More about Wilson's functional equation

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This paper is dedicated to János Aczél for his important contributions to the theory of functional equations on groups.

Abstract

Let G be a group with an involution $x \mapsto x^*$, let $\mu : G \to \mathbb{C}$ be a multiplicative function such that $\mu(xx^*) = 1$ for all $x \in G$, and let the pair $f, g : G \to \mathbb{C}$ satisfy that

$$f(xy) + \mu(y)f(xy^*) = 2f(x)g(y), \ \forall x, y \in G.$$

For G compact we obtain: If g is abelian, then f is abelian.
For G nilpotent we obtain: (1) If G is generated by its squares and $f \neq 0$, then g is abelian. (2) If g is abelian, but not a multiplicative function, then f is abelian.

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1 Prologue

Many trigonometric identities provide solutions to functional equations. The following two identities are relevant for us.

The identity $\cos(x + y) + \cos(x - y) = 2 \cos x \cos y$ is a starting point of d'Alembert's functional equation $g(x + y) + g(x - y) = 2g(x)g(y)$ for all $x, y \in \mathbb{R}$, in which $g : \mathbb{R} \to \mathbb{C}$ is the unknown function. d'Alembert's functional equation is also known as the cosine equation.

The identity $\sin(x + y) + \sin(x - y) = 2 \sin x \cos y$ is a starting point of Wilson's functional equation $f(x+y)+f(x-y) = 2f(x)g(y)$ for all $x, y \in \mathbb{R}$, in which the pair of functions $f, g : \mathbb{R} \to \mathbb{C}$ is the unknown.

Aczél's classic monograph [1] discusses the real valued, continuous solutions of d'Alembert's and Wilson's equations and contains references to earlier works.

The exponential-cosine equation $g(x+y)+e^{\lambda y}g(x-y) = 2g(x)g(y)$ where $\lambda \in \mathbb{C}$ is a constant, forms an extension of d'Alembert's functional equation, which is the case of $\lambda = 0$. Versions of it occur in Parnami, Singh and Vasudeva [15], and in Davison [5, Proposition 2.11]. Stetkær [19, p. 158] has some examples.

We shall study generalizations of d'Alembert's, the exponential-cosine and Wilson's functional equations, in which the domain of definition $\mathbb{R}$ of the unknown functions is replaced by a monoid M and the exponential function $e^{\lambda y}$ by a multiplicative function $\mu : M \to \mathbb{C}^*$.

2 Notation, definitions, terminology and setup

The notation and terminology introduced in Section 2 are kept throughout the paper, as is the setup described in Subsection 2.2.

2.1 General concepts

Definition 2.1. Let S be a semigroup.

A *multiplicative function* on S is a function $m : S \to \mathbb{C}$ such that $m(xy) = m(x)m(y)$ for all $x, y \in S$, while an *additive function* on S is a function $a : S \to \mathbb{C}$ such that $a(xy) = a(x) + a(y)$ for all $x, y \in S$. $\mu = 1$ is multiplicative on any semigroup. For any $\lambda \in \mathbb{C}$ the function $x \mapsto \exp(\lambda x)$ is multiplicative, and $x \mapsto \lambda x$ is additive, both on $S = \mathbb{R}$.

A *character* on a group G is a multiplicative function $\chi : G \to \mathbb{C}^*$, where $\mathbb{C}^*$ denotes the multiplicative group of non-zero complex numbers. We do not require characters to be unitary. As is well known, any non-zero multiplicative function on a group is a character (see [19, Lemma 3.4(a)]).

An *involution* of S is a map $x \mapsto x^*$ from S into S such that $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in S$. Note that $e^* = e$, if e is a neutral element of S. If S is a group, then the inversion map $x \mapsto x^{-1}$ is an involution of S.

Definition 2.2. Let S be a semigroup.

A function F on S is said to be *central*, iff $F(xy) = F(yx)$ for all $x, y \in S$. In group theory central functions are also called class functions.

A function F on S is said to be *abelian*, iff

$$F(x_1x_2\cdots x_n) = F(x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(n)})$$

for all $x_1, x_2 \dots, x_n \in S$, all permutations π of n elements and all $n = 2, 3, \dots, n$. We say that F is *non-abelian*, if it is not abelian.

Lemma 2.3. *Let M be a monoid, and let F be a function on M. Define*

$$Z(F) := \{z \in M \mid F(xyz) = F(xzy) \text{ for all } x, y \in M\}.$$

(a) $Z(F)$ is a submonoid of M, which can also be characterized as

$$Z(F) = \{z \in M \mid F(xzy) = F(zxy) \text{ for all } x, y \in M\}. \tag{1}$$

(b) If M is a group, then $Z(F)$ is a normal subgroup of M.

(c) F is abelian if and only if $M = Z(F)$.

Proof. Contained in [19, Lemmas B.4 and B.5]. □

Definition 2.4. If Ω is a topological space, we let $C(\Omega)$ denote the complex vector space of complex valued functions on Ω.

Given a continuous map $x \mapsto x^*$ of Ω into Ω such that $(x^*)^* = x$ for all $x \in \Omega$ and given a function $m \in C(\Omega)$ such that $m(x)m(x^*) = 1$ for all $x \in \Omega$, we define for any $F \in C(\Omega)$ the function $F^*(x) := m(x)F(x^*)$, $x \in \Omega$.

The map $F \mapsto F^*$ is an involution of $C(\Omega)$, i.e., an automorphism of the vector space $C(\Omega)$ such that $(F^*)^* = F$ for all $F \in C(\Omega)$.

We say that $F \in C(\Omega)$ is *even* iff $F^* = F$, and *odd* iff $F^* = -F$. The even/odd parts of F are $F_e := (F+F^*)/2$ and $F_o := (F-F^*)/2$, respectively. Once in a while the formula $F(x) = m(x)F^*(x^*)$ comes in handy.

2.2 Our setup

Throughout the paper M, $x \mapsto x^*$, μ and $F \mapsto F^*$ are fixed and mean the following.

