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Abstract

We consider an infinitely divisible random field in $\mathbb{R}^d$ given as an integral of a kernel function with respect to a Lévy basis. Under mild regularity conditions we derive central limit theorems for the moment estimators of the mean and the variogram of the field.

Keywords: central limit theorem; infinite divisibility; Lévy-based modelling

1 Introduction

In the present paper, we derive central limit theorems (CLTs) for mean and variogram estimators for a field $(X_i)_{i \in \mathbb{Z}^d}$, defined by

$$X_i = \int_{\mathbb{R}^d} f(s - i) \, M(\mathrm{d}s) \,, \tag{1.1}$$

where M is an infinitely divisible, independently scattered random measure on $\mathbb{R}^d$ and f is some kernel function.

Lévy-based models as defined in (1.1) provide a flexible modelling framework that recently has been used in a range of modelling contexts, including modelling of turbulent flows, growth processes, Cox point processes and brain imaging data [3, 6, 8, 9].

Due to the tractability of Lévy-based models, it has been possible to derive tail asymptotics for the supremum of such a field as well as asymptotics of excursion sets of the field. The case where M is a convolution equivalent measure is considered in [12, 14]. Results for random measures with regularly varying tails have been derived in [11] and refined in [1, 2].

In [4], CLTs are proved for a stationary random field $(X_i)_{i \in \mathbb{Z}^d}$, discretely defined by

$$X_i = u(M_{s+i} \,:\, s \in \mathbb{Z}^d) \,, \tag{1.2}$$

where $(M_i)_{i \in \mathbb{Z}^d}$ are independent and identically distributed (i.i.d.) random variables and u is a measurable function. In contrast, our set-up involves a random measure

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defined on $\mathbb{R}^d$. However, if we discretize the integral in (1.1), then our model is of the form (1.2) with u a linear function and the random variables M_i having an infinitely divisible distribution. Note that if this type of model is going to hold for finer and finer resolution, then the common distribution of the random variables M_i has to be infinitely divisible.

For a random field defined by (1.1), we prove in the present paper CLTs for

$$S_\Gamma = \sum_{i \in \Gamma} X_i \quad \text{and} \quad T_\Gamma(h) = \sum_{i \in \Gamma} (X_i - X_{i+h})^2,$$

where $h \in \mathbb{Z}^d$ and Γ is a finite subset of $\mathbb{Z}^d$ with number of elements increasing to infinity. We show under mild regularity conditions that both S_Γ and $T_\Gamma(h)$ are asymptotically normally distributed and give explicit expressions for the asymptotic variances. It turns out that the asymptotic variance of $T_\Gamma(h)$ is equal to an approximate variance, earlier derived in [13]. In the latter paper, the approximate variance was used in the assessment of the precision of a section spacing estimator in electron microscopy. More details are given in Section 5 below.

If we discretize the integral in (1.1), both S_Γ and $T_\Gamma(h)$ take the form

$$\sum_{i \in \Gamma} v \Big(\sum_{s \in \mathbb{Z}^d} a_s M_{s+i} \Big). \tag{1.3}$$

Here, $v(x) = x$ and $a_s = f(s)$ for S_Γ, while $v(x) = x^2$ and $a_s = f(s) - f(s-h)$ for $T_\Gamma(h)$. In [4], CLTs for random variables of the form (1.3) are discussed. In particular, a CLT for S_Γ is derived which is a discrete analogue of the CLT we obtain. For second-order properties, [4] considers

$$\sum_{i \in \Gamma} X_i X_{i+h},$$

but do not deal explicitly with the case where the mean of the field is unknown, as is done in our paper.

The present paper is organised as follows. In Section 2 we define the random field (1.1) and state the main result. In Section 3 we provide the proofs. An important tool in the proofs will be a CLT (Theorem 3.1) for m_n-dependent fields proved in [5]. Section 4 discusses the case where the kernel function is isotropic, while the results are used for estimation of sample spacing in Section 5. Cumulant formulae and a proof of a corollary to Ottaviani's inequality are deferred to two Appendices.

2   Preliminaries and main result

Consider an independently scattered random measure M on $\mathbb{R}^d$. Then for a sequence of disjoint sets $(A_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}^d$ in $\mathcal{B}(\mathbb{R}^d)$ the random variables $(M(A_n))_{n \in \mathbb{N}}$ are independent and satisfy $M(\cup A_n) = \sum M(A_n)$. Assume furthermore that $M(A)$ is infinitely divisible for all $A \in \mathcal{B}(\mathbb{R}^d)$. Then M is called a Lévy basis, see [3] and references therein.

