expanding both $(1-t)^{n-k}$ and $(1+t)^{n+k}$ into Taylor series around $t=0$ shows the claim. □

Recall the elements $\widetilde{T}_1, \widetilde{T}_2, \widetilde{T}_3 \in \mathfrak{k}$ from Proposition 5.1.6 and Remark 5.1.7 which span the ideal $\mathfrak{k}_1 \simeq \mathfrak{su}(2)$. The element $\widetilde{T}_2$ spans a maximal torus in $\mathfrak{k}_1$ and $\mathrm{ad}(\widetilde{T}_2)$ has eigenvalues $0, \pm 2i$. By the representation theory of $\mathfrak{su}(2)$, the action of $\widetilde{T}_2$ in every finite-dimensional representation of $\mathfrak{k}_1$ is diagonalizable with eigenvalues $2ik$, $k \in \mathbb{Z} \cup (\mathbb{Z} + \frac{1}{2})$, and the elements $\widetilde{T}_3 \pm i\widetilde{T}_1$ step between the subsequent eigenspaces. Lemma 5.2.5 shows that f_k is in fact an eigenfunction of $d\pi_{\min}(\widetilde{T}_2)$ to the eigenvalue $2ik$. We therefore compute the action of $d\pi_{\min}(\widetilde{T}_3 \pm i\widetilde{T}_1)$ on f_k.

Lemma 5.2.9. *For $-n \leq k \leq n$ we have*

$$d\pi_{\min}(2T_0 \pm i\sqrt{2}(C - \overline{D}))f_k = -3(k \mp n)f_{k\pm 1}.$$

Proof. For $f \in \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda)$ of the form

$$f(\lambda, a, x) = (\lambda - i\sqrt{2}a)^k (\lambda^2 + 2a^2)^{\frac{s_{\min} - k}{2}} \exp\left(-\frac{iaI_3}{\lambda R}\right) f_3(S, T)$$

with f_3 satisfying (5.2.5) and (5.2.6) a lengthy computation shows that

$$d\pi_{\min}(2T_0 \pm i\sqrt{2}(C - \overline{D}))f(\lambda, a, x) = (\lambda - i\sqrt{2}a)^{k\pm 1}(\lambda^2 + 2a^2)^{\frac{s_{\min} - (k\pm 1)}{2}} \exp\left(-\frac{iaI_3}{\lambda R}\right)$$
$$\times 3i\Big[T\partial_S^2 - S\partial_S\partial_T \mp T\partial_S \pm S\partial_T + s_{\min}\partial_T\Big] f_3(S, T).$$

If now $f_3(S, T) = \sum_m h_{k,m} K_m(r) e^{im\theta}$ with $(S, T) = (r\cos\theta, r\sin\theta)$ it can further be shown that

$$\Big[T\partial_S^2 - S\partial_S\partial_T \mp T\partial_S \pm S\partial_T + s_{\min}\partial_T\Big] f_3(S, T)$$
$$= i\sum_m \left[\frac{m + s_{\min}}{2} h_{k,m-1} + \frac{m - s_{\min}}{2} h_{k,m+1} \pm m h_{k,m}\right] K_m(r) e^{im\theta}.$$

Finally, it can be shown using Lemma 5.2.8 that

$$\frac{m + s_{\min}}{2} h_{k,m-1} + \frac{m - s_{\min}}{2} h_{k,m+1} \pm m h_{k,m} = (k \pm (s_{\min} + 1)) h_{k+1,m}.$$ □

Combining the various lemmas, we obtain Theorem 5.2.1.

Corollary 5.2.10. *The elements $w_0^2, w_1^2, w_2^2 \in K$ act on W in the following way:*

$$\pi_{\min}(w_0^2)f_k = (-1)^{n-k} f_{-k}, \qquad \pi_{\min}(w_1^2)f_k = (-1)^k f_k, \qquad \pi_{\min}(w_2^2)f_k = (-1)^n f_{-k}.$$

Proof. From Lemma 5.2.5 and 5.2.9 it follows that the $\mathfrak{su}(2)$-triple $\widetilde{T}_1, \widetilde{T}_2, \widetilde{T}_3$ acts on W by

$$d\pi_{\min}(\widetilde{T}_2)f_k = 2ikf_k, \qquad d\pi_{\min}(\widetilde{T}_3 \pm i\widetilde{T}_1)f_k = 2i(s_{\min} + 1 \mp k)f_{k\mp 1}. \tag{5.2.8}$$

Since $\mathfrak{k}_2$ acts trivially on W, we find

$$\pi_{\min}(w_0^2)f_k = \pi_{\min}(\exp(-\tfrac{\pi}{2}\widetilde{T}_1))f_k,$$
$$\pi_{\min}(w_1^2)f_k = \pi_{\min}(\exp(\tfrac{\pi}{2}\widetilde{T}_2))f_k,$$
$$\pi_{\min}(w_2^2)f_k = \pi_{\min}(\exp(\tfrac{\pi}{2}\widetilde{T}_3))f_k.$$

The formulas now follow from the representation theory of $\mathrm{SU}(2)$. □

5.3 The split cases $\mathfrak{g} = \mathfrak{e}_{6(6)}, \mathfrak{e}_{7(7)}, \mathfrak{e}_{8(8)}$

Assume that the Jordan algebra $\mathcal{J}$ is simple, non-Euclidean and split. Then the group G is split. By the classification we have $\mathfrak{g} \simeq \mathfrak{e}_{6(6)}$, $\mathfrak{e}_{7(7)}$ or $\mathfrak{e}_{8(8)}$.

The lowest K-type in this case turns out to be the trivial representation. It is spanned by a vector which is most easily described using a renormalization $\overline{K}_\alpha(x) = x^{-\frac{\alpha}{2}} K_\alpha(\sqrt{x})$ of the K-Bessel function (see Appendix A for details).

Theorem 5.3.1. *The space $W = \mathbb{C} f_0$ with*

$$f_0(\lambda, a, x) = (\lambda^2 + 2a^2)^{\frac{s_{\min}}{2}} \exp\left(-\frac{2ian(x)}{\lambda(\lambda^2 + 2a^2)}\right) \overline{K}_{-\frac{s_{\min}+1}{2}}\left(\frac{2R^2 I_2 + I_3^2 - RI_4 + 2R^3}{2R^2}\right),$$

where

$$R = \lambda^2 + 2a^2, \qquad I_2 = \omega(Jx, x), \qquad I_3 = \omega(\Psi(x), B), \qquad I_4 = \omega(\mu(x)Jx, Jx),$$

is a $\mathfrak{k}$-subrepresentation of $(d\pi_{\min}, \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda))$ isomorphic to the trivial representation.

Remark 5.3.2. The spherical vector f_0 has previously been found in [31] using case-by-case computations. Note that $-\frac{s_{\min}+1}{2}$ equals $\frac{1}{2}, 1, 2$ for $\mathfrak{g} = \mathfrak{e}_{6(6)}, \mathfrak{e}_{7(7)}, \mathfrak{e}_{8(8)}$.

We prove this result in several steps. The following lemma is proven in a similar way as Lemma 5.2.3:

Lemma 5.3.3. *The Lie algebra $\mathfrak{k}$ is generated by $\mathfrak{g}_0 \cap \mathfrak{k}$ and $\{v + \theta v : v \in \Lambda\}$.*

Lemma 5.3.4. *If $f \in \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda)$ is $(\mathfrak{g}_{(0,0)} \cap \mathfrak{k})$-invariant, it is of the form*

$$f(\lambda, a, x) = f_1(\lambda, a, I_2, I_3, I_4),$$

where

$$I_2 = \omega(Jx, x), \qquad I_3 = \omega(\Psi(x), B), \qquad I_4 = \omega(\mu(x)Jx, Jx).$$

Proof. In a non-Euclidean split Jordan algebra $\mathcal{J}$ of degree 3 the polynomials invariant under a maximal compact subgroup of the structure group are $\operatorname{tr}(\vartheta(x)x)$, $\det(x)$ and $\operatorname{tr}(\vartheta(x)x\vartheta(x)x)$. These are essentially I_2, I_3 and I_4. □

Lemma 5.3.5. *If $f \in \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda)$ is additionally invariant under $A - \overline{B}$, it is of the form*

$$f(\lambda, a, x) = \exp\left(-\frac{2ian(x)}{\lambda(\lambda^2 + 2a^2)}\right) f_2(R, I_2, I_3, I_4)$$

with

$$R = \lambda^2 + 2a^2.$$

Proof. We have

$$d\pi_{\min}(A - \overline{B}) = -\lambda\partial_A + 2a\partial_\lambda - \frac{iI_3}{\lambda^2}.$$

As in the quaternionic case, the method of characteristics shows the claim. □

Lemma 5.3.6. *If $f \in \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda)$ is additionally invariant under $\{\lambda\, d\pi_{\min}(v+\theta v)+2a\, d\pi_{\min}(B_\mu(v,B)+B_\mu(A,Jv)) : v \in \mathcal{J}\}$, it is of the form*

$$f(\lambda,a,x) = (\lambda^2+2a^2)^{\frac{s_{\min}}{2}} \exp\left(-\frac{2ian(x)}{\lambda(\lambda^2+2a^2)}\right) f_3(Z)$$

with

$$Z = \frac{2R^2I_2 + I_3^2 - RI_4 + 2R^3}{2R^2}.$$

Proof. We have

$$\begin{aligned}\lambda\, d\pi_{\min}&(v+\theta v) + 2a\, d\pi_{\min}(B_\mu(v,B)+B_\mu(A,Jv)) \\ &= -\omega(x,Jv)(\lambda\partial_\lambda + a\partial_A - s_{\min}) - (\lambda^2+2a^2)\partial_v + \partial_{\mu(x)Jv} + \frac{ia}{\lambda}\omega(\mu(x)v,B).\end{aligned}$$

Using

$$\begin{aligned}\omega(\mu(x)Jx, J\mu(x)Jv) &= \frac{1}{2}I_3\omega(\mu(x)v,B) - \frac{1}{2}I_4\omega(x,Jv),\\ \omega(\mu(x)\mu(x)Jv,B) &= 2I_3\omega(x,Jv),\\ \omega(x,J\mu(x)Jv) &= \omega(\mu(x)Jx,Jv)\end{aligned}$$

we find that this equals

$$\begin{aligned}\exp\left(-\frac{2ian(x)}{\lambda(\lambda^2+2a^2)}\right)\Bigg[&\omega(x,Jv)\Big(-2R\partial_R f_2 + 2R\partial_2 f_2 - 2I_3\partial_3 f_2 - 2I_4\partial_4 f_2\Big)\\ &+\omega(\mu(x)v,B)\Big(R\partial_3 f_2 + 2I_3\partial_4 f_2\Big)\\ &+\omega(\mu(x)Jx,Jv)\Big(-4R\partial_4 f_2 - 2\partial_2 f_2\Big)\Bigg],\end{aligned}$$

resulting in three first order differential equations for f_2. Solving all three using the method of characteristics yields

$$f_2(R,I_2,I_3,I_4) = R^{\frac{s_{\min}}{2}} f_3(Z).$$

□

Lemma 5.3.7. *The distribution $f \in \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda)$ is additionally invariant under $\{d\pi_{\min}(v+\theta v) : v \in \mathcal{J}\}$ if and only if the function $f_3(Z)$ solves the differential equation*

$$Zf_3''(Z) + \frac{1-s_{\min}}{2}f_3'(Z) - \frac{1}{4}f_3(Z) = 0.$$

Proof. We have

$$\begin{aligned}\lambda d\pi_{\min}(v+\theta v) = -\lambda^2\partial_v - \omega(x,Jv)\lambda\partial_\lambda + s_{\min}\omega(x,Jv) + \partial_{\mu(x)Jv}&\\ + ia\lambda\sum_{\alpha,\beta}\omega(B_\mu(A,Jv)\widehat{e}_\alpha,\widehat{e}_\beta)\partial_\alpha\partial_\beta&.\end{aligned}$$

Using

$$\sum_{\alpha,\beta}\omega(B_\mu(A,Jv)\widehat{e}_\alpha,\widehat{e}_\beta)\omega(B_\mu(e_\alpha,e_\beta)x,B) = s_{\min}\omega(x,Jv),$$

$$\sum_{\alpha,\beta}\omega(B_\mu(A,Jv)\widehat{e}_\alpha,\widehat{e}_\beta)\omega(\mu(x)e_\alpha,B)\omega(\mu(x)e_\beta,B) = 2I_3\omega(x,Jv),$$

$$\sum_{\alpha,\beta}\omega(B_\mu(A,Jv)\widehat{e}_\alpha,\widehat{e}_\beta)\omega(\mu(x)e_\alpha,B)\omega(x,Je_\beta) = \omega(\mu(x)Jx,Jv),$$

$$\sum_{\alpha,\beta}\omega(B_\mu(A,Jv)\widehat{e}_\alpha,\widehat{e}_\beta)\omega(\mu(x)e_\alpha,B)\omega(\mu(x)Jx,Je_\beta) = \frac{1}{2}I_3\omega(\mu(x)v,B) - \frac{1}{2}I_4\omega(x,Jv),$$

$$\sum_{\alpha,\beta}\omega(B_\mu(A,Jv)\widehat{e}_\alpha,\widehat{e}_\beta)\omega(e_\beta,Je_\alpha) = 0,$$

$$\sum_{\alpha,\beta}\omega(B_\mu(A,Jv)\widehat{e}_\alpha,\widehat{e}_\beta)\omega(x,Je_\alpha)\omega(x,Je_\beta) = \omega(\mu(x)v,B),$$

$$\sum_{\alpha,\beta}\omega(B_\mu(A,Jv)\widehat{e}_\alpha,\widehat{e}_\beta)\omega(x,Je_\alpha)\omega(\mu(x)Jx,Je_\beta) = \frac{1}{2}I_2\omega(\mu(x)v,B) + \frac{1}{2}I_3\omega(x,Jv),$$

$$\sum_{\alpha,\beta}\omega(B_\mu(A,Jv)\widehat{e}_\alpha,\widehat{e}_\beta)\omega(\mu(x)Je_\alpha,Je_\beta) = s_{\min}\omega(\mu(x)v,B),$$

$$\sum_{\alpha,\beta}\omega(B_\mu(A,Jv)\widehat{e}_\alpha,\widehat{e}_\beta)\omega(B_\mu(x,e_\beta)Jx,Je_\alpha) = -\frac{1}{2}\omega(\mu(x)v,B),$$

$$\begin{aligned}\sum_{\alpha,\beta}\omega(B_\mu(A,Jv)\widehat{e}_\alpha,\widehat{e}_\beta)\omega(\mu(x)Jx,Je_\alpha)\omega(\mu(x)Jx,Je_\beta) &= I_2I_3\omega(x,Jv) - \frac{1}{2}I_4\omega(\mu(x)v,B)\\ &\quad - I_3\omega(\mu(x)Jx,Jv)\end{aligned}$$

we find that

$$\lambda d\pi_{\min}(v+\theta v)f = \frac{ia\lambda}{R^2}\Big(I_3\omega(x,Jv) + R\omega(\mu(x)v,B)\Big)\Big(4Z\partial_Z^2 f_3 + 2(1-s_{\min})\partial_Z f_3 - f_3\Big). \qquad \square$$

By Appendix A, the unique tempered solution to this ordinary differential equation is the renormalized K-Bessel function $\overline{K}_\alpha(Z)$ with $\alpha = -\frac{s_{\min}+1}{2}$. This shows Theorem 5.3.1.

Corollary 5.3.8. *The elements $w_0^2, w_1^2, w_2^2 \in K$ act on W in the following way:*

$$\pi_{\min}(w_0^2)f_0 = \pi_{\min}(w_1^2)f_0 = \pi_{\min}(w_2^2)f_0 = f_0.$$

5.4 The case $\mathfrak{g} = \mathfrak{g}_{2(2)}$

Let $\mathfrak{g} = \mathfrak{g}_{2(2)}$, then $\mathcal{J} = \mathbb{R}C$ is one-dimensional and therefore, strictly speaking, not of rank three. We treat this case separately. For simplicity we write $x \in \mathcal{J}$ as $x = cC$ and $f(\lambda, a, x) = f(\lambda, a, c)$.

Theorem 5.4.1. *The space $W = \mathbb{R}f_{-1} \oplus \mathbb{R}f_0 \oplus \mathbb{R}f_1$ with*

$$f_0(\lambda, a, c) = (\lambda^2 + 2a^2)^{-\frac{1}{6}} \exp\left(-\frac{iac^3}{2\lambda(\lambda^2+2a^2)}\right)\overline{K}_{-\frac{1}{3}}(S),$$
$$f_{\pm1}(\lambda, a, c) = (\lambda \mp i\sqrt{2}a)(\lambda^2 + 2a^2)^{-\frac{7}{6}} \exp\left(-\frac{iac^3}{2\lambda(\lambda^2+2a^2)}\right)$$
$$\times \left[c\overline{K}_{-\frac{1}{3}}(S) \mp \frac{\sqrt{2}(2\lambda^2+4a^2+c^2)^2}{4(\lambda^2+2a^2)}\overline{K}_{\frac{2}{3}}(S)\right],$$

where

$$S = \frac{(2\lambda^2+4a^2+c^2)^3}{8(\lambda^2+2a^2)^2},$$

is a $\mathfrak{k}$-subrepresentation of $(d\pi_{\min}, \mathcal{D}'(\mathbb{R}^\times) \otimes \mathcal{S}'(\Lambda))$ isomorphic to the representation $\mathbb{C} \boxtimes S^2(\mathbb{C}^2)$ of $\mathfrak{k} \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

Remark 5.4.2. In [18] one vector in the K-type W is found explicitly, but the formula differs slightly from ours. In comparison to the other cases, our formula looks more natural than the one in [18] which contains an additional transcendental function.