M is a topological monoid. Its identity element is denoted e.

$x \mapsto x^*$ is a continuous involution of M.

$\mu \in C(M)$ is multiplicative and satisfies that $\mu(xx^*) = 1$ for all $x \in M$.

$F \mapsto F^*$ is the involution of $C(M)$, which is defined in Definition 2.4, i.e., by $F^*(x) := \mu(x)F(x^*)$ for $x \in M$ and $F \in C(M)$.

M is in some parts of the paper not just a monoid, but even a topological group. For mnemonic reasons there we use the letter G instead of M.

2.3 The two functional equations and their solutions

We shall study the functional equations (2) and (3) below. The classic functional equations on the real line in Section 1 are special cases.

d'Alembert's μ-functional equation is

$$g(xy) + \mu(y)g(xy^*) = 2g(x)g(y), \; x, y \in M, \tag{2}$$

in which the function $g : M \to \mathbb{C}$ is the unknown. A *solution* of (2) is a function $g \in C(M)$ satisfying (2). The solutions for which $g(e) = 1$ are named *d'Alembert μ-functions.*

Wilson's μ-functional equation is

$$f(xy) + \mu(y)f(xy^*) = 2f(x)g(y), \; x, y \in M, \tag{3}$$

in which the pair of functions $f, g : M \to \mathbb{C}$ is the unknown. A *solution* of (3) is an ordered pair $(f, g) \in C(M) \times C(M)$ of functions satisfying (3). It is said to be *abelian*, if both functions f and g are abelian.

For $g \in C(M)$ we let $\mathrm{Wil}_\mu(g)$ denote the complex vector space of *Wilson μ-functions* corresponding to g, defined by

$$\mathrm{Wil}_\mu(g) := \{f \in C(M) \mid (f, g) \text{ satisfies } (3)\}.$$

Most of the theory of (2) and (3) has been developed for $M = G$ a topological group, $\mu = 1$ and $x^* = x^{-1}$. In this case (2) and (3) reduce to *d'Alembert's functional equation* (4) and *Wilson's functional equation* (5):

$$g(xy) + g(xy^{-1}) = 2g(x)g(y), \; x, y \in G, \text{ and} \tag{4}$$

$$f(xy) + f(xy^{-1}) = 2f(x)g(y), \; x, y \in G. \tag{5}$$

Here and in the literature *d'Alembert functions* are solutions g of (4) such that $g(e) = 1$.

3 Introduction

3.1 Background

Let (f, g) be a solution of (3) on a monoid M, such that $f \neq 0$. In that case g is a d'Alembert μ-function (see Theorem 4.2). The two possibilities, g is non-abelian and g is abelian, are quite different.

When g is non-abelian, then the form of g is known, as is the set solutions $\mathrm{Wil}_\mu(g)$ of f corresponding to g (by Davison [5]): $\dim \mathrm{Wil}_\mu(g) = 4$, $\mathrm{Wil}_\mu(g)$ consists of linear combinations of translates of g etc. So the case of non-abelian g's has been taken care of. That is why we in the present paper focus on abelian g's.

When g is abelian, then g has the form $g = (\chi + \chi^*)/2$ for some non-zero multiplicative function χ on M (Proposition 4.1), so g is known. But the corresponding vector space $\mathrm{Wil}_\mu(g)$ is not known in general. As a step towards a study of $\mathrm{Wil}_\mu(g)$ we seek conditions that force f to be abelian, because formulas for the solutions of (3) are known in that case (Proposition 4.5 and Corollary 4.6).

Our paper's objective is twofold.

(I) To uncover basic properties of the solutions of versions (2) and (3) of d'Alembert's and Wilson's functional equations.

(II) For solutions (f, g) of (3) on a group G to find conditions on G and g that ensure that f is abelian, assuming g is abelian.

About objective (I). The setup described in Section 2.2 can be found in the literature, but most papers deal with the functional equations (4) and (5), that are simpler than (2) and (3). Our work on (I) is done in Section 4. The results of (I) are used in (II).

We restrict our discussion of (II) to compact and nilpotent groups G. This is done in Sections 5, 7 and 8. The compactness of G is a powerful condition, so much can be obtained with it. An argument for looking at nilpotent groups is the metamathematical statement that nilpotent groups are “almost abelian”, so that basic results on abelian groups should prevail. It turns out that nilpotency is not quite enough. Supplementary conditions are needed.

In the last 20 years d’Alembert’s and Wilson’s functional equations with an involution $x \mapsto x^*$ have been studied among others by Bouikhalene and Elqorachi [2], Davison [4, 5], Ebanks and Stetkær [6], Friis [10], Sinopoulos [16], Stetkær [17, 18, 19] and Yang [21]. We extend and use some of their results.

Remark 3.1. A functional equation similar to (3) is

$$f(xy) + f(\sigma(y)x) = 2f(x)g(y),\ x, y \in M,$$

where $\sigma : M \to M$ is a monoid automorphism such that $(\sigma \circ \sigma)(x) = x$ for all $x \in M$. We do not dicuss it here. For its solutions see Fadli, Zeglami and Kabbaj [8] or Fadli [9, Section 2.4].

3.2 Results of the paper

We study the functional equations (2) and (3) in the setup of Section 2.2, and on groups involution and group inversion may be different.

We obtain some basic results for the solutions on monoids (Corollary 4.3 and Proposition 4.4). We illuminate the relations between involution and $Z(f)$ (Section 4.2).

For compact groups an easy extension of the known results about the solutions of (5) to the solutions of (3) in Theorem 5.1 is new.

For nilpotent groups Theorems 7.1 and 8.1 are new. They display sufficient, algebraic conditions for the solutions of (2) and (3) to be abelian.

