DIMENSION-INDEPENDENT FUNCTIONAL INEQUALITIES BY TENSORIZATION AND PROJECTION ARGUMENTS

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Abstract. We study stability under tensorization and projection-type operations of gradient-type estimates and other functional inequalities for Markov semigroups on metric spaces. Using transportation-type inequalities obtained by F. Baudoin and N. Eldredge in 2021, we prove that constants in the gradient estimates can be chosen to be independent of the dimension. Our results are applicable to hypoelliptic diffusions on sub-Riemannian manifolds and some hypocoercive diffusions. As a byproduct, we obtain dimension-independent reverse Poincaré, reverse logarithmic Sobolev, and gradient bounds for Lie groups with a transverse symmetry and for non-isotropic Heisenberg groups.

Contents

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1. Introduction

This paper is aimed at exploring stability of functional inequalities satisfied by Markov semigroups on products of metric measure spaces. We focus on three key functional inequalities: gradient bounds, reverse Poincaré and reverse logarithmic Sobolev inequalities. These inequalities play a crucial role in establishing various analytical properties for the underlying Markov semigroup, including Liouville-type properties, Harnack-type inequalities, convergence towards the equilibrium distribution (if it exists), quasi-invariance, and more. In this context, the past three decades have witnessed significant progress in proving these functional inequalities on general metric measure spaces equipped with a Dirichlet form, see [20, 38] and references therein. These results are often interpreted within the framework of Gamma calculus introduced by Bakry and Émery [3] and curvature-dimension inequalities. Such techniques are widely applicable in the setting of Riemannian manifolds and elliptic diffusions.

However, for hypoelliptic diffusions several fundamental issues arise due to lack of such geometric methods in general, and in particular, for not having a Dirichlet form corresponding to the hypoelliptic differential generator. Such difficulties have been tackled by extending the work of Bakry-Émery to hypoelliptic settings, see [6] in the context of obtaining Villani's [39] hypocoercivity estimates, [15, 16] for gradient estimates of Kolmogorov diffusions, [14] for Langevin dynamics with singular potentials, [8] for gradient estimates of sub-elliptic heat kernels on SU(2), and using coupling for gradient estimates on the Heisenberg group [5]. One of the key tools in such a context is the generalized curvature-dimension condition, which implies reverse Poincaré, reverse logarithmic Sobolev , Li-Yau type gradient estimates for the associated Markov semigroups. For a detailed account on such techniques, we refer to [9, 13]. In general, the results obtained through these methods may lead to dimension-dependent functional inequalities. Besides geometric and probabilistic techniques such as coupling, functional inequalities in such degenerate settings can be approached by using structure of the underlying spaces as in [18, 22, 23, 27, 28, 36, 37].

The present work is closely related to [27, 28], where the authors employ tensorization and projection-type arguments to derive a dimension-independent logarithmic Sobolev inequality on the non-isotropic Heisenberg group. They further extend these ideas to homogeneous spaces, utilizing the tensorization property of the Dirichlet form associated with the heat semigroup on such manifolds to establish their results. As mentioned earlier, gradient bounds, reverse Poincaré, and reverse logarithmic Sobolev inequalities cannot be deduced in such a way when we do not have a natural Dirichlet form associated to the hypoelliptic operator. To address this issue, we adopt a different approach by invoking the duality-type results for these functional inequalities developed in [34, 35] and [12]. While the first two papers focus on the relationship between gradient bounds and Wasserstein metric, the latter provides equivalent formulations of reverse Poincaré and reverse logarithmic Sobolev inequalities through the lens of Wasserstein and Hellinger metrics, along with some entropy inequalities. This approach enables us to extend our results beyond manifolds, specifically to any path-connected metric measure space, also known as *length space*. Using simple inequalities involving the Wasserstein and Hellinger metrics, we show that (GB_p), (RPI), and (RLSI) in Theorem 2.1 extend to the product space, and the constants in these inequalities remain invariant when the spaces are identical. As a byproduct of our results, we show that these inequalities including Li-Yau type gradient estimates and parabolic Harnack inequalities can be easily deduced for sub-Riemannian manifolds obtained through tensorization and *sub-Riemannian submersion* as defined by Definition 3.1. In Section 4.3 we consider $(2n+m)$-dimensional Lie groups with transverse symmetry, which is a sub-Riemannian manifold with transverse symmetry introduced in [13]. In Theorem 4.4 we show that our approach yields functional inequalities sharper than those obtained using the curvature-dimension criterion in [13]. However, their results are proved for manifolds with a more general structure, whereas we only consider Lie groups. Our examples include Kolmogorov diffusions on $\mathbb{R}^d \times \mathbb{R}^d$, kinetic Fokker-Planck operators, and Carnot groups of step 2, in particular, to the non-isotropic Heisenberg groups, and the orthogonal groups $\mathrm{SO}(3), \mathrm{SO}(4)$.

The paper is structured as follows. In Section 2 we describe the framework for our main results. Section 3 presents preliminaries on sub-Riemannian manifolds, along with the implications of Theorem 2.1 within the context of sub-Riemannian geometry. Section 4 presents examples including Section 4.3 devoted to Lie groups with transverse symmetry.

2. Tensorization of functional inequalities on metric measure spaces

Let (X, d) be a complete, locally compact, separable metric space which is a length space. In particular, by the Hopf-Rinow theorem for length spaces (see e.g. [29, p. 9]), each pair of points can be joined by a minimizing geodesic, i. e. a rectifiable curve whose length is the distance between the points.

We assume that X is equipped with a *strong upper gradient*, which implies that for any measurable function $f : X \to \mathbb{R}$ we have the upper Lipschitz constant of f defined by

$$|\nabla f|(x) = \lim_{r \downarrow 0} \sup_{y : d(x,y) < r} \frac{|f(x) - f(y)|}{d(x,y)}. \tag{2.1}$$

We refer to [19] for some basic properties of length spaces and strong upper gradients, and for more details [2, p. 2] and [32, Section 6.2]. We denote by $\mathrm{Lip}_b(X)$ the Banach space of all Lipschitz functions on X endowed with the Lipschitz norm

$$\|f\|_{\mathrm{Lip}_b(X)} = \sup_{x\in X} |f(x)| + \sup_{x\neq y} \left| \frac{f(x)-f(y)}{d(x,y)} \right|,$$

by $\mathcal{B}_X$ we denote the Borel-σ-algebra on (X,d), and by $\mathcal{P}(X)$ the space of all probability measures on $(X,\mathcal{B}_X)$. For a Markov kernel $P : X \times \mathcal{B}_X \to [0,1]$, we denote by Pf and μP the usual action of P on bounded Borel functions and probability measures, that is,

$$\begin{aligned} Pf(x) &:= \int_X f(y)P(x,dy), \\ \mu P(A) &:= \int_X P(x,A)\mu(dx). \end{aligned}$$

We are interested in the following functional inequalities for P.

(1) *Gradient bound:* for $1 \leqslant p \leqslant 2$ and for all $f \in \mathrm{Lip}_b(X)$

$$|\nabla Pf| \leqslant C(P|\nabla f|^p)^{\frac{1}{p}}. \tag{GB$_p$}$$

(2) *Reverse Poincaré inequality:* for all $f \in \mathrm{Lip}_b(X)$

$$|\nabla Pf|^2 \leqslant C(P(f^2) - (Pf)^2). \tag{RPI}$$

(3) *Reverse logarithmic Sobolev inequality:* for all positive $f \in \mathrm{Lip}_b(X)$

$$Pf|\nabla \log Pf|^2 \leqslant C(P(f\log f) - Pf\log(Pf)), \tag{RLSI}$$

where C is a positive constant and may differ for each inequality.

We consider a collection of complete separable metric length spaces $\{(X_i,d_i)\}_{i=1}^n$ and let $X := X_1 \times \cdots \times X_n$ be endowed with the metric d defined by

$$d(x,y)^2 := \sum_{i=1}^n d_i(x_i,y_i)^2. \tag{2.2}$$

It is known that (X,d) is a complete separable metric length space. For Markov kernels P_i defined on (X_i,d_i), the tensor product P is defined as the Markov operator on $X = X_1 \times \cdots \times X_n$ such that for all $f \in B_b(X)$

$$\begin{aligned} Pf &:= P_1 \otimes \cdots \otimes P_n f(x_1,\dots,x_n) \\ &= \int_X f(z_1,\dots,z_n)P_1(x_1,dz_1)\cdots P_n(x_n,dz_n). \end{aligned}$$

We are now ready to state the main result of this section.

Theorem 2.1. *Let P_i be a Markov kernel on $(X_i,d_i), i=1,...,n$ that satisfies* (GB$_p$) *(resp.* (RPI), (RLSI)*) with a constant C_i. The, $P := \otimes_{i=1}^n P_i$ satisfies* (GB$_p$) *(resp.* (RPI), (RLSI)*) with constant $C = \max\{C_i : 1 \leqslant i \leqslant n\}$.*

By [12, Theorem 3.5] we know that (RPI) is equivalent to a Harnack-type inequality, thus we obtain the following corollary.

Corollary 2.2. *Let P_i be a Markov kernel on $(X_i, d_i), i = 1, ..., n$ such that for all bounded non-negative $f \in B_b(X_i)$ and $x_i, y_i \in X_i$,*

$$Pf(x_i) \leqslant Pf(y_i) + \sqrt{C_i} d_i(x_i, y_i) \sqrt{P(f^2)(x_i)}$$

for some constant $C_i > 0$. Then, for all $f \in B_b(X)$ and $x, y \in X$ we have

$$Pf(x) \leqslant Pf(y) + \sqrt{C} d(x, y) \sqrt{P(f^2)(x)} \tag{2.3}$$

with $C = \max\{C_i : 1 \leqslant i \leqslant n\}$.

2.1. **Proof of Theorem 2.1 using transportation inequalities.** Suppose as before that (X, d) is a complete, locally compact, separable length space. For any $1 \leqslant p < \infty$, the *L^p-Wasserstein distance* $W_p(\mu, \nu)$ between $\mu, \nu \in \mathcal{P}(X)$ is defined as

$$W_p(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu,\nu)} \left(\int_X d(x, y)^p \pi(dxdy) \right)^{\frac{1}{p}}, \tag{2.4}$$

where $\mathcal{C}(\mu, \nu)$ is the collection of all possible couplings of μ, ν. W_p defines a metric on the space of all probability measures though it may be infinite. When $W_p(\mu, \nu) < \infty$, the infimum in (2.4) is attained, that is, there exist random elements U, V taking values in X such that $U \sim \mu, V \sim \nu$ and $\mathbb{E}(d(U, V)^p) = W_p(\mu, \nu)^p$.

We define the L^p-Wasserstein space of order p as

$$\mathcal{W}_p(X) := \left\{ \mu \in \mathcal{P}(X), \int_X d(x, y)^p \mu(dy) \right\},$$

where x is arbitrary. This space does not depend on the point $x \in X$. We refer to [40, Chapter 6] for a discussion about Wasserstein distances and Wasserstein spaces.