We prove this result in several steps. First note that the Lie algebra $\mathfrak{k}$ splits into the direct sum of two ideals

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$$

with $\mathfrak{k}_1, \mathfrak{k}_2 \simeq \mathfrak{su}(2)$ given by

$$\mathfrak{k}_1 = \mathbb{R}T_1 \oplus \mathbb{R}T_2 \oplus \mathbb{R}T_3, \qquad \mathfrak{k}_2 = \mathbb{R}S_1 \oplus \mathbb{R}S_2 \oplus \mathbb{R}S_3,$$

where T_1, T_2, T_3 are as in Proposition 5.1.6 and

$$S_1 = 2T_0 + 3(E-F), \qquad S_2 = 3A + 2D + \theta(3A+2D), \qquad S_3 = 3B - 2C + \theta(3B-2C).$$

We expect to find the K-type $\mathbb{C} \boxtimes S^2(\mathbb{C}^2)$ on which $\mathfrak{k}_1$ acts trivially and $\mathfrak{k}_2 \simeq \mathfrak{su}(2)$ acts by the three-dimensional representation $S^2(\mathbb{C}^2)$. Writing $S^2(\mathbb{C}^2)$ as the direct sum of weight spaces relative to the maximal torus $\mathbb{R}S_2 \subseteq \mathfrak{k}_2$, we expect to find a vector of weight 0, i.e. an S_2-invariant vector.

Lemma 5.4.3. *If $f \in \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda)$ is $d\pi_{\min}(S_2 + T_2)$-invariant, it is of the form*

$$f(\lambda, a, x) = \exp\left(-\frac{iac^3}{2\lambda(\lambda^2+2a^2)}\right) f_1(R, c),$$

where $R = \lambda^2 + 2a^2$.

Proof. We have $S_2 + T_2 = 4(A - \overline{B})$ and

$$d\pi_{\min}(A - \overline{B}) = -\lambda\partial_A + 2a\partial_\lambda - \frac{iI_3}{\lambda^2},$$

where $I_3 = 2n(x) = \frac{1}{2}c^3$. Applying the method of characteristics shows the claim. □

Lemma 5.4.4. *If $f \in \mathcal{D}'(\mathbb{R}^\times)\widehat{\otimes}\mathcal{S}'(\Lambda)$ is additionally invariant under $d\pi_{\min}(S_2-3T_2)$ and $\lambda d\pi_{\min}(T_1)-a\,d\pi_{\min}(T_3)$, the function $f_1(R,c)$ satisfies*

$$\left[2R\partial_R\partial_C+\frac{1}{3}c\partial_C^2+\frac{4}{3}\partial_C+\frac{3c}{8R^2}(c^2+2R)(c^2-2R)\right]f_1=0$$

and

$$\left[\frac{2R}{3}\partial_C^2-4R^2\partial_R^2-6R\partial_R+\frac{2}{3}c\partial_C-\frac{2}{9}+\frac{8R^3-24c^2R^2-6c^4R+4c^6}{8R^2}\right]f_1=0.$$

Proof. This is an elementary computation. □

Inspired by the previous cases we make the Ansatz $f_1(R,c)=R^{-\frac{1}{6}}f_2(S)$ with

$$S=\frac{(2R+c^2)^3}{8R^2}.$$

Lemma 5.4.5. *A function of the form $f_1(R,c)=R^{-\frac{1}{6}}f_2(S)$ satisfies the differential equations in Lemma 5.4.4 if and only if f_2 satisfies*

$$Tf_2''(S)+\frac{2}{3}f_2'(S)-\frac{1}{4}f_2(S)=0.$$

Proof. Another elementary computation. □

From Appendix A we know that $f_2(S)=\overline{K}_{-\frac{1}{3}}(S)$ is the unique tempered solution to the above differential equation. This leads to the function $f_0(\lambda,a,x)$. We now apply $S_3\pm i\sqrt{2}S_1$ to f_0.

Lemma 5.4.6. *Let*

$$f(\lambda,a,x)=(\lambda^2+2a^2)^{-\frac{1}{6}}\exp\left(-\frac{iac^3}{2\lambda(\lambda^2+2a^2)}\right)f_2(S)$$

be invariant under T_1,T_2,T_3 and S_2. Then

$$d\pi_{\min}(S_3\pm i\sqrt{2}S_1)f=-8(\lambda\mp i\sqrt{2}a)(\lambda^2+2a^2)^{-\frac{7}{6}}\exp\left(-\frac{iac^3}{2\lambda(\lambda^2+2a^2)}\right)$$
$$\times\left(\frac{c}{2}f_2(S)\pm\frac{\sqrt{2}(2R+c^2)^2}{4R}f_2'(S)\right).$$

Proof. Since f is invariant under T_1 and T_3 we find that

$$d\pi_{\min}(S_1)f=8\,d\pi_{\min}(T)f\qquad\text{and}\qquad d\pi_{\min}(S_3)f=-8\,d\pi_{\min}(C-\overline{D}).$$

The rest is a simple computation. □

Since $\overline{K}'_{-\frac{1}{3}}(x)=-\frac{1}{2}\overline{K}_{\frac{2}{3}}(x)$ by (A.2.1) this leads to the functions f_1 and f_{-1}.

Proposition 5.4.7. *The functions f_{-1}, f_0 and f_1 are $\mathfrak{k}_1$-invariant and transform in the following way under the action of $\mathfrak{k}_2$:*

$$d\pi_{\min}(S_2)f_k = 4\sqrt{2}ikf_k \qquad d\pi_{\min}(S_3 \pm i\sqrt{2}S_1)f_0 = -4f_{\pm 1},$$
$$d\pi_{\min}(S_3 \pm i\sqrt{2}S_1)f_{\pm 1} = 0, \qquad d\pi_{\min}(S_3 \mp i\sqrt{2}S_1)f_{\pm 1} = 16f_0.$$

Proof. The first formula follows as in the proof of Lemma 5.4.3. The second formula is essentially Lemma 5.4.6. The third and the fourth formula are easily computed using

$$S_3 \pm i\sqrt{2}S_1 \equiv 8(-C + \overline{D} \pm i\sqrt{2}T_0) \mod \mathfrak{k}_1$$

and (A.2.1) and (A.2.2). □

Proof of Theorem 5.4.1. Note that the elements

$$\widetilde{S}_1 = \frac{1}{2}S_1, \qquad \widetilde{S}_2 = \frac{1}{2\sqrt{2}}S_2 \qquad \text{and} \qquad \widetilde{S}_3 = -\frac{1}{2\sqrt{2}}S_3$$

form an $\mathfrak{su}(2)$-triple, then the statement follows from Proposition 5.4.7. □

Corollary 5.4.8. *The elements $w_0^2, w_1^2, w_2^2 \in K$ act on W in the following way:*

$$\pi_{\min}(w_0^2)f_k = (-1)^{k-1}f_{-k}, \qquad \pi_{\min}(w_1^2)f_k = (-1)^k f_k, \qquad \pi_{\min}(w_2^2)f_k = -f_{-k}.$$

Proof. Since $\mathfrak{k}_1$ acts trivially on W, we find

$$\pi_{\min}(w_0^2)f_k = \pi_{\min}(\exp(\tfrac{\pi}{2}\widetilde{S}_1))f_k,$$
$$\pi_{\min}(w_1^2)f_k = \pi_{\min}(\exp(\tfrac{\pi}{2}\widetilde{S}_2))f_k,$$
$$\pi_{\min}(w_2^2)f_k = \pi_{\min}(\exp(-\tfrac{\pi}{2}\widetilde{S}_3))f_k.$$

The rest is an SU(2) computation. □

5.5 The case $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$

Let $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$, then $\mathcal{J} \simeq \mathbb{R}^{n-3}$ does not carry the structure of a Jordan algebra. In this case, it is not necessary to use the decomposition $\mathfrak{g}_{-1} = \mathbb{R}A \oplus \mathcal{J} \oplus \mathcal{J}^* \oplus \mathbb{R}B$ at all. We choose any Cartan involution θ on $\mathfrak{g}$ associated to a map $J : V \to V$ as in Lemma 1.8.1 and let $\mathfrak{k} = \mathfrak{g}^\theta \simeq \mathfrak{so}(n)$ denote the maximal compact subalgebra.

In contrast to the previous cases, we have a continuous family $(d\pi_{\min,r}, \mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda))$ ($r \in \mathbb{C}$) of representations, and for each $r \in \mathbb{C}$ we find two non-equivalent $(\mathfrak{g},K)$-submodules of $d\pi_{\min,r}$ which integrate to non-equivalent irreducible unitary representations of $L^2(\mathbb{R}^\times \times \Lambda)$. In this section we determine their lowest K-types.

Theorem 5.5.1. *Let $r \in \mathbb{C}$.*

(1) The space $W_{0,r} = \mathbb{C}f_{0,r}$ with

$$f_{0,r}(\lambda,x) = \overline{K}_{\frac{n-2r-2}{4}}(\lambda^2 + 4|x|^2)$$

is a $\mathfrak{k}$-subrepresentation of $(d\pi_{\min,r}, \mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda))$ isomorphic to the trivial representation.

(2) The space $W_{1,r} = \mathbb{C}g_{0,r} \oplus \mathbb{C}g_{1,r} \oplus \{g_{w,r} : w \in \Lambda^*\}$ *with*

$$g_{0,r}(\lambda, x) = \lambda \overline{K}_{\frac{n-2r}{4}}(\lambda^2 + 4|x|^2)$$
$$g_{1,r}(\lambda, x) = \overline{K}_{\frac{n-2r-4}{4}}(\lambda^2 + 4|x|^2)$$
$$g_{w,r}(\lambda, x) = \omega(x, w)\overline{K}_{\frac{n-2r}{4}}(\lambda^2 + 4|x|^2) \qquad (w \in \Lambda^*)$$

is a $\mathfrak{k}$*-subrepresentation of* $(d\pi_{\min}, \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda))$ *isomorphic to the standard representation of* $\mathfrak{k} \simeq \mathfrak{so}(n)$ *on* $\mathbb{C}^n$*.*

Proof. We first find the spherical vector $f_{0,r}$. If $f_{0,r}$ is invariant under $\mathfrak{k} \cap \mathfrak{g}_0 = \mathfrak{so}(n-2)$, it has to be of the form

$$f_{0,r}(\lambda, x) = f_1(\lambda, I_2) \qquad \text{with} \qquad I_2 = |x|^2 = \frac{1}{4}\omega(Jx, x).$$

For such $f_{0,r}$ the equation $d\pi_{\min,r}(v + \theta v)f_{0,r} = 0$ takes the form

$$\omega(x, Jv)\left(\frac{1}{2}\lambda\partial_2 - \partial_\lambda\right) f_1 = 0,$$

so that $f_1(\lambda, I_2) = f_2(\lambda^2 + 4I_2)$. Finally, the equation $d\pi_{\min,r}(w + \theta w) = 0$ reduces to

$$4zf_2'' + (n - 2r + 2)f_2' - f_2 = 0$$

which has by Appendix A the unique tempered solution $f_2(z) = \overline{K}_{\frac{n-2r-2}{4}}(z)$.
Now let us show that $g_{0,r}$, $g_{1,r}$ and the $g_{w,r}$'s span a finite-dimensional $\mathfrak{k}$-representation. Similar to (1) one verifies for $T \in \mathfrak{k} \cap \mathfrak{g}_0 \simeq \mathfrak{so}(p-2)$:

$$d\pi_{\min}(T)g_{w,r} = g_{Tw,r}, \qquad d\pi_{\min}(T)g_{0,r} = d\pi_{\min}(T)g_{1,r} = 0,$$

for $v \in \Lambda$:

$$d\pi_{\min}(v + \theta v)g_{0,r} = -g_{Jv,r}, \quad d\pi_{\min}(v + \theta v)g_{1,r} = 0, \quad d\pi_{\min}(v + \theta v)g_{w,r} = -\omega(v, w)g_{0,r},$$

and for $w \in \Lambda^*$:

$$d\pi_{\min}(w + \theta w)g_{0,r} = 0, \quad d\pi_{\min}(w + \theta w)g_{1,r} = -ig_{w,r}, \quad d\pi_{\min}(w + \theta w)g_{w',r} = i\omega(w', Jw)g_{1,r},$$

where we have used (A.2.1) and (A.2.2). □

5.6 The case $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$

For $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$ there is an additional $(\mathfrak{g}, K)$-module contained in $(d\pi_{\min}, \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda))$ which gives rise to a genuine irreducible unitary representation of the double cover $\widetilde{SL}(3, \mathbb{R})$ of $\mathrm{SL}(3, \mathbb{R})$. For this we choose $A, B \in V$ such that $JA = -B$ and $JB = A$ and use coordinates $x = aA \in \Lambda$.

Theorem 5.6.1. *The space* $W_{\frac{1}{2}} = \mathbb{C}h_{-\frac{1}{2}} \oplus \mathbb{C}h_{\frac{1}{2}}$ *with*

$$h_{\frac{1}{2}}(\lambda, a) = (|\lambda| - i\sqrt{2}\,\mathrm{sgn}(\lambda)a)^{\frac{1}{2}}\overline{K}_{\frac{1}{2}}(\lambda^2 + 2a^2),$$
$$h_{-\frac{1}{2}}(\lambda, a) = \mathrm{sgn}(\lambda)(|\lambda| + i\sqrt{2}\,\mathrm{sgn}(\lambda)a)^{\frac{1}{2}}\overline{K}_{\frac{1}{2}}(\lambda^2 + 2a^2)$$

is a $\mathfrak{k}$*-subrepresentation of* $(d\pi_{\min}, \mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda))$ *isomorphic to the representation* $S^1(\mathbb{C}^2)$ *of* $\mathfrak{k} = \mathfrak{so}(3) \simeq \mathfrak{su}(2)$.

Proof. The elements $U_1 = \sqrt{2}(A - \overline{B})$, $U_2 = \sqrt{2}(B + \overline{A})$ and $U_3 = -2(E - F)$ form an $\mathfrak{su}(2)$-triple, i.e. $[U_1, U_2] = 2U_3$, $[U_2, U_3] = 2U_1$ and $[U_3, U_1] = 2U_2$. With respect to the maximal torus $\mathbb{R}U_1$ in $\mathfrak{k}$ the vectors $U_2 \mp iU_3$ are root vectors:

$$[U_1, U_2 \mp iU_3] = \pm 2i(U_2 \mp iU_3).$$

The highest weight vector of a $\mathfrak{k}$-type isomorphic to $S^1(\mathbb{C}^2)$ solves the weight equation

$$d\pi_{\min}(U_1)h = \sqrt{2}(2a\partial_\lambda - \lambda\partial_A)h = ih.$$

Using the method of characteristics we find that

$$h(\lambda, a) = (|\lambda| - i\sqrt{2}\,\mathrm{sgn}(\lambda)a)^{\frac{1}{2}} \cdot u(\lambda^2 + 2a^2)$$

is a solution. The highest weight equation

$$d\pi_{\min}(U_2 - iU_3)h = 0$$

then gives

$$4zu''(z) + 6u'(z) - u(z) = 0.$$

By Appendix A the unique tempered solution is $u(z) = \overline{K}_{\frac{1}{2}}(z)$ which leads to the highest weight vector $h_{\frac{1}{2}}$. A straightforward computation using $\overline{K}_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2}}x^{-\frac{1}{2}}e^{-\sqrt{x}}$ (see Appendix A) shows that

$$d\pi_{\min}(U_2 + iU_3)h_{\frac{1}{2}} = -2h_{-\frac{1}{2}} \qquad \text{and} \qquad d\pi_{\min}(U_2 + iU_3)h_{-\frac{1}{2}} = 0. \qquad \square$$

Remark 5.6.2. We observe that these functions together with the ones from Theorem 5.5.1 in the case $n = 3$ agree (up to a change of coordinates) with the ones found by Torasso [53, Proposition 14, 15 & 16].

5.7 The case $\mathfrak{g} = \mathfrak{so}(p,q)$

The construction of a Cartan involution in Section 5.1 does not apply in the case $\mathfrak{g} \simeq \mathfrak{so}(p,q)$ since here $\mathcal{J}$ is not simple but the direct sum of a rank one Jordan algebra $\mathcal{J}_0 \simeq \mathbb{R}$ and a rank two Jordan algebra $\overline{\mathcal{J}} \simeq \mathbb{R}^{p-3,q-3}$. We therefore give a separate construction.

According to the decomposition $\mathcal{J} = \mathcal{J}_0 \oplus \overline{\mathcal{J}}$, the norm function decomposes into

$$n(x) = -\frac{1}{2}\omega(x_0, Q)\omega(\mu(\overline{x})P, B)$$

where $-\omega(\mu(\overline{x})P, B)$ is a quadratic form on $\overline{\mathcal{J}}$ of signature $(p-3, q-3)$, the norm function of the quadratic Jordan algebra $\overline{\mathcal{J}} \simeq \mathbb{R}^{p-3,q-3}$. The following result can be proven using the explicit decompositions in Appendix B.2:

Proposition 5.7.1. *Let $\vartheta : \overline{\mathcal{J}} \to \overline{\mathcal{J}}$ be a Jordan algebra automorphism such that the symmetric bilinear form $(v_1, v_2) \mapsto -\omega(B_\mu(v_1, \vartheta v_2)P, B)$ is positive definite. Then the map $J : V \to V$ given by*

$$\begin{aligned} JA &= -B, & JP &= -Q, & Jv &= -\sqrt{2}B_\mu(P,B)\vartheta v & (v &\in \overline{\mathcal{J}}), \\ JB &= A, & JQ &= P, & Jw &= \sqrt{2}\vartheta B_\mu(A,Q)w & (w &\in \overline{\mathcal{J}}^*), \end{aligned}$$

satisfies the conditions of Lemma 1.8.1.