3.3 How the paper is organized

Here is a brief outline of the paper. Sections 2 and 6 fix the terminology, notation, some definitions and setup of the paper. Sections 1 and 3 comprise the introduction with motivation and background. In Section 4 we derive some general results about solutions of (2) and (3), valid on any monoid. In particular we discuss $Z(f)$ for a solution (f, g) of (3). The short Section 5, which is independent of Sections 4, 6, 7 and 8, presents on compact groups those solutions of (3) for which g is abelian. In Section 7 and Section 8 we turn to our objective (II) for nilpotent groups.

4 General theory of (2) and (3)

We enforce the setup and notation from Section 2.2 throughout Section 4.

4.1 Connections between solutions of (2) and (3)

In Subsection 4.1 we present new results about the solutions of d'Alembert's and Wilson's μ-functional equations (2) and (3) on a monoid (Corollary 4.3 and Proposition 4.4). Not only do we find these results interesting in themselves, but we shall also use them later. In the present subsection we also recall some known formulas for the abelian solutions of (2) and (3) (Propositions 4.1 and 4.5 and Corollary 4.6).

Proposition 4.1 expresses the abelian solutions of d'Alembert's μ-functional equation (2) in terms of multiplicative functions. It extends Kannappan [11] about d'Alembert's functional equation (4), to (2).

Proposition 4.1. *The non-zero, abelian solutions g of d'Alembert's μ-functional equation* (2) *are the functions $g = (\chi + \chi^*)/2$, where $\chi \in C(M)$ ranges over the non-zero multiplicative functions on M. The multiplicative function χ is uniquely determined by g up to interchangeability with χ^*.*

Proof. See [19, Proposition 9.31]. The continuity of χ can be inferred from [19, Theorem 3.18(d)]. □

The theory of the solutions (f, g), $f \neq 0$, of Wilson's μ-functional equation (3) changed recently, when it was discovered that g had to be a d'Alembert μ-function, even on non-abelian semigroups, and thus was a well known object from the literature (see [20] for details and references). We formulate the discovery for monoids in Theorem 4.2, which we shall use repeatedly together with its consequences Corollary 4.3 and Proposition 4.4.

Theorem 4.2. *Let the pair (f, g) be a solution of Wilson's μ-functional equation* (3) *such that $f \neq 0$. Then g is a d'Alembert μ-function, i.e. $g \in Wil_\mu(g)$ and $g \neq 0$. Furthermore $g = g^*$ and g is central.*

Proof. $g \in \mathrm{Wil}_\mu(g)$ by [2, Proposition 2.1(2)] (only for groups) or [20] (for semigroups). Taking $y = e$ in (3) we see that $g(e) = 1$, so $g \neq 0$. For the remaining statements see [19, Proposition 9.17(a) and (b)]. □

The trigonometric identity $\sin(x+y)+\sin(x-y) = 2\sin x \cos y$ illustrates Theorem 4.2. It shows that the pair $x \mapsto (f(x), g(x)) := (\sin x, \cos x)$ of functions on $\mathbb{R}$ is a solution of Wilson's functional equation. Clearly $g(x) = \cos x$ satisfies d'Alembert's functional equation. Furthermore g is central $(g(x + y) = g(y + x))$ and even $(g(x) = g(-x))$.

Corollary 4.3 and Proposition 4.4 have not been noted before in the setting of non-abelian monoids. We shall use them to split the treatment of solutions (f, g) of (3) into cases, in which f is either even or odd.

Corollary 4.3. *If (f, g) is a solution of Wilson's μ-functional equation* (3)*, then $f_e, f_o, f^* \in Wil_\mu(g)$.*

Proof. The corollary is trivially true if $f = 0$, so we may assume that $f \neq 0$. In that case $g \in \mathrm{Wil}_\mu(g)$ by Theorem 4.2. Putting $x = e$ in (3) we get that $f_e = (f + f^*)/2 = f(e)g \in \mathrm{Wil}_\mu(g)$. By linearity $f_o = f - f_e \in \mathrm{Wil}_\mu(g)$ and $f^* = 2f_e - f \in \mathrm{Wil}_\mu(g)$. □

Proposition 4.4 states some properties of the even and odd parts of a function $f \in \mathrm{Wil}_\mu(g)$.

Proposition 4.4. *Let the pair (f, g) be a solution of Wilson's μ-functional equation* (3)*. Then*

(a) $f_e = cg$ for some constant $c \in \mathbb{C}$.

(b) (f_o, g) satisfies the symmetrized sine addition law

$$f_o(xy) + f_o(yx) = 2f_o(x)g(y) + 2f_o(y)g(x) \text{ for all } x, y \in M.$$

This reduces to the sine addition law

$$f_o(xy) = f_o(x)g(y) + f_o(y)g(x), \text{ when } x \in M \text{ and } y \in Z(f_o). \tag{6}$$

Proof. (a) was derived for $f \neq 0$ with $c = f(e)$ during the proof of Corollary 4.3. For $f = 0$, we can take $c = 0$.

(b) Replace f by f_o in (3). This is allowed by Corollary 4.3. Then interchange x and y and add the two identities, noting that

$$\mu(x)f_o(yx^*) = -\mu(x)f_o^*(yx^*) = -\mu(x)\mu(yx^*)f_o(xy^*) = -\mu(y)f_o(xy^*).$$

This gives the symmetrized sine addition law, which implies (6). □

When a solution (f, g) of (3) with $\mu = 1$ is abelian, then explicit formulas for f and g exist in the literature (see for example [19, Proposition 11.5] with the correction $a \circ \tau = -a$). Proposition 4.5 generalizes these formulas from $\mu = 1$ to any μ allowed by the setup of Section 2.2. It expresses the abelian solutions (f, g) of (3) in terms of multiplicative functions and special solutions of the sine addition law. Corollary 4.6 describes what the proposition becomes when the underlying monoid M is a group.

Proposition 4.5. *Let (f, g) be a solution of Wilson's μ-functional equation* (3) *such that f is abelian and $\neq 0$.*

Then there exists a non-zero multiplicative function $\chi \in C(M)$ of M such that $g = (\chi + \chi^*)/2$. The multiplicative function χ is uniquely determined by g up to interchangeability with χ^*.