Next we introduce the *Hellinger distance* between $\mu, \nu \in \mathcal{P}(X)$ defined by

$$\mathrm{He}_2(\mu, \nu)^2 := \int_X \left(\sqrt{\frac{d\mu}{dm}} - \sqrt{\frac{d\nu}{dm}} \right)^2 dm,$$

for some measure m with respect to which both μ, ν are absolutely continuous. The above definition is independent of m and in particular one can take $m = (\mu + \nu)/2$. Before going into the proof of Theorem 2.1, we prove the following lemma which will be crucial subsequently.

Lemma 2.3. *For each $i = 1, \ldots, n$, let $\mu_i, \nu_i \in \mathcal{P}(X_i)$. Then*

$$W_p(\mu_1 \otimes \cdots \otimes \mu_n, \nu_1 \otimes \cdots \otimes \nu_n)^2 \leqslant \sum_{i=1}^n W_p(\mu_i, \nu_i)^2, p \geqslant 2; \tag{2.5}$$

$$\mathrm{He}_2(\mu_1 \otimes \cdots \otimes \mu_n, \nu_1 \otimes \cdots \otimes \nu_n)^2 \leqslant \sum_{i=1}^n \mathrm{He}_2(\mu_i, \nu_i)^2, \tag{2.6}$$

where $\mu_1 \otimes \cdots \otimes \mu_n, \nu_1 \otimes \cdots \otimes \nu_n$ *are product measures on* (X, d).

Proof. For each $i = 1, \ldots, n$, let (U_i, V_i) be an optimal coupling such that $U_i \sim \mu_i, V_i \sim \nu_i$ and $\|d_i(U_i, V_i)\|_p = W_p(\mu_i, \nu_i)$, where for a random variable Z, $\|Z\|_p = (\mathbb{E}|Z|^p)^{1/p}$ denotes the L^p norm of Z. Without loss of generality, we can assume that $(U_i, V_i)_{i=1}^n$ are jointly independent pairs. Define $U = (U_1, \ldots, U_n), V = (V_1, \ldots, V_n)$. Now using the triangle inequality in the $L^{p/2}$ space, we have

$$\begin{aligned}\|d(U,V)\|_p^2 &= \left\| \sum_{i=1}^n d_i(U_i, V_i)^2 \right\|_{p/2} \\ &\leqslant \sum_{i=1}^n \|d_i(U_i, V_i)^2\|_{p/2} = \sum_{i=1}^n W_p(\mu_i, \nu_i)^2.\end{aligned}$$

Since $W_p(\mu_1 \otimes \cdots \otimes \mu_n, \nu_1 \otimes \cdots \otimes \nu_n) \leqslant \|d(U,V)\|_p$, then (2.5) follows.

For (2.6), let $m_1, \ldots, m_n$ be probability measures on (X, d) such that both μ_i, ν_i are absolutely continuous with respect to m_i for all $1 \leqslant i \leqslant n$. Writing $f_i := \frac{d\mu_i}{dm_i}, g_i := \frac{d\nu_i}{dm_i}$, we have

$$\begin{aligned}\mathrm{He}_2(\mu_1 \otimes \cdots \otimes \mu_n, \nu_1 \otimes \cdots \otimes \nu_n)^2 &= \int \left(\sqrt{f_1 \cdots f_n} - \sqrt{g_1 \cdots g_n} \right)^2 dm_1 \cdots dm_n \\ &= 2 - 2 \int \sqrt{f_1 \cdots f_n g_1 \cdots g_n} dm_1 \cdots dm_n \\ &= 2 - 2 \prod_{i=1}^n \int \sqrt{f_i g_i} dm_i \qquad (2.7) \\ &\leqslant \sum_{i=1}^n 2 \left(1 - \int \sqrt{f_i g_i} dm_i \right) \qquad (2.8) \\ &= \sum_{i=1}^n \mathrm{He}_2(\mu_i, \nu_i)^2,\end{aligned}$$

where (2.8) follows from (2.7) using by the elementary fact that

$$1 - x_1 \cdots x_n \leqslant \sum_{i=1}^n (1 - x_i) \text{ for } 0 \leqslant x_1, \ldots, x_n \leqslant 1.$$

□

Proof of Theorem 2.1. To prove stability of (GB_p) under tensorization, we use Kuwada's duality theorem [34, Thereom 2.2]. In this paper, the duality result was obtained under the assumption of a volume doubling property, which was later relaxed and generalized to Orlicz spaces in [35]. Moreover, from the proof of [34, Theorem 2.2], it suffices to show that for any $x, y \in X$

$$W_q(\delta_x P, \delta_y P) \leqslant C d(x, y), \tag{2.9}$$

where δ_a denotes the Dirac measure at $a \in X$ and $p^{-1} + q^{-1} = 1$. Now, for each $1 \leqslant i \leqslant n$, and $x_i, y_i \in X_i$ we have $W_q(\delta_{x_i} P_i, \delta_{y_i} P_i) \leqslant C_i d_i(x_i, y_i)$. Using Lemma 2.3(2.5) we conclude

$$\begin{aligned} W_q(\delta_x P, \delta_y P)^2 &= W_q(\otimes_{i=1}^n \delta_{x_i} P_i, \otimes_{i=1}^n \delta_{y_i} P_i)^2 \\ &\leqslant \sum_{i=1}^n W_q(\delta_{x_i} P_i, \delta_{y_i} P_i)^2 \\ &\leqslant \sum_{i=1}^n C_i^2 d_i(x_i, y_i)^2 \leqslant C^2 d(x, y)^2, \end{aligned}$$

where $C = \max\{C_i : 1 \leqslant i \leqslant n\}$.

Next, to prove stability of (RPI) under tensorization, we use the Hellinger-Kantorovich contraction criterion from [12, Theorem 3.7]. Following the argument in their proof, it is enough to show that for any $x, y \in X$

$$\mathrm{He}_2(\delta_x P, \delta_y P)^2 \leqslant \frac{C}{4} d(x, y)^2. \tag{2.10}$$

Since (RPI) holds for each P_i with constant C_i, [12, Theorem 3.7] implies that for all $x_i, y_i \in X_i$

$$\mathrm{He}_2(\delta_{x_i} P_i, \delta_{y_i} P_i)^2 \leqslant \frac{C_i}{4} d_i(x_i, y_i)^2.$$

Therefore, using an argument similar to the proof of (2.9) we conclude (2.10) with $C = \max\{C_i : 1 \leqslant i \leqslant n\}$.

Finally, to prove (RLSI) for P, we again resort to [12, Theorem 5.15] which proves the equivalence between a reverse logarithmic Sobolev inequality and a Wang-Harnack inequality. Using the above result, we have that for all $1 \leqslant i \leqslant n$ and $p > 1$

$$\begin{gathered} P_i f(x_i)^p \leqslant P_i f^p(y_i) \exp\left(\frac{p}{p-1} \frac{C_i d_i(x_i, y_i)^2}{4}\right) \\ \text{for all } x_i, y_i \in X_i, f \in \mathrm{Lip}_b(X_i), f > 0. \end{gathered} \tag{2.11}$$

Now, for any positive function $f \in \mathrm{Lip}_b(X)$, we note that $P_1 \otimes \cdots \otimes P_{n-1} f(x_1, \ldots, x_{n-1}, \cdot)$ is a positive $\mathrm{Lip}_b(X_n)$ function. As a result, applying

(2.11) we obtain

$$
\begin{aligned}
&Pf(x_1,\dots,x_n)^p = \left(\int_{X_n} P_1\otimes\cdots\otimes P_{n-1}f(x_1,\dots,x_{n-1},z)P_n(x_n,dz)\right)^p \\
&= \int_{X_n} \left(P_1\otimes\cdots\otimes P_{n-1}f(x_1,\dots,x_{n-1},z)\right)^p P_n(y_n,dz) \\
&\times \exp\left(\frac{p}{p-1}\frac{C_n d_n^2(x_n,y_n)}{4}\right).
\end{aligned}
$$

Iterating the above inequality for each of the variables $x_1,\dots,x_n$, we get

$$
\begin{aligned}
Pf(x)^p &\leqslant Pf^p(y)\exp\left(\frac{p}{p-1}\sum_{i=1}^n \frac{C_i d_i^2(x_i,y_i)}{4}\right) \\
&\leqslant Pf^p(y)\exp\left(\frac{p}{p-1}\frac{Cd^2(x,y)}{4}\right)
\end{aligned}
$$

with $C=\max\{C_i : 1\leqslant i\leqslant n\}$. Invoking [12, Theorem 5.15] we conclude stability of (RLSI) under tensorization. This completes the proof of the theorem. □

3. Applications to sub-Riemannian manifolds

Let M be a connected smooth manifold equipped with a sub-Riemannian structure $(\mathcal{H},g)$, where $\mathcal{H}$ is a horizontal distribution which satisfies the Hörmander's condition and which is equipped with an inner product $g(\cdot,\cdot)$. For any two points $x,y\in M$, the Carnot-Carathéodory distance between x,y is given by

$$
\begin{aligned}
d(x,y) := \inf\Big\{ \int_0^1 \|\sigma'(t)\|_{\mathcal{H}}dt : \sigma(0)=x, \sigma(1)=y, \\
\sigma'(t)\in\mathcal{H}(\sigma(t)) \text{ for all } 0\leqslant t\leqslant 1\Big\},
\end{aligned}
\tag{3.1}
$$

where for a horizontal vector u, $\|u\|_{\mathcal{H}} := \sqrt{g(u,u)}$. Throughout this section, we always make the following assumption when considering the Carnot-Carathéodory distance. For more on this property we refer to [1, Section 3.3.1]

Assumption 3.1. (M,d) *is a complete metric space.*

For any smooth function f on M, we denote the norm of the horizontal gradient by $\|\nabla_{\mathcal{H}}f\|^2_{\mathcal{H}} = g(\nabla_{\mathcal{H}}f,\nabla_{\mathcal{H}}f)$. We consider on M a second order, locally sub-elliptic operator $\Delta^M_{\mathcal{H}}$ which is compatible with the sub-Riemannian structure in the sense that for every $f\in\mathcal{C}^\infty_c(M)$

$$
\frac{1}{2}\Delta^M_{\mathcal{H}}(f^2) - f\Delta^M_{\mathcal{H}}f = \|\nabla_{\mathcal{H}}f\|^2_{\mathcal{H}}.
$$

We also assume that $\Delta_{\mathcal{H}}^M$ is symmetric with respect to some measure μ_M on M, that is for every $f, g \in \mathcal{C}_c^\infty(M)$,

$$\int_M f \Delta_{\mathcal{H}}^M g d\mu_M = \int_M g \Delta_{\mathcal{H}}^M f d\mu_M$$

We refer to [25,26] for a discussion of the role of a measure on sub-Riemannian manifolds, to [7, Section 1.3] for details about such symmetric locally subelliptic operators, and to [21, p. 950] about essential self-adjointness of such operators in the setting of Lie groups. By [7, Propositions 2.20 and 2.21], Assumption (3.1) implies that the operator $\Delta_{\mathcal{H}}^M$ is essentially self-adjoint on $\mathcal{C}_c^\infty(M)$ in the space $L^2(M, \mu_M)$. This implies that the unique self-adjoint (Friedrichs) extension of $\Delta_{\mathcal{H}}^M$ (which we still denote by $\Delta_{\mathcal{H}}^M$) is the generator of the Dirichlet form $\mathcal{E}_M$ obtained by closing the quadratic form $\int_M \|\nabla f\|_{\mathcal{H}}^2 d\mu_M$, $f \in \mathcal{C}_c^\infty(M)$. We call the semigroup corresponding to $\Delta_{\mathcal{H}}^M$ the *horizontal heat semigroup* P_t^M as in [7, Section 1.5].

