ON THE DIRECT INTEGRAL DECOMPOSITION IN BRANCHING LAWS FOR REAL REDUCTIVE GROUPS

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Abstract. The restriction of an irreducible unitary representation π of a real reductive group G to a reductive subgroup H decomposes into a direct integral of irreducible unitary representations τ of H with multiplicities $m(\pi,\tau)\in\mathbb{N}\cup\{\infty\}$. We show that on the smooth vectors of π, the direct integral is pointwise defined. This implies that $m(\pi,\tau)$ is bounded above by the dimension of the space $\operatorname{Hom}_H(\pi^\infty|_H,\tau^\infty)$ of intertwining operators between the smooth vectors, also called *symmetry breaking operators*, and provides a precise relation between these two concepts of multiplicity.

Introduction

Let G be a real reductive group and $H\subseteq G$ a reductive subgroup. Then, for any irreducible unitary representation $(\pi,\mathcal{H}_\pi)$ of G, its restriction to H decomposes into a direct integral of irreducible unitary representations $(\tau,\mathcal{H}_\tau)$ of H, i.e. there exists an H-equivariant unitary isomorphism

$$T:\mathcal{H}_\pi\to\int_{\widehat{H}}^{\oplus}\mathcal{H}_\tau\otimes\mathcal{M}_{\pi,\tau}\,d\mu_\pi(\tau), \tag{1}$$

where $\mathcal{M}_{\pi,\tau}$ is a family of Hilbert spaces (called *multiplicity spaces*) and μ_π a Borel measure on the unitary dual $\widehat{H}$ of H. Here, H acts only on the first factor of $\mathcal{H}_\tau\otimes\mathcal{M}_{\pi,\tau}$, so

$$m(\pi,\tau)=\dim\mathcal{M}_{\pi,\tau}\in\mathbb{N}\cup\{\infty\}$$

is the *multiplicity of τ in π*. Note that the function $\tau\mapsto m(\pi,\tau)$ is unique up to a set of measure zero, and in this sense the direct integral decomposition is unique.

The map T is in general not *pointwise defined*, i.e. there do not exist continuous linear maps $A_{\pi,\tau}:\mathcal{H}_\pi\to\mathcal{H}_\tau\otimes\mathcal{M}_{\pi,\tau}$ such that $T(v)_\tau=A_{\pi,\tau}(v)$ for μ_π-almost every $\tau\in\widehat{H}$. In fact, the existence of a non-zero H-equivariant continuous linear map $\mathcal{H}_\pi\to\mathcal{H}_\tau$ implies that τ occurs discretely inside $\pi|_H$, i.e. $\mu_\pi(\{\tau\})>0$. To also capture the continuous part of the decomposition, one has to restrict to a subspace of $\mathcal{H}_\pi$. We show that the dense subspace $\mathcal{H}_\pi^\infty$ of smooth vectors is sufficient for this purpose:

Theorem A. *The restriction of T to the smooth vectors $\mathcal{H}_\pi^\infty$ is pointwise defined, i.e. for every $\tau\in\widehat{H}$ there exists an H-equivariant continuous linear map $A_{\pi,\tau}^\infty:\mathcal{H}_\pi^\infty\to\mathcal{H}_\tau\otimes\mathcal{M}_{\pi,\tau}$ such that*

$$T(v)_\tau=A_{\pi,\tau}^\infty(v)\qquad\text{for }\mu_\pi\text{-almost every }\tau\in\widehat{H}.$$

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We remark that every H-equivariant continuous linear map $\mathcal{H}_\pi^\infty \to \mathcal{H}_\tau$ automatically maps into the smooth vectors $\mathcal{H}_\tau^\infty$ of τ, i.e. is contained in $\mathrm{Hom}_H(\mathcal{H}_\pi^\infty, \mathcal{H}_\tau^\infty)$. Such operators are also called *symmetry breaking operators* (see Kobayashi [5]). The space of symmetry breaking operators has been studied intensively in connection with finite multiplicity/multiplicity one statements (see e.g. [6, 10]) and branching laws for specific pairs of groups (G, H) (see e.g. [2, 3, 7, 9]). Theorem A shows that the direct integral decomposition (1) of an irreducible unitary representation can always be constructed in terms of symmetry breaking operators. This statement might have been common knowledge, but we could not find a reference in the literature.