(a) If $\chi \neq \chi^*$, then there exist constants $c_1, c_2 \in \mathbb{C}$ such that

$$f = c_1 \frac{\chi + \chi^*}{2} + c_2 \frac{\chi - \chi^*}{2}.$$

(b) If $\chi = \chi^*$, then $f = c\chi + h$, where $c \in \mathbb{C}$ and $h \in C(M)$ is an odd (i.e. $h^* = -h$) solution of the sine addition law

$$h(xy) = h(x)\chi(y) + h(y)\chi(x), \quad x, y \in M.$$

Proof. Since $f \neq 0$ and abelian, it follows from (3) that g is abelian. Furthermore, g is by Theorem 4.2 a non-zero solution of (2), so according to Proposition 4.1 it has the desired form $g = (\chi + \chi^*)/2$ for some non-zero multiplicative function $\chi \in C(G)$. The essential uniqueness of χ can be read from [19, Corollary 3.19].

For the statements about f it suffices to prove that the even and odd parts f_e and f_o of f have the forms indicated. For f_e this is immediate, because $f_e = cg = c(\chi + \chi^*)/2$ for some $c \in \mathbb{C}$ (by Proposition 4.4(a)). Left is f_o.

If $f_o = 0$, the statements are trivially true, so we may assume that $f_o \neq 0$. The pair (f_o, g) satisfies the sine addition formula (by Proposition 4.4(b) and the abelianness of f and hence of f_o), i.e.,

$$f_o(xy) = f_o(x)g(y) + f_o(y)g(x), \text{ for all } x, y \in M. \tag{7}$$

According to the theory of (7) (see [19, Theorem 4.1(b)]) there exist multiplicative functions χ_1 and χ_2 such that $g = (\chi_1 + \chi_2)/2$. By the essential uniqueness of the decomposition [19, Corollary 3.19] we have that $\chi_1 = \chi$ and $\chi_2 = \chi^*$ or vice versa. We work with the first possibility; the other can be handled in the same way, so we skip it.

(a) Let $\chi \neq \chi^*$. We see from [19, Theorem 4.1(c)] that $f_o = c(\chi_1 - \chi_2) = c(\chi - \chi^*)$ for some $c \in \mathbb{C}$. This proves (a) with $c_1 = 0$.

(b) Let $\chi = \chi^*$. Here $g = (\chi + \chi^*)/2 = \chi$, so (7) becomes

$$f_o(xy) = f_o(x)\chi(y) + f_o(y)\chi(x) \text{ for all } x, y \in M,$$

which is (b) with $h := f_o$. □

Corollary 4.6. *Let (f, g) be a solution of Wilson's μ-functional equation (3) on a topological group G such that f is abelian and $\neq 0$.*

Then there exists a character $\chi \in C(G)$ of G such that $g = (\chi + \chi^*)/2$. The character χ is uniquely determined by g up to interchangeability with χ^*.

(a) If $\chi \neq \chi^*$, then there exist constants $c_1, c_2 \in \mathbb{C}$ such that

$$f = c_1 \frac{\chi + \chi^*}{2} + c_2 \frac{\chi - \chi^*}{2}.$$

(b) If $\chi = \chi^*$, then $f = c\chi + a\chi$, where $c \in \mathbb{C}$ and $a \in C(G)$ is an additive function such that $a(x^*) = -a(x)$ for all $x \in G$.

Elqorachi et al [7, Theorems 1 and 2] derive Proposition 4.1 and Corollary 4.6 for discrete, abelian groups.

4.2 About $Z(f)$ in the general setting

In Subsection 4.2 we derive some properties of $Z(f)$ for solutions (f, g) of Wilson's μ-functional equation (3). This is a natural thing to do according to Lemma 2.3(c) and our objective (II) stated in Subsection 3.1.

The inversion $x \mapsto x^{-1}$ on a group G satisfies $xx^{-1} = x^{-1}x = e \in Z(G)$. This does of course not carry over to involutions $x \mapsto x^*$ of a monoid. Nevertheless Lemma 4.7(b) and (c) reveal that elements of the form xx^* have some centrality properties when taken together with Wilson μ-functions.

Lemma 4.7(d) ensures that $Z(f)$ is a monoid with involution (the involution of $Z(f)$ being the restriction of $x \mapsto x^*$ to $Z(f)$), when (f, g) is a solution of (3), so the restriction of (f, g) to $Z(f)$ is a solution of (3) on the monoid $Z(f)$ with its involution. This is used in the proof of Lemma 4.8. Lemma 4.7(b), (c) and (d) are new for monoids with an involution $x \mapsto x^*$.

Lemma 4.7. *If the pair $f, g : M \to \mathbb{C}$ is a solution of Wilson's μ-functional equation* (3)*, then f has the following properties.*

(a) $f(xyy^*) = f(x)g(yy^*)$ for all $x, y \in M$.

(b) $yy^* \in Z(f)$ for all $y \in M$.

(c) $f(xyy^*z) = f(xy^*yz)$ for all $x, y, z \in M$.

(d) $Z(f)^* = Z(f)$, so $Z(f)$ is invariant under the involution $x \mapsto x^*$ of M.

Proof. The lemma is trivially true, if $f = 0$, so we may assume $f \neq 0$. Then the conclusions of Theorem 4.2 hold, in particular that g is central.

(a) Replace y by yy^* in (3).

(b) comes from the following computation, in which $x, y, z \in M$. We first use that μ is a multiplicative function such that $\mu(yy^*) = 1$, next (3) and finally (a) to get

$$\begin{aligned} f(xyy^*z) &= f(xyy^*z) + \mu(yy^*z)f(x(yy^*z)^*) - \mu(z)f(xz^*yy^*) \\ &= 2f(x)g(yy^*z) - \mu(z)f(xz^*)g(yy^*). \end{aligned}$$

Noting that g is central, next that $g(zyy^*) = g(z)g(yy^*)$ (replace y by yy^* in d'Alembert's μ-functional equation (2)) and finally (3) we continue the computations as follows

$$\begin{aligned} &= 2f(x)g(zyy^*) - \mu(z)f(xz^*)g(yy^*) \\ &= 2f(x)g(z)g(yy^*) - \mu(z)f(xz^*)g(yy^*) \\ &= [2f(x) - \mu(z)f(xz^*)]g(yy^*) = f(xz)g(yy^*) = f(xzyy^*), \end{aligned}$$

which shows that $yy^* \in Z(f)$.