We fix ϑ as in the proposition and let $\mathcal{J}_1$ denote the $+1$ eigenspace and $\mathcal{J}_2$ the -1 eigenspace of ϑ in $\overline{\mathcal{J}}$. Then $\mathcal{J} = \mathcal{J}_0 \oplus \mathcal{J}_1 \oplus \mathcal{J}_2$, and using the symplectic form we obtain a dual decomposition $\mathcal{J}^* = \mathcal{J}_0^* \oplus \mathcal{J}_1^* \oplus \mathcal{J}_2^*$.

Lemma 5.7.2. *(1) $B_\mu(A,Q)$ and $B_\mu(P,B)$ map $\mathcal{J}_i$ and $\mathcal{J}_i^*$ to each other $(i = 1, 2)$.*

(2) $B_\mu(\mathcal{J}_1, \mathcal{J}_2) = 0$ and $\mu(\mathcal{J}_1), \mu(\mathcal{J}_2) \in \mathbb{R}B_\mu(A,Q)$.

Denote by θ the corresponding Cartan involution of $\mathfrak{g}$ and by $\mathfrak{k} = \mathfrak{g}^\theta$ the corresponding maximal compact subalgebra of $\mathfrak{g}$. Then

$$T_0 = B_\mu(A,Q) - B_\mu(P,B) \in \mathfrak{k}.$$

Proposition 5.7.3. *The Lie algebra $\mathfrak{k}$ decomposes into the sum of two ideals $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ with $\mathfrak{k}_1 \simeq \mathfrak{so}(p)$ and $\mathfrak{k}_2 \simeq \mathfrak{so}(q)$ given by*

$$\begin{aligned} \mathfrak{k}_1 = \mathbb{R}(2T_0 + \sqrt{2}(E-F)) \oplus \mathbb{R}(A - \sqrt{2}Q + \theta(A - \sqrt{2}Q)) + \mathbb{R}(B + \sqrt{2}P + \theta(B + \sqrt{2}P)) \\ \oplus \{B_\mu(v,B) + B_\mu(A,Jv) : v \in \mathcal{J}_1\} \oplus \{x + \theta x : x \in \mathcal{J}_1 \oplus \mathcal{J}_1^*\} \oplus \mathfrak{so}(p-3) \end{aligned}$$

and

$$\begin{aligned} \mathfrak{k}_2 = \mathbb{R}(2T_0 - \sqrt{2}(E-F)) \oplus \mathbb{R}(A + \sqrt{2}Q + \theta(A + \sqrt{2}Q)) + \mathbb{R}(B - \sqrt{2}P + \theta(B - \sqrt{2}P)) \\ \oplus \{B_\mu(v,B) + B_\mu(A,Jv) : v \in \mathcal{J}_2\} \oplus \{x + \theta x : x \in \mathcal{J}_2 \oplus \mathcal{J}_2^*\} \oplus \mathfrak{so}(q-3), \end{aligned}$$

where $\mathfrak{so}(p-3)$ resp. $\mathfrak{so}(q-3)$ denotes the ideal of $\mathfrak{k} \cap \mathfrak{g}_{(0,0)} \simeq \mathfrak{so}(p-3) \oplus \mathfrak{so}(q-3)$ which acts trivially on $\mathcal{J}_2 \simeq \mathbb{R}^{q-3}$ resp. $\mathcal{J}_1 \simeq \mathbb{R}^{p-3}$.

Proof. Note that

$$T_0A = -P, \qquad T_0P = \frac{1}{2}A, \qquad T_0Q = \frac{1}{2}B, \qquad T_0B = -Q.$$

The rest is along the same lines as the proof of Proposition 5.1.6. □

To state explicit formulas for K-finite vectors we first need the following result:

Lemma 5.7.4. *For $j \in \mathbb{N}$ and $k \in \mathbb{Z}$ there exists a unique family of polynomials $(p_{j,k,m})_{m=0,\ldots,j} \subseteq \mathbb{C}[S,T]$ satisfying*

(1) $(2\partial_S\partial_T + k\sqrt{2})p_{j,k,m} - T\partial_S p_{j,k,m-1} - 2S\partial_T p_{j,k,m+1} = 0$,

(2) $\partial_T^2 p_{j,k,m+1} + (m - T\partial_T)p_{j,k,m} = 0,$

(3) $p_{j,k,0}(S,T) = S^j.$

For the family of polynomials $(p_{j,k,m})_{j,k,m}$ the following identities hold:

(a) $\partial_S p_{j,k,m} \mp \sqrt{2}\partial_T p_{j,k,m+1} = (j \mp k)p_{j-1,k\pm1,m},$

(b) $\pm\frac{\sqrt{2}}{2}Tp_{j,k,m-1} + Sp_{j,k,m} \mp \sqrt{2}\partial_T p_{j,k,m} = p_{j+1,k\pm1,m},$

(c) $(S\partial_S - T\partial_T)p_{j,k,m} = (j - 2m)p_{j,k,m},$

(d) $-\frac{T^2}{2}p_{j,k,m-2} + (2m-1)p_{j,k,m-1} + S^2 p_{j,k,m} = p_{j+2,k,m},$

(e) $p_{j,k,m}(-S,T) = (-1)^j p_{j,-k,m}(S,T).$

Proof. We first show uniqueness. For this we write $p_m = p_{j,k,m}$ for short. Every polynomial p_m can be written as the sum of homogeneous polynomials

$$p_m = \sum_{\alpha\geq 0} p_m^\alpha$$

with p_m^α homogeneous of degree α. Let α be maximal with $p_m^\alpha \neq 0$ for some m. We claim that $\alpha = j$. By (3) we have $\alpha \geq j$, so we assume $\alpha > j$. Then (2) would imply $p_m^\alpha = c_m^\alpha S^{\alpha-m}T^m$ and (1) would imply

$$k\sqrt{2}c_m^\alpha - (\alpha - m + 1)c_{m-1}^\alpha - 2(m+1)c_{m+1}^\alpha = 0.$$

From (3) we know that $c_0^\alpha = 0$ and recursively we find $c_m^\alpha = 0$ for all m which is a contradiction, so $\alpha = j$ is maximal with the property that $p_m^\alpha \neq 0$ for some m. The previous argument also shows that $p_m^j = c_m^j S^{j-m}T^m$ with c_m^j uniquely determined by $c_0^j = 1$. For the lower order terms we observe that $p_0^\alpha = 0$ for $\alpha < j$. For fixed $\alpha < j$ equation (1) determines $\partial_T p_{m+1}^\alpha$ from p_{m-1}^α, p_m^α and $p_m^{\alpha+2}$, so p_{m+1}^α is unique modulo polynomials in S independent of T. This disambiguity is removed by (2) for $m > 0$.
Now let us prove existence by induction on j. For $j = 0$ we also have $k = m = 0$ and $p_{0,0,0} = 1$ by (3) which satisfies also (1) and (2). Next we note that the left hand side of (b) satisfies (1), (2) and (3) for j replaced by $j+1$ and k replaced by $k \pm 1$. Therefore, (b) can be used to recursively define the family $(p_{j+1,k,m})_{k,m}$ using $(p_{j,k,m})_{k,m}$ which establishes the existence part of the proof. Finally, the identities (a), (b), (c), (d) and (e) are proven by showing that the left hand side satisfies (1) and (2) for certain values of j and k (for (c) the term $2mp_{j,k,m}$ has to be moved to the left hand side first) and then using the previously established uniqueness result. □

Remark 5.7.5. It is easy to see that

$$p_{j,k,m}(S,T) = \text{const} \times S^{j-m}T^m + \text{const} \times S^{j-m-1}T^{m-1} + \cdots .$$

For $k = \pm j$ it is possible to find the coefficient of $S^{j-m}T^m$:

$$p_{j,\pm j,m}(S,T) = (\pm 1)^m 2^{-\frac{m}{2}} \binom{j}{m} S^{j-m}T^m + \text{lower order terms}.$$

However, we were not able to find a closed formula in general.

We assume from now on that $p \geq q \geq 3$. The lowest K-type turns out to be isomorphic to $\mathbb{C} \boxtimes \mathcal{H}^{\frac{p-q}{2}}(\mathbb{R}^q)$ as a representation of $\mathfrak{k} \simeq \mathfrak{so}(p) \oplus \mathfrak{so}(q)$, where $\mathcal{H}^\alpha(\mathbb{R}^n)$ denotes the space of homogeneous polynomials on $\mathbb{R}^n$ of degree α which are harmonic. Since only the subalgebra $\mathfrak{k}_2 \cap \mathfrak{g}_{(0,0)} \simeq \mathfrak{so}(q-3)$ acts geometrically in the representation $d\pi_{\min}$, it is helpful to use the following multiplicity-free branching rule:

$$\mathcal{H}^{\frac{p-q}{2}}(\mathbb{R}^q)|_{\mathfrak{so}(2)\oplus\mathfrak{so}(q-2)} \simeq \bigoplus_{j=0}^{\frac{p-q}{2}} \bigoplus_{\substack{k=-j \\ k\equiv j \mod 2}}^{j} \mathbb{C}_k \boxtimes \mathcal{H}^{\frac{p-q}{2}-j}(\mathbb{R}^{q-2}),$$

where $\mathbb{C}_k$ denotes the obvious character of $\mathfrak{so}(2)$, and further decompose

$$\mathcal{H}^{\frac{p-q}{2}-j}(\mathbb{R}^{q-2})|_{\mathfrak{so}(q-3)} \simeq \bigoplus_{\ell=0}^{\frac{p-q}{2}-j} \mathcal{H}^\ell(\mathbb{R}^{q-3}).$$

Together we find that

$$\mathcal{H}^{\frac{p-q}{2}}(\mathbb{R}^q)|_{\mathfrak{so}(2)\oplus\mathfrak{so}(q-3)} \simeq \bigoplus_{j=0}^{\frac{p-q}{2}} \left(\bigoplus_{\substack{k=-j \\ k\equiv j \mod 2}}^{j} \mathbb{C}_k \right) \boxtimes \left(\bigoplus_{\ell=0}^{\frac{p-q}{2}-j} \mathcal{H}^\ell(\mathbb{R}^{q-3}) \right).$$

For a distribution $f \in \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda)$ we write $f \otimes \mathcal{H}^\ell(\mathbb{R}^{q-3})$ for the space of distributions $f \otimes \varphi$, $\varphi \in \mathcal{H}^\ell(\mathbb{R}^{q-3})$, given by

$$(f \otimes \varphi)(\lambda, a, x) = f(\lambda, a, x)\varphi(x_2).$$

Here we define spherical harmonics with respect to the positive definite quadratic form $x_2 \mapsto \omega(\mu(x_2)P, B)$ on $\mathcal{J}_2 \simeq \mathbb{R}^{q-3}$.

We further recall the renormalized K-Bessel function $\overline{K}_\alpha(z)$ from Appendix A.

Theorem 5.7.6. *Let $\mathfrak{g} = \mathfrak{so}(p,q)$ with $p \geq q \geq 3$ and $p+q$ even. Then*

$$W = \bigoplus_{j=0}^{\frac{p-q}{2}} \bigoplus_{\substack{k=-j \\ k\equiv j \mod 2}}^{j} \bigoplus_{\ell=0}^{\frac{p-q}{2}-j} f_{j,k,\ell} \otimes \mathcal{H}^\ell(\mathbb{R}^{q-3})$$

with

$$f_{j,k,\ell}(\lambda, a, x) = (\lambda - i\sqrt{2}a)^k (\lambda^2 + 2a^2)^{-\frac{k+1}{2}} \exp\left(-\frac{2ian(x)}{\lambda(\lambda^2 + a^2)} \right) h_{j,k,\ell}(S, T, U)$$

and

$$h_{j,k,\ell}(S, T, U) = \sum_{m=0}^{j} p_{j,k,m}(S, T)(1 + S^2)^{\frac{q+2\ell+2m-4}{2}} \overline{K}_{\frac{q+2\ell+2m-4}{2}}(U)$$

with

$$S = \frac{I_1}{\sqrt{\lambda^2 + 2a^2}} \qquad \text{and} \qquad T = \frac{I_2^p + I_2^q - \sqrt{2}(\lambda^2 + 2a^2)}{\sqrt{\lambda^2 + 2a^2}}$$

and

$$I_1 = \omega(x_0, Q), \qquad I_2^p = -2\sqrt{2}|x_1|^2, \qquad I_2^q = 2\sqrt{2}|x_2|^2$$

is a $\mathfrak{k}$*-subrepresentation of* $(d\pi_{\min}, \mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda))$ *isomorphic to the representation* $\mathbb{C}\boxtimes\mathcal{H}^{\frac{p-q}{2}}(\mathbb{R}^q)$ *of* $\mathfrak{k} \simeq \mathfrak{so}(p)\oplus\mathfrak{so}(q)$.

We begin with the action of $\mathfrak{k}\cap\mathfrak{g}_{(0,0)} \simeq \mathfrak{so}(p-3)\oplus\mathfrak{so}(q-3)$.

Lemma 5.7.7. *If* $f \in \mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda)$ *generates under the action of* $\mathfrak{k}\cap\mathfrak{g}_{(0,0)} \simeq \mathfrak{so}(p-3)\oplus\mathfrak{so}(q-3)$ *a subrepresentation isomorphic to* $\mathbb{C}\boxtimes\mathcal{H}^\ell(\mathbb{R}^{q-3})$*, it has to be a linear combination of distributions of the form*

$$f(\lambda,a,x) = f_1(\lambda,a,I_1,I_2^p,I_2^q)\varphi(x_2),$$

where

$$I_1 = \omega(x,Q), \qquad I_2^p = \omega(\mu(x_1)P,B) = -2\sqrt{2}|x_1|^2, \qquad I_2^q = \omega(\mu(x_2)P,B) = 2\sqrt{2}|x_2|^2$$

and $\varphi \in \mathcal{H}^\ell(\mathbb{R}^{q-3})$.

Proof. The subalgebra $\mathfrak{k}\cap\mathfrak{g}_{(0,0)} \simeq \mathfrak{so}(p-3)\oplus\mathfrak{so}(q-3)$ acts on $\mathcal{J}_1\oplus\mathcal{J}_2 \simeq \mathbb{R}^{p-3}\oplus\mathbb{R}^{q-3}$ by the direct sum of the standard representations of $\mathfrak{so}(p-3)$ and $\mathfrak{so}(q-3)$. An $\mathfrak{so}(p-3)$-invariant distribution on $\mathcal{J}_1 \simeq \mathbb{R}^{p-3}$ only depends on $I_2^p = -2\sqrt{2}|x_1|^2$, and a distribution on $\mathcal{J}_2 \simeq \mathbb{R}^{q-3}$ that belongs to the isotypic component of $\mathcal{H}^\ell(\mathbb{R}^{q-3})$ has to be a linear combination of products of $\varphi \in \mathcal{H}^\ell(\mathbb{R}^{q-3})$ and distributions only depending on $I_2^q = 2\sqrt{2}|x_2|^2$. □

We note the following identities for derivatives of the invariants I_1, I_2^p, I_2^q in the directions P, $v\in\mathcal{J}_1$ and $w\in\mathcal{J}_2$:

$$\begin{aligned}
\partial_P I_1 &= 1, & \partial_P I_2^p &= 0, & \partial_P I_2^q &= 0,\\
\partial_v I_1 &= 0, & \partial_v I_2^p &= \sqrt{2}\omega(x,Jv), & \partial_v I_2^q &= 0,\\
\partial_w I_1 &= 0, & \partial_w I_2^p &= 0, & \partial_w I_2^q &= -\sqrt{2}\omega(x,Jv).
\end{aligned}$$

Further, we have

$$I_3 = \omega(\Psi(x),B) = 2n(x) = -I_1(I_2^p+I_2^q).$$

Lemma 5.7.8. *If* f *is additionally an eigenfunction of* $d\pi_{\min}(A-\overline{B})$ *to the eigenvalue* $ik\sqrt{2}$*, it is of the form*

$$f(\lambda,a,x) = (\lambda-i\sqrt{2}a)^k\exp\left(-\frac{iaI_3}{\lambda R}\right)f_2(R,I_1,I_2^p,I_2^q)\varphi(x_2),$$

where

$$R = \lambda^2+2a^2.$$

Proof. The method of characteristics applied to the first order equation

$$d\pi_{\min}(A-\overline{B})f = \left(-\lambda\partial_A + 2a\partial_\lambda - \frac{2in(x)}{\lambda^2}\right)f = ik\sqrt{2}f$$

shows the claim. □

Lemma 5.7.9. *If f is additionally invariant under $\{\lambda d\pi_{\min}(v+\theta v)+2a(B_\mu(v,B)+B_\mu(A,Jv)) : v\in\mathcal{J}_1\}\subseteq\mathfrak{k}_1$, it has to be of the form*

$$f(\lambda,a,x)=(\lambda-i\sqrt{2}a)^kR^{-\frac{k+1}{2}}\exp\left(-\frac{iaI_3}{\lambda R}\right)f_3\left(S,T,I_2^q\right)\varphi(x_2),$$

where

$$S=\frac{I_1}{R^{\frac{1}{2}}}\qquad\text{and}\qquad T=\frac{I_2^p+I_2^q-\sqrt{2}R}{R^{\frac{1}{2}}}.$$