For a random variable X let $\kappa_X(\lambda)$ denote its cumulant function $\log E(e^{i\lambda X})$. We shall assume that the Lévy basis is stationary and isotropic such that for $A \in \mathcal{B}(\mathbb{R}^d)$ the variable $M(A)$ has a Lévy–Khintchine representation given by

$$\kappa_{M(A)}(\lambda) = i\lambda a m_d(A) - \tfrac{1}{2}\lambda^2 \theta m_d(A) + \int_{A \times \mathbb{R}} \left(e^{i\lambda u} - 1 - i\lambda u 1_{[-1,1]}(u) \right) F(\mathrm{d}s, \mathrm{d}u), \quad (2.1)$$

where m_d is the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $a \in \mathbb{R}$, $\theta \geq 0$ and F is a measure on $\mathcal{B}(\mathbb{R}^d \times \mathbb{R})$ of the form

$$F(A \times B) = m_d(A)\rho(B). \quad (2.2)$$

In the following, we let M' be a so–called spot variable, i.e. M' is a random variable with the distribution of $M(A)$ for $A \in \mathcal{B}(\mathbb{R}^d)$ with $m_d(A) = 1$. We will assume that the Lévy measure ρ satisfies

$$\int_{[-1,1]} z^2 \, \rho(\mathrm{d}z) < \infty \quad \text{and} \quad \int_{[-1,1]^c} |z| \, \rho(\mathrm{d}z) < \infty.$$

The first assumption is needed for ρ to be a Lévy measure. The second assumption is equivalent to $E|M'| < \infty$ (see [15, Theorem 25.3]). Now assume that $f : \mathbb{R}^d \to [0, \infty)$ is a bounded kernel function satisfying

$$\int_{\mathbb{R}^d} f(s)\mathrm{d}s < \infty. \quad (2.3)$$

Let $\mathbb{Z}^d$ be the grid integer numbers in $\mathbb{R}^d$. We consider the family of random variables $(X_i)_{i \in \mathbb{Z}^d}$ defined by

$$X_i = \int_{\mathbb{R}^d} f(s - i) \, M(\mathrm{d}s). \quad (2.4)$$

See [10, Theorem 2.7] for existence of the integrals, where their conditions (i)–(iii) can be easily verified under the given assumptions on M and f. The field is stationary, and if furthermore f has the form $f(s) = g(\|s\|)$, where $\|\cdot\|$ is Euclidean norm, it will be isotropic. This special case is studied in Section 4.

We will be interested in estimating the mean and the variogram of the field. We have

$$\mu = \mathbb{E}X_0 = \mathbb{E}(M') \int_{\mathbb{R}^d} f(s) \, \mathrm{d}s$$

and for $h \in \mathbb{Z}^d$

$$\gamma(h) = \mathbb{E}(X_0 - X_h)^2 = \mathrm{Var}(M') \int_{\mathbb{R}^d} (f(s) - f(s - h))^2 \, \mathrm{d}s,$$

where identities from Appendix A have been applied. Note that due to stationarity, $\mu = \mathbb{E}X_i$ and $\gamma(h) = \mathbb{E}(X_i - X_{i+h})^2$ for all $i \in \mathbb{Z}^d$. For Γ a finite subset of $\mathbb{Z}^d$, we define

$$S_\Gamma = \sum_{i \in \Gamma} X_i, \quad (2.5)$$

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and for $h \in \mathbb{Z}^d$

$$T_\Gamma(h) = \sum_{i \in \Gamma} (X_i - X_{i+h})^2 \, . \tag{2.6}$$

Unbiased estimators for μ and $\gamma(h)$, respectively, are obtained as

$$\widehat{\mu} = \frac{1}{|\Gamma|} S_\Gamma \qquad \widehat{\gamma(h)} = \frac{1}{|\Gamma|} T_\Gamma \, ,$$

where $|\Gamma|$ denotes the number of elements in Γ. The main result gives asymptotic normality of these estimators for $|\Gamma|$ going to infinity.

Theorem 2.1. *Let $(\Gamma_n)_{n \geq 1}$ be a sequence of finite subsets of $\mathbb{Z}^d$ and assume that $|\Gamma_n|$ goes to infinity and $|\partial\Gamma_n|/|\Gamma_n|$ goes to 0. Let*

$$\sigma_S^2 = \operatorname{Var}(M') \sum_{i \in \mathbb{Z}^d} \int_{\mathbb{R}^d} f(s) f(s-i) \, ds \, .$$

If $\sigma_S^2 < \infty$, then

$$\frac{S_{\Gamma_n} - |\Gamma_n|\mu}{\sqrt{|\Gamma_n|}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_S^2) \, . \tag{2.7}$$

If furthermore $\mathbb{E}(M')^4 < \infty$, then

$$\frac{T_{\Gamma_n}(h) - |\Gamma_n|\gamma(h)}{\sqrt{|\Gamma_n|}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_T^2) \, , \tag{2.8}$$

where

$$\sigma_T^2 = \sum_{j \in \mathbb{Z}^d} \left[\kappa_4(M') \int_{\mathbb{R}^d} g(s)^2 g(s-j)^2 \, ds + 2 \operatorname{Var}(M')^2 \left(\int_{\mathbb{R}^d} g(s) g(s-j) \, ds \right)^2 \right] < \infty \, ,$$

$\kappa_4(M')$ is the fourth cumulant of M' and $g(s) = f(s) - f(s-h)$.

