Parameter-state estimation for mechanical systems with small model errors ⋆

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Abstract: The performance of algorithms for parameter-state estimation, often expressed within a Bayesian filtering context, can be hindered by the need to recompute the parametric system matrices at each parameter iteration step. This work aims to alleviate the associated computational burden by expressing model errors that stem from parametric uncertainties, as additive parametric terms in the state and observation equations. For this purpose errors in the eigenstructure of a parametrized mechanical system are propagated to the physical parameters and eventually modelled as additive terms by means of perturbation analysis. A state observer is derived under the assumption that the change between the current and a true model parameter is deterministic and known. An estimate of the possible parameter discrepancy is obtained by minimizing the value of a change detection test applied on a Kalman filter innovation sequence.

Keywords: Parameter estimation, change detection, mechanical vibrations, Bayesian filtering.

1. INTRODUCTION

The joint estimation of parameters and states has been an extensively studied problem within the adaptive control and fault detection and isolation (FDI) fields with applications including aircraft navigation (Schön et al., 2005), estimation of physical properties of structural systems (Chatzi and Smyth, 2009; Zghal et al., 2014; Dertimanis et al., 2019) and fault diagnosis in mechanical systems (Döhler et al., 2021; Tatsis et al., 2022).

The design of parameter-state estimation algorithms falls into three main categories. A classic approach is to consider an augmented system by appending the unknown parameters into the state vector, and to subsequently apply an Extended Kalman Filter. This, however, can result in biased, or divergent parameter estimates (Ljung, 1979). Alternate nonlinear Bayesian filters, such as the Unscented Kalman Filter (Julier et al., 1995; Chatzi and Smyth, 2009) or sequential Monte Carlo samplers (Doucet et al., 2000; Del Moral et al., 2006), such as the particle filter (PF), prove more efficient (Ching et al., 2006). However, their computational performance can be hindered by recomputing the system matrices at each parameter realization. A further approach concerns the design of observers, where parameters are modelled via known additive terms in the filter equations, allowing to establish an adaptive law for the joint parameter-state estimation (Zhang, 2002). In this case, parameters enter the system through an additional term, which is typically assumed to be independent of the current state. For identifying changes in the mechanical systems, however, this term is coupled with the system states (Döhler et al., 2017), rendering the application of the adaptive observers in this context non-trivial.

The purpose of this paper is to develop a simple approach for estimating changes in parameters of linear mechanical systems. The motivation behind this is to reduce the computational toll related to the PF-based methods and to ensure reliable numerical computation by avoiding tuning of the parameter vector covariance. The proposed approach is inspired by the recent work on FDI (Döhler et al., 2017), where a multiplicative change in the eigenstructure of a parametrized mechanical system is modelled as an additive one with perturbation analysis. The terms modelling the signature (profile) of the parameter change are used here to develop a simple Kalman filter. It is shown that a small perturbation in the parameter change introduces a small bias in the state estimates and in the innovation sequence. A generalized likelihood ratio (GLR) test to detect this bias is developed and used as a cost function, which is minimized to estimate the value of the change in the parameter, reflecting a model error. The paper is organized as follows. The model parametrization is defined in Section 3, the error signatures are derived in Section 4 and the test for the innovation bias is developed in Section 5.

2. PROBLEM STATEMENT

Consider an error-free, parametric linear time-invariant dynamic system of the form

$$x_{k+1} = F(\theta)x_k + B(\theta)u_k + w_k \ , \quad (1a)$$

$$y_k = H(\theta)x_k + D(\theta)u_k + v_k \ , \quad (1b)$$

where θ is the model parameter defined later, $x_k \in \mathbb{R}^n$ are the states, $y_k \in \mathbb{R}^r$ are the outputs, $u_k \in \mathbb{R}^u$ are the inputs, and $F(\theta) \in \mathbb{R}^{n\times n}$, $H(\theta) \in \mathbb{R}^{r\times n}$, $B(\theta) \in \mathbb{R}^{n\times u}$, $D(\theta) \in \mathbb{R}^{r\times u}$ are the parametric state transition, observation, input and feedthrough matrices, respectively. The process noise $w_k \in \mathbb{R}^n$ and the measurement noise $v_k \in \mathbb{R}^r$ are modelled as zero-mean, Gaussian and white processes, of covariance and cross-covariance matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{r\times r}$ and $S \in \mathbb{R}^{n\times r}$, respectively.

Let θ^0 denote some erroneously assumed value of the system parameter and let $\tilde{\theta}^\star$ denote an error such that

$$\theta = \theta^0 + \tilde{\theta}^\star.$$

It is hereby assumed that the parameter change $\tilde{\theta} = \theta - \theta^0$ is small, and takes a value $\tilde{\theta} = \epsilon\theta^1$. A GLR test to determine if the assumed change corresponds to $\tilde{\theta}^\star$ is derived in Section 5. Before precisely defining the parametrization, consider the system

$$x_{k+1} \approx F^0 x_k + B^0 u_k + \Xi_k \tilde{\theta} + w_k \ , \tag{2a}$$

$$y_k \approx H^0 x_k + D^0 u_k + \Phi_k \tilde{\theta} + v_k \ , \tag{2b}$$

in which Ξ_k and Φ_k are labelled as error signatures to be specified later. Typically Ξ_k and Φ_k characterize actuator and sensor faults respectively, or modelling uncertainties. Herein they both describe the influence of a small change in θ towards system (1a)–(1b), after Döhler et al. (2017). Matrices $F^0 \in \mathbb{R}^{n\times n}$, $H^0 \in \mathbb{R}^{r\times n}$, $B^0 \in \mathbb{R}^{n\times u}$, $D^0 \in \mathbb{R}^{r\times u}$ are the state transition, observation, input and feedthrough matrices obtained under θ^0 such that

$$F(\theta) \approx F(\theta^0) + \epsilon F(\theta^1) \triangleq F^0 + \epsilon F^1, \tag{3a}$$

$$H(\theta) \approx H(\theta^0) + \epsilon H(\theta^1) \triangleq H^0 + \epsilon H^1, \tag{3b}$$

$$B(\theta) \approx B(\theta^0) + \epsilon B(\theta^1) \triangleq B^0 + \epsilon B^1, \tag{3c}$$

$$D(\theta) \approx D(\theta^0) + \epsilon D(\theta^1) \triangleq D^0 + \epsilon D^1. \tag{3d}$$

In the next section, the change in model parameter θ is related to a change in a physical parameter of a mechanical system. The goal of this paper is then to estimate the change in the physical parametrization based on the current data, the assumed erroneous model F^0, B^0, H^0, D^0 and the error signatures Ξ_k, Φ_k. For this purpose a state observer is developed, for which it is shown that a small error in the physical parametrization introduces a small bias in the state estimate and consequently in the related innovation sequence. The estimation of the parameter error is then posed as an optimization problem, minimizing the value of a statistical test to detect bias in the innovation.