Proof. Applying $\lambda d\pi_{\min}(v+\theta v)+2a(B_\mu(v,B)+B_\mu(A,Jv))$ to $f(\lambda,a,x)$ as in Lemma 5.7.8 leads to the differential equation

$$2R\partial_R+I_1\partial_1+(I_2^p+I_2^q+\sqrt{2}R)\partial_{2p})f_2=-(k+1)f_2$$

which can be solved using the method of characteristics. □

Lemma 5.7.10. *If f is additionally invariant under $\{d\pi_{\min}(Jv-\overline{v}) : v\in\mathcal{J}_1\}\subseteq\mathfrak{k}_1$, the function $f_3(S,T,I_2^q)$ solves the following two partial differential equations:*

$$\Big(-2\partial_T^2-\sqrt{2}I_2^q\partial_{2q}^2-\tfrac{\sqrt{2}}{2}(2\ell+q-3)\partial_{2q}+(1+S^2)\Big)f_3=0,\tag{5.7.1}$$

$$\Big(2\partial_S\partial_T-ST+k\sqrt{2}\Big)f_3=0.\tag{5.7.2}$$

Proof. A lengthy computation involving Lemma 4.7.1 and Lemma 4.7.2 shows that

$$\begin{aligned}id\pi_{\min}(Jv-\overline{v})f&=\omega(x,Jv)(\lambda-i\sqrt{2}a)^kR^{-\frac{k+1}{2}}\exp\left(-\frac{iaI_3}{\lambda R}\right)\varphi(x_2)\\&\quad\times\Bigg[\Big(-2\partial_T^2-\sqrt{2}I_2^q\partial_{2q}^2-\tfrac{\sqrt{2}}{2}(2\ell+q-3)\partial_{2q}+(1+S^2)\Big)f_3\\&\qquad+R^{-\frac{1}{2}}\Big(-\sqrt{2}S\partial_S\partial_T+\tfrac{\sqrt{2}}{2}S^2T-kS\Big)f_3\Bigg].\end{aligned}$$

Since $R^{-\frac{1}{2}}$ is independent of S, T and I_2^q this implies two equations. □

We remark at this point that

$$\frac{1}{2}T^2+2\sqrt{2}I_2^q=\frac{1}{2R}(I_2^p+I_2^q)^2-\sqrt{2}(I_2^p-I_2^q)+R>0.$$

Together with a deeper analysis of the equations in Lemma 5.7.10 this leads us to introducing a new variable

$$U=(1+S^2)\Big(\frac{1}{2}T^2+2\sqrt{2}I_2^q\Big).$$

and making the Ansatz

$$f_3(S,T,I_2^q)=f_4(U)g_4(S).$$

Equation (5.7.1) applied to this gives

$$Uf_4''(U)+\tfrac{q+2\ell-2}{2}f_4'(U)-\tfrac{1}{4}f_4(U)=0$$

which has the solution $f_4(U) = \overline{K}_{\frac{q+2\ell-4}{2}}(U)$. Plugging this into (5.7.2) gives

$$2Tf_4'(U)\Big((1+S^2)g_4'(S) - (q+2\ell-4)Sg_4(S)\Big) + k\sqrt{2}f_4(U)g_4(S) = 0.$$

Since by (A.2.1) the functions $f_4(U)$ and $f_4'(U)$ are linearly independent, this equation can only have a non-trivial solution for $k = 0$. In this case

$$g_4(S) = (1+S^2)^{\frac{q+2\ell-4}{2}}$$

solves the equation and we obtain the functions $f_{0,0,\ell}(\lambda,a,x)$, $\ell \geq 0$. To investigate whether these functions are K-finite we apply $\mathfrak{k}_2 \simeq \mathfrak{so}(q)$ to $f_{0,0,\ell}$. For this we decompose $\mathfrak{k}_2$ according to Proposition 5.7.3 and compute the action of each part on a distribution $f(\lambda,a,x)$ of the form given in Lemma 5.7.9.

Lemma 5.7.11. *For a distribution $f \in \mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda)$ of the form*

$$f(\lambda,a,x) = (\lambda - i\sqrt{2}a)^k R^{-\frac{k+1}{2}} \exp\left(-\frac{iaI_3}{\lambda R}\right) f_3\left(S,T,I_2^q\right)\varphi(x_2)$$

with $f_3(S,T,I_2^q)$ satisfying (5.7.1) and (5.7.2) we have for $v \in \mathcal{J}_2$:

$$\begin{aligned} id\pi_{\min}(Jv - \overline{v})f &= (\lambda - i\sqrt{2}a)^k R^{-\frac{k+1}{2}} \exp\left(-\frac{iaI_3}{\lambda R}\right) \\ &\times \left[\omega(x,Jv)\varphi\Big(4\partial_T^2 + \sqrt{2}S\partial_S\partial_{2q} - \sqrt{2}T\partial_T\partial_{2q} + \frac{\sqrt{2}}{2}(2\ell+q-p-2)\partial_{2q} - 2S^2\Big)f_3\right. \\ &\qquad\left. + \partial_v\varphi\Big(-S\partial_S + T\partial_T + 2I_2^q\partial_{2q} + \frac{p+q+2\ell-8}{2}\Big)f_3\right], \end{aligned}$$

$$\begin{aligned} d\pi_{\min}(2i(B_\mu(A,Jv) + B_\mu(v,B)) \mp \sqrt{2}(v+\overline{Jv}))f &= (\lambda - i\sqrt{2}a)^{k\pm1} R^{-\frac{(k\pm1)+1}{2}} \exp\left(-\frac{iaI_3}{\lambda R}\right) \\ &\times \left[\omega(x,Jv)\varphi\Big(-2\partial_S\partial_{2q} \mp 4\partial_T \pm \sqrt{2}T\partial_{2q} + 2\sqrt{2}S\Big)f_3\right. \\ &\qquad\left. + \partial_v\varphi\Big(\sqrt{2}\partial_S \mp T\Big)f_3\right], \end{aligned}$$

and

$$\begin{aligned} d\pi_{\min}(-P + \overline{Q} \mp i\sqrt{2}T_0)f &= (\lambda - i\sqrt{2}a)^{k\pm1} R^{-\frac{(k\pm1)+1}{2}} \exp\left(-\frac{iaI_3}{\lambda R}\right)\varphi(x_2) \\ &\times \Big(\pm\sqrt{2}T\partial_T^2 \pm 2\sqrt{2}I_2^q\partial_T\partial_{2q} + (1+S^2)\partial_S + (-ST \pm \sqrt{2}\tfrac{p+q+2\ell-6}{2})\partial_T \\ &\qquad - 2SI_2^q\partial_{2q} \mp \tfrac{\sqrt{2}}{2}(1+S^2)T - \tfrac{p+q+2\ell\mp2k-8}{2}S\Big)f_3. \end{aligned}$$

Applying these operators to $f_{0,0,\ell}$ suggests that the $\mathfrak{k}$-representation generated by $f_{0,0,0}$ consists of functions $f(\lambda,a,x)$ as in Lemma 5.7.9 with

$$f_3(S,T,I_2^q)=\sum_m p_m(S,T)(1+S^2)^{\frac{q+2\ell+2m-4}{2}}\overline{K}_{\frac{q+2\ell+2m-4}{2}}(U) \tag{5.7.3}$$

for some polynomials $p_m(S,T)$.

Lemma 5.7.12. *A function $f_3(S,T,I_2^q)$ of the form (5.7.3) solves the equations (5.7.1) and (5.7.2) in Lemma 5.7.10 if and only if the family of polynomials $p_m(S,T)$ satisfies (1) and (2) in Lemma 5.7.4.*

Proof. From (A.2.1) and (A.2.2) it follows easily that

$$\begin{aligned}\partial_S(1+S^2)^\alpha\overline{K}_\alpha(U)&=-S(1+S^2)^{\alpha-1}\overline{K}_{\alpha-1}(U),\\ \partial_T(1+S^2)^\alpha\overline{K}_\alpha(U)&=-\frac{T}{2}(1+S^2)^{\alpha+1}\overline{K}_{\alpha+1}(U).\end{aligned}$$

Using these identities the proof is an easy computation. □

This motivates the definition of the functions $f_{j,k,\ell}$ in Theorem 5.7.6. We finally calculate how the Lie algebra $\mathfrak{k}_2\simeq\mathfrak{so}(q)$ acts on $f_{j,k,\ell}$. For this note that

$$\omega(x,Jv)\varphi=\varphi_v^++I_2^q\varphi_v^-$$

with

$$\varphi_v^+=\omega(x,Jv)\varphi+\frac{\sqrt{2}I_2^q\partial_v\varphi}{q+2\ell-5}\in\mathcal{H}^{\ell+1}(\mathbb{R}^{q-3}),\qquad \varphi_v^-=-\frac{\sqrt{2}\partial_v\varphi}{q+2\ell-5}\in\mathcal{H}^{\ell-1}(\mathbb{R}^{q-3}).$$

Proposition 5.7.13. *For all $j,\ell\geq 0$, $k\in\mathbb{Z}$ and $\varphi\in\mathcal{H}^\ell(\mathbb{R}^{q-3})$ the function $f_{j,k,\ell}\otimes\varphi$ is $\mathfrak{k}_1$-invariant and satisfies*

$$\begin{aligned}d\pi_{\min}(T)f_{j,k,\ell}\otimes\varphi&=f_{j,k,\ell}\otimes(-\partial_{Tx}\varphi)\qquad (T\in\mathfrak{k}_2\cap\mathfrak{g}_{(0,0)}\simeq\mathfrak{so}(q-3)),\\ d\pi_{\min}(A-\overline{B})f_{j,k,\ell}\otimes\varphi&=ik\sqrt{2}f_{j,k,\ell}\otimes\varphi,\end{aligned}$$

and

$$\begin{aligned}d\pi_{\min}(2i(B_\mu(A,Jv)+B_\mu(v,B))\mp\sqrt{2}(v+\overline{Jv}))f_{j,k,\ell}\otimes\varphi&=2\sqrt{2}(j\mp k)f_{j-1,k\pm1,\ell+1}\otimes\varphi_v^+\\ &\quad+\Big[(q+j\mp k+2\ell-5)f_{j+1,k\pm1,\ell-1}+(j\mp k)f_{j-1,k\pm1,\ell-1}\Big]\otimes\varphi_v^-,\\ id\pi_{\min}(Jv-\overline{v})f_{j,k,\ell}\otimes\varphi&=(p-q-2j-2\ell)f_{j,k,\ell+1}\otimes\varphi_v^+\\ &\quad+\tfrac{1}{2\sqrt{2}}\Big[(p-q-2\ell-2j)f_{j+2,k,\ell-1}+(p+q+2\ell-2j-10)f_{j,k,\ell-1}\Big]\otimes\varphi_v^-,\\ d\pi_{\min}(-P+\overline{Q}\mp i\sqrt{2}T_0)f_{j,k,\ell}\otimes\varphi&=\Big[(j\mp k)f_{j-1,k\pm1,\ell}-\tfrac{p-q-2j-2\ell}{2}f_{j+1,k\pm1,\ell}\Big]\otimes\varphi.\end{aligned}$$

Proof. With Lemma 5.7.4 and Lemma 5.7.11 this is now an easy, though longish, computation using

$$\partial_{2q}(1+S^2)^\alpha\overline{K}_\alpha(U)=-\sqrt{2}(1+S^2)^{\alpha+1}\overline{K}_{\alpha+1}(U).$$

□

This proves Theorem 5.7.6.

5.8 The case $\mathfrak{g} = \mathfrak{so}(p,3)$

For $q = 3$ we note that $\mathcal{H}^\ell(\mathbb{R}^{q-3}) = \{0\}$ for $\ell > 0$, so that the lowest K-type is spanned by $f_{j,k,0}$ ($0 \leq j \leq \frac{p-3}{2}$, $-j \leq k \leq j$, $k \equiv j \mod 2$). However, these functions cannot form a basis of $W \simeq \mathcal{H}^{\frac{p-3}{2}}(\mathbb{R}^3) \simeq S^{p-3}(\mathbb{C}^2)$ since $\dim W = p-2$, so the functions $f_{j,k,0}$ for fixed k have to be linearly dependent.

Lemma 5.8.1. *For $q = 3$ and fixed $k \in \mathbb{Z}$ we have*

$$h_{j,k,0}(S,T) = i^{j+k}\sqrt{\frac{\pi}{2}}(1+S^2)^{-\frac{1}{2}}(\sqrt{1+S^2}-S)^k.$$

Proof. Since $q = 3$ we have $I_2^q = 0$ and therefore equation (5.7.1) becomes

$$\Big(-2\partial_T^2 + (1+S^2)\Big)f_3 = 0$$

which has the unique tempered solution $f_3(S,T) = g_4(S)e^{\sqrt{\frac{1+S^2}{2}}T}$. Plugging this into (5.7.2) gives

$$g_4'(S) + \left(\frac{S}{1+S^2} + \frac{k}{\sqrt{1+S^2}}\right)g_4(S) = 0$$

which has the solution

$$g_4(S) = \text{const} \times (1+S^2)^{-\frac{1}{2}}(\sqrt{1+S^2}-S)^k.$$

To find the constant we multiply both $h_{j,k,0}(S,T)$ and $f_3(S,T)$ by $(1+S^2)^{\frac{1}{2}}$ and let $S \to \pm i$ (for the unique analytic extensions of the functions). Using the explicit formulas for the K-Bessel function at half-integer parameters (A.3.1) and (A.3.2) shows that for $h_{j,k,0}(S,T) = \sum_{m=0}^j p_{j,k,m}(S,T)(1+S^2)^{m-\frac{1}{2}}\widetilde{K}_{m-\frac{1}{2}}(U)$ only the summand for $m = 0$ survives. Comparing this with $(\sqrt{1+S^2}-S)^k e^{-\sqrt{U}}$ at $S = \pm i$. Shows the claim. □

Remark 5.8.2. It should be possible to obtain this expression for $h_{j,k,0}(S,T)$ from the explicit formulas for the K-Bessel function of half-integer parameters (A.3.1) and (A.3.2) once a closed formula for the polynomials $p_{j,k,m}(S,T)$ in Lemma 5.7.4 is known.

If we let

$$f_k(\lambda,a,x) = (\lambda - i\sqrt{2}a)^k R^{-\frac{k+1}{2}} \exp\left(-\frac{iaI_3}{\lambda R}\right) \times (1+S^2)^{-\frac{1}{2}}(\sqrt{1+S^2}+S)^{-k}\exp\left(\sqrt{\frac{1+S^2}{2}}T\right), \quad (5.8.1)$$

then Proposition 5.7.13 can be reformulated as

$$d\pi_{\min}(-P+\overline{Q} \mp i\sqrt{2}T_0)f_k = \left(\pm\frac{p-3}{2} - k\right)f_{k\pm1}.$$

We note that $(\lambda - i\sqrt{2}a)^k = \operatorname{sgn}(\lambda)^k(|\lambda| - i\sqrt{2}\operatorname{sgn}(\lambda)a)^k$ and, in view of the case $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$, we put for even $p \in 2\mathbb{N}$, $p \geq 4$, and $k \in \mathbb{Z} + \frac{1}{2}$:

$$f_k(\lambda, a, x) = \operatorname{sgn}(\lambda)^{k-\frac{1}{2}}(|\lambda| - i\sqrt{2}\operatorname{sgn}(\lambda)a)^k R^{-\frac{k+1}{2}} \exp\left(-\frac{iaI_3}{\lambda R}\right) \times (1+S^2)^{-\frac{1}{2}}(\sqrt{1+S^2}+S)^{-k} \exp\left(\sqrt{\frac{1+S^2}{2}}T\right). \quad (5.8.2)$$

Then it is easy to see that f_k is still $\mathfrak{k}_1$-invariant. The subalgebra $\mathfrak{k}_2 \simeq \mathfrak{so}(3) \simeq \mathfrak{su}(2)$ is spanned by the $\mathfrak{su}(2)$-triple

$$T_1 = \tfrac{\sqrt{2}}{2}A + Q + \theta(\tfrac{\sqrt{2}}{2}A + Q), \qquad T_2 = \tfrac{\sqrt{2}}{2}B - P + \theta(\tfrac{\sqrt{2}}{2}B - P), \qquad T_3 = \sqrt{2}T_0 - (E - F),$$

and it follows from the computations in Section 5.7 that

$$d\pi_{\min}(T_1)f_k = 2ikf_k, \qquad d\pi_{\min}(T_2 \mp iT_3)f_k = (\pm(p-3) - 2k)\, f_{k\pm 1}.$$

This shows:

Theorem 5.8.3. *Let $p \geq 3$ be arbitrary. Then the space*

$$W = \text{span}\left\{f_k : k = -\frac{p-3}{2}, -\frac{p-3}{2}+1, \ldots, \frac{p-3}{2}\right\}$$

with f_k as in (5.8.1) resp. (5.8.2) is a $\mathfrak{k}$-subrepresentation of $(d\pi_{\min}, \mathcal{D}'(\mathbb{R}^\times)\widehat{\otimes}\mathcal{S}'(\Lambda))$ isomorphic to the representation $\mathbb{C} \boxtimes S^{p-3}(\mathbb{C}^2)$ of $\mathfrak{k} \simeq \mathfrak{so}(p) \oplus \mathfrak{su}(2)$.

Chapter 6

L^2-models for minimal representations

After having exhibited explicit K-finite vectors in the representation $d\pi_{\min}$ of $\mathfrak{g}$ on $\mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda)$, we show in this section that these vectors generate an irreducible $(\mathfrak{g}, K)$-module which integrates to an irreducible unitary representation of G (or a covering) on $L^2(\mathbb{R}^\times \times \Lambda)$.

6.1 Integration of the $(\mathfrak{g}, K)$-module

Let W be the irreducible $\mathfrak{k}$-subrepresentation of $(d\pi_{\min}, \mathcal{D}'(\mathbb{R}^\times)\,\widehat{\otimes}\,\mathcal{S}'(\Lambda))$ constructed in Chapter 5. For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, W may be one of the K-types $W_{0,r}$ or $W_{1,r}$, where we now assume that $r \in i\mathbb{R}$ (in order for the Lie algebra representation $d\pi_{\min,r}$ to be infinitesimally unitary on $L^2(\mathbb{R}^\times \times \Lambda)$). For $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$ we additionally allow $W_{\frac{1}{2}}$. Consider the $\mathfrak{g}$-subrepresentation generated by W:

$$\overline{W} = d\pi_{\min}(U(\mathfrak{g}))W \subseteq \mathcal{D}'(\mathbb{R}^\times \times \Lambda).$$

Then by standard arguments, $\overline{W}$ is a $(\mathfrak{g}, K)$-module (see e.g. [25, Lemma 2.23]).