(c) Let $x, y, z \in M$. Using that $yy^* \in Z(f)$ by (b) we get from (a) that $f(xyy^*z) = f(xzyy^*) = f(xz)g(yy^*)$. Thus it suffices to show that $g(yy^*) = g(y^*y)$. But this holds, since g is central.

(d) We will first derive the inclusion $Z(f)^* \subseteq Z(f)$. So we shall for any $z \in Z(f)$ show that $z^* \in Z(f)$, i.e., that $f(xz^*y) = f(xyz^*)$ for all $x, y \in M$. Using (3), $z \in Z(f)$ and the fact that g is central we compute that

$$\begin{aligned} f(xz^*y) &= f(xz^*y) + \mu(z^*y)f(x(z^*y)^*) - \mu(z^*y)f(xy^*z) \\ &= 2f(x)g(z^*y) - \mu(z^*y)f(xzy^*) = 2f(x)g(yz^*) - \mu(z^*y)f(xzy^*) \\ &= f(xyz^*) + \mu(yz^*)f(x(yz^*)^*) - \mu(z^*y)f(xzy^*) \\ &= f(xyz^*) + [\mu(yz^*) - \mu(z^*y)]f(xzy^*) = f(xyz^*) \text{ as desired.} \end{aligned}$$

Applying $x \mapsto x^*$ to $Z(f)^* \subseteq Z(f)$ gives the converse inclusion. □

Lemma 4.8 enters the proof of Theorem 8.1, which is our main results about Wilson's μ-functional equation (3) on nilpotent groups, even though Lemma 4.8 deals with monoids, not nilpotent groups.

Lemma 4.8. *Let $f, g : M \to \mathbb{C}$ be a solution of Wilson's μ-functional equation (3) with $g = (\chi + \chi^*)/2$, where χ is a multiplicative function on M such that $\chi \neq \chi^*$. Then there exist constants $c_1, c_2 \in \mathbb{C}$ such that*

$$f = c_1 \frac{\chi + \chi^*}{2} + c_2 \frac{\chi - \chi^*}{2} \text{ on } Z(f).$$

Proof. We first prove the

Claim. If $x_0 \in M$, then the sub-monoid M_0 of M generated by x_0, x_0^* and $Z(f)$ is invariant under the involution $x \mapsto x^*$, and $f|_{M_0}$ is abelian.

Proof (of the claim). Let $x_0 \in M$. It follows from Lemma 4.7(d) that M_0 is invariant under the involution $x \mapsto x^*$, so M_0 is a monoid and equipped with the given involution. That $f|_{M_0}$ is abelian means according to Lemma 2.3 that $Z(f|_{M_0}) = M_0$. Now, $Z(f|_{M_0})$ is a sub-monoid of M_0, so it suffices to show that it contains the generating elements of M_0, i.e., that $x_0 \in Z(f|_{M_0})$, $x_0^* \in Z(f|_{M_0})$ and $Z(f) \subseteq Z(f|_{M_0})$. However $x_0 \in Z(f|_{M_0})$ implies that $x_0^* \in Z(f|_{M_0})$ due to Lemma 4.7(d), and it is trivial that $Z(f) \subseteq Z(f|_{M_0})$ (by the definitions), so it only remains to verify that $x_0 \in Z(f|_{M_0})$, i.e., that $f(xx_0y) = f(xyx_0)$ for all $x, y \in M_0$. So we shall be able to pass x_0 from the position in the middle to one at the very right, i.e. from the expression $f(xx_0y)$ to $f(xyx_0)$, without changing the value of f. Now $y \in M_0$ is a product $y = y_1y_2 \cdots y_n$, where each factor y_j is either x_0, x_0^* or an element of $Z(f)$. Clearly we can pass x_0 from left to right through a factor x_0 and a factor from $Z(f)$ (by the definition of $Z(f)$). That it can also be passed through a factor x_0^* is contained in Lemma 4.7(c). This finishes the proof of the claim. □

The conclusion of the lemma is trivially true, if $f|_{Z(f)} = 0$, so we may in the rest of the proof assume that $f|_{Z(f)} \neq 0$. Choose $x_0 \in M$ such that $\chi(x_0) \neq \chi^*(x_0)$, and consider the sub-monoid M_0 of M generated by x_0, x_0^* and $Z(f)$. By the claim it is invariant under involution, so $f|_{M_0}$ is an abelian solution of Wilson's μ-functional equation on M_0. Note that $\chi^*|_{M_0} = (\chi|_{M_0})^*$, so that $g|_{M_0} = (\chi|_{M_0} + \chi^*|_{M_0})/2 = (\chi|_{M_0} + (\chi|_{M_0})^*)/2$. We have that $\chi|_{M_0} \neq (\chi|_{M_0})^*$ by choice of x_0, so according to Proposition 4.5(a) there are constants $c_1, c_2 \in \mathbb{C}$ such that

$$\begin{aligned} f|_{M_0} &= c_1 \frac{\chi|_{M_0} + (\chi|_{M_0})^*}{2} + c_1 \frac{\chi|_{M_0} - (\chi|_{M_0})^*}{2} \\ &= c_1 \frac{\chi|_{M_0} + \chi^*|_{M_0}}{2} + c_1 \frac{\chi|_{M_0} - \chi^*|_{M_0}}{2}, \end{aligned}$$

or equivalently that

$$f = c_1 \frac{\chi + \chi^*}{2} + c_1 \frac{\chi - \chi^*}{2} \text{ on } M_0.$$

This proves the lemma, since $M_0 \supseteq Z(f)$ by the definition of M_0. □

5 Wilson's μ-functions on compact groups

The setup from Section 2.2 is enforced throughout this Section 5. Furthermore $M = G$ is a topological group here. We incorporate the Hausdorff property in the definition of a compact group.