Theorem A implies an upper bound for the multiplicity $m(\pi, \tau)$ in terms of the dimension of the space $\mathrm{Hom}_H(\pi^\infty|_H, \tau^\infty) = \mathrm{Hom}_H(\mathcal{H}_\pi^\infty, \mathcal{H}_\tau^\infty)$ of symmetry breaking operators between the smooth vectors of π and τ:

Corollary B. *For μ_π-almost every $\tau \in \widehat{H}$:*

$$m(\pi, \tau) \leq \dim \mathrm{Hom}_H(\pi^\infty|_H, \tau^\infty).$$

We remark that, although this upper bound is attained for some representations τ, the right hand side might be strictly larger in many cases. This means that not every non-trivial symmetry breaking operator contributes to the decomposition of the unitary representation.

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1. Direct integrals of Hilbert spaces

We briefly recall the construction of direct integrals of Hilbert spaces following the exposition in [8, Section 8].

Let $(\mathcal{H}_\lambda)_{\lambda\in\Lambda}$ be a family of Hilbert spaces indexed by a second-countable topological space Λ and let μ be a σ-finite Borel measure on Λ. Denote by $\langle\cdot,\cdot\rangle_\lambda$ the inner product on $\mathcal{H}_\lambda$. We identify elements $s \in \prod_{\lambda\in\Lambda} \mathcal{H}_\lambda$ with sections $s : \Lambda \to \bigsqcup_{\lambda\in\Lambda} \mathcal{H}_\lambda$ satisfying $s(\lambda) \in \mathcal{H}_\lambda$ for every $\lambda \in \Lambda$. Suppose we are given a subspace $\mathcal{F} \subseteq \prod_{\lambda\in\Lambda} \mathcal{H}_\lambda$, called the space of *measurable sections*, satisfying:

- For all $s, t \in \mathcal{F}$ the map $\Lambda \to \mathbb{C}$, $\lambda \mapsto \langle s(\lambda), t(\lambda)\rangle_\lambda$ is measurable.
- If $s \in \prod_{\lambda\in\Lambda} \mathcal{H}_\lambda$ is such that $\lambda \mapsto \langle s(\lambda), t(\lambda)\rangle_\lambda$ is measurable for all $t \in \mathcal{F}$, then $s \in \mathcal{F}$.
- There exists a countable subset $(s_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$ such that $\{s_n(\lambda) : n \in \mathbb{N}\}$ spans a dense subspace of $\mathcal{H}_\lambda$ for every $\lambda \in \Lambda$.

The family $(\mathcal{H}_\lambda)_{\lambda\in\Lambda}$ together with the measure μ and the subspace $\mathcal{F}$ of measurable sections is called a *measurable family of Hilbert spaces*. The direct integral of such a family is defined as the Hilbert space

$$\int_\Lambda^\oplus \mathcal{H}_\lambda \, d\mu(\lambda) = \left\{ s \in \mathcal{F} : \int_\Lambda \langle s(\lambda), s(\lambda)\rangle_\lambda < \infty \right\} \Big/ \sim$$

of square integrable sections modulo the subspace of sections which are zero almost everywhere. This Hilbert space carries the obvious inner product.

A continuous linear map $T : E \to \int_\Lambda^\oplus \mathcal{H}_\lambda \, d\mu(\lambda)$ from a topological vector space E into a direct integral is said to be *pointwise defined* if there exists a continuous linear map $T_\lambda : E \to \mathcal{H}_\lambda$ for every $\lambda \in \Lambda$ such that for every $v \in E$:

$$T(v)(\lambda) = T_\lambda(v) \qquad \text{for almost every } \lambda \in \Lambda.$$

Theorem 1 (Gelfand–Kostyuchenko, see e.g. [1, Theorem 1.5])**.** *Every Hilbert–Schmidt operator $T:\mathcal{H}\to\int_\Lambda^\oplus \mathcal{H}_\lambda\,d\mu(\lambda)$ is pointwise defined.*