3. MODEL AND PARAMETRIZATION

Let $\eta \in \mathbb{R}^p$ denote a stiffness parameter of the physical model. It is assumed that the dynamic system can be described by an ordinary differential equation of motion

$$\mathcal{M}\ddot{q}(t) + \mathcal{C}\dot{q}(t) + \mathcal{K}(\eta)q(t) = S_p u(t) \tag{4}$$

where t denotes continuous time, and $\mathcal{M}$, $\mathcal{C}$, $\mathcal{K}(\eta) \in \mathbb{R}^{m\times m}$ denote the parametric mass, damping and stiffness matrices, respectively. Vectors $q(t) \in \mathbb{R}^m$ and $u(t) \in \mathbb{R}^u$ respectively contain the continuous-time displacements and the external forces and $S_p \in \mathbb{R}^{m\times u}$ is a selection matrix. Let system (4) be observed by sensors measuring accelerations, velocities or displacements, at r degrees of freedom (DOF) of the structure, collected in the vector

$$y(t) = C_a\ddot{q}(t) + C_v\dot{q}(t) + C_d q(t) + \tilde{v}(t) \tag{5}$$

where $y(t) \in \mathbb{R}^r$ is the output vector, $\tilde{v}(t) \in \mathbb{R}^r$ denotes the measurement noise, and matrices C_a, C_v, $C_d \in \mathbb{R}^{r\times m}$ select the respective type of the output at the measurement DOFs. Sampled at time instants $t = k\tau$, where τ is the time step and k is an integer, (4) and (5) can be represented by a discrete-time state-space model (1a)–(1b) of order $n = 2m$ with system matrices of shape

$$F = \exp(F_c\tau), \quad F_c = \begin{bmatrix} 0 & 0 \\ -\mathcal{M}^{-1}\mathcal{K} & -\mathcal{M}^{-1}\mathcal{C} \end{bmatrix},$$

$$H = \begin{bmatrix} C_d - C_a\mathcal{M}^{-1}\mathcal{K} & C_v - C_a\mathcal{M}^{-1}\mathcal{C} \end{bmatrix},$$

$$B = (F - I_n)\, F_c^{-1} B_c, \quad B_c = \begin{bmatrix} 0 \\ \mathcal{M}^{-1}S_p \end{bmatrix},$$

$$D_c = C_a\mathcal{M}^{-1}S_p,$$

where the parameter θ was dropped for simplicity. It is assumed that the model errors relate to changes in the stiffness properties of the structural elements, which generate changes in the eigenstructure of the linear system (4). Due to this assumption $D = D^0$.

The eigenstructure of the mechanical system (4) contains m complex conjugate eigenvalue μ_i and eigenvector ψ_i pairs that satisfy

$$\left(\mathcal{M}\mu_i^2 + \mathcal{C}\mu_i + \mathcal{K}\right)\psi_i = 0, \quad i = 1\ldots 2m,$$

and relate to the eigenstructure of the state matrix $F\lambda_i = \lambda_i\phi_i$ where $\lambda_i = \exp(\mu_i\tau)$ and $\phi_i = \begin{bmatrix}\psi_i \\ \mu_i\psi_i\end{bmatrix}$. Define $\mu \triangleq [\mu_1 \ \ldots \mu_m] \in \mathbb{C}^m$ and $\lambda \triangleq [\lambda_1 \ \ldots \lambda_m] \in \mathbb{C}^m$ to contain one of each $n = 2m$ eigenvalues and $\psi \triangleq [\psi_1 \ \ldots \psi_m] \in \mathbb{C}^{m\times m}$ to contain the corresponding eigenvectors. The parameter $\theta \in \mathbb{R}^{2m+2m^2}$ is then defined as

$$\theta \triangleq \begin{bmatrix} \Re(\mu) \\ \Im(\mu) \\ \mathrm{vec}(\Re(\psi)) \\ \mathrm{vec}(\Im(\psi)) \end{bmatrix}, \tag{6}$$

where $\Re(\cdot)$ and $\Im(\cdot)$ denote respectively the real and the imaginary part of a complex variable.

The change in θ can be related to a change in η with sensitivity analysis. The linearization with the first-order Taylor series writes

$$\tilde{\theta} \approx \mathcal{J}_\eta^\theta \tilde{\eta},$$

where

$$\mathcal{J}_\eta^\theta = \begin{bmatrix}\mathcal{J}_{\eta_1}^\theta & \ldots & \mathcal{J}_{\eta_p}^\theta\end{bmatrix} \in \mathbb{R}^{2m+2m^2\times p}$$

contains $i = 1\ldots p$ derivatives of real and imaginary parts of μ and ϕ with respect to the physical parameter

$$\mathcal{J}_{\eta_i}^\theta = \begin{bmatrix} \mathcal{J}_{\eta_i}^{\Re(\mu)} \\ \mathcal{J}_{\eta_i}^{\Im(\mu)} \\ \mathcal{J}_{\eta_i}^{\Re(\psi)} \\ \mathcal{J}_{\eta_i}^{\Im(\psi)} \end{bmatrix}$$

in which $\mathcal{J}_{\eta_i}^{\Re(\mu)}$, $\mathcal{J}_{\eta_i}^{\Im(\mu)}$, $\mathcal{J}_{\eta_i}^{\Re(\psi)}$, $\mathcal{J}_{\eta_i}^{\Im(\psi)}$ contain partial derivatives of the eigenstructure w.r.t. η_i which can be obtained analytically after Heylen et al. (1997).

4. STATE ESTIMATION

In this section, the error signatures are developed and their coupling with the states is accounted for in the development of a Kalman filter setup.