3.1. **Sub-Riemannian manifolds obtained by tensorization and projection.**

We introduce the concept of sub-Riemannian submersion, which is a generalization of Riemannian submersion for Riemannian manifolds.

Definition 3.1. A mapping $\pi : (M, \mathcal{H}_M, g_M) \longrightarrow (N, \mathcal{H}_N, g_N)$ between two sub-Riemannian manifolds is called a *sub-Riemannian submersion* if

(i). π is a submersion between differentiable manifolds M and N;
(ii). For any $x \in M$, $\mathcal{H}_M(x) \subset \ker(d\pi_x)^\perp$;
(iii). For all $x \in M$, $d\pi_x : \mathcal{H}_M(x) \to \mathcal{H}_N(\pi(x))$ is an isometry.

As we usually consider sub-Riemannian manifolds equipped with a measure symmetrizing for the corresponding sub-Laplacian, we usually assume that sub-Riemannian submersion satisfies the following assumption.

Assumption 3.2. *Suppose $\pi : M \longrightarrow N$ is a sub-Riemannian submersion between sub-Riemannian manifolds $\pi : (M, \mathcal{H}_M, g_M) \longrightarrow (N, \mathcal{H}_N, g_N)$, $\Delta_{\mathcal{H}}^M$ and $\Delta_{\mathcal{H}}^N$ are sub-Laplacians compatible with the corresponding sub-Riemannian structures on M and N respectively. If μ_M is a symmetrizing measure for $\Delta_{\mathcal{H}}^M$, and μ_N is a symmetrizing measure for $\Delta_{\mathcal{H}}^N$, we assume that the measure μ_N is the pushforward of μ_M by π, i.e. $\mu_N = \pi_* \mu_M$.*

Example. If (M, g_M) and (N, g_N) are Riemannian manifolds, then a Riemannian submersion $\pi : (M, g_M) \to (N, g_N)$ induces a sub-Riemannian submersion $\pi : (M, \mathcal{H}_M, g_M) \longrightarrow (N, \mathcal{H}_N, g_N)$, where $\mathcal{H}_M$ is the horizontal space of the submersion which we assume to satisfy Hörmander's condition and $\mathcal{H}_N$ is the whole tangent bundle of N. The horizontal Laplacian Δ^M on M, as defined on [11, p. 70], is then a sub-Laplacian on M with symmetrizing measure μ_M, where μ_M is the Riemannian volume measure of M. In this case π intertwines the horizontal Laplacian Δ^M with the Laplace-Beltrami operator Δ^N of N if and only if π is harmonic, see [24, Theorem 1] and the proof of [11, Theorem 4.1.10.]. It follows for instance that for Riemannian submersions with totally geodesic fibers the Assumption 3.2 is satisfied when Δ^M is the horizontal Laplacian and Δ^N the Laplace-Beltrami operator.

We also consider the product of sub-Riemannian manifolds $(M_i, \mathcal{H}_i, g_i)_{i=1}^n$ where the horizontal distribution on the product space $M = M_1 \times \cdots \times M_n$ is given by $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ equipped with the sub-Riemannian metric $g = g_1 \oplus \cdots \oplus g_n$. It can be easily verified that $(M, \mathcal{H}, g)$ is a sub-Riemannian manifold. Moreover, the horizontal sub-Laplacian on $(M, \mathcal{H}, g)$ is defined by

$$\Delta_{\mathcal{H}}^M = \Delta_{\mathcal{H}_1}^{M_1} \oplus \cdots \oplus \Delta_{\mathcal{H}_n}^{M_n}. \tag{3.2}$$

The next two theorems show the stability of functional inequalities under tensorization and sub-Riemannian submersions of manifolds. In addition to the functional inequalities (GB_p), (RPI) and (RLSI), we also consider the following *Li-Yau type inequality* on sub-Riemannian manifolds. For all $t > 0$ and positive $f \in \mathcal{C}_c^\infty(M)$

$$C_1(t) \|\nabla_{\mathcal{H}} \log P_t^M f\|_{\mathcal{H}}^2 \leqslant \frac{\Delta_{\mathcal{H}} P_t^M f}{P_t^M f} + C_2(t), \tag{LYI}$$

where $C_1(t), C_2(t)$ are positive, possibly time-dependent constants.

Theorem 3.2. *Let $(M_i, \mathcal{H}_i, g_i, \Delta_{\mathcal{H}_i}^{M_i}, \mu_{M_i})$, $i = 1, ..., n$ be sub-Riemannian manifolds such that* (GB_p) *(resp.* (RPI), (RLSI)*) holds for each horizontal heat semigroup $(P_t^{(i)})_{t \geqslant 0}$ on M_i with a constant $C_i(t)$. Then the horizontal heat semigroup $(P_t^M)_{t \geqslant 0}$ on $M = M_1 \times \cdots \times M_n$ generated by $\Delta_{\mathcal{H}}^M$ satisfies* (GB_p) *(resp.* (RPI), (RLSI)*) with the constant $C(t) = \max\{C_i(t) : 1 \leqslant i \leqslant n\}$.*

Moreover, if (LYI) *holds for each $(P_t^{(i)})_{t \geqslant 0}$ with constants $C_1^{(i)}(t)$ and $C_2^{(i)}(t)$, then $(P_t^M)_{t \geqslant 0}$ satisfies* (LYI) *as well with $C_1(t) = \min\{C_1^{(i)}(t) : 1 \leqslant i \leqslant n\}$ and $C_2(t) = \sum_{i=1}^n C_2^{(i)}(t)$.*

Theorem 3.3. *Let $(M, \mathcal{H}_M, g_M, \Delta_{\mathcal{H}}^M, \mu_M)$, $(N, \mathcal{H}_N, g_N, \Delta_{\mathcal{H}}^N, \mu_N)$ be two sub-Riemannian manifolds such that* (GB_p) *(resp.* (RPI), (RLSI),(LYI)*) holds for M with constant $C(t)$. Assume that there exists a sub-Riemannian submersion from M to N. Then* (GB_p) *(resp.* (RPI), (RLSI),(LYI)*) also holds for N with the same constants.*

In the following section we prove some simple results on Carnot-Carathéodory metric which will be useful in the proof of the above theorems. If we think of such sub-Riemannian manifolds as Dirichlet space, we can think of such tensor products as tensor products of Dirichlet forms, see [17, Section 2.1, Proposition 2.1.2].

3.2. **Carnot-Carathéodory metric on manifolds obtained by tensorization and submersions.** The following lemma states that the Carnot-Carathéodory metric is preserved under both tensorization and sub-Riemannian submersions.

Lemma 3.4. *(1) Let $(M_i, \mathcal{H}_i, g_i)_{i=1}^n$ be a collection of sub-Riemannian manifolds with horizontal distributions $(\mathcal{H}_i)_{i=1}^n$. If d_i denotes the Carnot-Carathéodory metric on M_i, then the Carnot-Carathéodory metric on $M_1 \times \cdots \times M_n$ with horizontal distribution $(\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n, g_1 \oplus \cdots \oplus g_n)$ is given by*

$$d(x,y) = \left(\sum_{i=1}^n d_i(x_i, y_i)^2 \right)^{\frac{1}{2}}. \tag{3.3}$$

(2) Let $(M, \mathcal{H}_M, g_M), (N, \mathcal{H}_N, g_N)$ be two sub-Riemannian manifolds with Carnot-Carathéodory metrics d_M, d_N respectively. Assume that $\pi : M \to N$ is a sub-Riemannian submersion. Then, for any $x, y \in N$, there exists $a \in \pi^{-1}(x), b \in \pi^{-1}(y)$ such that

$$d_M(a,b) = d_N(x,y).$$

Proof. We prove item (1) for $n = 2$ and the general case will follow by induction. Consider any horizontal curve $\sigma : [0,1] \to M_1 \times M_2$ such that $\sigma(0) = x, \sigma(1) = y$. Writing $\sigma = (\sigma_1, \sigma_2)$, it is evident that $\sigma_i'(t) \in \mathcal{H}_i(\sigma_i(t))$ and $\sigma_i(0) = x_i, \sigma_i(1) = y_i$ for $i = 1, 2$. Therefore, by orthogonality of the horizontal distributions, we have for all $t \in [0,1]$

$$\|\sigma'(t)\|_{\mathcal{H}}^2 = \|\sigma_1'(t)\|_{\mathcal{H}_1}^2 + \|\sigma_2'(t)\|_{\mathcal{H}_2}^2.$$

Let us write $l_1 = \int_0^1 \|\sigma_1'(t)\|_{\mathcal{H}_1} dt, l_2 = \int_0^1 \|\sigma_2'(t)\|_{\mathcal{H}_2} dt$. Then we have

$$\begin{aligned}
&\int_0^1 \|\sigma'(t)\|_{\mathcal{H}} dt \\
&= \int_0^1 \sqrt{\|\sigma_1'(t)\|_{\mathcal{H}_1}^2 + \|\sigma_2'(t)\|_{\mathcal{H}_2}^2} dt \\
&\geqslant \frac{l_1}{\sqrt{l_1^2 + l_2^2}} \int_0^1 \|\sigma_1'(t)\|_{\mathcal{H}_1} dt + \frac{l_2}{\sqrt{l_1^2 + l_2^2}} \int_0^1 \|\sigma_2'(t)\|_{\mathcal{H}_2} dt \\
&= \sqrt{l_1^2 + l_2^2}
\end{aligned}$$

This shows that $d(x,y) \geqslant \sqrt{d_1^2(x_1,y_1) + d_2^2(x_2,y_2)}$. For the inequality in the opposite direction, we consider unit speed horizontal geodesics γ_1, γ_2 on M_1, M_2 respectively such that $\gamma_i(0) = x_i, \gamma_i(d_i(x_i,y_i)) = y_i$ for $i = 1, 2$. The existence of such geodesics are guaranteed by Assumption 3.1. Let us define $\sigma_i : [0,1] \to M_i$ such that $\sigma_i(t) = \gamma_i(t d_i(x_i, y_i))$. Then for $\sigma := (\sigma_1, \sigma_2)$, we have

$$\int_0^1 \|\sigma'(t)\|_{\mathcal{H}} dt = \int_0^1 \sqrt{\|\sigma_1'(t)\|_{\mathcal{H}_1}^2 + \|\sigma_2'(t)\|_{\mathcal{H}_2}^2} dt = \sqrt{d_1^2(x_1,y_1) + d_2^2(x_2,y_2)}.$$

This shows that $d(x,y) \leqslant \sqrt{d_1^2(x_1,y_1) + d_2^2(x_2,y_2)}$ and the proof of item (1) is complete.