It is known for solutions (f, g) of (5) on a compact group G that f is abelian, if g is. This follows from the explicit solution formulas of Yang [21, Theorem 3.3] or Stetkær [19, Theorem 11.9(b)]. Theorem 5.1 extends the results about (5) just mentioned to the more general equation (3). It expresses each solution of (3) (with $f \neq 0$) in terms of a continuous, unitary character χ of G. The method of the proof fits nicely with our objective (II) stated in Section 3.1.

Theorem 5.1. *Let the pair (f, g) be a solution of (3) on a compact group G, such that $f \neq 0$ and g is abelian. Then there exist a unitary character $\chi \in C(G)$ of G and constants $c_1, c_2 \in \mathbb{C}$ such that*

$$g = \frac{\chi + \chi^*}{2} \text{ and } f = c_1 \frac{\chi + \chi^*}{2} + c_2 \frac{\chi - \chi^*}{2}.$$

The character χ is uniquely determined by g up to interchangeability with χ^*.

Proof. Proposition 4.4(a) says that $f_e = cg$ for some $c \in \mathbb{C}$, so f_e is abelian. Proposition 4.4(b) says that f_o satisfies the symmetrized sine addition law, which implies that f_o is abelian, since G is compact (by [6, Lemma 11]). It follows that $f = f_e + f_o$ is abelian. We obtain the rest of the proof from Corollary 4.6, because on a compact group the only continuous, additive function is 0, and any continuous character is unitary (see [19, Exercise 2.5 and Lemma 3.4(a)]). □

The proof of Theorem 5.1 does not extend to compact monoids, because the reference [6] uses the Haar measure of G.

6 About nilpotent groups

Section 6 recalls standard notation and terminology from group theory and presents some examples. Throughout the section G denotes a group, but unlike in the rest of the paper it need not possess a topology here.

Definition 6.1. The *commutator* $[x, y]$ between $x \in G$ and $y \in G$ is $[x, y] := xyx^{-1}y^{-1} \in G$. For $A, B \subseteq G$ we let $[A, B]$ denote the subgroup of G generated by $\{[a, b] \in G \mid a \in A, b \in B\}$.

Definition 6.2. The *descending central series* $\mathcal{C}^0G$, $\mathcal{C}^1G$, ..., $\mathcal{C}^kG$, ... of G is defined inductively by $\mathcal{C}^0G := G$ and $\mathcal{C}^kG := [G, \mathcal{C}^{k-1}G]$ for $k = 1, 2, \ldots$ The group G is said to be *nilpotent*, if $\mathcal{C}^nG = \{e\}$ for some $n \in \{0, 1, 2, \ldots\}$.

Properties of the descending central series are known from group theory. For example $\mathcal{C}^kG$ is a normal subgroup of G, and $\mathcal{C}^0G \supseteq \mathcal{C}^1G \supseteq \cdots \supseteq \mathcal{C}^kG \supseteq \ldots$ for all $k = 0, 1, 2, \ldots$

Examples 6.3. Many interesting and important groups are nilpotent.

(i) All abelian groups are nilpotent.

(ii) Any subgroup of the group of upper triangular matrices with 1 along the diagonal is nilpotent. A particular case is the Heisenberg group

$$H_3 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

It has a natural involution *, different from group inversion, viz.,

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}^* := \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \text{ for } x, y, z \in \mathbb{R}.$$

(iii) The quaternion group $Q_8 := \{\pm 1, \pm i, \pm j, \pm k\}$ is nilpotent.

(iv) Any nilpotent group is solvable, but the $(ax + b)$-*group*

$$G := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a > 0,\ b \in \mathbb{R} \right\}$$

is a solvable group which is not nilpotent.

7 Pre-d'Alembert functions on nilpotent groups

The setup in Section 2.2 with $M = G$ being a topological group is enforced throughout Section 7. The main result of the section is Theorem 7.1.

Theorem 7.1 implies that the d'Alembert μ-functions on a nilpotent group are abelian, if the group satisfies the supplementary condition that it is generated by its squares. The Heisenberg group is an example satisfying this supplementary condition, while the quaternion group Q_8 shows that it can not be omitted (see Examples 7.2(a) and (b)).

We formulate Theorem 7.1 for pre-d'Alembert functions. Any d'Alembert μ-function is a pre-d'Alembert function (by [19, Proposition 9.17(c)]), so the theorem holds for d'Alembert μ-functions.

Theorem 7.1. *If G is nilpotent and generated by its squares, then all pre-d'Alembert functions on G are abelian.*

Proof. Let $g : G \to \mathbb{C}$ be a pre-d'Alembert function on G. We shall show that g is abelian, which by Lemma 2.3 means that $G = Z(g)$.

We recall that $d(x) := 2g(x)^2 - g(x^2)$, $x \in G$, is a character on G (for this see Davison [5, Proposition 2.10] or [19, Proposition 8.11]). If $g(z)^2 \neq d(z)$ for some $z \in Z(g)$, then g is abelian (according to [19, Proposition 8.14(a)]). Left is the case $g(z)^2 = d(z)$ for all $z \in Z(g)$. Here $g(xz) = g(x)g(z)$ for all $x \in G$ and $z \in Z(g)$ by [19, Proposition 8.14(b)]. In particular $g|_{Z(g)}$ is a character of $Z(g)$.

Claim. If H is a subgroup of G such that $[G, H] \subseteqq Z(g)$, then $H \subseteqq Z(g)$.

Proof (of the claim). Let H be a subgroup of G such that $[G, H] \subseteqq Z(g)$. We shall prove that $h \in Z(g)$ for all $h \in H$, so we let $h_0 \in H$ be arbitrary, but fixed.