Substituting $F(\theta)$ with (3a) in (1a), $H(\theta)$ with (3b) in (1b) and $B(\theta)$ with (3c) in (1a) writes

$$x_{k+1} \approx F^0 x_k + \epsilon F^1 x_k + B^0 u_k + \epsilon B^1 u_k + w_k \ ,$$

$$y_k \approx H^0 x_k + \epsilon H^1 x_k + D u_k + v_k \ .$$

Proposition 1. It holds that

$$x_{k+1} \approx F^0 x_k + B^0 u_k + \Xi_k \tilde{\eta} + w_k \ ,$$
$$y_k \approx H^0 x_k + D u_k + \Phi_k \tilde{\eta} + w_k \ ,$$

where $\Xi_k = \Psi_k + \Upsilon_k$ and $\Psi_k \in \mathbb{R}^{n\times p}$, $\Phi_k \in \mathbb{R}^{r\times p}$ are the error signatures obtained from the eigenstructure of the erroneous system after Döhler et al. (2017), see Appendix A. Matrix $\Upsilon_k \in \mathbb{R}^{n\times p}$ is the error signature of the input matrix; for derivation, see Appendix B.

Proposition 2. It holds that

$$x_{k+1} \approx F^0 x_k + B^0 u_k + (\check{\Psi} X_k + \Upsilon_k)\tilde{\eta} + w_k,$$
$$y_k \approx H^0 x_k + D u_k + \check{\Phi} X_k \tilde{\eta} + v_k,$$

where $X_k = I_p \otimes x_k \in \mathbb{R}^{np\times p}$, $I_p \in \mathbb{R}^{p\times p}$ is an identity matrix and $\check{\Psi} \in \mathbb{R}^{n\times np}$, $\check{\Phi} \in \mathbb{R}^{r\times np}$, see Appendix C.

The discrete-time model from Proposition 2 can be rewritten as

$$x_{k+1} \approx F^0 x_k + B^0 u_k + \overline{\Psi} X_k^{\mathrm{vec}} + \Upsilon_k \tilde{\eta} + w_k, \tag{7a}$$
$$y_k \approx H^0 x_k + D u_k + \overline{\Phi} X_k^{\mathrm{vec}} + v_k, \tag{7b}$$

where $\overline{\Psi} = \tilde{\eta}^T \otimes \check{\Psi}$ and $\overline{\Phi} = \tilde{\eta}^T \otimes \check{\Phi}$, and X_k^{vec} denotes the vectorized X_k. By treating $\tilde{\eta}$ as a known term, the classical Kalman filter can be applied to (7a)–(7b), where special care must be taken to account for the covariance between the additive error terms and the states in the state error covariance propagation. Let K_k be a Kalman predictor gain to be specified later. A recursive filter to obtain one step-ahead state estimates $\hat{x}_{k+1|k}$ reads

$$\hat{x}_{k+1|k} = F^0 \hat{x}_{k|k-1} + B^0 u_k + \overline{\Psi} \hat{X}_{k|k-1}^{\mathrm{vec}} + \Upsilon_k \tilde{\eta} + K_k \left(y_k - H^0 \hat{x}_{k|k-1} - \overline{\Phi} \hat{X}_{k|k-1}^{\mathrm{vec}} - D u_k \right) \tag{8}$$
$$K_k = G_{k|k-1} \Sigma_{k|k-1}^{-1} \tag{9}$$

where $G_{k|k-1}$ and $\Sigma_{k|k-1}$ are derived in Appendix D.

5. PARAMETER ESTIMATION

In the previous section, a state observer for system subjected to model errors was designed under the assumption that the change in the model parameter is known. In practice, the value of $\tilde{\eta}$ is unknown; a decision whether $\tilde{\eta}$ is equal to a true value of the parameter change $\tilde{\eta}^\star$ can be taken by analyzing the innovation sequence for bias. For this, two hypotheses are defined

$$\mathrm{H}_0 : \tilde{\eta} = \tilde{\eta}^\star \quad \text{(model error matched)},$$
$$\mathrm{H}_1 : \tilde{\eta} = \tilde{\eta}^\star + \tilde{\eta}^\bullet \quad \text{(model error mismatched)} \ ,$$

where $\tilde{\eta}^\bullet$ denotes a change of the same order as $\tilde{\eta}^\star$.

5.1 Model error change detection test

The GLR test proposed hereafter follows the derivation of the recursive change detection test in Zhang and Basseville (2014). To detect a change in $\tilde{\eta}^\star$ assume H_1, i.e., that the currently assumed model change is erroneous. Then, the state vector is a linear combination of the nominal state $\hat{x}^\star_{k|k-1}$ and a small additive error

$$\hat{x}_{k|k-1} \approx \hat{x}^\star_{k|k-1} + \Gamma_{k|k-1} \tilde{\eta}^\bullet \tag{10}$$

where $\Gamma_{k|k-1} \in \mathbb{R}^{n\times m}$ is an error gain that is derived later. A small perturbation in the system state introduces an additive bias in the innovation sequence

$$\begin{aligned}\tilde{y}_k &\triangleq y_k - H^0(\hat{x}^\star_{k|k-1} + \Gamma_{k|k-1}\tilde{\eta}^\bullet) \\ &\quad - \check{\Phi}(\hat{X}^\star_{k|k-1} + \Gamma^{\tilde{\eta}^\bullet}_{k|k-1})(\tilde{\eta}^\star + \tilde{\eta}^\bullet) - D u_k \\ &\approx \tilde{y}^\star_k + M_k \tilde{\eta}^\bullet \ ,\end{aligned} \tag{11}$$

where $\tilde{y}^\star_k$ denotes the innovation sequence under H_0, $M_k\tilde{\eta}^\bullet$ is a bias term with $M_k = -H^0\Gamma_{k|k-1} - \check{\Phi}\hat{X}^\star_{k|k-1}$, $\hat{X}^\star_{k|k-1} = I_p \otimes \hat{x}^\star_{k|k-1}$ and $\Gamma^{\tilde{\eta}^\bullet}_{k|k-1} = I_p \otimes \Gamma_{k|k-1}\tilde{\eta}^\bullet$. The gain $\Gamma_{k+1|k}$ is obtained based on (10) by analyzing the one step-ahead prediction of the nominal state