It should be noted that the assumptions of the theorem

$$|\Gamma_n| \to \infty \quad \text{and} \quad |\partial\Gamma_n|/|\Gamma_n| \to 0$$

are equivalent with the (less intuitive, but more computationally convenient) property

$$|\Gamma_n| \to \infty \quad \text{and} \quad |(\Gamma_n - j) \cap \Gamma_n|/|\Gamma_n| \to 1$$

for all $j \in \mathbb{Z}^d$, where $\Gamma + j$ denotes translation of Γ in direction $j \in \mathbb{Z}^d$, i.e.

$$\Gamma + j = \{i + j \mid i \in \Gamma\} \, .$$

3   Proofs

An important tool in the proof of our main result, Theorem 2.1, is a CLT for m_n–dependent random fields that may be found in [5, Theorem 2]. This theorem is also used in the proofs of the CLTs, derived in [4]. However, our proof of the CLT for S_Γ is simpler and takes advantage of the linearity of the Lévy–based model (1.1), while our proof of the CLT for $T_\Gamma(h)$ follows a different route than the one adopted in [4] for the second–order property $\sum_{i \in \Gamma} X_i X_{i+h}$.

Recall that a random field $(Z_i)_{i \in \mathbb{Z}^d}$ is said to be m_n–dependent, if for any $A, B \subset \mathbb{Z}^d$ with $|A|, |B| < \infty$ and $\inf\{\|a - b\| \; : \; a \in A \,, b \in B\} > m_n$, $(Z_a)_{a \in A}$ and $(Z_b)_{b \in B}$ are independent. The CLT for m_n–dependent fields is here stated as

Theorem 3.1. *Let $(\Gamma_n)_{n \geq 1}$ be a sequence of finite subsets of $\mathbb{Z}^d$ with $|\Gamma_n| \to \infty$ as $n \to \infty$ and let $(m_n)_{n \geq 1}$ be a sequence of positive integers. For each $n \geq 1$, let $\{U_n(j)\,,\, j \in \mathbb{Z}^d\}$ be an m_n–dependent random field with $\mathbb{E}U_n(j) = 0$ for all $j \in \mathbb{Z}^d$. Assume that $\mathbb{E}(\sum_{j \in \Gamma_n} U_n(j))^2 \to \sigma^2$ as $n \to \infty$ with $\sigma^2 < \infty$. Then $\sum_{j \in \Gamma_n} U_n(j)$ converges in distribution to a Gaussian random variable with mean zero and variance σ^2, if there exists a finite constant $c > 0$ such that for any $n \geq 1$,*

$$\sum_{j \in \Gamma_n} \mathbb{E}U_n(j)^2 \leq c,$$

and for any $\epsilon > 0$, it holds that $\lim_{n \to \infty} L_n(\epsilon) = 0$, where

$$L_n(\epsilon) = m_n^{2d} \sum_{j \in \Gamma_n} \mathbb{E}\big(U_n(j)^2 \mathbf{1}_{\{|U_n(j)| \geq \epsilon m_n^{-2d}\}}\big).$$

Proof of Theorem 2.1 for S_{Γ_n}. Let $(m_n)_{n \geq 1}$ be a sequence of integers increasing to infinity. The choice of the sequence will be specified below.

Let for $i \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, $C_n(i)$ be the ball in $\mathbb{R}^d$ with center in i and radius $m_n/3$. Let

$$\overline{X}_i^n = \int_{C_n(i)} f(s - i) \, M(\mathrm{d}s)$$

and let furthermore $\mu_n = \mathbb{E}(\overline{X}_0^n) = \mathbb{E}(M') \int_{C_n(0)} f(s) \, \mathrm{d}s$ and $\overline{S}_{\Gamma_n} = \sum_{i \in \Gamma_n} \overline{X}_i^n$. The proof will be divided into two steps. The first step is to show that

$$\lim_{n \to \infty} \frac{\|S_{\Gamma_n} - |\Gamma_n|\mu - (\overline{S}_{\Gamma_n} - |\Gamma_n|\mu_n)\|_2}{\sqrt{|\Gamma_n|}} = 0 \,. \tag{3.1}$$

The second step is to show, using Theorem 3.1, that with an appropriately chosen sequence $(m_n)_{n \geq 1}$, we have

$$\frac{\overline{S}_{\Gamma_n} - |\Gamma_n|\mu_n}{\sqrt{|\Gamma_n|}} \xrightarrow{\;\mathcal{D}\;} \mathcal{N}(0, \sigma_S^2) \tag{3.2}$$

To prove (3.1), we find, using Appendix A,

$$
\frac{\|S_{\Gamma_n} - |\Gamma_n|\mu - (\overline{S}_{\Gamma_n} - |\Gamma_n|\mu_n)\|_2^2}{|\Gamma_n|}
$$

$$
= \frac{1}{|\Gamma_n|} \mathbb{E}\left[\left(\sum_{i \in \Gamma_n}\left[\int_{C_n(i)^c} f(s-i)\,M(\mathrm{d}s) - \mathbb{E}(M')\int_{C_n(i)^c} f(s-i)\,\mathrm{d}s\right]\right)^2\right]
$$

$$
= \frac{1}{|\Gamma_n|} \sum_{i \in \Gamma_n}\sum_{j \in \Gamma_n} \mathrm{Cov}\left[\int_{C_n(i)^c} f(s-i)\,M(\mathrm{d}s), \int_{C_n(j)^c} f(s-j)\,M(\mathrm{d}s)\right]
$$

$$
= \frac{\mathrm{Var}(M')}{|\Gamma_n|} \sum_{i \in \Gamma_n}\sum_{j \in \Gamma_n} \int_{C_n(i)^c \cap C_n(j)^c} f(s-i)\,f(s-j)\,\mathrm{d}s
$$