Step 1. We prove that the map $x \mapsto g([x, h_0])$ is a character on G.

We take $x_3 = h_0 \in H$ in the formula $[x_1x_2, x_3] = x_1[x_2, x_3]x_1^{-1}[x_1, x_3]$, which is valid for all $x_1, x_2, x_3 \in G$. By assumption $[x_2, h_0] \in Z(g)$, so

$$\begin{aligned} g([x_1x_2, h_0]) &= g\left(x_1[x_2, h_0]x_1^{-1}[x_1, h_0]\right) = g\left(x_1\,[x_2, h_0]\,(x_1^{-1}[x_1, h_0])\right) \\ &= g\left(x_1\,(x_1^{-1}\,[x_1, h_0])\,[x_2, h_0]\right) = g\left([x_1, h_0][x_2, h_0]\right). \end{aligned}$$

This equals $g([x_1, h_0])\, g([x_2, h_0])$, because $g|_{Z(g)}$ is a character of $Z(g)$, and $[x_1, h_0], [x_2, h_0] \in Z(g)$. This finishes the proof of Step 1.

Step 2. We prove that $g([x, h_0]) = 1$ for all $x \in G$.

For any $x \in G$ we obtain by help of Step 1 that $g([x^2, h_0]) = g([x, h_0])^2 = d([x, h_0]) = 1$, where the last equality is due to the fact that d is a character of G, so that it assumes the value 1 on any commutator. Thus the character $x \mapsto g([x, h_0])$ assumes the value 1 on any square and hence equals 1 on the subgroup of G generated by the squares. By assumption this subgroup is all of G. This finishes the proof of Step 2.

Step 3. We finally prove that $h_0 \in Z(g)$ by showing that $h_0^{-1} \in Z(g)$. This suffices, since $Z(g)$ is a group (Lemma 2.3). Using that $g(xz) = g(x)g(z)$ for all $x \in G$ and $z \in Z(g)$ we find for any $x, y \in G$ by the help of Step 2 that

$$\begin{aligned} g(xyh_0^{-1}) &= g\left(xh_0^{-1}y[y^{-1}, h_0]\right) = g\left(xh_0^{-1}y\right) g\left([y^{-1}, h_0]\right) \\ &= g(xh_0^{-1}y) \cdot 1 = g(xh_0^{-1}y), \end{aligned}$$

which expresses that $h_0^{-1} \in Z(g)$. This finishes the proof of the claim. □

G is nilpotent, so $[G, C^{n-1}G] = C^n G = \{e\}$ in the notation of Definition 6.2. Thus $C^{n-1}G$ satisfies the condition on H of the claim. We conclude that $C^{n-1}G \subseteqq Z(g)$. From $[G, C^{n-2}G] = C^{n-1}G \subseteqq Z(g)$ we conclude that $C^{n-2}G$ satisfies the condition on H in the claim, so also $C^{n-2}G \subseteqq Z(g)$. Proceeding inductively we end up with $C^0G = G \subseteqq Z(g)$, which means that $G = Z(g)$ as desired. □

Examples 7.2. (a) Theorem 7.1 holds for nilpotent, connected Lie groups, because any connected Lie group is generated by its squares. The Heisenberg group H_3 is an example.

(b) The quaternion group Q_8 is a nilpotent group, which is not generated by its squares. There is a non-abelian d'Alembert function on it, viz. $g_0 : Q_8 \to \mathbb{R}$ given by $g_0(\pm 1) = \pm 1$, $g_0(\pm i) = g_0(\pm j) = g_0(\pm k) = 0$ (this is computed in [19, Example 9.10]). Thus the assumption that G is generated by its squares can not be omitted from Theorem 7.1.

Remarks 7.3. The comments (1)-(6) below discuss some of the literature related to our work in this paper about abelian d'Alembert functions.

(1) Corovei [3, Theorem 2] says the following: If G is a nilpotent group all of whose elements have (finite) odd order, then every d'Alembert function on G is abelian.

(2) Precursors to Theorem 7.1 are Stetkær [17, Proposition V.5] and Stetkær [18, Theorem 7.2], in which G is step 2 nilpotent instead of just nilpotent.

(3) Friis [10, Corollary 2.8] showed that d'Alembert functions on connected, nilpotent Lie groups are abelian, cfr. Example 7.2(a).

(4) By Davison [4, Proposition 5.5] d'Alembert functions on 2-divisible, nilpotent groups are abelian. He notes in [4, Corollaries 5.6 and 5.7] that this encompasses both (1) and (3). His result is contained in our Theorem 7.1, because (i) obviously any 2-divisible group is generated by its squares, and (ii) we deal with pre-d'Alembert functions, not just the smaller set of d'Alembert functions.

(5) Theorem 7.1 for d'Alembert functions and for solutions of d'Alembert's long functional equation can be found in the literature as Stetkær [19, Theorem 9.25 and Proposition 10.6].

(6) Other criteria for pre-d'Alembert functions to be abelian are derived in Stetkær [19, Section 8.5]. One is that G is a solvable, connected Lie group like the $(ax+b)$-group. This extends Example 7.2(a), because nilpotent groups are solvable.

8 Wilson's μ-functions on nilpotent groups

The setup from Section 2.2 with $M = G$ being a topological group is enforced throughout this section. Its main result is Theorem 8.1.

Let (f, g) be a solution of (3) on a nilpotent group G, such that $g = (\chi+\chi^*)/2$ for some character χ of G. Theorem 8.1 shows that f in this case is a linear combination of χ and χ^*, when $\chi \neq \chi^*$. In particular that f is abelian in accordance with our objective (II) in Section 3.1. However, when $\chi = \chi^*$, then an example shows that f can be non-abelian (Remark 8.2(b)).