$$\begin{aligned}\hat{x}^\star_{k+1|k} &\approx F^0(\hat{x}^\star_{k|k-1} + \Gamma_{k|k-1}\tilde{\eta}^\bullet) + B^0 u_k + \Upsilon_k(\tilde{\eta}^\star + \tilde{\eta}^\bullet) \\ &\quad + \check{\Psi}(\hat{X}^\star_{k|k-1} + \Gamma^{\tilde{\eta}^\bullet}_{k|k-1})(\tilde{\eta}^\star + \tilde{\eta}^\bullet) + K_k\tilde{y}_k - \Gamma_{k+1|k}\tilde{\eta}^\bullet \\ &\approx F^0\hat{x}^\star_{k|k-1} + B^0 u_k + \check{\Psi}\hat{X}^\star_{k|k-1}\tilde{\eta}^\star + \Upsilon_k\tilde{\eta}^\star + K_k\tilde{y}^\star_k \\ &\quad + \big((F^0 - K_k H^0)\Gamma_{k|k-1} \\ &\quad + (\check{\Psi} - K_k\check{\Phi})\hat{X}^\star_{k|k-1} + \Upsilon_k - \Gamma_{k+1|k}\big)\tilde{\eta}^\bullet.\end{aligned} \tag{12}$$

Note that due to the small error assumption the terms with ϵ^2 are omitted in (11) and (12). The recursion (12) yields a one step-ahead prediction of the nominal state independent of the error perturbation

$$\hat{x}^\star_{k+1|k} \approx F^0\hat{x}^\star_{k|k-1} + \check{\Psi}\hat{X}^\star_{k|k-1}\tilde{\eta}^\star + B^0 u_k + \Upsilon_k\tilde{\eta}^\star + K_k\tilde{y}^\star_k$$

when

$$\Gamma_{k+1|k} \triangleq (F^0 - K_k H^0)\Gamma_{k|k-1} + (\check{\Psi} - K_k\check{\Phi})\hat{X}^\star_{k|k-1} + \Upsilon_k.$$

As a small perturbation in the model error results in a small bias in the innovation sequence, the change detection hypothesis can be written as

$$\mathrm{H}_0 : \tilde{y}_k \sim \mathcal{N}(0, \Sigma_{\tilde{y}_k})$$
$$\mathrm{H}_1 : \tilde{y}_k \sim \mathcal{N}(M_k\tilde{\eta}^\bullet, \Sigma_{\tilde{y}_k}),$$

where $\Sigma_{\tilde{y}_k}$ is the innovation covariance. When $\tilde{\eta} = \tilde{\eta}^\star$, the innovation covariance can be obtained as

$$\Sigma_{\tilde{y}_k} = H^0 P^x_{k|k-1} {H^0}^T + \overline{\Phi} P^X_{k|k-1} \overline{\Phi}^T + R + \overline{\Phi} P^{Xx}_{k|k-1} {H^0}^T + H^0 P^{xX}_{k|k-1} \overline{\Phi}^T \ .$$

A decision between H_0 and H_1 boils down to monitoring the expected value of the innovation sequence. For this purpose, a GLR test confronting $\tilde{\eta} \neq \tilde{\eta}^\star$ against $\tilde{\eta} = \tilde{\eta}^\star$ based on the statistical properties of $\tilde{y}_k$ is performed. The test statistics writes (Basseville et al., 1993)

$$\mathrm{GLR}(\tilde{\eta}) = d_k^T E_k^{-1} d_k \ , \tag{13}$$

where d_k and E_k are obtained recursively

$$d_k = d_{k-1} + M_k^T \Sigma^{-1}_{\tilde{y}_k} \tilde{y}_k,$$
$$E_k = E_{k-1} + M_k^T \Sigma^{-1}_{\tilde{y}_k} M_k$$

with $d_0 = 0$, $E_0 = 0$. Under H_0, it is well known that $\mathrm{GLR}(\tilde{\eta})$ follows a central χ^2_p distribution with degrees of freedom equal to the number of tested parameters. The quantile $q_{\chi^2_p}$ of this distribution, satisfying $\int_0^{q_{\chi^2_p}} f_{\chi^2_p}(x)dx = \gamma$, where γ is a desired confidence level, can be used to define threshold containing all $\tilde{\eta}$ complying with $\tilde{\eta}^\star$ under the test statistics.

5.2 Change estimation

Finding the optimal $\tilde{\eta}$ can be then formulated as an optimization problem, where the value of the GLR test is minimized over the parameter space Ω

$$\tilde{\eta} = \underset{\tilde{\eta}\in\Omega}{\arg\min}\ \mathrm{GLR}(\tilde{\eta}). \tag{14}$$

The Covariance Matrix Adaptation Evolution Strategy (CMA-ES) (Hansen and Ostermeier, 2001) is adapted as the optimization approach. An acceptance region based on the $q_{\chi^2_p}$ is defined similarly as in Greś et al. (2021a)

$$\Omega = \{\tilde{\eta} : \mathrm{GLR}(\tilde{\eta}) \leq q_{\chi^2_p}\}, \quad (15)$$

which comprises all the model change vectors $\tilde{\eta}$ complying with the unknown nominal change $\tilde{\eta}^\star$ under the statistical distribution of the GLR test.

6. NUMERICAL APPLICATION

In this section, the proposed parameter estimation scheme is applied on the illustrative example of a 6 DOF mechanical chain-like system that for any consistent set of units is modeled with spring stiffness $k_1 = k_3 = k_5 = 100$ and $k_2 = k_4 = k_6 = 200$, mass $m_i = 1/20$ and a proportional damping matrix such that all modes have a damping ratio of 2%. The chain is illustrated in Figure 1. The system is excited by a white noise signal acting at all DOFs. The acceleration outputs are simulated at DOFs 1, 3 and 5 at a sampling frequency of 50 Hz, and white measurement noise with 5% of the standard deviation of the output is added to each response measurement.

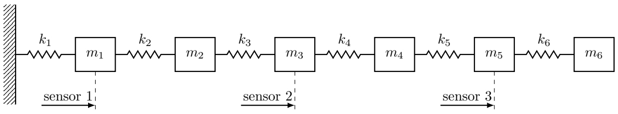

Fig. 1. 6 DOF chain system sketch.