Proposition 6.1.1. $\overline{W} \subseteq L^2(\mathbb{R}^\times \times \Lambda)$.

Proof. We first observe that $W \subseteq L^2(\mathbb{R}^\times \times \Lambda)$. In fact, this follows from the asymptotic behavior of the K-Bessel function (see Section A.1). For the more general statement $\overline{W} \subseteq L^2(\mathbb{R}^\times \times \Lambda)$ first note that for the Lie algebra $\mathfrak{q}$ of any standard parabolic subgroup of G we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ which implies $U(\mathfrak{g}) = U(\mathfrak{q})U(\mathfrak{k})$ by the Poincare–Birkhoff–Witt Theorem. For $\mathfrak{q}$ we may for instance choose

$$\mathfrak{q} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_{(-1,1)} \oplus \mathfrak{g}_{(0,0)}.$$

Then $\overline{W} = d\pi_{\min}(U(\mathfrak{q}))W$ can be computed using the identities (A.2.1) and (A.2.2), the rest is technicality. □

Lemma 6.1.2. *The action of $\mathfrak{g}$ on $\overline{W}$ is infinitesimally unitary with respect to the inner product on $L^2(\mathbb{R}^\times \times \Lambda)$.*

Proof. A heuristic proof is given by the observations in Remark 4.5.5. A rigorous proof can be obtained using the formulas for $d\pi_{\min}(X)$ $(X \in \mathfrak{g})$ and integrating by parts, showing directly that $d\pi_{\min}(X)$ is symmetric on $L^2(\mathbb{R}^\times \times \Lambda)$. □

To finally integrate $\overline{W}$ to a group representation, we make use of the following statement about its restriction to $\overline{P}_0$:

Lemma 6.1.3. *The representation of $\overline{P}_0 = M_0 A \overline{N}$ on $L^2(\mathbb{R}^\times \times \Lambda)$ given by*

$$\varpi(\overline{n}_{(z,t)})f(\lambda,x) = e^{i\lambda t}e^{i(\omega(z'',x)+\frac{1}{2}\lambda\omega(z',z''))}f(\lambda,x-\lambda z'),$$
$$\varpi(m)f(\lambda,x) = (\mathrm{id}^*_{\mathbb{R}^\times} \otimes \omega_{\mathrm{met},\lambda^{-1}}(m))f(\lambda,x),$$
$$\varpi(\exp(sH))f(\lambda,x) = e^{-\frac{\dim\Lambda+2}{2}s}f(e^{-2s}\lambda, e^{-s}x)$$

is unitary and decomposes into the direct sum of irreducible subrepresentations $L^2(\mathbb{R}_+ \times \Lambda) \oplus L^2(\mathbb{R}_- \times \Lambda)$. Moreover, if ϖ extends to a unitary representation of $\overline{P} = MA\overline{N}$ on $L^2(\mathbb{R}^\times \times \Lambda)$, then this extension is irreducible.

Proof. Using the isomorphism

$$L^2(\mathbb{R}^\times \times \Lambda) \simeq L^2(\mathbb{R}^\times, L^2(\Lambda))$$

we observe that $\varpi|_{\overline{N}}$ is given by

$$(\varpi(\overline{n})f)(\lambda) = \widetilde{\sigma}_\lambda[f(\lambda)] \qquad (f \in L^2(\mathbb{R}^\times, L^2(\Lambda))),$$

where $\widetilde{\sigma}_\lambda$ is the representation of $\overline{N}$ on $L^2(\Lambda)$ given by $\widetilde{\sigma}_\lambda(\overline{n}_{(z,t)}) = \sigma_{\lambda^{-1}}(\overline{n}_{(\lambda z,\lambda^2 t)})$. Since σ_λ is irreducible for every $\lambda \in \mathbb{R}^\times$, it follows from Schur's Lemma that any intertwining operator $T : L^2(\mathbb{R}^\times, L^2(\Lambda)) \to L^2(\mathbb{R}^\times, L^2(\Lambda))$ is of the form $Tf(\lambda,x) = t(\lambda)f(\lambda,x)$ for some measurable function t on $\mathbb{R}^\times$. Now, T also commutes with A which implies $t(e^{-2s}\lambda) = t(\lambda)$ for all $s \in \mathbb{R}$, whence $t(\lambda)$ is constant on $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively.
Now assume that ϖ extends to $\overline{P}$ and let $m_0 \in M$ with $\chi(m_0) = -1$ (which exists by Theorem 1.9.1 since G is non-Hermitian). The operator $\varpi(m_0)$ satisfies

$$\varpi(m_0) \circ \varpi(\overline{n}_{(0,t)}) = \varpi(\overline{n}_{(0,-t)}) \circ \varpi(m_0),$$

where $\varpi(\overline{n}_{(0,t)})f(\lambda,x) = e^{i\lambda t}f(\lambda,x)$. It follows that $\varpi(m_0)f(\lambda,x) = m(\lambda)U_xf(-\lambda,x)$ for some function $m(\lambda)$ and a unitary operator U on $L^2(\Lambda)$. If now T also commutes with $\varpi(m_0)$, it follows that $t(-\lambda) = t(\lambda)$ which implies that t is constant on $\mathbb{R}^\times$ and hence T is a scalar multiple of the identity. □

Theorem 6.1.4. *The $(\mathfrak{g},K)$-module $\overline{W}$ integrates to an irreducible unitary representation $\pi_{\min}$ of the universal cover $\widetilde{G}$ of G on $L^2(\mathbb{R}^\times \times \Lambda)$. This representation is minimal in the sense that its annihilator in $U(\mathfrak{g}_\mathbb{C})$ is the Joseph ideal.*

Proof. Using Proposition 6.1.1, Lemma 6.1.2 and Lemma 6.1.3, it follows along the same lines as in [25, Proposition 2.27] that $\overline{W}$ is admissible. It therefore integrates to a representation $\pi_{\min}$ of G. This representation is unitary on a Hilbert space $\mathcal{H} \subseteq L^2(\mathbb{R}^\times \times \Lambda)$ by Proposition 6.1.1 and Lemma 6.1.2. On the other hand, its restriction to $\overline{P}$ is given by the action in Lemma 6.1.3 which is irreducible on $L^2(\mathbb{R}^\times \times \Lambda)$. This implies $\mathcal{H} = L^2(\mathbb{R}^\times \times \Lambda)$ and $\pi_{\min}$ is irreducible. That the representation is minimal follows from Proposition 4.5.7. □

For $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ we write $\pi_{\min,\varepsilon,r}$ for the representation with underlying $(\mathfrak{g},K)$-module $W_{\varepsilon,r}$, $r \in i\mathbb{R}$, and in the case $n = 3$ we write $\pi_{\min,\frac{1}{2}}$ for the representation with underlying $(\mathfrak{g},K)$-module $W_{\frac{1}{2}}$.

6.2 Action of Weyl group elements

The results from the previous section can also be phrased in a different way. The parabolic subgroup $\overline{P}$ acts unitarily and irreducibly on $L^2(\mathbb{R}^\times \times \Lambda)$ by the representation ϖ (see Lemma 6.1.3). Theorem 6.1.4 shows that this representation extends to some covering group of G. This point of view was used in [32, Theorem 2] and [48, Proposition 4.2] in order to construct the above L^2-models for the split groups $\mathrm{SO}(n,n)$, $E_{6(6)}$, $E_{7(7)}$, $E_{8(8)}$ and $G_{2(2)}$. There it is shown that ϖ can be extended to an irreducible unitary representation $\pi_{\min}$ of G by defining $\pi_{\min}$ on the representative of a certain Weyl group element w_1 and checking the Chevalley relations (see Section 4.2 for the definition of w_1). This technique does not easily generalize to the case of non-split groups. However, after having constructed the L^2-model in a different way, we can obtain the action of w_1 as a corollary.

The answer depends on the eigenvalues of $d\pi_{\min}(A-\overline{B})$ on the lowest K-type W. Note that in all cases, W is spanned by functions of the form

$$f(\lambda,a,x) = (\lambda - i\sqrt{2}a)^k g(\lambda^2+2a^2,x) \qquad (\lambda>0)$$

with either $k \in \mathbb{Z}$ or $k \in \mathbb{Z}+\frac{1}{2}$. On such functions, $d\pi_{\min}(A-\overline{B})$ acts by $ik\sqrt{2}$. We refer to the *integer case* if $k \in \mathbb{Z}$ and to the *half-integer case* if $k \in \mathbb{Z}+\frac{1}{2}$. From the constructions in Chapter 5 it follows that:

- For $\mathfrak{g} = \mathfrak{e}_{6(2)}, \mathfrak{e}_{6(6)}, \mathfrak{e}_{7(-5)}, \mathfrak{e}_{7(7)}, \mathfrak{e}_{8(-24)}, \mathfrak{e}_{8(8)}, \mathfrak{g}_{2(2)}$ the representation $\pi_{\min}$ belongs to the integer case.
- For $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ the representations $\pi_{\min,0,r}$ and $\pi_{\min,1,r}$ ($r \in i\mathbb{R}$) belong to the integer case, and for $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$ the representation $\pi_{\min,\frac{1}{2}}$ belongs to the half-integer case.
- For $\mathfrak{g} = \mathfrak{so}(p,q)$ the representation $\pi_{\min}$ belongs to the integer case if $p,q \geq 3$, $p+q$ even, and it belongs to the half-integer case if $p \geq q = 3$, p even.

Theorem 6.2.1. *The element w_1 acts in the L^2-model of the minimal representation by*

$$\pi_{\min}(w_1)f(\lambda,a,x) = e^{-i\frac{n(x)}{\lambda a}} f(\sqrt{2}a, -\tfrac{\lambda}{\sqrt{2}}, x) \times \begin{cases} 1 & \text{in the integer case,} \\ \varepsilon(a\lambda) & \text{in the half-integer case,} \end{cases} \tag{6.2.1}$$

where

$$\varepsilon(x) = \begin{cases} 1 & \text{for } x>0, \\ i & \text{for } x<0. \end{cases}$$

Proof. Let A denote the unitary operator on $L^2(\mathbb{R}^\times \times \Lambda)$ given by (6.2.1). Then it is an easy computation to show that

$$A \circ d\pi_{\min}(X) \circ A^{-1} = d\pi_{\min}(\mathrm{Ad}(w_1)X) \qquad \text{for all } X \in \mathfrak{g}.$$

It follows that $A\circ\pi_{\min}(w_1)^{-1}$ is a $\mathfrak{g}$-intertwining unitary operator on $L^2(\mathbb{R}^\times\times\Lambda)$ and therefore has to be a scalar multiple of the identity by Schur's Lemma. To find the scalar, we apply both A and $\pi_{\min}(w_1)$ to a vector in the lowest K-type W. Both can be computed using the explicit description of W in Chapter 5. □

Remark 6.2.2. As mentioned above, the formula for $\pi_{\min}(w_1)$ can be found in [32, Theorem 2] and [48, Proposition 4.2] for the cases $G=\mathrm{SO}(n,n)$, $E_{6(6)}$, $E_{7(7)}$, $E_{8(8)}$ and $G_{2(2)}$. Note that these are all integer cases. In the half-integer case $G=\widetilde{\mathrm{SL}}(3,\mathbb{R})$ Torasso obtained the formula in [53, Lemme 16]. In fact, he even obtained the action of the whole one-parameter subgroup $\exp(\mathbb{R}(A-\overline{B}))$ which should also be possible in general using the same methods as in Theorem 6.2.1.

Remark 6.2.3. The restriction ϖ of $\pi_{\min}$ to $\overline{P}$ together with the action $\pi_{\min}(w_1)$ of w_1 determines the representation $\pi_{\min}$ uniquely since $\overline{P}$ and w_1 generate G. This philosophy was advocated in [36] where a different L^2-model for the minimal representation of $\mathrm{O}(p,q)$ was explicitly determined on a maximal parabolic subgroup and the representative of a non-trivial Weyl group element. Theorem 6.2.1 can be seen as an analogue of their result for our L^2-models.

Remark 6.2.4. It would be interesting to also find explicit formulas for the action of the Weyl group element w_0. One possible way to achieve this is by the help of an additional element

$$w_2=\exp\left(\frac{\pi}{2\sqrt{2}}(B+\overline{A})\right).$$

We have the identity

$$w_2=w_1w_0w_1^{-1},$$

so that $\pi_{\min}(w_0)$ and $\pi_{\min}(w_2)$ can be computed from each other using the previously obtained formula for $\pi_{\min}(w_1)$. Moreover, in all cases except $\mathfrak{g}\simeq\mathfrak{sl}(n,\mathbb{R})$ the elements w_1 and w_2 are conjugate via M_0 which acts in $\pi_{\min}$ via the metaplectic representation. It should be possible to use this in order to obtain a formula for $\pi_{\min}(w_2)$ and then also for $\pi_{\min}(w_0)$.

Using the same technique as in Theorem 6.2.1 we can obtain the action of w_0^2, w_1^2 and w_2^2 which are all contained in M, but may lie in different connected components of M.

Proposition 6.2.5. *The elements $w_0^2,w_1^2,w_2^2\in M$ act in the L^2-model of the minimal representation in the following way:*

(1) In the quaternionic cases $\mathfrak{g}=\mathfrak{e}_{6(2)},\mathfrak{e}_{7(-5)},\mathfrak{e}_{8(-24)}$ we have

$$\pi_{\min}(w_0^2)f(\lambda,a,x)=(-1)^nf(\lambda,-a,-x),$$
$$\pi_{\min}(w_1^2)f(\lambda,a,x)=f(-\lambda,-a,x),$$
$$\pi_{\min}(w_2^2)f(\lambda,a,x)=(-1)^nf(-\lambda,a,-x),$$

where $n=-s_{\min}-1=1,2,4$, i.e. the lowest K-type has dimension $2n+1$.

(2) In the split cases $\mathfrak{g}=\mathfrak{e}_{6(6)},\mathfrak{e}_{7(7)},\mathfrak{e}_{8(8)}$ we have

$$\pi_{\min}(w_0^2)f(\lambda,a,x)=f(\lambda,-a,-x),$$
$$\pi_{\min}(w_1^2)f(\lambda,a,x)=f(-\lambda,-a,x),$$
$$\pi_{\min}(w_2^2)f(\lambda,a,x)=f(-\lambda,a,-x).$$

(3) *In the case* $\mathfrak{g} = \mathfrak{g}_{2(2)}$ *we have*

$$\begin{aligned}\pi_{\min}(w_0^2)f(\lambda,a,x) &= -f(\lambda,-a,-x),\\ \pi_{\min}(w_1^2)f(\lambda,a,x) &= f(-\lambda,-a,x),\\ \pi_{\min}(w_2^2)f(\lambda,a,x) &= -f(-\lambda,a,-x).\end{aligned}$$

(4) *In the case* $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ *we have for* $\varepsilon = 0,1$, $r \in i\mathbb{R}$

$$\begin{aligned}\pi_{\min,\varepsilon,r}(w_0^2)f(\lambda,a,x) &= (-1)^\varepsilon f(\lambda,-a,-x),\\ \pi_{\min,\varepsilon,r}(w_1^2)f(\lambda,a,x) &= f(-\lambda,-a,x),\\ \pi_{\min,\varepsilon,r}(w_2^2)f(\lambda,a,x) &= (-1)^\varepsilon f(-\lambda,a,-x).\end{aligned}$$

(5) *In the case* $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$ *we have*

$$\begin{aligned}\pi_{\min,\frac{1}{2}}(w_0^2)f(\lambda,a) &= -i\operatorname{sgn}(\lambda)f(\lambda,-a),\\ \pi_{\min,\frac{1}{2}}(w_1^2)f(\lambda,a) &= if(-\lambda,-a),\\ \pi_{\min,\frac{1}{2}}(w_2^2)f(\lambda,a) &= -\operatorname{sgn}(\lambda)f(-\lambda,a).\end{aligned}$$

(6) *In the case* $\mathfrak{g} = \mathfrak{so}(p,q)$ *with either* $p \geq q \geq 4$ *and* $p+q$ *even or* $p \geq q = 3$ *we have*

$$\begin{aligned}\pi_{\min}(w_0^2)f(\lambda,a,x) &= (-i)^{p-q}\operatorname{sgn}(\lambda)^{p-q}f(\lambda,-a,-x),\\ \pi_{\min}(w_1^2)f(\lambda,a,x) &= f(-\lambda,-a,x)\times\begin{cases}1 & \text{for } p+q \text{ even},\\ i & \text{for } p+q \text{ odd},\end{cases}\\ \pi_{\min}(w_2^2)f(\lambda,a,x) &= (-i)^{p-q}\operatorname{sgn}(\lambda)^{p-q}f(-\lambda,a,-x)\times\begin{cases}1 & \text{for } p+q \text{ even},\\ i & \text{for } p+q \text{ odd}.\end{cases}\end{aligned}$$