To prove item (2), we first claim that for any $a, b \in M$, $d_M(a,b) \geqslant d_N(\pi(a), \pi(b))$. Indeed, for any horizontal curve $\sigma : [0,1] \to M$ with

$\sigma(0)=a,\sigma(1)=b$, $\pi\circ\sigma$ defines a horizontal curve on N such that $\pi\circ\sigma(0)=\pi(a),\pi\circ\sigma(1)=\pi(b)$. Moreover, for any $t\in[0,1]$, $\|(\pi\circ\sigma)'(t)\|_{\mathcal{H}_N}=\|d\pi_{\sigma(t)}(\sigma'(t))\|_{\mathcal{H}_N}=\|\sigma'(t)\|_{\mathcal{H}_M}$. Thus

$$\begin{aligned}d_M(a,b)&=\inf\left\{\int_0^1\|(\pi\circ\sigma)'(t)\|_{\mathcal{H}_N}dt:\sigma(0)=a,\sigma(1)=b,(\pi\circ\sigma)'\in\mathcal{H}_N\right\}\\&\geqslant d_N(\pi(a),\pi(b)).\end{aligned}$$

Now let $x,y\in N$. Then, any horizontal curve $\gamma:[0,1]\to N$ with $\gamma(0)=x,\gamma(1)=y$ possesses a horizontal lift $\widehat{\gamma}:[0,1]\to M$ such that $\pi\circ\widehat{\gamma}=\gamma$. Let us choose γ such that $d_N(x,y)=\int_0^1\|\gamma'(t)\|_{\mathcal{H}_N}dt$. This implies that

$$d_N(x,y)\leqslant d_M(\widehat{\gamma}(0),\widehat{\gamma}(1))\leqslant\int_0^1\|\widehat{\gamma}'(t)\|_{\mathcal{H}_M}dt=\int_0^1\|\gamma'(t)\|_{\mathcal{H}_N}dt=d_N(x,y).$$

Choosing $a=\widehat{\gamma}(0),b=\widehat{\gamma}(1)$ completes the proof of item (2). □

$$C_1^{(i)}(t)\|\nabla_{\mathcal{H}_i}\log P_t^Mf\|_{\mathcal{H}_i}^2\leqslant\frac{\Delta_{\mathcal{H}_i}^{M_i}P_t^Mf}{P_t^Mf}+C_2^{(i)}(t).\tag{3.4}$$

Now, (3.2) implies that $\sum_{i=1}^n\Delta_{\mathcal{H}_i}^{M_i}P_t^Mf=\Delta_{\mathcal{H}}^MP_t^Mf$, while the orthogonality of the horizontal distributions $(\mathcal{H}_i)_{i=1}^n$ implies

$$\sum_{i=1}^n\|\nabla_{\mathcal{H}_i}\log P_t^Mf\|_{\mathcal{H}_i}^2=\|\nabla_{\mathcal{H}}\log P_t^Mf\|_{\mathcal{H}}^2.$$

Therefore, the proof of (LYI) is concluded by adding the inequalities in (3.4) for each $1\leqslant i\leqslant n$.

3.3. **Proof of Theorem 3.3.** Denoting the horizontal heat semigroups on M,N by P^M and P^N respectively, we first claim that for all $f\in\mathcal{C}_c^\infty(N)$ and $t\geqslant0$

$$P_t^M(f\circ\pi)=P_t^Nf\circ\pi.\tag{3.5}$$

Since π is a sub-Riemannian submersion, then for any $x\in M$, $d\pi_x:\mathcal{H}_M\to\mathcal{H}_N$ is an isometry, which similarly to [27, Equation (4.3)] and [28, Equation (3.4)] leads to

$$\|\nabla_{\mathcal{H}}^M(f\circ\pi)\|_{\mathcal{H}_M}^2=\|\nabla_{\mathcal{H}}^Nf\|_{\mathcal{H}_N}^2\circ\pi,\tag{3.6}$$

where $\nabla_{\mathcal{H}}^M$ (resp. $\nabla_{\mathcal{H}}^N$) denotes the horizontal gradient on $(M,\mathcal{H}_M)$ (resp. $N,\mathcal{H}_N$). Moreover, since $\mu_N=\pi_*\mu_M$ we deduce that for any $f\in\mathcal{C}_c^\infty(N)$

$$\int_M\|\nabla_{\mathcal{H}}^M(f\circ\pi)\|_{\mathcal{H}_M}^2d\mu_M=\int_N\|\nabla_{\mathcal{H}}^Nf\|_{\mathcal{H}_N}^2d\mu_N.$$

As a consequence, one has for any $f \in \mathcal{C}^\infty(N)$, $\Delta_{\mathcal{H}}^M(f \circ \pi) = (\Delta_{\mathcal{H}}^N f) \circ \pi$. Therefore, for any $t \geqslant 0$ we have $P_t^M(f \circ \pi) = (P_t^N f) \circ \pi$, which proves our claim.

Now assume that (GB_p) holds for some $p \geqslant 1$. Then for any $t \geqslant 0$ and $f \in \mathcal{C}^\infty(H)$, using (3.6) and (3.5) we deduce

$$\begin{aligned}
\|\nabla_{\mathcal{H}}^N P_t^N f\|_{\mathcal{H}_N}^p \circ \pi &= \|\nabla_{\mathcal{H}}^M (P_t^N f \circ \pi)\|_{\mathcal{H}_M}^p \\
&= \|\nabla_{\mathcal{H}}^M P_t^M (f \circ \pi)\|_{\mathcal{H}_M}^p \\
&\leqslant C(p,t) P_t^M \|\nabla_{\mathcal{H}}^M (f \circ \pi)\|_{\mathcal{H}_M} \\
&= C(p,t) P_t^M (\|\nabla_{\mathcal{H}}^N f\|_{\mathcal{H}_N}^p \circ \pi) \\
&= C(p,t) P_t^N \|\nabla_{\mathcal{H}}^N f\|_{\mathcal{H}_N}^p \circ \pi.
\end{aligned}$$

Since π is surjective, (GB_p) follows for P_t^N. Stability of (RPI), (RLSI) and (LYI) under the mapping π is a direct consequence of (3.5).

4. Examples

4.1. **Kolmogorov diffusion on Euclidean spaces.**

We consider a Kolmogorov-type diffusion operator on $\mathbb{R}^d \times \mathbb{R}^d$ given by

$$Lf(x,y) = \sum_{j=1}^d x_j \frac{\partial f}{\partial y_j}(x,y) + \sum_{j=1}^d \sigma_j^2 \frac{\partial^2 f}{\partial x_j^2}(x,y), \tag{4.1}$$

where $\sigma_j^2 > 0$ for all $j = 1, \ldots, d$.

Proposition 4.1. *For all $t \geqslant 0$ and $f \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d)$ we have*

$$\|\nabla_x P_t f\|^p \leqslant P_t \left(\sum_{i=1}^d \left| \frac{\partial f}{\partial x_i} + t \frac{\partial f}{\partial y_i} \right|^2 \right)^{\frac{p}{2}} \quad \text{for all } p \geqslant 1, \tag{4.2}$$

$$\sum_{i=1}^d \left(\frac{\partial P_t f}{\partial x_i} - \frac{1}{2} t \frac{\partial P_t f}{\partial y_i} \right)^2 + \frac{t^2}{12} \left(\frac{\partial P_t f}{\partial y_i} \right)^2 \leqslant \frac{1}{\sigma^2 t} (P_t f^2 - (P_t f)^2), \tag{4.3}$$

$$\begin{aligned}
&\sum_{i=1}^d \left(\frac{\partial \log P_t f}{\partial x_i} - \frac{t}{2} \frac{\partial \log P_t f}{\partial y_i} \right)^2 + \frac{t^2}{12} \left(\frac{\partial \log P_t f}{\partial y_i} \right)^2 \\
&\leqslant \frac{1}{\sigma^2 t P_t f} (P_t(f \log f) - P_t f \log(P_t f)),
\end{aligned} \tag{4.4}$$

where $\sigma^2 := \min\{\sigma_j^2 : 1 \leqslant j \leqslant d\}$.

Proof. We first observe that by Jensen's inequality, it suffices to prove (4.2) for $p = 1$. For each $i = 1, \ldots, d$, let $P^{(i)}$ denote the semigroup generated by

$$L_i = x_i \frac{\partial}{\partial y_i} + \sigma_i^2 \frac{\partial^2}{\partial x_i^2}.$$

Then for each $t>0$, $P_t = P_t^{(1)} \otimes \cdots \otimes P_t^{(d)}$. Now for $d=1$, [15, Proposition 2.10] implies that for any $i=1,\dots,d$,

$$|\nabla_{x_i} P_t^{(i)} f| \leqslant P_t^{(i)} \left(\left| \frac{\partial f}{\partial x_i} + \frac{t}{2}\frac{\partial f}{\partial y_i} \right| \right). \tag{4.5}$$

Let $a_1,\dots,a_d \in \mathbb{R}$ be such that $\sum_{i=1}^d a_i^2 = 1$. Then using (4.5) followed by the Cauchy-Schwarz inequality we have

$$\sum_{i=1}^d a_i |\nabla_{x_i} P_t f| \leqslant P_t \left(\sum_{i=1}^d a_i \left| \frac{\partial f}{\partial x_i} + \frac{t}{2}\frac{\partial f}{\partial y_i} \right| \right) \leqslant P_t \left(\sum_{i=1}^d \left| \frac{\partial f}{\partial x_i} + \frac{t}{2}\frac{\partial f}{\partial y_i} \right|^2 \right)^{\frac{1}{2}}.$$

Optimizing with respect to $a_1,\dots,a_d$ yields the inequality in (4.2). Next, to prove (4.3) and (4.4), it is known from [15] that for any $d \geqslant 1$, the control distance associated to the squared gradient $\Gamma_t(f) = \sum_{i=1}^d (\frac{\partial f}{\partial x_i} - \frac{t}{2}\frac{\partial f}{\partial y_i})^2 + \frac{t^2}{12}\sum_{i=1}^d (\frac{\partial f}{\partial y_i})^2$ on $\mathbb{R}^d \times \mathbb{R}^d$ is given by

$$d_t((x_1,y_1),(x_2,y_2))^2 = 4\|x_1-x_2\|^2 + \frac{12}{t}\langle x_1-x_2, y_1-y_2\rangle + \frac{12}{t^2}\|y_1-y_2\|^2.$$

As a result, for any $f \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\Gamma_t(f)(x) = \lim_{r\to 0} \sup_{y: d_t(x,y)\leqslant r} \left| \frac{f(x)-f(y)}{d_t(x,y)} \right|.$$

Again, invoking [15, Proposition 2.7, 2.8] for $d=1$, both (4.3) and (4.4) follow from Theorem 2.1. □

4.2. **Kinetic Fokker-Planck equations.** Consider the stochastic differential equation on $\mathbb{R}^d \times \mathbb{R}^d$

$$\begin{cases} dX_t = Y_t dt \\ dY_t = dW_t - \nabla V(X_t)dt - Y_t dt, \end{cases} \tag{4.6}$$

where $\{W_t\}_{t\geqslant 0}$ is an $\mathbb{R}^d$-valued Brownian motion and $V(x) = \sum_{i=1}^d V_i(x_i)$ with $V_i \geqslant 0$. The semigroup associated with the equation is denoted P_t. Under the assumption of polynomial growth of the potentials V_i as in [39, Theorem A.8], that is,

$$|V_i''(x)| \leqslant C(1+V_i'(x)) \text{ for all } 1\leqslant i\leqslant n \text{ and for some constant } C, \tag{4.7}$$

we have the following result.