To validate the proposed approach the chain system is assumed to be approximated via a model with assumed error, which is then estimated via the minimization of the GLR test of (13). The erroneous system matrices are modelled with 10% change in k_2, k_3, k_5 such that $\tilde{\eta}^\star = [20\ 10\ 10]^T$. In the example below, the noise covariance matrices are $Q = B^0 B^{0T}$, $R = D^0 D^{0T}$, $S = B^0 D^{0T}$ and the initial state $x_0 = 0$ with the initial covariance $P_0 = I_{12}$. First, the state observer developed in Section 4 is validated. Assuming that $\tilde{\eta}$ is known, the states estimated with (8) are compared with the states of the system without errors. This comparison is illustrated in Figure 2, where it can be observed that the state estimate matches the true state well.

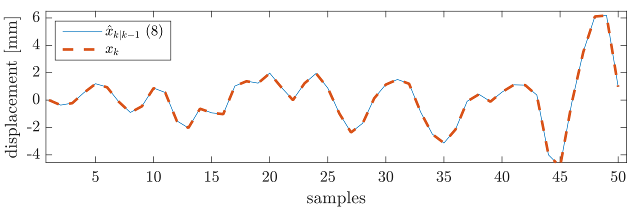

Fig. 2. The first component of the state vector and its estimate.

Next, the theoretical properties of the proposed GLR test are verified. For this purpose two cases of the model parameter change are considered; the first case where $\tilde{\eta}$ corresponds to $\tilde{\eta}^\star$ and the second case where $\tilde{\eta}$ corresponds to $\tilde{\eta}^\star$ with a small perturbation added. This comparison is illustrated in Figure 3, where it can be observed that the GLR test matches the theoretical threshold when $\tilde{\eta} = \tilde{\eta}^\star$.

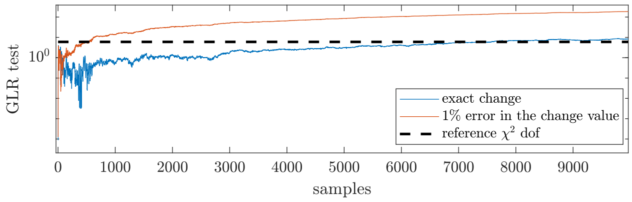

Fig. 3. GLR test statistics.

Lastly, the estimation of $\tilde{\eta}$ is validated. For this purpose 50 data sets are generated under different white noise inputs and for each $\tilde{\eta}$ is estimated. The histograms of the estimates are depicted in Figure 4.

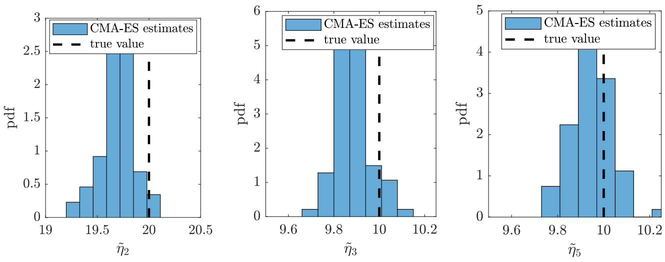

Fig. 4. Monte Carlo histograms of estimates of $\tilde{\eta}_2$, $\tilde{\eta}_3$, $\tilde{\eta}_5$.

It is observed that the spread of the histograms is narrower compared to a similar GLR-based method (Greś et al., 2021b) and a small bias in the mean is likely a consequence of errors in the approximation of parameter change terms.

7. CONCLUSIONS

We propose a simple method to estimate a small parameter change in linear time invariant mechanical systems. The developed approach avoids recomputing the parametric system matrices, which is beneficial for applications to large-scale systems. Numerical application has shown that the change in stiffness parameters of chain system is estimated with a reasonable accuracy.

Appendix A. STATE AND OBSERVATION ERROR SIGNATURES (Döhler et al., 2017)

Matrices Ψ_k and Φ_k are obtained by first-order perturbation of $F(\theta)$ and $H(\theta)$. Assume that $F(\theta)$ is diagonalizable $F(\theta) = T(\theta)\Lambda(\theta)T^{-1}(\theta)$ and that $H(\theta)$ can be written as $H(\theta) = T(\theta)\Lambda(\theta)T^{-1}(\theta)$, where

$$A(\theta) = \begin{bmatrix} \mathrm{diag}(\Re(\lambda)) & \mathrm{diag}(\Im(\lambda)) \\ -\mathrm{diag}(\Im(\lambda)) & \mathrm{diag}(\Re(\lambda)) \end{bmatrix},$$

$$A_c(\theta) = \begin{bmatrix} \mathrm{diag}(\Re(\mu)) & \mathrm{diag}(\Im(\mu)) \\ -\mathrm{diag}(\Im(\mu)) & \mathrm{diag}(\Re(\mu)) \end{bmatrix},$$

$$T(\theta) = \begin{bmatrix} \Re(\psi) & \Im(\psi) \\ \Re(\psi\ \mathrm{diag}(\mu)) & \Re(\psi\ \mathrm{diag}(\mu)) \end{bmatrix}.$$

It holds

$$A(\theta) = A(\theta^0) + \epsilon A(\theta^1) \triangleq A^0 + \epsilon A^1,$$
$$A_c(\theta) = A_c(\theta^0) + \epsilon A_c(\theta^1) \triangleq A_c^0 + \epsilon A_c^1,$$
$$T(\theta) = T(\theta^0) + \epsilon T(\theta^1) \triangleq T^0 + \epsilon T^1,$$

where

$$A^0 \triangleq \begin{bmatrix} \mathrm{diag}(\Re(\lambda^0)) & \mathrm{diag}(\Im(\lambda^0)) \\ -\mathrm{diag}(\Im(\lambda^0)) & \mathrm{diag}(\Re(\lambda^0)) \end{bmatrix},$$

$$A_c^0 \triangleq \begin{bmatrix} \mathrm{diag}(\Re(\mu^0)) & \mathrm{diag}(\Im(\mu^0)) \\ -\mathrm{diag}(\Im(\mu^0)) & \mathrm{diag}(\Re(\mu^0)) \end{bmatrix},$$

$$T^0 \triangleq \begin{bmatrix} \Re(\psi^0) & \Im(\psi^0) \\ \Re(\psi^0\ \mathrm{diag}(\mu^0)) & \Re(\psi^0\ \mathrm{diag}(\mu^0)) \end{bmatrix}$$

and λ^0, μ^0, ψ^0 are obtained from the eigenstructure of F^0 analogously to λ, μ, ψ defined in Section 3.