Proof. As in the proof of Theorem 6.2.1 we define a unitary operator A on $L^2(\mathbb{R}^\times \times \Lambda)$ by the right hand side for one of the elements $m = w_0^2, w_1^2, w_2^2$ and show that it satisfies

$$A \circ d\pi_{\min}(X) \circ A^{-1} = d\pi_{\min}(\operatorname{Ad}(m)X) \qquad \text{for all } X \in \mathfrak{g}$$

using the adjoint action of m which was computed in Section 1.2 and Lemma 4.2.2. Then, thanks to Schur's Lemma, $\pi_{\min}(m) = \text{const} \times A$. To show that $\pi_{\min}(m) = A$ we apply $\pi_{\min}(m)$ and A to a vector in the lowest K-type. Using the explicit formulas for vectors in the lowest K-type W from Chapter 5 one can compute A on W, and the identification of W with a finite-dimensional $\mathfrak{k}$-representation allows to compute $\pi_{\min}(m) = \exp(d\pi_{\min}(X))$ where $X = \pi(E-F)$ for $m = w_0^2$, $X = \frac{\pi}{\sqrt{2}}(A - \overline{B})$ for $m = w_1^2$ and $X = \frac{\pi}{\sqrt{2}}(B + \overline{A})$ for $m = w_2^2$. The latter only requires the representation theory of $\mathfrak{su}(2)$, more precisely if $U_1, U_2, U_3 \in \mathfrak{su}(2)$ form an $\mathfrak{su}(2)$-triple and V_n is an irreducible representation of $\mathfrak{su}(2)$ of dimension n with basis $v_0, v_1, \ldots, v_n$ such that

$$U_1 \cdot v_k = i(n-2k)v_k, \qquad (U_2 + iU_3)\cdot v_k = -2i(n-k)v_{k+1}, \quad (U_2 - iU_3)\cdot v_k = -2ikv_{k-1},$$

then

$$\exp(\tfrac{\pi}{2}U_1)\cdot v_k = i^{n-2k}v_k, \quad \exp(\tfrac{\pi}{2}U_2)\cdot v_k = i^{-n}v_{n-k}, \quad \exp(\tfrac{\pi}{2}U_3)\cdot v_k = (-1)^{n-k}v_{n-k}.$$

In all cases except $\mathfrak{g} \simeq \mathfrak{so}(p,q)$, $p \geq q \geq 4$, $p+q$ even, the lowest K-type is an irreducible representation of $\mathfrak{su}(2)$, and for $\mathfrak{g} \simeq \mathfrak{so}(p,q)$ it is sufficient to consider the vectors $f = f_{0,0,\frac{p-q}{2}} \otimes \varphi$, $\varphi \in \mathcal{H}^{\frac{p-q}{2}}(\mathbb{R}^{q-3})$ which are invariant under $\mathfrak{so}(p) \oplus \mathfrak{so}(3) \subseteq \mathfrak{k}$ with $\mathfrak{so}(3) \subseteq \mathfrak{so}(q)$ spanned by

$$\begin{aligned} T_1 &= \tfrac{\sqrt{2}}{2}A + Q + \theta(\tfrac{\sqrt{2}}{2}A + Q) && \equiv \sqrt{2}(A - \overline{B}) \mod \mathfrak{so}(p), \\ T_2 &= \tfrac{\sqrt{2}}{2}B - P + \theta(\tfrac{\sqrt{2}}{2}B - P) && \equiv \sqrt{2}(B + \overline{A}) \mod \mathfrak{so}(p), \\ T_3 &= \sqrt{2}T_0 - (E - F) && \equiv -2(E - F) \mod \mathfrak{so}(p). \end{aligned}$$ □

The knowledge of $\pi_{\min}(m)$ for $m = w_0^2, w_1^2, w_2^2 \in M$ allows us to obtain information about the induction parameter ζ for the corresponding degenerate principal series representation $I(\zeta,\nu)$ that contains $\pi_{\min}$ as a subrepresentation. We exclude the case $\mathfrak{g} = \mathfrak{so}(p,q)$ since here ζ is infinite-dimensional (see Section 4.7 for details).

Corollary 6.2.6. *Assume that $\pi_{\min}$ is a subrepresentation of the degenerate principal series $I(\zeta,\nu)$.*

(1) In the quaternionic cases $\mathfrak{g} = \mathfrak{e}_{6(2)}, \mathfrak{e}_{7(-5)}, \mathfrak{e}_{8(-24)}$ the character ζ of M/M_0 satisfies

$$\zeta(w_0^2) = 1, \qquad \zeta(w_1^2) = \zeta(w_2^2) = (-1)^n, \qquad \text{where } n = -s_{\min} - 1.$$

(2) In the split cases $\mathfrak{g} = \mathfrak{e}_{6(6)}, \mathfrak{e}_{7(7)}, \mathfrak{e}_{8(8)}$ the character ζ of M/M_0 satisfies

$$\zeta(w_0^2) = \zeta(w_1^2) = \zeta(w_2^2) = 1.$$

(3) In the case $\mathfrak{g} = \mathfrak{g}_{2(2)}$ the character ζ of M/M_0 satisfies

$$\zeta(w_0^2) = 1, \qquad \zeta(w_1^2) = \zeta(w_2^2) = -1.$$

(4) In the case $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ with $\pi_{\min} = \pi_{\min,0,r}$, $r \in i\mathbb{R}$, the character ζ of M satisfies

$$\zeta(w_0^2) = \zeta(w_1^2) = \zeta(w_2^2) = 1,$$

and for $\pi_{\min} = \pi_{\min,1,r}$, $r \in i\mathbb{R}$, it satisfies

$$\zeta(w_0^2) = -1, \qquad \text{and either} \begin{cases} \zeta(w_1^2) = 1 \quad \text{and} \quad \zeta(w_2^2) = -1 \quad \text{or} \\ \zeta(w_1^2) = -1 \quad \text{and} \quad \zeta(w_2^2) = 1. \end{cases}$$

(5) In the case $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$ with $\pi_{\min} = \pi_{\min,\frac{1}{2}}$ the representation ζ is the unique irreducible two-dimensional representation of the quaternion group $M \simeq \{\pm 1, \pm i, \pm j, \pm k\}$.

Proof. By Frobenius reciprocity, the lowest K-type W as determined in Chapter 5 is contained in the degenerate principal series $\pi_{\zeta,\nu}$ if and only if

$$\mathrm{Hom}_{M\cap K}(W|_{M\cap K}, \zeta|_{M\cap K}) \neq \{0\}. \tag{6.2.2}$$

(1) In the quaternionic cases $\mathfrak{g} = \mathfrak{e}_{6(2)}, \mathfrak{e}_{7(-5)}, \mathfrak{e}_{8(-24)}$, the lowest K-type has to contain a non-zero vector $f \in W$ such that $d\pi_{\min}(\mathfrak{m} \cap \mathfrak{k})f = 0$ and $\pi_{\min}(w_i^2)f = \zeta(w_i^2)f$, $i = 0, 1, 2$. The first condition implies that $d\pi_{\min}(T_1)f = 0$, and with (5.2.8) it follows that

$$f = \text{const} \times \sum_{\substack{k=-n \\ k \equiv n \mod 2}}^{n} \binom{n}{\frac{k+n}{2}} f_k.$$

Acting by $\pi_{\min}(w_i^2)$ using Corollary 5.2.10 and comparing with $\zeta(w_i^2)$ shows the claim.

(2) In the split cases $\mathfrak{g} = \mathfrak{e}_{6(6)}, \mathfrak{e}_{7(7)}, \mathfrak{e}_{8(8)}$, the lowest K-type W is the trivial representation of K, hence the character ζ has to be trivial on $M \cap K$ by (6.2.2).

(3) In the case $\mathfrak{g} = \mathfrak{g}_{2(2)}$, the lowest K-type has to contain a non-zero vector $f \in W$ such that $d\pi_{\min}(\mathfrak{m} \cap \mathfrak{k})f = 0$ and $\pi_{\min}(w_i^2)f = \zeta(w_i^2)f$, $i = 0, 1, 2$. The first condition implies that $d\pi_{\min}(S_1)f = 0$, and with Proposition 5.4.7 it follows that $f = \text{const} \times (f_1 + f_{-1})$. Acting by $\pi_{\min}(w_i^2)$ using Corollary 5.4.8 and comparing with $\zeta(w_i^2)$ shows the claim.

(4) In the case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ with $\pi_{\min} = \pi_{\min,0,r}$ resp. $\pi_{\min,1,r}$, the lowest K-type is the trivial representation $\mathbb{C}$ of $K = \mathrm{SO}(n)$ resp. the standard representation $\mathbb{C}^n$ of $K = \mathrm{SO}(n)$. In the first case, it is clear that $\zeta|_{M \cap K}$ must be the trivial representation. In the second case, the lowest K-type must contain a non-zero vector f such that $d\pi_{\min}(\mathfrak{m} \cap \mathfrak{k})f = 0$ and $\pi_{\min}(w_i^2)f = \zeta(w_i^2)f$, $i = 0, 1, 2$. The first condition implies that $f = c_0 f_0 + c_1 f_1$. By Proposition 6.2.5 we find

$$\begin{aligned} &\pi_{\min,1}(w_0^2)f_0 = -f_0, && \pi_{\min,1}(w_1^2)f_0 = -f_0, && \pi_{\min,1}(w_2^2)f_0 = f_0, \\ &\pi_{\min,1}(w_0^2)f_1 = -f_1, && \pi_{\min,1}(w_1^2)f_1 = f_1, && \pi_{\min,1}(w_2^2)f_1 = -f_1, \end{aligned}$$

which implies $\zeta(w_0^2) = -1$ and either $c_0 = 0$ or $c_1 = 0$. The claim follows.

(5) In the case $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$ the restriction of the lowest K-type $W \simeq \mathbb{C}^2$ to $M \subseteq K$ is irreducible and two-dimensional. □

Remark 6.2.7. For the adjoint group $G = G_{2(2)}$, the subgroup M has two connected components (see e.g. [28, Section 2]) and since $\chi(w_1^2) = \chi(w_2^2) = -1$ it follows that $w_1^2, w_2^2 \in M$ are contained in the non-trivial component. Therefore, Corollary 6.2.6 determines the character ζ completely in this case. We believe that a similar statement is true for the other cases. Note that even for the non-linear group $G = \widetilde{\mathrm{SL}}(3, \mathbb{R})$ for which M has 8 connected components, the elements w_0^2, w_1^2, w_2^2 generate the component group M/M_0 and hence any representation ζ of M is uniquely determined by $\zeta(w_0^2), \zeta(w_1^2), \zeta(w_2^2)$.

Finally, we are able to describe the precise principal series embedding $\pi_{\min} \hookrightarrow \pi_{\zeta,\nu}$. Recall that for $u \in I(\zeta, \nu)^{\Omega_\mu(\mathfrak{m})}$ (or the corresponding subrepresentations in the cases $\mathfrak{g} \simeq \mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{so}(p, q)$) it was shown that

$$\widehat{u}(\lambda, x, y) = \xi_{-\lambda,0}(x)u_0(\lambda, y) + \xi_{-\lambda,1}(x)u_1(\lambda, y)$$

for some $u_0, u_1 \in \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda)$. Recall further the map

$$\Phi_\delta : \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda) \to \mathcal{D}'(\mathbb{R}^\times) \widehat{\otimes} \mathcal{S}'(\Lambda), \quad \Phi_\delta u(\lambda, x) = \operatorname{sgn}(\lambda)^\delta |\lambda|^{-s_{\min}} u(\lambda, \tfrac{x}{\lambda}),$$

then it was shown in Sections 4.5 that $u \mapsto \Phi_\delta u_\varepsilon$ is $\mathfrak{g}$-intertwining from $d\pi_{\zeta,\nu}$ to $d\pi_{\min}$ for any $\delta, \varepsilon \in \mathbb{Z}/2\mathbb{Z}$. Note that u_ε could be zero. We determine for which $\delta, \varepsilon \in \mathbb{Z}/2\mathbb{Z}$ the map $u \mapsto \Phi_\delta u_\varepsilon$ is G-intertwining from $\pi_{\zeta,\nu}$ to $\pi_{\min}$. Note that the case $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ is excluded since here $\widehat{u}(\lambda, x, y) = u_0(\lambda, y)$, so there is no $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ to determine.

Corollary 6.2.8. *Let $u \in I(\zeta,\nu)^{\Omega_\mu(\mathfrak{m})}$, then*

$$\widehat{u}(\lambda, x, y) = \xi_{-\lambda,\varepsilon}(x) u_\varepsilon(\lambda, y)$$

and the map $u \mapsto \Phi_\delta u_\varepsilon$ is G-intertwining from $\pi_{\zeta,\nu}$ to $\pi_{\min}$, where

(1) In the quaternionic cases $\mathfrak{g} = \mathfrak{e}_{6(2)}, \mathfrak{e}_{7(-5)}, \mathfrak{e}_{8(-24)}$ we have

$$\delta = \varepsilon = n = -s_{\min} - 1.$$

(2) In the split cases $\mathfrak{g} = \mathfrak{e}_{6(6)}, \mathfrak{e}_{7(7)}, \mathfrak{e}_{8(8)}$ we have

$$\delta = \varepsilon = 0.$$

(3) In the case $\mathfrak{g} = \mathfrak{g}_{2(2)}$ we have

$$\delta = \varepsilon = 1.$$

Proof. Let $u \in I(\zeta,\nu)$ such that

$$\widehat{u}(\lambda, x, y) = \xi_{-\lambda,0}(x) u_0(\lambda, y) + \xi_{-\lambda,1}(x) u_1(\lambda, y).$$

For $m \in M$ we have by Proposition 2.5.5

$$\widehat{\pi}_{\zeta,\nu}(m)\widehat{u}(\lambda, x, y) = \zeta(m)\Big(\omega_{\text{met},-\lambda}(m)\xi_{-\chi(m)\lambda,0}(x) \cdot \omega_{\text{met},\lambda}(m) u_0(\chi(m)\lambda, y) + \omega_{\text{met},-\lambda}(m)\xi_{-\chi(m)\lambda,1}(x) \cdot \omega_{\text{met},\lambda}(m) u_1(\chi(m)\lambda, y)\Big). \tag{6.2.3}$$

On the other hand, if $d\rho_{\min}$ integrates to the group representation $\rho_{\min}$ it follows that

$$\widehat{\pi}_{\zeta,\nu}(m)\widehat{u}(\lambda, x, y) = \xi_{-\lambda,0}(x) \cdot \rho_{\min}(m) u_0(\lambda, y) + \xi_{-\lambda,1}(x) \cdot \rho_{\min}(m) u_1(\lambda, y). \tag{6.2.4}$$

We compare (6.2.3) and (6.2.4) for $m = w_0^2, w_1^2, w_2^2$. Note that for $m = w_0^2, w_1^2, w_2^2$ the action $\omega_{\text{met},\lambda}(m)$ can be computed using (2.4.1) and (2.5.1) as well as the adjoint action $\text{Ad}(m)$ on $V = \mathfrak{g}_{-1}$ which is known by Section 1.2 and Lemma 4.2.2:

$$\omega_{\text{met},\lambda}(w_0^2) u(a, y) = \pm u(-a, -y),$$
$$\omega_{\text{met},\lambda}(w_1^2) u(a, y) = \pm u(a, -y),$$
$$\omega_{\text{met},\lambda}(w_2^2) u(a, y) = \pm u(-a, y).$$

Moreover, we have

$$\chi(w_0^2) = 1 \qquad \text{and} \qquad \chi(w_1^2) = \chi(w_2^2) = -1.$$

Finally, $\rho_{\min}(m) = \Phi_\delta^{-1} \circ \pi_{\min}(m) \circ \Phi_\delta$ with $\pi_{\min}(m)$ given by Proposition 6.2.5.

(1) In the quaternionic cases $\mathfrak{g} = \mathfrak{e}_{6(2)}, \mathfrak{e}_{7(-5)}, \mathfrak{e}_{8(-24)}$, comparing (6.2.3) and (6.2.4) shows that

$$\zeta(w_0^2) = (-1)^{\varepsilon+n}, \qquad \zeta(w_1^2) = (-1)^{\delta}, \qquad \zeta(w_2^2) = (-1)^{\delta+\varepsilon+n},$$

where $n = -s_{\min} - 1$ and

$$\widehat{u}(\lambda, x, y) = \xi_{-\lambda,\varepsilon}(x) u_\varepsilon(y).$$

Comparing with Corollary 6.2.6 shows $\delta = \varepsilon = n$.

(2) In the split cases $\mathfrak{g} = \mathfrak{e}_{6(6)}, \mathfrak{e}_{7(7)}, \mathfrak{e}_{8(8)}$, comparing (6.2.3) and (6.2.4) shows that

$$\zeta(w_0^2) = (-1)^{\varepsilon}, \qquad \zeta(w_1^2) = (-1)^{\delta}, \qquad \zeta(w_2^2) = (-1)^{\delta+\varepsilon}.$$

Comparing with Corollary 6.2.6 shows $\delta = \varepsilon = 0$.

(3) In the case $\mathfrak{g} = \mathfrak{g}_{2(2)}$, comparing (6.2.3) and (6.2.4) shows that

$$\zeta(w_0^2) = (-1)^{\varepsilon+1}, \qquad \zeta(w_1^2) = (-1)^{\delta}, \qquad \zeta(w_2^2) = (-1)^{\delta+\varepsilon+1}.$$

Comparing with Corollary 6.2.6 shows $\delta = \varepsilon = 1$. □

Appendix A

The K-Bessel function

We collect some basic information about the K-Bessel function $K_\alpha(z)$ and its renormalization $\overline{K}_\alpha(z)$.