$$
\leq \frac{\mathrm{Var}(M')}{|\Gamma_n|} \sum_{i \in \Gamma_n}\sum_{j \in \mathbb{Z}^d} \int_{C_n(i)^c \cap C_n(j)^c} f(s-i)\,f(s-j)\,\mathrm{d}s
$$

$$
= \mathrm{Var}(M') \sum_{j \in \mathbb{Z}^d} \int_{C_n(0)^c \cap C_n(j)^c} f(s)\,f(s-j)\,\mathrm{d}s
$$

$$
\to 0\,,
$$

where the convergence is a result of dominated convergence, since

$$
\sum_{j \in \mathbb{Z}^d} \int_{\mathbb{R}^d} f(s)\,f(s-j)\,\mathrm{d}s < \infty\,, \tag{3.3}
$$

and $C_n(0)^c \cap C_n(j)^c \to \emptyset$ as $n \to \infty$.

In order to prove (3.2), define for $n \in \mathbb{N}$ and $j \in \mathbb{Z}^d$

$$
U_n(j) = \frac{\overline{X}_j^n - \mu_n}{\sqrt{|\Gamma_n|}},
$$

such that

$$
\sum_{j \in \Gamma_n} U_n(j) = \frac{\overline{S}_{\Gamma_n} - |\Gamma_n|\mu_n}{\sqrt{|\Gamma_n|}}.
$$

Then, clearly $(U_n(j))_{j \in \mathbb{Z}^d}$ is m_n–dependent with $\mathbb{E}U_n(j) = 0$ for all $n \geq 1$ and $j \in \mathbb{Z}^d$. Furthermore, cf. Appendix A,

$$
\mathbb{E}\left[\left(\sum_{i \in \Gamma_n} U_n(i)\right)^2\right]
$$

$$
= \frac{1}{|\Gamma_n|} \sum_{i \in \Gamma_n}\sum_{j \in \Gamma_n} \mathrm{Cov}\left[\int_{C_n(i)} f(s-i)\,M(\mathrm{d}s), \int_{C_n(j)} f(s-j)\,M(\mathrm{d}s)\right]
$$

$$
= \frac{\mathrm{Var}(M')}{|\Gamma_n|} \sum_{i \in \Gamma_n}\sum_{j \in \Gamma_n} \int_{C_n(i) \cap C_n(j)} f(s-i)\,f(s-j)\,\mathrm{d}s
$$

$$
= \mathrm{Var}(M') \sum_{j \in \mathbb{Z}^d} \frac{|(\Gamma_n - j) \cap \Gamma_n|}{|\Gamma_n|} \int_{C_n(0) \cap C_n(j)} f(s)\,f(s-j)\,\mathrm{d}s
$$

$$
\to \mathrm{Var}(M') \sum_{j \in \mathbb{Z}^d} \int_{\mathbb{R}^d} f(s)\,f(s-j)\,\mathrm{d}s,
$$

as $n \to \infty$, by dominated convergence. Here, we have used that (3.3) is fulfilled, $|(\Gamma_n - j) \cap \Gamma_n|/|\Gamma_n| \to 1$ for all j and $C_n(0) \cap C_n(j) \to \mathbb{R}^d$ as $n \to \infty$. Furthermore, we find, cf. Appendix A,

$$\sum_{i \in \Gamma_n} \mathbb{E}[U_n(i)^2] = \operatorname{Var}(\overline{X}_0^n) = \operatorname{Var}(M') \int_{C_n(0)} f(s)^2 \, \mathrm{d}s \le \operatorname{Var}(M') \int_{\mathbb{R}^d} f(s)^2 \, \mathrm{d}s < \infty \, .$$

Now define

$$Y = \sup_{n \ge 1} |\overline{X}_0^n - \mu_n| \, .$$

Note that the sequence $(\overline{X}_0^n)_{n \ge 1}$ has independent increments and that

$$\sup_{n \ge 1} \mathbb{E}\big[(\overline{X}_0^n - \mu_n)^2 \big] = \sup_{n \ge 1} \operatorname{Var}(M') \int_{C_n(0)} f(s)^2 \, \mathrm{d}s \le \operatorname{Var}(M') \int_{\mathbb{R}^d} f(s)^2 \, \mathrm{d}s < \infty \, .$$

Then, $\mathbb{E}Y^2 < \infty$, due to Corollary B.2. Furthermore,

$$
\begin{aligned}
L_n(\epsilon) &= m_n^{2d} \sum_{j \in \Gamma_n} \mathbb{E}\big[U_n^2(j) \mathbf{1}_{\{|U_n(j)| \ge \epsilon m_n^{-2d}\}} \big] \\
&= m_n^{2d} \mathbb{E}\big[(\overline{X}_0^n - \mu_n)^2 \mathbf{1}_{\{|\overline{X}_0^n - \mu_n| \ge \epsilon \sqrt{|\Gamma_n|} m_n^{-2d}\}} \big] \\
&\le m_n^{2d} \mathbb{E}\big[Y^2 \mathbf{1}_{\{Y \ge \epsilon \sqrt{|\Gamma_n|} m_n^{-2d}\}} \big] \\
&= m_n^{2d} \mathbb{E}\big[Y^2 \mathbf{1}_{\{Y \ge \epsilon \sqrt{|\Gamma_n|} m_n^{-2d}, Y \le |\Gamma_n|^{1/4}\}} \big] + m_n^{2d} \mathbb{E}\big[Y^2 \mathbf{1}_{\{Y \ge \epsilon \sqrt{|\Gamma_n|} m_n^{-2d}, Y > |\Gamma_n|^{1/4}\}} \big] \\
&\le m_n^{2d} \sqrt{|\Gamma_n|} \mathbb{P}\big[Y \ge \epsilon \sqrt{|\Gamma_n|} m_n^{-2d} \big] + m_n^{2d} \mathbb{E}\big[Y^2 \mathbf{1}_{\{Y > |\Gamma_n|^{1/4}\}} \big] \\
&\le \frac{m_n^{6d}}{\epsilon^2 \sqrt{|\Gamma_n|}} \mathbb{E}Y^2 + m_n^{2d} \psi(|\Gamma_n|^{1/4}) \, ,
\end{aligned}
$$