Lemma 2. *Assume that $T:\mathcal{H}\to\int_\Lambda^\oplus \mathcal{H}_\lambda\,d\mu(\lambda)$ is pointwise defined and has dense image, then almost every $T_\lambda:\mathcal{H}\to\mathcal{H}_\lambda$ has dense image.*

Proof. Let $(s_n)\subseteq\int_\Lambda^\oplus \mathcal{H}_\lambda\,d\mu(\lambda)$ as in the definition of the direct integral. For each n we let

$$\Lambda_n=\{\lambda\in\Lambda:s_n(\lambda)\notin\overline{T_\lambda(\mathcal{H})}\}.$$

We first show that every Λ_n has measure zero. Assume to the contrary that some Λ_n has positive measure. Then

$$\begin{aligned}\|s_n-T(v)\|^2&=\int_\Lambda\|s_n(\lambda)-T_\lambda(v)\|^2\,d\mu(\lambda)\geq\int_{\Lambda_n}\|s_n(\lambda)-T_\lambda(v)\|^2\,d\mu(\lambda)\\&=\int_{\Lambda_n}\|s_n(\lambda)_\perp\|^2+\|s_n(\lambda)_\|-T_\lambda(v)\|^2\,d\mu(\lambda)\ \geq\int_{\Lambda_n}\|s_n(\lambda)_\perp\|^2\,d\mu(\lambda),\end{aligned}$$

where $s_n(\lambda)_\|$ resp. $s_n(\lambda)_\perp$ denotes the orthogonal projection of $s_n(\lambda)$ to $\overline{T_\lambda(\mathcal{H})}$ resp. $T_\lambda(\mathcal{H})^\perp$. The latter integral is positive since $s_n(\lambda)_\perp\neq 0$ on Λ_n which is of positive measure. This shows that $\|s_n-T(v)\|\geq c>0$ for all $v\in\mathcal{H}$, contradicting the fact that T has dense image.
We have shown that Λ_n is of measure zero for every n, hence the countable union $\bigcup_n\Lambda_n$ has measure zero. This implies for almost every λ that $s_n(\lambda)\in\overline{T_\lambda(\mathcal{H})}$ for all n, so by the assumption that $(s_n(\lambda))_n$ spans a dense subspace of $\mathcal{H}_\lambda$ for every λ, we obtain that $\overline{T_\lambda(\mathcal{H})}=\mathcal{H}_\lambda$ for almost every λ. □

2. Sobolev norms on unitary representations

Let G be a real reductive group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ a Cartan decomposition. Denote by $\Delta_\mathfrak{k}\in\mathcal{U}(\mathfrak{k})$ the Casimir element of $\mathfrak{k}$ with respect to an invariant inner product on $\mathfrak{k}$. For an irreducible unitary representation $(\pi,\mathcal{H})$ of G we write $(\pi^\infty,\mathcal{H}^\infty)$ for the subrepresentation on the Fréchet space of smooth vectors. For every $N\in\mathbb{N}$,

$$\|v\|_N^2=\sum_{j=0}^N\|(d\pi(\Delta_\mathfrak{k}^j)v\|^2\qquad(v\in\mathcal{H}^\infty)$$

defines a continuous norm $\|\cdot\|_N$ on $\mathcal{H}^\infty$ which dominates the Hilbert space norm $\|\cdot\|$ of $\mathcal{H}$. Therefore, the completion $\mathcal{H}^N$ of $\mathcal{H}^\infty$ with respect to the norm $\|\cdot\|_N$ naturally embeds into $\mathcal{H}$ and yields a scale of Hilbert spaces

$$\mathcal{H}^\infty\subseteq\ldots\subseteq\mathcal{H}^{N+1}\subseteq\mathcal{H}^N\subseteq\ldots\subseteq\mathcal{H}^0=\mathcal{H}.$$

Lemma 3. *For $4N>\dim\mathfrak{k}$ the embedding $\mathcal{H}^N\hookrightarrow\mathcal{H}$ is Hilbert–Schmidt.*