A.1 The differential equation and asymptotics

The differential equation

$$x^2u''(x) + xu'(x) - (x^2+\alpha^2)u(x) = 0$$

has the two linearly independent solutions $I_\alpha(x)$ and $K_\alpha(x)$. While the I-Bessel function $I_\alpha(x)$ grows exponentially as $x \to \infty$, the K-Bessel function $K_\alpha(x) = K_{-\alpha}(x)$ has the asymptotics

$$K_\alpha(x) = \sqrt{\frac{\pi}{2x}}e^{-x}\left(1+\mathcal{O}\left(\frac{1}{x}\right)\right) \qquad \text{as } x \to \infty.$$

Near $x = 0$ it behaves as follows:

$$K_\alpha(x) = \begin{cases} \frac{\Gamma(|\alpha|)}{2}(\frac{x}{2})^{-|\alpha|} + o(x^{-|\alpha|}) & \text{for } \alpha \neq 0, \\ -\log(\frac{x}{2}) + o(\log(\frac{x}{2})) & \text{for } \alpha = 0. \end{cases}$$

For our purposes it is more convenient to work with the renormalization

$$\overline{K}_\alpha(x) = x^{-\frac{\alpha}{2}}K_\alpha(\sqrt{x}).$$

Then $\overline{K}_\alpha(x)$ solves the differential equation

$$zu'' + (\alpha+1)u' - \frac{1}{4}u = 0. \tag{A.1.1}$$

A.2 Identities

For the derivative of $K_\alpha(x)$ the following two identities hold:

$$K'_\alpha(x) = \frac{\alpha}{x}K_\alpha(x) - K_{\alpha+1}(x) = -\frac{\alpha}{x}K_\alpha(x) - K_{\alpha-1}(x).$$

They imply in particular the three term recurrence relation

$$2\alpha K_\alpha(x) = x(K_{\alpha+1}(x) - K_{\alpha-1}(x)).$$

These can be reformulated in terms of the renormalization $\overline{K}_\alpha$:

$$\overline{K}'_\alpha(x) = -\frac{1}{2}\overline{K}_{\alpha+1}(x), \tag{A.2.1}$$

$$x\overline{K}'_\alpha(x) = -\frac{1}{2}\overline{K}_{\alpha-1}(x) - \alpha\overline{K}_\alpha(x), \tag{A.2.2}$$

as well as

$$x\overline{K}_{\alpha+1}(x) = 2\alpha\overline{K}_\alpha(x) + \overline{K}_{\alpha-1}(x). \tag{A.2.3}$$

A.3 Half-integer parameters

For $\alpha \in \mathbb{N} + \frac{1}{2}$ the K-Bessel function degenerates to a product of a rational function and an exponential function:

$$K_\alpha(x) = \sqrt{\frac{\pi}{2}}\frac{e^{-x}}{\sqrt{x}}\sum_{j=0}^{|\alpha|-\frac{1}{2}}\frac{(j+|\alpha|-\frac{1}{2})!}{j!(-j+|\alpha|-\frac{1}{2})!}(2x)^{-j}.$$

For the renormalized function $\overline{K}_\alpha(x)$ with $\alpha = -\frac{1}{2}$ this implies

$$\overline{K}_{-\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2}}e^{-\sqrt{x}} \tag{A.3.1}$$

and for $\alpha = \frac{1}{2} + n$

$$\overline{K}_{n+1/2}(x) = \sqrt{\frac{\pi}{2}}x^{-n-\frac{1}{2}}2^{-n}e^{-\sqrt{x}}\sum_{j=0}^{n}\frac{(2n-j)!}{j!(n-j)!}(2\sqrt{x})^j. \tag{A.3.2}$$

Appendix B

Examples

For a few examples we provide explicit information about the structure of the Lie algebra $\mathfrak{g}$, the symplectic vector space V and its invariants.

B.1 $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$

Let $G = \mathrm{SL}(n, \mathbb{R})$. Put

$$E = \begin{pmatrix} 0 & 0 & 1 \\ & \mathbf{0}_{n-2} & 0 \\ & & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & & \\ 0 & \mathbf{0}_{n-2} & \\ 1 & 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & & \\ & \mathbf{0}_{n-2} & \\ & & -1 \end{pmatrix},$$

then $\mathrm{ad}(H)$ has eigenvalues $0, \pm 1, \pm 2$ on $\mathfrak{g}$ and, in the above block notation:

$$\mathfrak{g}_{-2} = \begin{pmatrix} 0 & & \\ 0 & 0 & \\ \star & 0 & 0 \end{pmatrix}, \mathfrak{g}_{-1} = \begin{pmatrix} 0 & & \\ \star & 0 & \\ 0 & \star & 0 \end{pmatrix}, \mathfrak{g}_0 = \begin{pmatrix} \star & & \\ & \star & \\ & & \star \end{pmatrix}, \mathfrak{g}_1 = \begin{pmatrix} 0 & \star & 0 \\ & 0 & \star \\ & & 0 \end{pmatrix}, \mathfrak{g}_2 = \begin{pmatrix} 0 & 0 & \star \\ & 0 & 0 \\ & & 0 \end{pmatrix}.$$

Further,

$$M = \left\{ \begin{pmatrix} a & & \\ & g & \\ & & b \end{pmatrix} : g \in \mathrm{GL}(n-2, \mathbb{R}), |a| = |b|, ab\det(g) = 1 \right\}.$$

We parameterize $\mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \simeq \mathfrak{g}_{-1}$ by

$$(x, y) \mapsto \begin{pmatrix} 0 & & \\ x & \mathbf{0}_n & \\ 0 & y^\top & 0 \end{pmatrix},$$

then $\omega((x, y), (x', y')) = x'^\top y - x^\top y'$ and

$$\mu(x, y) = \begin{pmatrix} \frac{x^\top y}{2} & & \\ & -xy^\top & \\ & & \frac{x^\top y}{2} \end{pmatrix}, \qquad \Psi(x, y) = \frac{x^\top y}{2}(x, -y), \qquad Q(x, y) = \frac{(x^\top y)^2}{4}.$$

The choice of Cartan involution $\theta(X) = -X^\top$ gives

$$J(x, y) = (-y, x) \qquad \text{and} \qquad ((x, y)|(x', y')) = \frac{1}{4}(x^\top x' + y^\top y').$$

B.2 $\mathfrak{g} = \mathfrak{so}(p,q)$

Let $G = \mathrm{SO}_0(p,q)$, where $\mathrm{SO}(p,q) = \{g \in \mathrm{SL}(p+q,\mathbb{R}\} : g^\top \mathbf{1}_{p,q} g = \mathbf{1}_{p,q}\}$ with $\mathbf{1}_{p,q} = \mathrm{diag}(\mathbf{1}_p, -\mathbf{1}_q)$. Then $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$. Put

$$H = \begin{pmatrix} & & \mathbf{1}_2 \\ & \mathbf{0}_{p+q-4} & \\ \mathbf{1}_2 & & \end{pmatrix}$$

then $\mathrm{ad}(H)$ has eigenvalues $0, \pm 1, \pm 2$ on $\mathfrak{g}$ and

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} 0 & x & & a & b \\ -x & 0 & & b & d \\ & & T & & \\ a & b & & 0 & x \\ b & d & & -x & 0 \end{pmatrix} : x,a,b,d \in \mathbb{R}, T \in \mathfrak{so}(p-2,q-2) \right\},$$

$$\mathfrak{g}_{\pm 1} = \left\{ \begin{pmatrix} & V & \mp W^\top & \\ -V^\top & & & \pm V^\top \\ \mp W & & & W \\ & \pm V & -W^\top & \end{pmatrix} : \begin{array}{l} V \in M(2\times(p-2),\mathbb{R}), \\ W \in M((q-2)\times 2,\mathbb{R}) \end{array} \right\},$$

$$\mathfrak{g}_{\pm 2} = \left\{ \begin{pmatrix} 0 & x & & 0 & \mp x \\ -x & 0 & & \pm x & 0 \\ & & \mathbf{0}_{p+q-4} & & \\ 0 & \pm x & & 0 & -x \\ \mp x & 0 & & x & 0 \end{pmatrix} : x \in \mathbb{R} \right\}.$$

The map

$$\mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{m}, \quad \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \begin{pmatrix} 0 & \frac{b-c}{2} & & a & \frac{b+c}{2} \\ -\frac{b-c}{2} & & & \frac{b+c}{2} & -a \\ & & \mathbf{0}_{p+q-4} & & \\ a & \frac{b+c}{2} & & 0 & \frac{b-c}{2} \\ \frac{b+c}{2} & -a & & -\frac{b-c}{2} & 0 \end{pmatrix}$$

is an isomorphism onto an ideal $\mathfrak{m}_0 \simeq \mathfrak{sl}(2,\mathbb{R})$ of $\mathfrak{m}$. Further,

$$M = \left\{ \begin{pmatrix} \frac{g+g^{-\top}}{2} & & \frac{g-g^{-\top}}{2} \\ & h & \\ \frac{g-g^{-\top}}{2} & & \frac{g+g^{-\top}}{2} \end{pmatrix} : g \in \mathrm{SL}^\pm(2,\mathbb{R}), h \in \mathrm{SO}(p-2,q-2), \det(g) = \chi(h) \right\},$$

where $\chi : \mathrm{SO}(p-2,q-2) \to \{\pm 1\}$ is the non-trivial character of $\mathrm{SO}(p-2,q-2)$.

We identify $V = \mathfrak{g}_{-1} \simeq \mathbb{R}^{2\times(p-2)} \times \mathbb{R}^{(q-2)\times 2}$ by mapping a matrix of the above form to (V,W). We choose

$$E = \frac{1}{2} \begin{pmatrix} J & & -J \\ & \mathbf{0}_{p+q-4} & \\ J & & -J \end{pmatrix} \quad \text{and} \quad F = -\frac{1}{2} \begin{pmatrix} J & & J \\ & \mathbf{0}_{p+q-4} & \\ -J & & -J \end{pmatrix},$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then

$$\omega((V,W),(V',W')) = 2\big(v_1^\top v_2' - v_1'^\top v_2 - w_1^\top w_2' + w_1'^\top w_2\big),$$

where $V = (v_1, v_2)^\top$ and $W = (w_1, w_2)$ with $v_1, v_2 \in \mathbb{R}^{p-2}$, $w_1, w_2 \in \mathbb{R}^{q-2}$. Further,

$$\mu(V,W) = \begin{pmatrix} & -a & & & -b & c \\ a & & & & c & b \\ & & 2V^\top JV & 2V^\top JW^\top & & \\ & & -2WJV & -2WJW^\top & & \\ -b & c & & & & -a \\ c & b & & & a & \end{pmatrix},$$

with

$$a = \frac{|v_1|^2 + |v_2|^2 - |w_1|^2 - |w_2|^2}{2}, \quad b = v_1^\top v_2 - w_1^\top w_2, \quad c = \frac{|v_1|^2 - |v_2|^2 - |w_1|^2 + |w_2|^2}{2},$$

and

$$\begin{aligned} \Psi(V,W) &= \big((VV^\top - W^\top W)JV, WJ(W^\top W - VV^\top)\big), \\ &= \big((w_1^\top w_2 - v_1^\top v_2)v_1 + (|v_1|^2 - |w_1|^2)v_2, (|w_2|^2 - |v_2|^2)v_1 + (v_1^\top v_2 - w_1^\top w_2)v_2, \\ &\qquad (w_1^\top w_2 - v_1^\top v_2)w_1 + (|v_1|^2 - |w_1|^2)w_2, (|w_2|^2 - |v_2|^2)w_1 + (v_1^\top v_2 - w_1^\top w_2)w_2\big) \\ Q(V,W) &= -(|v_1|^2 - |w_1|^2)(|v_2|^2 - |w_2|^2) + (v_1^\top v_2 - w_1^\top w_2)^2 \end{aligned}$$

For $p, q > 2$ the group G is non-Hermitian and we choose

$$O = ((v_0, 0), (0, w_0))$$

for some fixed $v_0 \in \mathbb{R}^{p-2}$, $w_0 \in \mathbb{R}^{q-2}$ with $|v_0| = |w_0| = 1$. Then

$$A = \frac{1}{2}((v_0, -v_0), (-w_0, w_0)) \qquad \text{and} \qquad B = \frac{1}{2}((v_0, v_0), (w_0, w_0))$$

and

$$\begin{aligned} \mathcal{J} &= \{((\lambda v_0 + a, \lambda v_0 - a), (-\lambda w_0 - b, -\lambda w_0 + b)) : \lambda \in \mathbb{R}, v_0^\top a = w_0^\top b = 0\}, \\ \mathcal{J}^* &= \{((\lambda v_0 + a, -\lambda v_0 + a), (\lambda w_0 + b, -\lambda w_0 + b)) : \lambda \in \mathbb{R}, v_0^\top a = w_0^\top b = 0\}. \end{aligned}$$

Identifying $\mathcal{J}$ with $\mathbb{R} \times v_0^\perp \times w_0^\perp \subseteq \mathbb{R} \times \mathbb{R}^{p-2} \times \mathbb{R}^{q-2}$ by mapping $((\lambda v_0 + a, \lambda v_0 - a), (-\lambda w_0 - b, -\lambda w_0 + b))$ to (λ, a, b) we find that

$$n(\lambda, a, b) = 4\lambda(|a|^2 - |b|^2).$$

We choose

$$P = \tfrac{1}{2\sqrt{2}}((v_0, v_0), (-w_0, -w_0)) \qquad \text{and} \qquad Q = \tfrac{1}{2\sqrt{2}}((-v_0, v_0), (-w_0, w_0)),$$

then $\omega(P,Q)=1$ and $\mathcal{J}=\mathbb{R}P\oplus\overline{\mathcal{J}}$ with

$$\overline{\mathcal{J}}=\{((a,-a),(-b,b)):v_0^\top a=w_0^\top b=0\}\simeq\mathbb{R}^{p-3,q-3}.$$

We further choose $\vartheta:\mathcal{J}\to\mathcal{J}$ to be

$$\vartheta((a,-a),(-b,b))=((a,-a),(b,-b)),$$

then the corresponding map $J:V\to V$ in Proposition 5.7.1 is given by

$$J(V,W)=(JV,WJ).$$

B.3 $\mathfrak{g}=\mathfrak{g}_{2(2)}$

The structure theory of $\mathfrak{g}=\mathfrak{g}_{2(2)}$ is treated in detail in [50]. In this case V can be identified with the space of binary cubics

$$V=S^3(\mathbb{R}^2)=\{p=aX^3+3bX^2Y+3cXY^2+dY^3:a,b,c,d\in\mathbb{R}\}$$

with symplectic form

$$\omega(p,p')=ad'-da'-3bc'+3cb'$$

and

$$Q(p)=\frac{1}{4}(a^2d^2-3b^2c^2-6abcd+4b^3d+4ac^3).$$

The action of $\mathfrak{m}\simeq\mathfrak{sl}(2,\mathbb{R})$ on $V=S^3(\mathbb{R}^2)$ is induced by the natural action of $\mathfrak{sl}(2,\mathbb{R})$ on $\mathbb{R}^2$. One possible choice of A, B, C and D is

$$A=\sqrt{2}X^3,\quad B=\sqrt{2}Y^3,\quad C=-\frac{3}{\sqrt{2}}X^2Y,\quad D=\frac{3}{\sqrt{2}}XY^2.$$

Appendix C

A meromorphic family of distributions

Let $\mathfrak{g}$ be non-Hermitian and $\mathfrak{g} \not\simeq \mathfrak{sl}(n,\mathbb{R}), \mathfrak{so}(p,q)$. In Theorem 4.4.2 it is shown that the space of $\mathfrak{m}$-invariant distribution vectors in the metaplectic representation $(\omega_{\text{met},\lambda}, L^2(\Lambda))$ is two-dimensional and spanned by the two distributions

$$\xi_{\lambda,\varepsilon}(a,x) = \operatorname{sgn}(a)^\varepsilon |a|^{s_{\min}} e^{-i\lambda\frac{n(x)}{a}} \qquad (\varepsilon \in \mathbb{Z}/2\mathbb{Z}).$$

It is a priori not clear that this formula defines a distribution on Λ since both $|a|^{s_{\min}}$ and $e^{-i\lambda\frac{n(x)}{a}}$ have a singularity at $a = 0$ and $|a|^{s_{\min}}$ is only locally integrable for $s_{\min} > -1$. In this section we show that $\xi_{\lambda,\varepsilon}$ is the special value of a meromorphic family of distributions at a regular point.

Fix $\lambda \in \mathbb{R}^\times$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. For $s \in \mathbb{C}$ put

$$\psi_{s,\varepsilon}(a,x) = \operatorname{sgn}(a)^\varepsilon |a|^s e^{-i\lambda\frac{n(x)}{a}}.$$

Then $\psi_{s,\varepsilon} = \xi_{\lambda,\varepsilon}$ for $s = s_{\min} = -\frac{1}{6}(\dim\Lambda + 2)$. For $\operatorname{Re}(s) > -1$ the function $\psi_{s,\varepsilon}$ is locally integrable and tempered and hence defines a tempered distribution $\psi_{s,\varepsilon} \in \mathcal{S}'(\Lambda)$. We show that $\psi_{s,\varepsilon}$ extends meromorphically to $s \in \mathbb{C}$ and that $s = s_{\min}$ is not a pole of this meromorphic extension.

C.1 Some preliminary formulas

Let $(e_\alpha)_\alpha$ be a basis of $\mathcal{J} = \mathfrak{g}_{(0,-1)}$ and denote by $(\widehat{e}_\alpha)_\alpha$ the dual basis of $\mathfrak{g}_{(-1,0)}$ with respect to the symplectic form ω, i.e. $\omega(e_\alpha, \widehat{e}_\beta) = \delta_{\alpha\beta}$.