where $\psi(x) = \mathbb{E}(Y^2 1_{\{Y \ge x\}})$. Note that $\psi(x) \to 0$ as $x \to \infty$. With $[\cdot]$ denoting the integer part, now choose the sequence $(m_n)_{n \ge 1}$ as $m_n = [|\Gamma_n|^{\frac{1}{24d}}]$ if $\psi(|\Gamma_n|^{1/4}) = 0$ and

$$m_n = \min\big\{ \big[\psi(|\Gamma_n|^{1/4})^{-\frac{1}{4d}} \big], \big[|\Gamma_n|^{\frac{1}{24d}} \big] \big\},$$

otherwise. With this choice it is seen that $L_n(\epsilon) \to 0$ as $n \to \infty$. Thus we have from Theorem 3.1 that

$$\frac{\overline{S}_{\Gamma_n} - |\Gamma_n| \mu_n}{\sqrt{|\Gamma_n|}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_S^2) \, .$$

Combining this with (3.1) yields the desired result. $\qquad \square$

In the proof of Theorem 2.1 for $T_{\Gamma_n}(h)$, we need the following lemma, the proof of which follows from straightforward calculations, using that f is bounded and non–negative.

Lemma 3.2. *Let for $h \in \mathbb{Z}^d$*

$$g(s) = f(s) - f(s - h), \qquad s \in \mathbb{R}^d.$$

Then, for $s \in \mathbb{R}^d$, $j \in \mathbb{Z}^d$,

$$|g(s)g(s-j)| \le v_j(s),$$
$$g(s)^2 g(s-j)^2 \le Cv_j(s),$$

where

$$v_j(s) = f(s)f(s-j) + f(s)f(s-h-j) + f(s-h)f(s-j) + f(s-h)f(s-h-j)$$

and $C < \infty$ is an appropriately chosen constant. If

$$\sum_{j \in \mathbb{Z}^d} \int_{\mathbb{R}^d} f(s)f(s-j)\, \mathrm{d}s < \infty,$$

then

$$\sum_{j \in \mathbb{Z}^d} \int_{\mathbb{R}^d} v_j(s)\, \mathrm{d}s < \infty \quad and \quad \sum_{j \in \mathbb{Z}^d} \left(\int_{\mathbb{R}^d} v_j(s)\, \mathrm{d}s \right)^2 < \infty.$$

Proof of Theorem 2.1 for $T_{\Gamma_n}(h)$. First note that it follows from Lemma 3.2 that $\sigma_S^2 < \infty$ and $\mathbb{E}[(M')^4] < \infty$ implies $\sigma_T^2 < \infty$.

For ease of notation, we shall now consider the field $(Y_i)_{i \in \mathbb{Z}^d}$, where $Y_i = X_i - X_{i+h}$. Then, for all $i \in \mathbb{Z}^d$, $\mathbb{E}Y_i = 0$ and

$$Y_i = \int_{\mathbb{R}^d} \big(f(s-i) - f(s-i-h) \big) M(\mathrm{d}s) = \int_{\mathbb{R}^d} g(s-i)\, M(\mathrm{d}s),$$

where $g(s) = f(s) - f(s-h)$ as previously. Let $(m_n)_{n \ge 1}$ be (another) sequence of integers increasing to infinity. Let $C_n(i)$ and $\overline{X}_i^n$ be defined as in the proof of Theorem 2.1 for S_{Γ_n}. Let

$$\overline{Y}_i^n = \overline{X}_i^n - \overline{X}_{i+h}^n = \int_{\mathbb{R}^d} g_n(s-i)\, M(\mathrm{d}s),$$

where $g_n(s) = \mathbf{1}_{C_n(0)}(s)f(s) - \mathbf{1}_{C_n(h)}(s)f(s-h)$. Then $g_n(s) \to g(s)$ for all $s \in \mathbb{R}^d$. Note that $\mathbb{E}\overline{Y}_i^n = 0$ and let furthermore $\gamma(h)_n = \mathbb{E}(\overline{Y}_0^n)^2$. Also define $\overline{T}_{\Gamma_n}(h) = \sum_{i \in \Gamma_n} (\overline{Y}_i^n)^2$.