The error signature related to the state matrix is obtained as $\Psi_k = \Psi_k^1 + \Psi_k^2 + \Psi_k^3$ where:

- $\Psi_k^1 = \begin{bmatrix} 0 & 0 & \bar{l}_k^T \otimes I_m & \underline{l}_k^T \otimes I_m \\ \Psi_k^{(1,1)} & \Psi_k^{(1,2)} & \Psi_k^{(1,3)} & \Psi_k^{(1,4)} \end{bmatrix}$, where

$$\Psi_k^{(1,1)} = \Re(\psi^0)\overline{L}_k + \Im(\psi^0)\underline{L}_k$$
$$\Psi_k^{(1,2)} = -\Im(\psi^0)\overline{L}_k + \Re(\psi^0)\underline{L}_k$$
$$\Psi_k^{(1,3)} = \left(\overline{L}_k\Re(\mu^0) + \underline{L}_k\Im(\mu^0)\right)^T \otimes I_m$$
$$\Psi_k^{(1,4)} = \left(-\overline{L}_k\Im(\mu^0) + \underline{L}_k\Re(\mu^0)\right)^T \otimes I_m$$

  with

$$\begin{bmatrix} \bar{l}_k \\ \underline{l}_k \end{bmatrix} \triangleq A^0(T^0)^{-1}x_k,\ \overline{L}_k \triangleq \mathrm{diag}(\bar{l}_k),\ \underline{L}_k \triangleq \mathrm{diag}(\underline{l}_k).$$

- $\Psi_k^2 \triangleq -T^0A^0(T^0)^{-1}P_k$, where P_k is defined analogously to Ψ_k^1 by respectively replacing $\bar{l}_k$, $\underline{l}_k$, $\overline{L}_k$, $\underline{L}_k$ by $\bar{h}_k$, $\underline{h}_k$, $\overline{H}_k$, $\underline{H}_k$, where

$$\begin{bmatrix} \bar{h}_k \\ \underline{h}_k \end{bmatrix} \triangleq (T^0)^{-1}x_k,\ \overline{H}_k \triangleq \mathrm{diag}(\bar{h}_k),\ \underline{H}_k \triangleq \mathrm{diag}(\underline{h}_k).$$

- $\Psi_k^3 \triangleq [O_k\ 0_{n,2m^2}]$, where

$$O_k = \tau T^0 \begin{bmatrix} \overline{H}_k & \underline{H}_k \\ \underline{H}_k & -\overline{H}_k \end{bmatrix} \begin{bmatrix} \mathrm{diag}(\Re(\lambda^0)) & -\mathrm{diag}(\Im(\lambda^0)) \\ \mathrm{diag}(\Im(\lambda^0)) & \mathrm{diag}(\Re(\lambda^0)) \end{bmatrix}.$$

The error signature related to the observation matrix is obtained as $\Phi_k = C_a(\Phi_k^1 + \Phi_k^2 + \Phi_k^3)$ where:

- $\Phi_k^1 = \left[\Phi_k^{(1,1)}\ \Phi_k^{(1,2)}\ \Phi_k^{(1,3)}\ \Phi_k^{(1,4)}\right]$, where $\Phi_k^{(1,1)}$, $\Phi_k^{(1,2)}$, $\Phi_k^{(1,3)}$, $\Phi_k^{(1,4)}$ are obtained analogously to $\Psi_k^{(1,1)}$, $\Psi_k^{(1,2)}$, $\Psi_k^{(1,3)}$, $\Psi_k^{(1,4)}$ by substituting $\overline{L}_k$ and $\underline{L}_k$ with $\overline{S}_k$ and $\underline{S}_k$ such that

$$\begin{bmatrix} \bar{s}_k \\ \underline{s}_k \end{bmatrix} \triangleq A_c^0(T^0)^{-1}x_k,\ \overline{S}_k \triangleq \mathrm{diag}(\bar{s}_k),\ \underline{S}_k \triangleq \mathrm{diag}(\underline{s}_k).$$

- $\Phi_k^2 = -S_mT^0A_c^0(T^0)^{-1}P_k(T^0)^{-1}$
- $\Phi_k^3 = S_mT^0 \begin{bmatrix} \overline{H}_k & \underline{H}_k & 0_{m,2m^2} \\ \underline{H}_k & -\overline{H}_k & 0_{m,2m^2} \end{bmatrix}$.

and $S_m = [0_{m,m}\ I_m]$.

Appendix B. INPUT MATRIX PERTURBATION

Consider the following perturbation for the input matrix

$$B \approx B^0 + \epsilon(B^{(1,1)} - B^{(1,2)}),$$

where matrices $B^{(1,1)}$ and $B^{(1,2)}$ are obtained using first-order perturbation theory as

$$\begin{aligned} B^{(1,1)} &= T^1A^0A_c^{0^{-1}}T^{0^{-1}} - T^0A^1A_c^{0^{-1}}T^{0^{-1}} \\ &- T^0A^0A_c^{0^{-1}}A_c^1A_c^{0^{-1}}T^{0^{-1}} \\ &- T^0A^0A_c^{0^{-1}}T^{0^{-1}}T^1T^{0^{-1}}, \\ B^{(1,2)} &= T^1A_c^{0^{-1}}T^{0^{-1}} - T^0A_c^{0^{-1}}A_c^1A_c^{0^{-1}}T^{0^{-1}} \\ &- T^0A_c^{0^{-1}}T^{0^{-1}}T^1T^{0^{-1}}. \end{aligned}$$

Detailed derivation of the above is skipped for brevity.