Lemma C.1.1. *For all $x \in \mathcal{J}$:*

$$\Psi(\mu(x)B) = 4n(x)^2 B.$$

Proof. By the $\mathfrak{m}$-invariance of Ψ and Lemma 4.3.2 we have

$$\begin{aligned}\Psi(\mu(x)B) &= B_\Psi(\mu(x)B, \mu(x)B, \mu(x)B)\\ &= \mu(x)B_\Psi(B, \mu(x)B, \mu(x)B) - 2B_\Psi(B, \mu(x)^2B, \mu(x)B)\\ &= \mu(x)B_\Psi(B, \mu(x)B, \mu(x)B) + 8n(x)B_\Psi(B, x, \mu(x)B)\\ &= \frac{1}{2}\mu(x)\Big(\mu(x)B_\Psi(B, B, \mu(x)B) - B_\Psi(B, B, \mu(x)^2B)\Big)\\ &\qquad + 4n(x)\Big(\mu(x)B_\Psi(B, x, B) - B_\Psi(B, \mu(x)x, B)\Big).\end{aligned}$$

From the bigrading it follows that

$$B_\Psi(B, B, \mu(x)B) = 0 \qquad \text{and} \qquad B_\Psi(B, B, \mu(x)^2B) = -4n(x)B_\Psi(B, B, x) = 0.$$

Further, $\mu(x)x = -3\Psi(x) = -3n(x)A$ and $B_\Psi(A, B, B) = \frac{1}{3}B$ and the claim follows. □

Lemma C.1.2. *For all $x \in \mathcal{J}$:*

$$\sum_\alpha B_\mu(A, \widehat{e}_\alpha)B_\mu(e_\alpha, B)x = \left(\frac{1}{2} + \frac{\dim \mathfrak{g}_{(0,-1)}}{6}\right) x.$$

Proof. Let $y \in \mathfrak{g}_{(-1,0)}$, then by Lemma 4.3.1

$$\begin{aligned}\omega\left(\sum_\alpha B_\mu(A, \widehat{e}_\alpha)B_\mu(e_\alpha, B)x, y\right) &= -\sum_\alpha \omega(B_\mu(B, e_\alpha)x, B_\mu(A, \widehat{e}_\alpha)y)\\ &= -\sum_\alpha \omega(B_\mu(B, x)e_\alpha, B_\mu(A, y)\widehat{e}_\alpha)\\ &= \operatorname{tr}(B_\mu(A, y) \circ B_\mu(x, B)|_{\mathfrak{g}_{(0,-1)}})\\ &= \left(\frac{1}{2} + \frac{\dim \mathfrak{g}_{(0,-1)}}{6}\right)\omega(x, y).\end{aligned}$$

Since y was arbitrary and the symplectic form is non-degenerate on $\mathfrak{g}_{(0,-1)} \times \mathfrak{g}_{(-1,0)}$, the desired identity follows. □

Lemma C.1.3. *For all $x \in \mathcal{J}$:*

$$\sum_\alpha B_\Psi(\mu(x)B, B_\mu(x, e_\alpha)B, \widehat{e}_\alpha) = \left(\frac{1}{2} + \frac{\dim \mathfrak{g}_{(0,-1)}}{6}\right) n(x)B.$$

Proof. It follows from the bigrading that the left hand side is contained in $\mathfrak{g}_{(-2,1)}$ and hence a scalar multiple of B. Therefore, it suffices to compute the quantity

$$\begin{aligned}&\omega\left(A, \sum_\alpha B_\Psi(\mu(x)B, B_\mu(x, e_\alpha)B, \widehat{e}_\alpha)\right)\\ &= \sum_\alpha \omega(\mu(x)B, B_\Psi(A, \widehat{e}_\alpha, B_\mu(x, e_\alpha)B))\\ &= -\frac{1}{3}\sum_\alpha \omega(\mu(x)B, B_\mu(A, \widehat{e}_\alpha)B_\mu(x, e_\alpha)B) - \frac{1}{6}\sum_\alpha \omega(\mu(x)B, B_\tau(A, \widehat{e}_\alpha)B_\mu(x, e_\alpha)B).\end{aligned}$$

The second sum vanishes since

$$B_\tau(A,\widehat{e}_\alpha)B_\mu(x,e_\alpha)B = \frac{1}{2}\omega(A,B_\mu(x,e_\alpha)B)\widehat{e}_\alpha + \frac{1}{2}\omega(\widehat{e}_\alpha,B_\mu(x,e_\alpha)B)A$$

and $\omega(A,\mathfrak{g}_{(-1,0)}) = \omega(\mathfrak{g}_{(-1,0)},\mathfrak{g}_{(-1,0)}) = 0$. For the first sum we have by Lemma C.1.2

$$\begin{aligned}\sum_\alpha \omega(\mu(x)B,B_\mu(A,\widehat{e}_\alpha)B_\mu(e_\alpha,x)B) &= \sum_\alpha \omega(\mu(x)B,B_\mu(A,\widehat{e}_\alpha)B_\mu(e_\alpha,B)x)\\ &= \left(\frac{1}{2}+\frac{\dim\mathfrak{g}_{(0,-1)}}{6}\right)\omega(\mu(x)B,x)\\ &= -\left(3+\dim\mathfrak{g}_{(0,-1)}\right)n(x),\end{aligned}$$

so the claim follows. □

Lemma C.1.4. *For all $x\in\mathcal{J}$:*

$$\sum_{\alpha,\beta} B_\Psi(\widehat{e}_\alpha,\widehat{e}_\beta,B_\mu(e_\alpha,e_\beta)B) = \frac{\dim\mathfrak{g}_{(0,-1)}}{6}\left(\frac{1}{2}+\frac{\dim\mathfrak{g}_{(0,-1)}}{6}\right)B.$$

Proof. From the bigrading it follows that the left hand side is contained in $\mathfrak{g}_{(-2,1)}$ and hence has to be a scalar multiple of B. Therefore, it suffices to compute

$$\begin{aligned}\omega\left(A,\sum_{\alpha,\beta} B_\Psi(\widehat{e}_\alpha,\widehat{e}_\beta,B_\mu(e_\alpha,e_\beta)B)\right) &= \sum_{\alpha,\beta}\omega(\widehat{e}_\beta,B_\Psi(A,\widehat{e}_\alpha,B_\mu(e_\alpha,B)e_\beta)\\ &= -\frac{1}{3}\sum_{\alpha,\beta}\omega(\widehat{e}_\beta,B_\mu(A,\widehat{e}_\alpha)B_\mu(e_\alpha,B)e_\beta) - \frac{1}{6}\sum_{\alpha,\beta}\omega(\widehat{e}_\beta,B_\tau(A,\widehat{e}_\alpha)B_\mu(e_\alpha,B)e_\beta).\end{aligned}$$

The second sum vanishes as in the proof of Lemma C.1.3, and the first sum evaluates by Lemma C.1.2 to

$$\begin{aligned}-\frac{1}{3}\sum_{\alpha,\beta}\omega(\widehat{e}_\beta,B_\mu(A,\widehat{e}_\alpha)B_\mu(e_\alpha,B)e_\beta) &= -\frac{1}{3}\left(\frac{1}{2}+\frac{\dim\mathfrak{g}_{(0,-1)}}{6}\right)\sum_\beta \omega(\widehat{e}_\beta,e_\beta)\\ &= \frac{\dim\mathfrak{g}_{(0,-1)}}{3}\left(\frac{1}{2}+\frac{\dim\mathfrak{g}_{(0,-1)}}{6}\right).\end{aligned}$$

□

C.2 A Bernstein–Sato identity

We now show a Bernstein–Sato identity which expresses $\psi_{s,\varepsilon}$ in terms of $\psi_{t,\delta}$ for $t\in\{s+1,s+2\}$, $\delta\in\mathbb{Z}/2\mathbb{Z}$ and hence can be used to meromorphically extend $\psi_{s,\varepsilon}$ to $s\in\mathbb{C}$.

Lemma C.2.1. *We have the following formulas for derivatives of $\psi_{s,\varepsilon}$:*

$$\begin{aligned}\partial_A\psi_{s,\varepsilon} &= s\psi_{s-1,\varepsilon+1} + i\lambda n(x)\psi_{s-2,\varepsilon},\\ \partial_A^2\psi_{s,\varepsilon} &= s(s-1)\psi_{s-2,\varepsilon} + 2i(s-1)\lambda n(x)\psi_{s-3,\varepsilon+1} - \lambda^2 n(x)^2\psi_{s-4,\varepsilon},\\ \omega(A,\Psi(\tfrac{\partial}{\partial x}))\psi_{s,\varepsilon} &= i\lambda^3 n(x)^2\psi_{s-3,\varepsilon+1} - 3\left(\tfrac{1}{2}+\tfrac{\dim\mathfrak{g}_{(0,-1)}}{6}\right)\lambda^2 n(x)\psi_{s-2,\varepsilon}\\ &\quad - i\lambda\tfrac{\dim\mathfrak{g}_{(0,-1)}}{3}\left(\tfrac{1}{2}+\tfrac{\dim\mathfrak{g}_{(0,-1)}}{6}\right)\psi_{s-1,\varepsilon+1}\end{aligned}$$

Proof. The first two identities follow by direct computation. For the third identity note that

$$\omega(A,\Psi(\tfrac{\partial}{\partial x}))\psi_{s,\varepsilon} = \sum_{\alpha,\beta,\gamma} \omega(A, B_\Psi(\widehat{e}_\alpha,\widehat{e}_\beta,\widehat{e}_\gamma))\partial_{e_\alpha}\partial_{e_\beta}\partial_{e_\gamma}\psi_{s,\varepsilon}.$$

To compute the derivatives of $\psi_{s,\varepsilon}$ we use formula (4.4.1) for the derivatives of $n(x)$ and find

$$\begin{aligned}
\partial_{e_\alpha}\psi_{s,\varepsilon} &= \tfrac{i\lambda}{2}\omega(\mu(x)e_\alpha,B)\psi_{s-1,\varepsilon+1},\\
\partial_{e_\alpha}\partial_{e_\beta}\psi_{s,\varepsilon} &= -\tfrac{\lambda^2}{4}\omega(\mu(x)e_\alpha,B)\omega(\mu(x)e_\beta,B)\psi_{s-2,\varepsilon}+i\lambda\omega(B_\mu(x,e_\beta)e_\alpha,B)\psi_{s-1,\varepsilon+1},\\
\partial_{e_\alpha}\partial_{e_\beta}\partial_{e_\gamma}\psi_{s,\varepsilon} &= -\tfrac{i\lambda^3}{8}\omega(\mu(x)e_\alpha,B)\omega(\mu(x)e_\beta,B)\omega(\mu(x)e_\gamma,B)\psi_{s-3,\varepsilon+1}\\
&\quad -\tfrac{\lambda^2}{2}\omega(B_\mu(x,e_\alpha)e_\beta,B)\omega(\mu(x)e_\gamma,B)\psi_{s-2,\varepsilon}\\
&\quad -\tfrac{\lambda^2}{2}\omega(B_\mu(x,e_\alpha)e_\gamma,B)\omega(\mu(x)e_\beta,B)\psi_{s-2,\varepsilon}\\
&\quad -\tfrac{\lambda^2}{2}\omega(B_\mu(x,e_\beta)e_\gamma,B)\omega(\mu(x)e_\alpha,B)\psi_{s-2,\varepsilon}\\
&\quad +i\lambda\omega(B_\mu(e_\alpha,e_\beta)e_\gamma,B)\psi_{s-1,\varepsilon+1}.
\end{aligned}$$

Then

$$\begin{aligned}
\omega(A,\Psi(\tfrac{\partial}{\partial x}))\psi_{s,\varepsilon} &= \tfrac{i\lambda^3}{8}\omega(A,\Psi(\mu(x)B))\psi_{s-3,\varepsilon+1}\\
&\quad -\tfrac{3\lambda^2}{2}\sum_\alpha \omega(A,B_\Psi(\widehat{e}_\alpha,B_\mu(x,e_\alpha)B,\mu(x)B))\psi_{s-2,\varepsilon}\\
&\quad -i\lambda\sum_{\alpha,\beta}\omega(A,B_\Psi(\widehat{e}_\alpha,\widehat{e}_\beta,B_\mu(e_\alpha,e_\beta)B))\psi_{s-1,\varepsilon+1},
\end{aligned}$$

and evaluating the three terms with Lemma C.1.1, Lemma C.1.3 and Lemma C.1.4 proves the third identity. □

Combining the derivatives in the previous lemma immediately shows:

Proposition C.2.2 (Bernstein–Sato identity)**.** *The following Bernstein–Sato identity holds:*

$$\begin{aligned}
\omega(A,\Psi(\tfrac{\partial}{\partial x}))\psi_{s+1,\varepsilon+1}+i\lambda\partial_A^2\psi_{s+2,\varepsilon}-\left(3\left(\tfrac{1}{2}+\tfrac{\dim\mathfrak{g}_{(0,-1)}}{6}\right)+2(s+1)\right)i\lambda\partial_A\psi_{s+1,\varepsilon+1}\\
=-\left(s+\tfrac{\dim\mathfrak{g}_{(0,-1)}}{6}+\tfrac{3}{2}\right)\left(s+\tfrac{\dim\mathfrak{g}_{(0,-1)}}{3}+1\right)i\lambda\psi_{s,\varepsilon}.
\end{aligned}$$

As a consequence, $\psi_{s,\varepsilon}$ extends to a meromorphic family of distributions which is regular in the right half plane $\{s\in\mathbb{C}:\operatorname{Re}s>-\min(\frac{3}{2}+\frac{\dim\mathfrak{g}_{(0,-1)}}{6},1+\frac{\dim\mathfrak{g}_{(0,-1)}}{3})\}$. In particular, $\psi_{s,\varepsilon}$ is regular at $s=s_{\min}=-(\frac{1}{2}+\frac{\dim\mathfrak{g}_{(0,-1)}}{6})$.

Appendix D

Tables

The following classification of simple real Lie algebras with Heisenberg parabolic subalgebras is due to Cheng [3]. We also include the maximal compact subalgebra $\mathfrak{k}$, the Levi factor $\mathfrak{m}$ of the maximal parabolic subgroup, half the dimension of the symplectic vector space $\mathfrak{g}_1$ and, whenever it exists, the isomorphism class of the Jordan algebra $\mathcal{J}$ studied in Section 4.2.

Type	$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{m}$	$\frac{1}{2}\dim \mathfrak{g}_1$	$\mathcal{J}$
AI	$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{so}(n)$	$\mathfrak{gl}(n-2,\mathbb{R})$	$n-2$	$\mathbb{R}^{n-3}$
AIII	$\mathfrak{su}(p,q)$	$\mathfrak{s}(\mathfrak{u}(p)\oplus\mathfrak{u}(q))$	$\mathfrak{u}(p-1,q-1)$	$p+q-2$	–
BDI	$\mathfrak{so}(p,q)$	$\mathfrak{so}(p)\oplus\mathfrak{so}(q)$	$\mathfrak{so}(p-2,q-2)\oplus\mathfrak{sl}(2,\mathbb{R})$	$p+q-4$	$\mathbb{R}\oplus\mathbb{R}^{p-3,q-3}$
CI	$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{u}(n)$	$\mathfrak{sp}(n-1,\mathbb{R})$	$n-1$	–
DIII	$\mathfrak{so}^*(2n)$	$\mathfrak{u}(n)$	$\mathfrak{so}^*(2n-4)\oplus\mathfrak{su}(2)$	$2n-4$	–
EI	$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(4)$	$\mathfrak{sl}(6,\mathbb{R})$	10	$\mathrm{Herm}(3,\mathbb{C}_s)$
EII	$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(6)\oplus\mathfrak{su}(2)$	$\mathfrak{su}(3,3)$	10	$\mathrm{Herm}(3,\mathbb{C})$
EIII	$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10)\oplus\mathfrak{u}(2)$	$\mathfrak{su}(1,5)$	10	–
EV	$\mathfrak{e}_{7(7)}$	$\mathfrak{su}(8)$	$\mathfrak{so}(6,6)$	16	$\mathrm{Herm}(3,\mathbb{H}_s)$
EVI	$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}(12)\oplus\mathfrak{su}(2)$	$\mathfrak{so}^*(12)$	16	$\mathrm{Herm}(3,\mathbb{H})$
EVII	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_6\oplus\mathfrak{u}(1)$	$\mathfrak{so}(2,10)$	16	–
EVIII	$\mathfrak{e}_{8(8)}$	$\mathfrak{so}(16)$	$\mathfrak{e}_{7(7)}$	28	$\mathrm{Herm}(3,\mathbb{O}_s)$
EIX	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_7\oplus\mathfrak{su}(2)$	$\mathfrak{e}_{7(-25)}$	28	$\mathrm{Herm}(3,\mathbb{O})$
FI	$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}(3)\oplus\mathfrak{su}(2)$	$\mathfrak{sp}(3,\mathbb{R})$	7	$\mathrm{Herm}(3,\mathbb{R})$
G	$\mathfrak{g}_{2(2)}$	$\mathfrak{su}(2)\oplus\mathfrak{su}(2)$	$\mathfrak{sl}(2,\mathbb{R})$	2	$\mathbb{R}$

Table D.1: Simple real Lie algebras possessing a Heisenberg parabolic subalgebra

We further list the constants $\mathcal{C}(\mathfrak{m}')$ for each case (see [1, §8.10]). Note that $\mathcal{C}(\mathfrak{m}')$ only depend on the complexifications $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ and $\mathfrak{m}'_{\mathbb{C}}$ of $\mathfrak{m}'$.

$\mathfrak{g}_{\mathbb{C}}$	$\mathfrak{m}_{\mathbb{C}}$	$\mathcal{C}(\mathfrak{m}'_{\mathbb{C}})$
$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{sl}(n-2,\mathbb{C}) \oplus \mathfrak{gl}(1,\mathbb{C})$	$\mathcal{C}(\mathfrak{sl}(n-2,\mathbb{C})) = 1$
		$\mathcal{C}(\mathfrak{gl}(1,\mathbb{C})) = \frac{n}{2}$
$\mathfrak{so}(n,\mathbb{C})$	$\mathfrak{so}(n-4,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$	$\mathcal{C}(\mathfrak{so}(n-4,\mathbb{C})) = 2$
		$\mathcal{C}(\mathfrak{sl}(2,\mathbb{C})) = \frac{n-4}{2}$
$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(n-1,\mathbb{C})$	$\frac{1}{2}$
$\mathfrak{e}_6(\mathbb{C})$	$\mathfrak{sl}(6,\mathbb{C})$	3
$\mathfrak{e}_7(\mathbb{C})$	$\mathfrak{so}(12,\mathbb{C})$	4
$\mathfrak{e}_8(\mathbb{C})$	$\mathfrak{e}_7(\mathbb{C})$	6
$\mathfrak{f}_4(\mathbb{C})$	$\mathfrak{sp}(3,\mathbb{C})$	5/2
$\mathfrak{g}_2(\mathbb{C})$	$\mathfrak{sl}(2,\mathbb{C})$	5/3

Table D.2: Special values $\mathcal{C}(\mathfrak{m}')$

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