As for S_{Γ_n}, the first step will be to show that

$$\lim_{n \to \infty} \frac{\| T_{\Gamma_n}(h) - |\Gamma_n|\gamma(h) - (\overline{T}_{\Gamma_n}(h) - |\Gamma_n|\gamma(h)_n) \|_2}{\sqrt{|\Gamma_n|}} = 0. \qquad (3.4)$$

The second step is to show, using Theorem 3.1, that with an appropriately chosen sequence $(m_n)_{n \ge 1}$, we have

$$\frac{\overline{T}_{\Gamma_n}(h) - |\Gamma_n|\gamma(h)_n}{\sqrt{|\Gamma_n|}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_T^2). \qquad (3.5)$$

To prove (3.4), we find

$$\frac{\|T_{\Gamma_n}(h) - |\Gamma_n|\gamma(h) - (\overline{T}_{\Gamma_n}(h) - |\Gamma_n|\gamma(h)_n)\|_2^2}{|\Gamma_n|}$$

$$= \frac{1}{|\Gamma_n|}\mathbb{E}\left[\left(\sum_{i\in\Gamma_n}\left[Y_i^2 - \mathbb{E}Y_i^2 - ((\overline{Y}_i^n)^2 - \mathbb{E}(\overline{Y}_i^n)^2)]\right)^2\right]\right]$$

$$= \frac{1}{|\Gamma_n|}\sum_{i,j\in\Gamma_n}\mathbb{E}\left[(Y_i^2 - \mathbb{E}Y_i^2)(Y_j^2 - \mathbb{E}Y_j^2)\right]$$

$$+ \frac{1}{|\Gamma_n|}\sum_{i,j\in\Gamma_n}\mathbb{E}\left[\left((\overline{Y}_i^n)^2 - \mathbb{E}(\overline{Y}_i^n)^2\right)\left((\overline{Y}_j^n)^2 - \mathbb{E}(\overline{Y}_j^n)^2\right)\right]$$

$$- 2\frac{1}{|\Gamma_n|}\sum_{i,j\in\Gamma_n}\mathbb{E}\left[(Y_i^2 - \mathbb{E}Y_i^2)\left((\overline{Y}_j^n)^2 - \mathbb{E}(\overline{Y}_j^n)^2\right)\right]. \tag{3.6}$$

We will show convergence of the three terms in (3.6) separately. Using the calculation formulae from Appendix A, the first term equals

$$\frac{1}{|\Gamma_n|}\sum_{i,j\in\Gamma_n}\left[\kappa_4(M')\int_{\mathbb{R}^d}g(s-i)^2 g(s-j)^2\,\mathrm{d}s\right.$$

$$\left. + 2\operatorname{Var}(M')^2\left(\int_{\mathbb{R}^d}g(s-i)g(s-j)\,\mathrm{d}s\right)^2\right]$$

$$= \sum_{j\in\mathbb{Z}^d}\frac{|\Gamma_n\cap(\Gamma_n-j)|}{|\Gamma_n|}\left[\kappa_4(M')\int_{\mathbb{R}^d}g(s)^2 g(s-j)^2\,\mathrm{d}s\right.$$

$$\left. + 2\operatorname{Var}(M')^2\left(\int_{\mathbb{R}^d}g(s)g(s-j)\,\mathrm{d}s\right)^2\right].$$

By dominated convergence, this converges to σ_T^2, using that σ_T^2 is finite and that all terms in σ_T^2 are non–negative. The second term in (3.6) equals

$$\sum_{j\in\mathbb{Z}^d}\frac{|\Gamma_n\cap(\Gamma_n-j)|}{|\Gamma_n|}\left[\kappa_4(M')\int_{\mathbb{R}^d}g_n(s)^2 g_n(s-j)^2\,\mathrm{d}s\right.$$

$$\left. + 2\operatorname{Var}(M')^2\left(\int_{\mathbb{R}^d}g_n(s)g_n(s-j)\,\mathrm{d}s\right)^2\right].$$

By dominated convergence and Lemma 3.2, applied to g_n instead of g, but keeping the upper bound as $v_j(s)$, this also converges to σ_T^2. Similarly, the third term of (3.6) converges to $-2\sigma_T^2$, and we have the desired result in (3.4).

Now define for $n\in\mathbb{N}$ and $j\in\mathbb{Z}^d$

$$V_n(j) = \frac{(\overline{Y}_j^n)^2 - \gamma(h)_n}{\sqrt{|\Gamma_n|}}.$$

We want to show that

$$\sum_{j\in\Gamma_n}V_n(j) \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma_T^2),$$

using Theorem 3.1. Then by construction $\mathbb{E}V_n(j) = 0$ for all $n \geq 1$ and $j \in \mathbb{Z}^d$, and for n such that $m_n/3 > \|h\|$, the field $(V_n(j))_{j \in \mathbb{Z}^d}$ is m_n–dependent. Furthermore

$$\mathbb{E}\Big[\Big(\sum_{i \in \Gamma_n} V_n(i)\Big)^2\Big] = \frac{1}{|\Gamma_n|}\mathbb{E}\Big[\Big(\sum_{i \in \Gamma_n} \big[(\overline{Y}_i^n)^2 - \mathbb{E}(\overline{Y}_i^n)^2\big]\Big)^2\Big]$$

$$= \frac{1}{|\Gamma_n|} \sum_{i,j \in \Gamma_n} \mathbb{E}\big[\big((\overline{Y}_i^n)^2 - \mathbb{E}(\overline{Y}_i^n)^2\big)\big((\overline{Y}_j^n)^2 - \mathbb{E}(\overline{Y}_j^n)^2\big)\big],$$