Proposition 4.2. *Assume that* (4.7) *holds. Then, there exists a dimension-independent constant* c *such that for all* $f \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d)$ *and* $t>0$,

$$\|\nabla P_t f\|^2 \leqslant \frac{c}{(t\wedge 1)^3} P_t f^2. \tag{4.8}$$

The constant c only depends on the constant C in (4.7). It is worth noting that both in the pointwise bound (see [30]) and the L^2 bound (see [39, Theorem A.8]), the constants appearing on the right hand side of the inequalities of the form (4.8) are dimension-dependent.

Proof. Let $P^{(i)}$ denote the semigroup associated to the one-dimensional diffusion

$$\begin{aligned} dX_t^{(i)} &= dY_t^{(i)} \\ dY_t^{(i)} &= dW_t^{(i)} - V_i'(X_t)dt - Y_t^{(i)}dt. \end{aligned}$$

Then $P = P^{(1)} \otimes \cdots \otimes P^{(d)}$. Invoking [30, Corollary 3.3], for each $1 \leqslant i \leqslant d$ we have

$$\|\nabla_{(x_i,y_i)} P_t^{(i)} f\|^2 \leqslant \frac{c}{(t \wedge 1)^3} P_t^{(i)} f^2.$$

Hence the proposition follows from Theorem 2.1. □

Proposition 4.3. *Assume that there exist constants $m, M > 0$, such that $\sqrt{M} - \sqrt{m} \leqslant 1$ and*

$$m \leqslant \nabla^2 V \leqslant M.$$

There exist dimension-independent constants $c_1, c_2 > 0$ such that for all $f \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d)$ and $t > 0$,

$$\|\nabla P_t f\|^2 \leqslant c_1 e^{-c_2 t} P_t(\|\nabla f\|^2) \tag{4.9}$$

Proof. The proposition follows from Theorem 2.1 and [6, Theorem 2.12]. □

4.3. **Lie groups with transverse symmetries.** Denote by G a real or complex Lie group, by $\mathfrak{g} \cong T_e\mathrm{G}$ its Lie algebra identified with the tangent space at the identity e. For each $A \in \mathfrak{g}$, let $\widetilde{A}$ denote the unique extension of A to a left-invariant vector field on G. For $n \geqslant 1$ and $1 \leqslant m \leqslant n$ we consider a $2n+m$-dimensional Lie group G whose Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ has an orthonormal basis $\{X_1, \ldots, X_{2n}, Z_1, \ldots, Z_m\}$ such that for $1 \leqslant l \leqslant m$

$$[X_i, X_j] = \sum_{l=1}^{m} A_{l;i,j} Z_l \quad \text{and} \quad [X_i, Z_l] = \sum_{j=1}^{2n} R_{l;i,j} X_j,$$

where the family of matrices $\{A_l, R_l\}_{l=1}^n$ satisfies the following conditions

(A) $\{A_l, R_l\}_{l=1}^m$ are $2n \times 2n$ skew-symmetric matrices.
(B) $\{A_l\}_{l=1}^m$ are linearly independent and of full rank.
(C) $A_l A_k = A_k A_l$, $R_l R_k = R_k R_l$, $A_l R_k = R_k A_l$ for all $1 \leqslant l, k \leqslant m$.

Note that these assumptions imply that the corresponding left-invariant vector fields $(\widetilde{X}_i)_{i=1}^{2n}$ define a sub-Riemannian structure on G with the horizontal distribution $\mathcal{H}_\mathrm{G} = \mathrm{Span}\{\widetilde{X}_i : 1 \leqslant i \leqslant 2n\}$, and the sub-Riemannian metric $(g_x(\cdot,\cdot))_{x\in\mathrm{G}}$ given by

$$g_x(\widetilde{X}(x), \widetilde{Y}(x)) = \langle X, Y \rangle \text{ for all } x \in M, \; X, Y \in \mathcal{H}_\mathrm{G}.$$

We consider the sub-Laplacian on G defined by

$$\Delta_{\mathcal{H}}^{\mathrm{G}} = \sum_{i=1}^{2n} \widetilde{X}_i^2.$$

We note that G is a sub-Riemannian manifold with a transverse symmetry as described in [13].

Example. Let us consider $n = m = 1$. Then,

- the 3-dimensional Heisenberg group $\mathbb{H}$ corresponds to the case when $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $R_1 = \mathbf{0}_2$.
- SU(2), the group of all 2×2 unitary matrices with determinant equal to 1, corresponds to the case when $A_1 = R_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
- SL(2), the group of all invertible real matrices with determinant equal to 1, corresponds to the case when $A_1 = -R_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

We start by explaining that while the techniques introduced in [13] work in this setting, the constants one gets are dimension-dependent. In terms of the notation in the aforementioned paper, it follows that for all $f \in \mathcal{C}^\infty(\mathrm{G})$

$$\mathcal{R}(f) = \sum_{i=1}^{2n} \sum_{j=1}^{2n} \left(\sum_{k=1}^{2n} A_{l;i,k} R_{l;k,j} \right) X_i f X_j f + \frac{1}{2} \sum_{1 \leqslant i \leqslant j \leqslant 2n} \left(\sum_{l=1}^{m} A_{l;i,j} Z_l f \right)^2$$

$$\mathcal{T}(f) = \sum_{l=1}^{m} \sum_{i=1}^{2n} \left(\sum_{j=1}^{2n} A_{l;i,j} X_j f \right)^2 = \sum_{l=1}^{m} \| A_l \nabla_{\mathcal{H}} f \|_{\mathcal{H}}^2.$$

A simple computation shows that for all $f \in \mathcal{C}^\infty(\mathrm{G})$, one has

$$\mathcal{R}(f) \geqslant \rho \| \nabla_{\mathcal{H}} f \|_{\mathcal{H}}^2 + \gamma \| \nabla_{\mathcal{V}} f \|_{\mathcal{V}}^2,$$
$$\mathcal{T}(f) \leqslant \kappa \| \nabla_{\mathcal{V}} f \|_{\mathcal{V}}^2,$$

where ρ is the minimum eigenvalue of

$$\Lambda = \sum_{l=1}^{m} A_l R_l, \tag{4.10}$$

and

$$\gamma = \frac{1}{2} \inf_{\|x\|=1} \sum_{1 \leqslant i \leqslant j \leqslant 2n} \left(\sum_{l=1}^{m} A_{l;i,j} x_j \right)^2, \quad \kappa = \sup_{\|x\|=1} \sum_{l=1}^{m} \| A_l x \|^2. \tag{4.11}$$

As a result, [13, Theorem 2.19] implies that G satisfies the generalized curvature-dimension inequality $\mathrm{CD}(\rho, \gamma, \kappa, 2n)$ with respect to the sub-Laplacian $\Delta_{\mathcal{H}}^{\mathrm{G}}$. However, in general the parameters ρ, γ and κ depend on the dimension on G, see Remark 4.5 and Proposition 4.6 below.

To this end, we explain our method which leads to gradient estimates for the heat kernel $P_t^{\mathrm{G}} = e^{t\Delta_{\mathcal{H}}^{\mathrm{G}}}$ on G that do not depend on γ, κ, and n. First, we show that the Lie group G can be reduced to a product of 3-dimensional model groups introduced in [4]. We note that Λ defined in (4.10) is a symmetric matrix which is unitary equivalent to a diagonal matrix of the form

$$D = \begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_n \end{pmatrix}$$

where $D_i = \rho_i I_2$, I_2 being the 2×2 identity matrix, see Lemma 4.7 for details. For each $1 \leqslant i \leqslant n$, let $\mathbb{M}(\rho_i)$ denote the 3-dimensional model space as introduced in [4], that is, $\mathbb{M}(\rho_i)$ is a simply connected Lie group whose Lie algebra is given by $\mathfrak{m}_i = \mathrm{Span}\{X'_{2i-1}, X'_{2i}, Z'_i\}$ such that

$$[X'_{2i-1}, X'_{2i}] = Z'_i, \quad [X'_{2i-1}, Z'_i] = -\rho_i X'_{2i}, \quad [X'_{2i}, Z'_i] = \rho_i X'_{2i-1}, \tag{4.12}$$

and $\left\{X'_{2i-1}, X'_{2i}, Z'_i\right\}$ forms an orthonormal basis for $\mathfrak{m}_i$. Denoting

$$\mathbb{M} = \mathbb{M}(\rho_1) \times \cdots \times \mathbb{M}(\rho_n), \tag{4.13}$$

we observe that $\mathbb{M}$ is also a connected Lie group with Lie algebra $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_n$. Due to the existence of a sub-Riemannian submersion from $\mathbb{M}$ to G, thanks to Proposition 4.9), we obtain the following dimension independent inequalities for the horizontal heat semigroup $(P_t^{\mathrm{G}})_{t\geqslant 0}$.