The error signature related to the input matrix is can be obtained as $\Upsilon_k = -\Upsilon_k^1 + \Upsilon_k^2 + \Upsilon_k^3 + \Upsilon_k^4 + \Upsilon_k^5$ where:

- Υ_k^1 is defined analogously to Ψ_k^1 by respectively replacing $\bar{l}_k$, $\underline{l}_k$, $\overline{L}_k$, $\underline{L}_k$ by $\bar{g}_k$, $\underline{g}_k$, $\overline{G}_k$, $\underline{G}_k$, where

$$\begin{bmatrix} \bar{g}_k \\ \underline{g}_k \end{bmatrix} = A_c^{0^{-1}}T^{0^{-1}}B_cu_k,$$

  and $\overline{G}_k \triangleq \mathrm{diag}(\bar{g}_k)$, $\underline{G}_k \triangleq \mathrm{diag}(\underline{g}_k)$.
- Υ_k^2 is defined analogously to Ψ_k^1 by respectively replacing $\bar{l}_k$, $\underline{l}_k$, $\overline{L}_k$, $\underline{L}_k$ by $\bar{d}_k$, $\underline{d}_k$, $\overline{D}_k$, $\underline{D}_k$, where

$$\begin{bmatrix} \bar{d}_k \\ \underline{d}_k \end{bmatrix} = A^0A_c^{0^{-1}}T^{0^{-1}}B_cu_k,$$

  and $\overline{D}_k \triangleq \mathrm{diag}(\bar{d}_k)$, $\underline{D}_k \triangleq \mathrm{diag}(\underline{d}_k)$.
- $\Upsilon_k^3 \triangleq \left(T^0A_c^{0^{-1}} - T^0A^0A_c^{0^{-1}}\right)\begin{bmatrix} \overline{G}_k & \underline{G}_k & 0_{m,2m^2} \\ \underline{G}_k & -\overline{G}_k & 0_{m,2m^2} \end{bmatrix}$
- $\Upsilon_k^4 \triangleq [J_k\ 0_{n,2m^2}]$, where

$$J_k = \tau T^0 \begin{bmatrix} \overline{G}_k & \underline{G}_k \\ \underline{G}_k & -\overline{G}_k \end{bmatrix} \begin{bmatrix} \mathrm{diag}(\Re(\lambda^0)) & -\mathrm{diag}(\Im(\lambda^0)) \\ \mathrm{diag}(\Im(\lambda^0)) & \mathrm{diag}(\Re(\lambda^0)) \end{bmatrix}$$

- $\Upsilon_k^5 \triangleq (T^0A_c^{0^{-1}}T^{0^{-1}} - T^0A^0A_c^{0^{-1}}T^{0^{-1}})\overline{P}_k$, where $\overline{P}_k$ is defined analogously to Ψ_k^1 by respectively replacing $\bar{l}_k$, $\underline{l}_k$, $\overline{L}_k$, $\underline{L}_k$ by $\bar{t}_k$, $\underline{t}_k$, $\overline{T}_k$, $\underline{T}_k$, where

$$\begin{bmatrix} \bar{t}_k \\ \underline{t}_k \end{bmatrix} = T^{0^{-1}}B_cu_k,\ \overline{T}_k \triangleq \mathrm{diag}(\bar{t}_k),\ \underline{T}_k \triangleq \mathrm{diag}(\underline{t}_k).$$

Appendix C. PROOF OF PROPOSITION 2

The error signature for the state matrix can be written as

$$\Psi_k = \left[\Psi_k\mathcal{J}_{\eta_1}^{\theta} \dots \Psi_k\mathcal{J}_{\eta_p}^{\theta}\right] = \left[\check{\Psi}_1x_k \dots \check{\Psi}_px_k\right] = \check{\Psi}X_k\ ,$$

where $\check{\Psi} = \left[\check{\Psi}_1 \dots \check{\Psi}_p\right]$, $X_k = I_p \otimes x_k$. Each $\check{\Psi}_i$ is obtained as $\check{\Psi}_i = \check{\Psi}_i^1 + \check{\Psi}_i^2 + \check{\Psi}_i^3$, where:

- $\check{\Psi}_i^1$ is factored from $\Psi_k^1\mathcal{J}_{\eta_i}^{\theta}$ as

$$\check{\Psi}_i^1 = \begin{bmatrix} \left[\bar{\mathcal{J}}_{\eta_i}^{\Re(\psi)}\ \bar{\mathcal{J}}_{\eta_i}^{\Im(\psi)}\right] \\ \check{\Psi}_i^{(1,1)} + \check{\Psi}_i^{(1,2)} + \check{\Psi}_i^{(1,3)} + \check{\Psi}_i^{(1,4)} \end{bmatrix} A^0(T^0)^{-1},$$

  where

$$\bar{\mathcal{J}}_{\eta_i}^{\Re(\psi)} = \left[\Re\left(\frac{\partial\psi_1}{\partial\eta_i}\right) \dots \Re\left(\frac{\partial\psi_m}{\partial\eta_i}\right)\right],$$
$$\bar{\mathcal{J}}_{\eta_i}^{\Im(\psi)} = \left[\Im\left(\frac{\partial\psi_1}{\partial\eta_i}\right) \dots \Im\left(\frac{\partial\psi_m}{\partial\eta_i}\right)\right],$$

  and

$$\check{\Psi}_i^{(1,1)} = \left[\Re(\psi^0)\mathrm{diag}(\mathcal{J}_{\eta_i}^{\Re(\mu)})\ \Im(\psi^0)\mathrm{diag}(\mathcal{J}_{\eta_i}^{\Re(\mu)})\right],$$
$$\check{\Psi}_i^{(1,2)} = \left[-\Im(\psi^0)\mathrm{diag}(\mathcal{J}_{\eta_i}^{\Im(\mu)})\ \Re(\psi^0)\mathrm{diag}(\mathcal{J}_{\eta_i}^{\Im(\mu)})\right],$$
$$\check{\Psi}_i^{(1,3)} = \left[\bar{\mathcal{J}}_{\eta_i}^{\Re(\psi)}\mathrm{diag}(\Re(\mu^0))\ \bar{\mathcal{J}}_{\eta_i}^{\Re(\psi)}\mathrm{diag}(\Im(\mu^0))\right],$$
$$\check{\Psi}_i^{(1,4)} = \left[-\bar{\mathcal{J}}_{\eta_i}^{\Im(\psi)}\mathrm{diag}(\Im(\mu^0))\ \bar{\mathcal{J}}_{\eta_i}^{\Im(\psi)}\mathrm{diag}(\Re(\mu^0))\right].$$