Proof. Note that we may assume G to be connected by decomposing $\mathcal{H}$ into the direct sum of finitely many irreducible representations of the identity component G_0 of G.
Next, decompose $\mathcal{H}=\bigoplus_{\sigma\in\widehat{K}}\mathcal{H}[\sigma]$ into K-isotypic components. Then $d\pi(\Delta)|_{\mathcal{H}[\sigma]}$ is a scalar multiple of the identity. To describe the scalar, we identify an irreducible representation σ of K with its highest weight in $i\mathfrak{t}^*$ with respect to a maximal torus $\mathfrak{t}\subseteq\mathfrak{k}$ and a system of positive roots. Then

$$d\pi(\Delta_\mathfrak{k})|_{\mathcal{H}[\sigma]}=-(|\sigma+\rho_\mathfrak{k}|^2-|\sigma|^2)\,\mathrm{id}_{\mathcal{H}[\sigma]},$$

where $|\cdot|$ denotes the norm on $\mathfrak{t}^*$ induced from the same invariant inner product on $\mathfrak{k}$ that was used for the construction of the Casimir element $\Delta_{\mathfrak{k}}$. It follows that the square of the Hilbert–Schmidt norm of the embedding $\mathcal{H}^N \hookrightarrow \mathcal{H}$ is given by

$$\sum_{\sigma\in\widehat{K}} \frac{\dim\mathcal{H}[\sigma]}{\sum_{j=0}^N(|\sigma+\rho_{\mathfrak{k}}|^2-|\sigma|^2)^{2j}} \leq C\sum_{\sigma\in\widehat{K}}(1+|\sigma|)^{-4N}\dim\mathcal{H}[\sigma]$$

for some $C=C_N>0$, where $\rho_{\mathfrak{k}}\in i\mathfrak{t}^*$ is half the sum of all positive roots. By a result of Harish–Chandra [4, Theorem 4] combined with the Weyl Dimension Formula, there exist $C', C''>0$ such that

$$\dim\mathcal{H}[\sigma] \leq C'(\dim\sigma)^2 \leq C''(1+|\sigma|)^{\dim\mathfrak{k}-\dim\mathfrak{t}} \qquad \text{for all } \sigma\in\widehat{K}.$$

Hence, the square of the Hilbert–Schmidt norm of the embedding $\mathcal{H}^N \hookrightarrow \mathcal{H}$ can be bounded by a multiple of

$$\sum_{\sigma\in\widehat{K}}(1+|\sigma|)^{\dim\mathfrak{k}-\dim\mathfrak{t}-4N}.$$

Summation is over the highest weight lattice in $i\mathfrak{t}^*$, hence the sum is finite if and only if $\dim\mathfrak{k}-\dim\mathfrak{t}-4N<-\dim\mathfrak{t}$. $\square$

3. Proof of the main results

Combining Lemma 3 with Theorem 1 shows that the restriction of T to the Sobolev completion $\mathcal{H}_\pi^N$ is pointwise defined for N sufficiently large. Composing with the continuous linear embedding $\mathcal{H}_\pi^\infty \hookrightarrow \mathcal{H}_\pi^N$ shows Theorem A.

Now let $A_{\pi,\tau}^\infty:\mathcal{H}_\pi^\infty\to\mathcal{H}_\tau\otimes\mathcal{M}_{\pi,\tau}$, $\tau\in\widehat{H}$, be as in Theorem A. For an orthonormal basis $(w_\alpha)_{\alpha=1,\ldots,m(\pi,\tau)}$ of $\mathcal{M}_{\pi,\tau}$, $m(\pi,\tau)=\dim\mathcal{M}_{\pi,\tau}\in\mathbb{N}\cup\{\infty\}$, write

$$A_{\pi,\tau}^\infty(v)=\sum_\alpha A_{\pi,\tau,\alpha}^\infty(v)\otimes w_\alpha \qquad (v\in\mathcal{H}^\infty)$$

with $A_{\pi,\tau,\alpha}\in\operatorname{Hom}_H(\mathcal{H}_\pi^\infty,\mathcal{H}_\tau)$. If $A_{\pi,\tau}^\infty$ has dense image, then the operators $A_{\pi,\tau,\alpha}^\infty$ have to be linearly independent, so $m(\pi,\tau)=\dim\mathcal{M}_{\pi,\tau}\leq\dim\operatorname{Hom}_H(\mathcal{H}_\pi^\infty,\mathcal{H}_\tau)$. Lemma 2 implies that this is the case for almost every τ.

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