Theorem 4.4. *The horizontal heat semigroup $(P_t^{\mathrm{G}})_{t\geqslant 0}$ on* G *satisfies* (RPI), (RLSI), *and* (LYI). *More precisely, one has for all $t > 0$ and $f \in \mathcal{C}_c^\infty(\mathrm{G})$,*

$$\|\nabla_{\mathcal{H}} P_t^{\mathrm{G}} f\|_{\mathcal{H}}^2 \leqslant \frac{5+\rho^- t}{2t}(P_t^{\mathrm{G}} f^2 - (P_t^{\mathrm{G}} f)^2) \tag{4.14}$$

and when $f > 0$,

$$P_t^{\mathrm{G}} f \|\nabla_{\mathcal{H}} \log P_t^{\mathrm{G}} f\|_{\mathcal{H}}^2 \leqslant \frac{5+\rho^- t}{t}\left(P_t^{\mathrm{G}}(f\log f) - P_t^{\mathrm{G}} f \log(P_t^{\mathrm{G}} f)\right), \tag{4.15}$$

$$\|\nabla_{\mathcal{H}} \log P_t^{\mathrm{G}} f\|_{\mathcal{H}}^2 \leqslant \left(4 - \frac{2\rho t}{3}\right)\frac{\Delta_{\mathcal{H}}^{\mathrm{G}} P_t^{\mathrm{G}} f}{P_t^{\mathrm{G}} f} + \frac{n\rho^2}{3}t - 4n\rho + \frac{16n}{t}, \tag{4.16}$$

where Λ is defined in (4.10)*, ρ is the minimum eigenvalue of Λ, and $\rho^- = \max\{0, -\rho\}$. Additionally, when Λ is non-negative definite, we have the following results.*

(1) $(P_t^{\mathrm{G}})_{t\geqslant 0}$ satisfies (GB$_p$)*, that is, for all $p > 0$, $f \in \mathcal{C}_c^\infty(\mathrm{G})$ and $t > 0$,*

$$\|\nabla_{\mathcal{H}} P_t^{\mathrm{G}} f\|_{\mathcal{H}} \leqslant C_p e^{-t\rho}\left(P_t^{\mathrm{G}} \|\nabla_{\mathcal{H}} f\|^p\right)^{\frac{1}{p}}, \tag{4.17}$$

In fact, one has

$$\|\nabla_{\mathcal{H}} P_t^{\mathrm{G}} f\|_{\mathcal{H}}^2 + \|\nabla_{\mathcal{V}} P_t^{\mathrm{G}} f\|_{\mathcal{V}}^2 \leqslant C(P_t^{\mathrm{G}} \|\nabla_{\mathcal{H}} f\|_{\mathcal{H}}^2 + P_t^{\mathrm{G}} \|\nabla_{\mathcal{V}} f\|_{\mathcal{V}}^2) \tag{4.18}$$

for some positive constant C independent of n.

(2) For all $0 < s < t$ *and nonnegative function* $f \in \mathcal{C}_b(\mathrm{G})$ *we have*

$$P_t^{\mathrm{G}} f(x) \leqslant P_t^{\mathrm{G}} f(y) \left(\frac{t}{s}\right)^{8n} \exp\left(\frac{4d^2(x,y)}{t-s}\right). \tag{PHI}$$

Remark 4.5. We note that (RPI), (RLSI) and (LYI) for G could also be obtained using the generalized curvature-dimension inequality using the results in [9]. More precisely, [9, Proposition 3.1 and 3.2] shows that the generalized curvature-dimension inequality $\mathrm{CD}(\rho, \gamma, \kappa, \infty)$ implies

$$tP_t^{\mathrm{G}} \|\nabla_{\mathcal{H}} \log P_t^{\mathrm{G}} f\|_{\mathcal{H}}^2 \leqslant \left(1 + \frac{2\kappa}{\gamma} + \rho^-\right) \left(P_t^{\mathrm{G}}(f \log f) - (P_t^{\mathrm{G}} f)(\log P_t^{\mathrm{G}} f)\right) \tag{4.19}$$

$$t\|\nabla_{\mathcal{H}} P_t^{\mathrm{G}} f\|_{\mathcal{H}}^2 \leqslant \frac{1}{2}\left(1 + \frac{2\kappa}{\gamma} + \rho^-\right)(P_t^{\mathrm{G}} f^2 - (P_t^{\mathrm{G}} f)^2), \tag{4.20}$$

and [13, Theorem 6.1] implies the Li-Yau estimate

$$\begin{aligned}&\|\nabla_{\mathcal{H}} \log P_t^{\mathrm{G}} f\|_{\mathcal{H}}^2 \\ &\leqslant \left(1 + \frac{3\kappa}{2\gamma} - \frac{2\rho t}{3}\right) \frac{\Delta_{\mathcal{H}} P_t^{\mathrm{G}} f}{P_t^{\mathrm{G}} f} + \frac{n\rho^2 t}{3} - n\rho\left(1 + \frac{3\kappa}{2\gamma}\right) + \frac{n(1 + \frac{3\kappa}{2\gamma})^2}{t}.\end{aligned}$$

In the next proposition we show that the constants in Theorem 4.4 are sharper.

Proposition 4.6. *Let* ρ, γ *and* κ *be as in* (4.11)*. Then for any* $n \geqslant 1$ *and* $m = n$, $\frac{\kappa}{\gamma} \geqslant 2$. *Moreover, for any* $1 \leqslant m \leqslant n$, *one can choose a family of skew-symmetric commuting matrices* $\{A_l\}_{1 \leqslant l \leqslant m}$ *such that* $\frac{\kappa}{\gamma} = \frac{2(n^m - 1)}{n(n-1)}$.

To prove the above results, we first observe that the vector fields $\{\widetilde{X}_i\}_{i=1}^{2n}$ can be *decoupled* without changing the Lie algebra structure and the sub-Laplacian.

Lemma 4.7. *Without loss of generality we can assume that*

$$\begin{aligned} R_l &= \mathbf{0}_{2n-2r} \oplus \begin{pmatrix} 0 & -\mu_{1l} \\ \mu_{1l} & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & -\mu_{rl} \\ \mu_{rl} & 0 \end{pmatrix} \\ A_l &= \begin{pmatrix} 0 & -\lambda_{1l} \\ \lambda_{1l} & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & -\lambda_{nl} \\ \lambda_{nl} & 0 \end{pmatrix} \end{aligned} \tag{4.21}$$

for some $0 \leqslant r \leqslant n$ *and real numbers* λ_{il}, μ_{il}. *Moreover, for each* $1 \leqslant l \leqslant m$, *there exist* $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant r$ *such that* $\lambda_{il} \neq 0, \mu_{jl} \neq 0$.

Remark 4.8. We note that the linear independence condition in (B) equivalent to the full column rank of the matrix $(\lambda_{il})_{1 \leqslant i \leqslant n, 1 \leqslant l \leqslant m}$.

Proof. Observe that $(A_l, R_l)_{1 \leqslant l \leqslant m}$ is a family of normal matrices. As a result, [33, Theorem 2.5.15] implies that there exists a $2n \times 2n$ orthogonal matrix U such that for all $1 \leqslant l \leqslant m$

$$UA_lU^\top = \begin{pmatrix} 0 & -\lambda_{1l} \\ \lambda_{1l} & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & -\lambda_{nl} \\ \lambda_{nl} & 0 \end{pmatrix}$$
$$UR_lU^\top = \mathbf{0}_{2n-2r} \oplus \begin{pmatrix} 0 & \mu_{1l} \\ -\mu_{1l} & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \mu_{rl} \\ -\mu_{rl} & 0 \end{pmatrix},$$

where $\mathbf{0}_k$ denotes the zero matrix of dimension $k \times k$. Writing $U = (U_{ij})_{1 \leqslant i,j \leqslant 2n}$, let us now define

$$Y_i = \sum_{j=1}^{2n} U_{ij} X_j.$$

After a simple computation, it follows that

$$[Y_i, Y_j] = \begin{cases} \sum_{l=1}^m \lambda_{sl} Z_l & \text{if } i = 2s-1, j = 2s, 1 \leqslant s \leqslant n \\ 0 & \text{otherwise.} \end{cases} \tag{4.22}$$

and

$$[Y_i, Z_l] = \begin{cases} 0 & \text{if } 1 \leqslant i \leqslant 2r \\ -\mu_{sl} Y_{2s} & \text{if } i = 2s-1, r < s \leqslant n \\ \mu_{sl} Y_{2s-1} & \text{if } i = 2s,\ r < s \leqslant n \\ 0 & \text{otherwise} \end{cases} \tag{4.23}$$

Moreover, $\{\widetilde{Y}_i\}_{i=1}^{2n}$ is also a left-invariant orthonormal frame in G. Therefore from [25, Theorem 3.6] one gets

$$\Delta_{\mathcal{H}}^{\mathrm{G}} = \sum_{i=1}^{2n} \widetilde{X}_i^2 = \sum_{i=1}^{2n} \widetilde{Y}_i^2.$$

This proves the proposition. □

Proposition 4.9. *Let* $\mathbb{M}$ *be defined as in* (4.13)*. Then there exists a sub-Riemannian submersion* $\Phi : \mathbb{M} \to \mathrm{G}$.

Proof. We note that for each $1 \leqslant i \leqslant n$, $\rho_i = \sum_{l=1}^m \lambda_{il}\mu_{il}$ is an eigenvalue of Λ. Writing $\mathfrak{m}_i = \operatorname{Span}\{X'_{2i-1}, X'_{2i}, Z'_i\}$ as the Lie algebra of $\mathbb{M}_i$, we define the following linear map $\phi : \oplus_{i=1}^n \mathfrak{m}_i \to \mathfrak{g}$ such that

$$\phi(X'_{2i-1}) = X_{2i-1}, \quad \phi(X'_{2i}) = X_{2i}, \quad \phi(Z'_i) = \sum_{l=1}^m \lambda_{il} Z_l.$$

From Remark 4.8 it follows that ϕ is a surjective linear map. We note that for any $1 \leqslant i \leqslant n$ and $1 \leqslant l \leqslant m$ one gets

$$\begin{aligned} \phi([X'_{2i-1}, X'_{2i}]) &= \phi(Z'_i) \\ &= \sum_{l=1}^{m} \lambda_{il} Z_l \\ &= [X_{2i-1}, X_{2i}] \\ &= [\phi(X'_{2i-1}), \phi(X'_{2i})]. \end{aligned} \tag{4.24}$$

Also

$$\begin{aligned} \phi([X'_{2i-1}, Z'_i]) &= -\rho_i \phi(X'_{2i}) \\ &= -\rho_i X_{2i} \\ &= -\sum_{l=1}^{m} \lambda_{il}\mu_{il} X_{2i} \\ &= [\phi(X'_{2i-1}), \phi(Z'_i)]. \end{aligned} \tag{4.25}$$

This shows that $\phi : \oplus_{i=1}^{n} \mathfrak{m}_i \to \mathfrak{g}$ is a surjective Lie algebra homomorphism. Since $\mathbb{M}$ is also connected, there exists a surjective Lie group homomorphism $\Phi : \mathbb{M} \to \mathrm{G}$ such that $\phi = (d\Phi)_e$, e being the identity element of $\mathbb{M}$. Moreover, ϕ maps an orthonormal basis of $\mathcal{H}_{\mathbb{M}}$ to the same of $\mathcal{H}_{\mathrm{G}}$, which proves that Φ is a sub-Riemannian submersion. $\square$

In the next lemma, we show that the model spaces $\mathbb{M}(\rho_i)$ can be replaced by $\mathbb{H}$, $\mathrm{SU}(2)$ or $\widetilde{\mathrm{SL}}(2)$ depending on ρ_i, where $\widetilde{\mathrm{SL}}(2)$ is the universal cover of $\mathrm{SL}(2)$. This lemma will be useful to prove (4.17). Let $P^{(i)}$ denote the horizontal heat semigroup generated by $\Delta_i = (X'_{2i-1})^2 + (X'_{2i})^2$ on $\mathbb{M}(\rho_i)$.