- $\check{\Psi}_i^2$ is factored from $\Psi_k^2 \mathcal{J}_{\eta_i}^{\theta}$ as

$$\check{\Psi}_i^2 = -T^0 A^0 (T^0)^{-1} \check{P}_i (T^0)^{-1},$$

where

$$\check{P}_i = \begin{bmatrix} \begin{bmatrix} \mathcal{J}_{\eta_i}^{\Re(\psi)} & \mathcal{J}_{\eta_i}^{\Im(\psi)} \end{bmatrix} \\ \check{\Psi}_i^{(1,1)} + \check{\Psi}_i^{(1,2)} + \check{\Psi}_i^{(1,3)} + \check{\Psi}_i^{(1,4)} \end{bmatrix}.$$

- $\check{\Psi}_i^3$ is factored from $\Psi_k^3 \mathcal{J}_{\eta_i}^{\theta}$ as

$$\check{\Psi}_i^3 = \tau T^0 \begin{bmatrix} \mathrm{diag}(\Re(\lambda^0)) & \mathrm{diag}(\Im(\lambda^0)) \\ -\mathrm{diag}(\Im(\lambda^0)) & \mathrm{diag}(\Re(\lambda^0)) \end{bmatrix} \times \begin{bmatrix} \mathrm{diag}(\mathcal{J}_{\eta_i}^{\Re(\mu)}) & \mathrm{diag}(\mathcal{J}_{\eta_i}^{\Im(\mu)}) \\ -\mathrm{diag}(\mathcal{J}_{\eta_i}^{\Im(\mu)}) & \mathrm{diag}(\mathcal{J}_{\eta_i}^{\Re(\mu)}) \end{bmatrix} (T^0).$$

Matrix $\check{\Phi}$ can be factored from the error signature Φ_k analogously to $\check{\Psi}$. Let $\check{\Phi} = \begin{bmatrix} \check{\Phi}_1 & \dots & \check{\Phi}_p \end{bmatrix}$, where $\check{\Phi}_i = \check{\Phi}_i^1 + \check{\Phi}_i^2 + \check{\Phi}_i^3$. Each $\check{\Phi}_i$ is obtained as $\check{\Phi}_i = \check{\Phi}_i^1 + \check{\Phi}_i^2 + \check{\Phi}_i^3$, where:

- $\check{\Phi}_i^1 = \left[\check{\Psi}_i^{(1,1)} + \check{\Psi}_i^{(1,2)} + \check{\Psi}_i^{(1,3)} + \check{\Psi}_i^{(1,4)} \right] A_c^0 (T^0)^{-1}$
- $\check{\Phi}_i^2 = -S_m T^0 A_c^0 (T^0)^{-1} \check{P}_i (T^0)^{-1}$
- $\check{\Phi}_i^3 = S_m T^0 \begin{bmatrix} \mathrm{diag}(\mathcal{J}_{\eta_i}^{\Re(\mu)}) & \mathrm{diag}(\mathcal{J}_{\eta_i}^{\Im(\mu)}) \\ -\mathrm{diag}(\mathcal{J}_{\eta_i}^{\Im(\mu)}) & \mathrm{diag}(\mathcal{J}_{\eta_i}^{\Re(\mu)}) \end{bmatrix} (T^0)^{-1}$

Appendix D. KALMAN FILTER DERIVATION

Let $P_{k+1|k} \triangleq \mathrm{E}\left(\tilde{x}_{k+1|k} \tilde{x}_{k+1|k}^T\right)$ be the covariance of the state error $\tilde{x}_{k+1|k} \triangleq x_{k+1} - \hat{x}_{k+1|k}$, and let $P_{k|k-1}^x \triangleq \mathrm{E}\left(\tilde{x}_{k|k-1} \tilde{x}_{k|k-1}^T\right)$, $P_{k|k-1}^X \triangleq \mathrm{E}\left(\tilde{X}_{k|k-1}^{\mathrm{vec}} (\tilde{X}_{k|k-1}^{\mathrm{vec}})^T\right)$, $P_{k|k-1}^{xX} \triangleq \mathrm{E}\left(\tilde{x}_{k|k-1} (\tilde{X}_{k|k-1}^{\mathrm{vec}})^T\right)$ respectively be the auto and cross-covariance of $\tilde{x}_{k|k-1}$ and $\tilde{X}_{k|k-1}^{\mathrm{vec}}$. These matrices write

$$\begin{aligned} P_{k+1|k} &= (F^0 - K_k H^0) P_{k|k-1}^x (F^0 - K_k H^0)^T \\ &+ (\overline{\Psi} - K_k \overline{\Phi}) P_{k|k-1}^X (\overline{\Psi} - K_k \overline{\Phi})^T \\ &+ (F^0 - K_k H^0) P_{k|k-1}^{xX} (\overline{\Psi} - K_k \overline{\Phi})^T \\ &+ (\overline{\Psi} - K_k \overline{\Phi}) P_{k|k-1}^{Xx} (F^0 - K_k H^0)^T \\ &+ Q + K_k R K_k^T + K_k S^T + S K_k^T, \\ P_{k|k-1}^{xX} &= \begin{bmatrix} P_{k|k-1}^x & 0_{n,np} & P_{k|k-1}^x & \dots & 0_{n,np} & P_{k|k-1}^x \end{bmatrix} \\ P_{k|k-1}^X &= \begin{bmatrix} P_{k|k-1}^{Xx} & 0_{np^2,np} & P_{k|k-1}^{Xx} & \dots & 0_{np^2,np} & P_{k|k-1}^{Xx} \end{bmatrix}, \end{aligned}$$

where $P_{k|k-1}^{Xx} = (P_{k|k-1}^{xX})^T$. The Kalman gain $K_k = G_{k|k-1} \Sigma_{k|k-1}^{-1}$ is obtained by minimizing $\mathrm{Tr}\left(P_{k+1|k}\right)$ where

$$\begin{aligned} \Sigma_{k|k-1} &= H^0 P_{k|k-1}^x H^{0^T} + R + \overline{\Phi} P_{k|k-1}^X \overline{\Phi}^T \\ &+ \overline{\Phi} P_{k|k-1}^{Xx} H^{0^T} + H^0 P_{k|k-1}^{xX} \overline{\Phi}^T, \\ G_{k|k-1} &= F^0 P_{k|k-1}^x H^{0^T} + S + \overline{\Psi} P_{k|k-1}^X \overline{\Phi}^T \\ &+ F^0 P_{k|k-1}^{xX} \overline{\Phi}^T + \overline{\Psi} P_{k|k-1}^{Xx} H^{0^T}. \end{aligned}$$

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