which equals the second term of (3.6) and therefore has limit σ_T^2. Additionally, using Appendix A,

$$\sum_{i \in \Gamma_n} \mathbb{E}[V_n(i)^2] = \mathbb{E}\Big[\big[(\overline{Y}_0^n)^2 - \mathbb{E}(\overline{Y}_0^n)^2\big]^2\Big]$$

$$= \kappa_4(M')\Big(\int_{\mathbb{R}^d} g_n(s)^4\,\mathrm{d}s\Big) + 2\operatorname{Var}(M')^2\Big(\int_{\mathbb{R}^d} g_n(s)^2\,\mathrm{d}s\Big)^2$$

$$\leq \kappa_4(M')\Big(\int_{\mathbb{R}^d} v_0(s)^2\,\mathrm{d}s\Big) + 2\operatorname{Var}(M')^2\Big(\int_{\mathbb{R}^d} v_0(s)\,\mathrm{d}s\Big)^2 < \infty$$

with v_0 as defined in Lemma 3.2. Now define

$$Z = \big(\sup_{n \geq 1} |\overline{X}_0^n| + \sup_{n \geq 1} |\overline{X}_h^n|\big)^2 + C,$$

where

$$C = 2\operatorname{Var}(M')\int_{\mathbb{R}^d} f(s)^2\,\mathrm{d}s.$$

Using Appendix A, we find that $0 \leq \gamma(h)_n \leq C$ for all $n \geq 1$. Also note that the sequence $(\overline{X}_0^n)_{n \geq 1}$ has independent increments and that for $k = 1, 2, 3, 4$, cf. Appendix A,

$$\sup_{n \geq 1} \kappa_k\big(\overline{X}_0^n\big) = \sup_{n \geq 1} \kappa_k(M')\int_{C_n(0)} f(s)^k\,\mathrm{d}s \leq \kappa_k(M')\int_{\mathbb{R}^d} f(s)^k\,\mathrm{d}s < \infty.$$

Thus also $\sup_{n \geq 1} \mathbb{E}\big(\overline{X}_0^n\big)^4 < \infty$. By stationarity, also $\sup_{n \geq 1} \mathbb{E}\big(\overline{X}_h^n\big)^4 < \infty$, and thus $\mathbb{E}Z^2 < \infty$, due to Corollary B.2. Then

$$L_n(\epsilon) = m_n^{2d} \sum_{j \in \Gamma_n} \mathbb{E}\big[V_n^2(j)\mathbf{1}_{\{|V_n(j)| \geq \epsilon m_n^{-2d}\}}\big]$$

$$= m_n^{2d}\mathbb{E}\big[\big((\overline{X}_0^n - \overline{X}_h^n)^2 - \gamma(h)_n\big)^2\mathbf{1}_{\{|(\overline{X}_0^n - \overline{X}_h^n)^2 - \gamma(h)_n| \geq \epsilon\sqrt{|\Gamma_n|}m_n^{-2d}\}}\big]$$

$$\leq m_n^{2d}\mathbb{E}\big[Z^2\mathbf{1}_{\{Z \geq \epsilon\sqrt{|\Gamma_n|}m_n^{-2d}\}}\big].$$

Now, it follows that $L_n(\epsilon) \to 0$ by choosing $(m_n)_{n \geq 1}$ as in the proof of Theorem 2.1 for S_{Γ_n}. Thus, we can conclude from Theorem 3.1

$$\frac{\overline{T}_{\Gamma_n} - |\Gamma_n|\gamma(h)_n}{\sqrt{|\Gamma_n|}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_T^2),$$

and combining this with (3.4) gives the desired convergence.

$\square$

4 Isotropic model

Now make the assumption that the observations in the field have the form

$$X_i = \int_{\mathbb{R}^d} f(\|s - i\|)\, M(\mathrm{d}s) \tag{4.1}$$

for $i \in \mathbb{Z}^d$, where $f : [0, \infty) \to [0, \infty)$ satisfies

$$\int_{\mathbb{R}^d} f(\|s\|)\, \mathrm{d}s < \infty\,.$$

This is equivalent to assuming that $f(s)$ in (2.4) only depends on s through $\|s\|$. With this assumption, the field is isotropic, so

$$\mathbb{E}(X_i - X_j)^2 = \gamma(\|i - j\|)$$

for a variogram function $\gamma : [0, \infty) \to [0, \infty)$. Estimation of $\mu = \mathbb{E}X_0$ and the corresponding asymptotic properties are unchanged, compared to the more general framework in Section 2. Also estimation of $\gamma(\|h\|)$ for $h \in \mathbb{Z}^d$ can be carried out as before, and the asymptotic results are still valid.