Lemma 4.10. *For each $1 \leqslant i \leqslant n$, there exists a Lie group isomorphism $\pi_i : \mathbb{M}(\rho_i) \to H_i$, where*

$$H_i = \begin{cases} \mathrm{SU}(2) & \text{if } \rho_i > 0 \\ \widetilde{\mathrm{SL}}(2) & \text{if } \rho_i < 0 \\ \mathbb{H} & \text{if } \rho_i = 0 \end{cases}$$

such that $P_t^{(i)}(f \circ \pi_i) = Q_{t\alpha_i}^{(i)} f \circ \pi_i$ for all $t \geqslant 0$ and $f \in \mathcal{C}^\infty(H_i)$, where $\alpha_i = \mathbb{1}_{\{\rho_i = 0\}} + |\rho_i| \mathbb{1}_{\{\rho_i \neq 0\}}$.

Proof. When $\rho_i = 0$, it is easy to see that $\mathfrak{m}_i$ is isometrically isomorphic to the Lie algebra of $\mathbb{H}$. For $\rho_i \neq 0$, let us define $Y_{2i-1} = \frac{X'_{2i-1}}{\sqrt{|\rho_i|}}, Y_{2i} = \mathrm{sign}(\rho_i) \frac{X'_{2i}}{\sqrt{|\rho_i|}}, W_i = \frac{Z'_i}{\rho_i}$. Then one has

$$[Y_{2i-1}, Y_{2i}] = W_i,\ [Y_{2i-1}, W_i] = -\mathrm{sign}(\rho_i) Y_{2i},\ [Y_{2i}, W_i] = \mathrm{sign}(\rho_i) Y_{2i-1}. \tag{4.26}$$

Considering $\left\{\frac{X'_{2i-1}}{\sqrt{|\rho_i|}}, \operatorname{sign}(\rho_i)\frac{X'_{2i}}{\sqrt{|\rho_i|}}\right\}$ as an orthonormal basis of SU(2) or $\widetilde{\mathrm{SL}}(2)$, the lemma follows from Theorem 3.3. □

Proof of Theorem 4.4. We first show (4.14) and (4.15). By Theorem 3.2, Theorem 3.3 and Theorem 4.9, it suffices to show the validity of (RPI), (RLSI) for each $\mathbb{M}(\rho_i)$, which follows from the generalized curvature-dimension inequality proved in [13, Proposition 2.1] in conjunction with [9, Proposition 3.1, 3.2]. For (4.16), using [13, Remark 6.2], we note that for any i, $\mathbb{M}(\rho_i)$ satisfies $\mathrm{CD}(\rho, \frac{1}{2}, \frac{1}{2})$. As a result, (4.16) follows from [13, Theorem 6.1, Eq. (6.1)] and Theorem 3.2, Theorem 3.3. To prove (1), we note that the nonnegativity of Λ enforces that $\rho_i \geqslant 0$ for each $i = 1, \dots, d$. As a result, Lemma 4.10 implies that $\mathbb{M}(\rho_i) \cong \mathbb{H}$ or SU(2) according to $\rho_i = 0$ or $\rho_i > 0$. Using Driver-Melcher inequality [22] or H. Q. Li inequality [36] for $\rho_i = 0$ and Lemma 4.10 together with [8, Theorem 4.10] for $\rho_i > 0$, it follows that for all $\rho_i \geqslant 0$, $p > 1$ and $f \in \mathcal{C}_c^\infty(\mathbb{M}(\rho_i))$

$$\|\nabla_{\mathcal{H}}^{\mathbb{M}(\rho_i)} P_t^{\mathbb{M}(\rho_i)} f\|_{\mathcal{H}}^p \leqslant C_p^{(i)} e^{-\rho_i t} P_t^{\mathbb{M}(\rho_i)} \|\nabla_{\mathcal{H}}^{\mathbb{M}(\rho_i)} f\|_{\mathcal{H}}^p. \tag{4.27}$$

Therefore (4.17) follows from Theorem 3.2 and Theorem 3.3 as well. For (4.18), we first observe that $[X_i, Z_l] = 0$ for any $1 \leqslant i \leqslant 2n$ and $1 \leqslant l \leqslant m$. Therefore, for any $t \geqslant 0$, $\nabla_{\mathcal{V}} P_t^{\mathrm{G}} = P_t^{\mathrm{G}} \nabla_{\mathcal{V}}$, which entails that for all $f \in \mathcal{C}^\infty(\mathrm{G})$,

$$\|\nabla_{\mathcal{V}} P_t^{\mathrm{G}} f\|^2 \leqslant P_t^{\mathrm{G}} \|\nabla_{\mathcal{V}} f\|^2,$$

proving (4.18). To prove (2), we again resort to [13, Theorem 7.1] to argue that for any $1 \leqslant i \leqslant n$, $0 < s < t$ and nonnegative $f \in \mathcal{C}_b(\mathbb{M}(\rho_i))$,

$$P_s^{(i)} f(x_i) \leqslant P_t^{(i)} f(y_i) \left(\frac{t}{s}\right)^8 \exp\left(\frac{4d_i^2(x_i, y_i)}{t-s}\right), \quad \forall x_i, y_i \in \mathbb{M}(\rho_i),$$

where d_i denotes the Carnot-Carathéodory metric on $\mathbb{M}(\rho_i)$. As a result, (PHI) follows from an argument very similar to the proof of Wang-Harnack inequality in Theorem 2.1 accompanied with Lemma 3.4 and Theorem 4.9. □

Proof of Proposition 4.6. As noted in Lemma 4.7, without loss of generality we can assume that

$$A_l = \begin{pmatrix} 0 & -\lambda_{1l} \\ \lambda_{1l} & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & -\lambda_{nl} \\ \lambda_{nl} & 0 \end{pmatrix}.$$

Therefore, κ and γ becomes, respectively,

$$\kappa = \sup_{\|\mathbf{x}\|=1} \sum_{j=1}^n \sum_{l=1}^m \lambda_{jl}^2 (x_{2j-1}^2 + x_{2j}^2) = \max\left\{1 \leqslant j \leqslant n : \sum_{l=1}^m \lambda_{jl}^2\right\},$$

$$\gamma = \frac{1}{2} \inf_{\|\mathbf{z}\|=1} \sum_{i=1}^{n} \sum_{l=1}^{m} \lambda_{il}^2 z_l^2 = \min \left\{ 1 \leqslant l \leqslant m : \sum_{i=1}^{n} \lambda_{il}^2 \right\}.$$

When $m = n$, the above computation implies that $\kappa \geqslant 2\gamma$, which proves the first statement of the proposition.

Let us now choose $\lambda_{il} = i^{l-1}$, $1 \leqslant i \leqslant n, 1 \leqslant l \leqslant m$. Note that this choice of λ_{il} ensures that $A_1, \ldots, A_m$ are linearly independent. Now, with this choice of $(A_l)_{1\leqslant l \leqslant m}$, we have

$$\kappa = \max \left\{ 1 \leqslant j \leqslant n : \sum_{l=1}^{m} j^{l-1} \right\} = \frac{n^m - 1}{n-1},$$
$$\gamma = \frac{1}{2} \min \left\{ 1 \leqslant l \leqslant m : \sum_{i=1}^{n} i^{l-1} \right\} = \frac{n}{2}.$$

As a result, $\kappa/\gamma = \frac{2(n^m-1)}{n(n-1)}$, which completes the proof of the proposition. □

The next example is a special case where we assume $R_l = \mathbf{0}_{2n}$ for all $1 \leqslant l \leqslant m$.

4.3.1. *Step 2 homogeneous Carnot group.* Consider the following step 2 homogeneous Carnot group $\mathbb{G}_{n,m} = \mathbb{R}^{2n} \times \mathbb{R}^m$ with the group operation

$$(\mathbf{x}, \mathbf{y}) \star (\mathbf{x}', \mathbf{y}') = (\mathbf{x} + \mathbf{x}', y_1 + \langle A_1 \mathbf{x}, \mathbf{x}' \rangle, \ldots, y_m + \langle A_m \mathbf{x}, \mathbf{x}' \rangle), \tag{4.28}$$

where $\mathbf{x} \in \mathbb{R}^{2n}$, $\mathbf{y} \in \mathbb{R}^m$, and $(A_l)_{1\leqslant l\leqslant m}$ is a family of commuting, linearly independent skew-symmetric real matrices. The linear independence of $(A_l)_{1\leqslant l\leqslant m}$ implies that the Lie algebra of $\mathbb{G}_{n,m}$ is generated by the left invariant vector fields $\{X_1, \ldots, X_{2n}\}$ where

$$X_i = \frac{\partial}{\partial x_i} - \frac{1}{2} \sum_{l=1}^{m} \langle A_l \mathbf{x}, \mathbf{e}_i \rangle \frac{\partial}{\partial y_l}. \tag{4.29}$$

After some simple computations, it follows that for any $1 \leqslant i \neq j \leqslant 2n$,

$$[X_i, X_j] = \sum_{l=1}^{n} A_{l;i,j} \frac{\partial}{\partial y_l},$$

where $A_{l;i,j}$ denotes the $(i,j)^{th}$ entry of A_l. Also, $[X_i, \frac{\partial}{\partial y_l}] = 0$ for all $1 \leqslant i \leqslant 2n$ and $1 \leqslant l \leqslant m$. In particular, when $m = 1$ and

$$A_1 = \begin{pmatrix} S & & \\ & \ddots & \\ & & S \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then $\mathbb{G}_{n,1} = \mathbb{H}_{2n+1}$, the Heisenberg group of dimension $2n + 1$.