Theorem 8.1. *Let G be a nilpotent group. Let (f, g) be a solution with $f \neq 0$ of Wilson's μ-functional equation (3) on G. If g is abelian, but not a character, then there exist a character $\chi \in C(G)$ with $\chi \neq \chi^*$ and constants $c_1, c_2 \in \mathbb{C}$ such that*

$$f = c_1 \frac{\chi+\chi^*}{2} + c_2 \frac{\chi-\chi^*}{2} \quad \text{and} \quad g = \frac{\chi+\chi^*}{2}.$$

The character χ is uniquely determined by g up to interchangeability with χ^*.

Proof. g is a non-zero solution of d'Alembert's μ-functional equation (Theorem 4.2). Being abelian, g has (by Proposition 4.1) the form $g = (\chi+\chi^*)/2$, where $\chi \in C(G)$ is a character on G. Here $\chi \neq \chi^*$, because g by assumption is not a character. Furthermore χ is by Proposition 4.1 uniquely determined by g up to interchangeability with χ^*.

It suffices to prove that f is abelian, because in this case the formula for f can be read from Proposition 4.5(a). Proposition 4.4(a) says that $f_e = cg$ for some constant $c \in \mathbb{C}$, so f_e is abelian, because by hypothesis so is g. It is left to show that f_o is abelian.

The pair (f_o, g) satisfies (3) by Corollary 4.3, so from Lemma 4.8 we get that $f_o = c_2(\chi - \chi^*)/2$ on $Z(f_o)$ for some $c_2 \in \mathbb{C}$. From this we make the observation that $f_o = 0$ on $Z(f_o) \cap [G, G]$, using that characters of G are identically 1 on the commutator group $[G, G]$.

We use the notation from Definition 6.2 throughout the rest of the proof.

Claim. Let $k \in [1, 2, \ldots, n]$. If $\mathcal{C}^k G \subseteq Z(f_o)$, then $\mathcal{C}^{k-1} G \subseteq Z(f_o)$.

Proof (of the claim). We shall for any $z \in \mathcal{C}^{k-1}G$ prove that $z \in Z(f_o)$, i.e., that $f_o(xyz) = f_o(xzy)$ for all $x, y \in G$. Now

$$f_o(xyz) = f_o(xzy[y^{-1}, z^{-1}]).$$

Here $[y^{-1}, z^{-1}] \in [G, \mathcal{C}^{k-1}G] = \mathcal{C}^k G \subseteq Z(f_o)$ (by the assumption of the claim), so by the sine addition law (6) we infer that

$$f_o(xyz) = f_o(xzy)g([y^{-1}, z^{-1}]) + f_o([y^{-1}, z^{-1}])g(xzy). \tag{8}$$

We note that $g = (\chi + \chi^*)/2 = 1$ on commutators, so that $g([y^{-1}, z^{-1}]) = 1$, while from $[y^{-1}, z^{-1}] \in Z(f_o) \cap [G, G]$ we get that $f_o\left([y^{-1}, z^{-1}]\right) = 0$ by the observation above in this proof. Thus (8) becomes

$$f_o(xyz) = f_o(xzy) \cdot 1 + 0 \cdot g(xzy) = f_o(xzy),$$

which finishes the proof of the claim. □

Using the claim inductively starting from $\mathcal{C}^n G = \{e\} \subseteq Z(f_o)$ we get that $\mathcal{C}^0 G = G \subseteq Z(f_o)$, which means by Lemma 2.3 that f_o is abelian. □

Remarks 8.2. (a) Let (f, g) be a solution of (3) on a group G, not necessarily nilpotent. If g is a character, then we may reduce (3) to the version (9) below of Jensen's functional equation. Indeed, $F := f/g$ satisfies

$$F(xy) + F(xy^*) = 2F(x) \text{ for all } x, y \in G. \tag{9}$$

Proof. (9) is trivially true, if $f = 0$, so we may assume that $f \neq 0$. Then $g = g^*$ by Theorem 4.2, which means that $g(y) = \mu(y)g(y^*)$ for all $y \in G$. Dividing (3) by $g(xy) = g(x)g(y)$ we obtain (9). □

To get information about $f = Fg$ we may apply our knowledge of (9) (for instance from Ng [12, 13, 14] and/or Stetkær [19, Chapter 12]).

(b) The hypothesis of Theorem 8.1 that g is not a character can in general not be omitted. Jensen's functional equation on the Heisenberg group provides a counter example for (5). For details see [19, Example 12.4]. As we have seen in Theorem 5.1, the hypothesis is not needed on compact groups.

(c) Theorem 8.1 was derived for the functional equation (5) as [18, Theorem 10.1] for step 2 nilpotent groups.

(d) Theorem 8.1 does not remain true, if the word nilpotent is replaced by the word solvable. The $(ax + b)$-group harbours a counter example for (5). See [19, Example 11.14] for details.

Combining Theorems 7.1 and 8.1 we obtain

Corollary 8.3. *Let G be nilpotent and generated by its squares. If (f, g) is a solution of* (3) *such that $f \neq 0$ and g is not a character of G, then there exist a character $\chi \in C(G)$ with $\chi \neq \chi^*$ and constants $c_1, c_2 \in \mathbb{C}$ such that*

$$f = c_1 \frac{\chi + \chi^*}{2} + c_2 \frac{\chi - \chi^*}{2} \quad \textit{and} \quad g = \frac{\chi + \chi^*}{2}.$$

The character χ is uniquely determined by g up to interchangeability with χ^*.

Proof. Theorem 4.2 implies that g is a non-zero solution of d'Alembert's μ-functional equation. By Theorem 7.1 g is abelian, so it has according to Proposition 4.1 the form $g = (\chi + \chi^*)/2$ for some character $\chi \in C(G)$ of G. Here $\chi \neq \chi^*$, because g is not a character. We get the essential uniqueness of χ from [19, Corollary 3.19]. We conclude that f has the desired form from Theorem 8.1. □

Friis [10, Theorem 3.4] got the formulas of Corollary 8.3 for the functional equation (5) assuming that G is a connected, nilpotent Lie group and that $g \neq 1$.

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