Proposition 4.11. *For any $p \geqslant 1$, There exists a positive constants C such that for all $t > 0$ and $f \in \mathcal{C}_c^\infty(\mathbb{G}_{n,m})$*

$$\|\nabla_{\mathcal{H}} P_t^{\mathbb{G}_{n,m}} f\|_{\mathcal{H}} \leqslant C P_t^{\mathbb{G}_{n,m}} \|\nabla_{\mathcal{H}} f\|_{\mathcal{H}} \tag{4.30}$$

$$\|\nabla_{\mathcal{H}} P_t^{\mathbb{G}_{n,m}} f\|_{\mathcal{H}}^2 \leqslant \frac{1}{t}(P_t^{\mathbb{G}_{n,m}} f^2 - (P_t^{\mathbb{G}_{n,m}} f)^2), \tag{4.31}$$

and for all $f \in \mathcal{C}_c^\infty(\mathbb{G}_{n,m})$ with $f > 0$,

$$P_t^{\mathbb{G}_{n,m}} f \|\nabla_{\mathcal{H}} P_t^{\mathbb{G}_{n,m}} f\|_{\mathcal{H}}^2 \leqslant \frac{5}{t}(P_t^{\mathbb{G}_{n,m}}(f \ln f) - P_t^{\mathbb{G}_{n,m}} f \ln(P_t^{\mathbb{G}_{n,m}} f)) \tag{4.32}$$

$$\|\nabla_{\mathcal{H}} \ln P_t^{\mathbb{G}_{n,m}} f\|_{\mathcal{H}}^2 \leqslant 4\frac{\Delta_{\mathcal{H}} P_t^{\mathbb{G}_{n,m}} f}{P_t^{\mathbb{G}_{n,m}} f} + \frac{16n}{t} \tag{4.33}$$

Remark 4.12. We note that (4.31) for such groups has already been studied in [41, Theorem 2.1] using a derivative type formula for a class of diffusion Markov semigroups. However, the constants obtained there grow with the dimension of the group while our results show that the constants can be chosen independent of the dimension. This can be used to extend our results to infinite dimensions.

Remark 4.13. When $\mathbb{G}_{n,m} = \mathbb{H}_{2n+1}$, the Heisenberg group of dimension $2n+1$, Baudoin and Bonnefont [10, Corollary 2.7] showed that (RPI) holds with $C(t) = \frac{n+1}{2nt}$. Clearly, this estimate is sharper than (4.31) when $n \geqslant 2$. In general, for any two-step Carnot group, the optimal choice for $C(t)$ is bounded above by $\frac{2n+2m}{2} = n + m$, see [10, Proposition 2.6], which grows with the dimension of the group.

Remark 4.14. When $m = 1$, $\mathbb{G}_{n,m}$ is a non-isotropic Heisenberg group, for which the gradient bound (4.30) has been obtained in [37,42]. See also [27,28] for logarithmic Sobolev inequalities on the non-isotropic Heisenberg groups.

Proof of Proposition 4.11. In terms of the notation in Theorem 4.4, we note that $\Lambda = 0$. Therefore, (4.30), (4.32) and (4.33) follows from Theorem 4.4. To prove (4.31), we note that Theorem 4.9 implies that there exists a sub-Riemannian submersion from $\mathbb{H}^n$ to $\mathbb{G}_{n,m}$. Also, the optimal reverse Poincaré inequality for the Heisenberg group $\mathbb{H}$ has been proved in [10, Corollary 2.7]. As a result, (4.31) is a direct consequence of Theorem 3.2 and Theorem 3.3. □

4.4. **Hypoelliptic heat equation on** SO(3)**.** The Lie group SO(3) is the group of 3×3 real orthogonal matrices of determinant 1. A basis of the Lie algebra $\mathfrak{so}(3)$ is $\{X_1, X_2, Z\}$, where

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

They satisfy the following commutation rules

$$[X_1, X_2] = -Z, \quad [X_1, Z] = X_2, \quad [X_2, Z] = -X_1. \tag{4.34}$$

Let $\widetilde{X}_1, \widetilde{X}_2$ denote the left-invariant vector fields corresponding to X_1, X_2. Then, $\widetilde{X}_1, \widetilde{X}_2$ satisfy the Hörmander's condition and therefore SO(3) is a sub-Riemannian manifold. We consider the sub-Laplacian on SO(3) defined by

$$\Delta_{\mathcal{H}}^{\mathrm{SO}(3)} = \widetilde{X}_1^2 + \widetilde{X}_2^2.$$

Using the notations from Section 4.3, we note that $A_1 = R_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. As a consequence, we have the following result.

Corollary 4.15. *The horizontal heat semigroup $P_t^{\mathrm{SO}(3)}$ generated by $\Delta_{\mathcal{H}}^{\mathrm{SO}(3)}$ satisfies* (4.14), (4.15), (4.16) *and* (4.17) *with $\rho = 1$.*

4.5. **Hypoelliptic heat equation on** SO(4)**.** The Lie algebra of SO(4) is given by

$$\mathfrak{so}(4) = \{A \in \mathfrak{gl}_4(\mathbb{R}) : \ A + A^\top = 0\},$$

endowed with the inner product $\langle A, B\rangle_{\mathfrak{so}(4)} = \operatorname{tr}(AB^\top)$. Let $E_{i,j} \in \mathfrak{gl}_4(\mathbb{R})$ denote the matrix whose $(i,j)^{th}$ entry equals 1 and the rest of the entries are equal to 0. We consider

$$X_j = E_{j+1,1} - E_{1,j+1}, \quad 1 \leqslant j \leqslant 3,$$

and let us write

$$Z_1 = [X_2, X_3], \ Z_2 = [X_3, X_1], \ Z_3 = [X_1, X_2].$$

Then, a simple computation shows that

$$\begin{aligned} &[Z_1, Z_2] = Z_3, \quad [Z_2, Z_3] = Z_1, \quad [Z_3, Z_1] = Z_2, \\ &[Z_i, X_j] = \epsilon_{ijk} X_k, \end{aligned} \tag{4.35}$$

where

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{otherwise} \end{cases}$$

Moreover, $\{X_1, X_2, X_3, Z_1, Z_2, Z_3\}$ is an orthonormal basis for $\mathfrak{so}(4)$. Writing the corresponding left-invariant vector fields by $\widetilde{X}_i, \widetilde{Z}_i$ for $i = 1, 2, 3$, the Laplace-Beltrami operator on SO(4) is given by

$$\Delta^{\mathrm{SO}(4)} = \widetilde{X}_1^2 + \widetilde{X}_2^2 + \widetilde{X}_3^2 + \widetilde{Z}_1^2 + \widetilde{Z}_2^2 + \widetilde{Z}_3^2.$$

Let us consider the horizontal distribution $\mathcal{H}_{\mathrm{SO}(4)}$ on SO(4) equipped with the inner product

$$g_x(\widetilde{X}, \widetilde{Y}) = \operatorname{tr}(XY^\top),$$

and generated by the orthonormal frame $\{\widetilde{X}_1, \widetilde{X}_2, \widetilde{Z}_1, \widetilde{Z}_2\}$. In this case, the sub-Laplacian is given by

$$\Delta_{\mathcal{H}}^{\mathrm{SO}(4)} = \widetilde{X}_1^2 + \widetilde{X}_2^2 + \widetilde{Z}_1^2 + \widetilde{Z}_2^2$$

We note that the Let $\left(P_t^{\mathrm{SO}(4)}\right)_{t\geqslant 0}$ denote the horizontal heat semigroup on SO(4) generated by $\Delta_{\mathcal{H}}^{\mathrm{SO}(4)}$. Let us also recall the 3-dimensional model space $\mathbb{M}(\rho), \rho\in\mathbb{R}$ introduced in Section 4.3. Then, we have the following result.

Proposition 4.16. *There exists a Lie group homomorphism* $\Phi:\mathbb{M}(\sqrt{2})\times\mathbb{M}(\sqrt{2})\to\mathrm{SO}(4)$ *such that* $(d\Phi)_I=\phi$. *Moreover,* Φ *is a Riemannian submersion, and for all* $f\in C^\infty(\mathrm{SO}(4))$ *and* $t\geqslant 0$,

$$P_t^{\mathrm{SO}(4)}(f\circ\Phi)=Q_tf\circ\Phi, \tag{4.36}$$

where $(Q_t)_{t\geqslant 0}$ *denotes the horizontal heat semigroup on* $\mathbb{M}(\sqrt{2})\times\mathbb{M}(\sqrt{2})$.

Proof. For each $i=1,2,3$, we define

$$U_i=\frac{1}{\sqrt{2}}(X_i+Z_i),\quad V_i=\frac{1}{\sqrt{2}}(-X_i+Z_i).$$

It is easily seen that U_1,U_2,U_3,V_1,V_2,V_3 form an orthonormal basis for $\mathfrak{so}(4)$ and that

$$\begin{gathered}[U_1,U_2]=\sqrt{2}U_3,\ [U_2,U_3]=\sqrt{2}U_1,\ [U_3,U_1]=\sqrt{2}U_2,\\ [V_1,V_2]=\sqrt{2}V_3,\ [V_2,V_3]=\sqrt{2}V_1,\ [V_3,V_1]=\sqrt{2}V_2,\\ [U_i,V_j]=0.\end{gathered}$$

Denoting the Lie algebra of $\mathbb{M}(\sqrt{2})$ by $\mathfrak{m}(\sqrt{2})$, we consider the linear map $\phi:\mathfrak{m}(\sqrt{2})\oplus\mathfrak{m}(\sqrt{2})\to\mathfrak{so}(4)$ such that $\phi(W_i)=U_i,\phi(W_i')=V_i$, where $\{W_1,W_2,W_3,W_1',W_2',W_3'\}$ is an orthonormal basis for $\mathfrak{m}(\sqrt{2})\oplus\mathfrak{m}(\sqrt{2})$ satisfying $[W_i,W_i']=0$ for $i=1,2,3$. Then clearly, $\phi:\mathfrak{m}(\sqrt{2})\oplus\mathfrak{m}(\sqrt{2})\to\mathfrak{so}(4)$ is a Lie algebra isomorphism. Since ϕ maps an orthonormal basis of $\mathfrak{m}(\sqrt{2})\oplus\mathfrak{m}(\sqrt{2})$ to the same of $\mathfrak{so}(4)$, ϕ is an isometry. Recalling that $\mathbb{M}(\sqrt{2})\times\mathbb{M}(\sqrt{2})$ is simply connected, [31, Theorem 5.6] ensures the existence of the unique Lie group homomorphism Φ satisfying $\Phi(e^X)=e^{\phi(X)}$ for all $X\in\mathfrak{m}(\sqrt{2})\oplus\mathfrak{m}(\sqrt{2})$. Hence $(d\Phi)_I=\phi$. Since ϕ is an isomorphism, it follows that Φ is surjective. Also, ϕ being a isometry between the horizontal distributions of $\mathbb{M}(\sqrt{2})\times\mathbb{M}(\sqrt{2})$ and SO(4), we conclude that $\Phi:\mathbb{M}(\sqrt{2})\times\mathbb{M}(\sqrt{2})\to\mathrm{SO}(4)$ is a sub-Riemannian submersion. Therefore, the rest of the proof follows from Theorem 3.3. □

The following Corollary is a direct consequence of Theorem 3.2 and Theorem 3.3.

Corollary 4.17. *The horizontal heat semigroup* $P_t^{\mathrm{SO}(4)}$ *satisfies* (4.14), (4.15), (4.16) *and* (4.17) *with* $\rho=\sqrt{2}$.

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