However, we can improve the variogram estimator by taking into account that $\gamma(\|h\|)$ only depends on $\|h\|$. For this, we introduce

$$T'_\Gamma(d_0) = \frac{1}{|h(d_0)|} \sum_{i \in \Gamma} \sum_{h \in h(d_0)} (X_i - X_{i+h})^2\,,$$

where Γ is a finite subset of $\mathbb{Z}^d$, $d_0 > 0$ and $h(d_0) = \{h \in \mathbb{Z}^d : \|h\| = d_0\}$. We have

Theorem 4.1. *Let $(\Gamma_n)_{n \geq 1}$ be a sequence of finite subsets of $\mathbb{Z}^d$ and assume that $|\Gamma_n|$ goes to infinity and $|\partial\Gamma_n|/|\Gamma_n|$ goes to 0. If*

$$\sum_{i \in \mathbb{Z}^d} \int_{\mathbb{R}^d} f(\|s\|) f(\|s - i\|)\, \mathrm{d}s < \infty$$

and $\mathbb{E}(M')^4 < \infty$, then

$$\frac{T'_{\Gamma_n}(d_0) - |\Gamma_n|\gamma(d_0)}{\sqrt{|\Gamma_n|}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2_{T'})\,,$$

where

$$\sigma^2_{T'} = \frac{1}{|h(d_0)|^2} \sum_{j \in \mathbb{Z}^d} \sum_{h,h' \in h(d_0)} \left[\kappa_4(M') \int_{\mathbb{R}^d} g_h(s)^2 g_{h'}(s - j)^2\, \mathrm{d}s \right.$$
$$\left. + 2\operatorname{Var}(M')^2 \left(\int_{\mathbb{R}^d} g_h(s) g_{h'}(s - j)\, \mathrm{d}s \right)^2 \right] < \infty\,,$$

and $g_h(s) = f(\|s\|) - f(\|s - h\|)$.

The proof of Theorem 4.1 follows exactly the same lines as the second part of the proof of Theorem 2.1. Additional notation is only needed to take care of the extra summation in the definition of T'_Γ.

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5   Estimation of sample spacing

Assume that $d = 3$ and consider the isotropic case again. Let the random field $(X_t)_{t \in \mathbb{R}^3}$ be defined as

$$X_t = \int_{\mathbb{R}^3} f(\|s - t\|) \, M(\mathrm{d}s) \,.$$

Suppose that the field is only observed in grid points within two parallel planes with an unknown distance that we want to estimate. Due to the isotropy, we can without loss of generality assume that observations are made in the two planes

$$\mathbb{Z}^2 = \{(z_1, z_2, 0) \, : \, z_1, z_2 \in \mathbb{Z}\}$$

and

$$\mathbb{Z}^2 + h_0 = \{(z_1, z_2, d_0) \, : \, z_1, z_2 \in \mathbb{Z}\} \,,$$

where $h_0 = (0, 0, d_0)$. The aim is then to estimate d_0. Note that $\mathbb{Z}^2 + h_0 \subset \mathbb{Z}^3$ only if $d_0 \in \mathbb{Z}$, which is not assumed. The idea will be to estimate $\gamma(d_0)$, the variogram evaluated in d_0, only based on pairs $(i, i + h_0)$, where $i \in \mathbb{Z}^2$ and $i + h_0 \in \mathbb{Z}^2 + h_0$. For a finite subset $\Gamma \subset \mathbb{Z}^2$, we can estimate $\gamma(d_0)$ by $\widehat{\gamma(d_0)} = \frac{1}{|\Gamma|} \tilde{T}_\Gamma$, where

$$\tilde{T}_\Gamma = \sum_{i \in \Gamma} (X_i - X_{i+h_0})^2 \,.$$

Assuming that the variogram function $d \mapsto \gamma(d)$ is known and strictly increasing, an estimator for d_0 is obtained by

$$\hat{d}_0 = \gamma^{-1}(\widehat{\gamma(d_0)}) \,.$$

This estimator is studied in [13], where an expression for the variance of $\tilde{T}_\Gamma$ is derived. Furthermore an approximate expression for the variance of $\hat{d}_0$ is found using a Taylor approximation.

Below we state a theorem, giving asymptotic normality of $\tilde{T}_\Gamma$. In fact, apart from the scaling, the asymptotic variance is equal to the approximation to the variance suggested in [13].

A result showing asymptotic normality of $\tilde{T}_\Gamma$ can directly be translated into asymptotic confidence intervals for $\hat{d}_0$. If also $d \mapsto \gamma(d)$ is assumed to be differentiable, asymptotic normality of $\hat{d}_0$ is easily derived using the delta method.

Theorem 5.1. *Let $(\Gamma_n)_{n \geq 1}$ be a sequence of finite subsets of $\mathbb{Z}^2$ and assume that $|\Gamma_n|$ goes to infinity and that $|\partial\Gamma_n|/|\Gamma_n| \to 0$ for all $j \in \mathbb{Z}^2$. Assume furthermore that $\mathbb{E}(M')^4 < \infty$ and*

$$\sum_{i \in \mathbb{Z}^2} \int_{\mathbb{R}^3} f(\|s\|) f(\|s - i\|) \, \mathrm{d}s < \infty \,.$$

Then,

$$\frac{\tilde{T}_{\Gamma_n} - |\Gamma_n|\gamma(d_0)}{\sqrt{|\Gamma_n|}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_{\tilde{T}}^2) \,,$$

where

$$\sigma_{\hat{T}}^2 = \sum_{j \in \mathbb{Z}^2} \left[\kappa_4(M') \int_{\mathbb{R}^d} g(s)^2 g(s-j)^2 \, \mathrm{d}s \right.$$
$$\left. + 2 \operatorname{Var}(M')^2 \left(\int_{\mathbb{R}^d} g(s) g(s-j) \, \mathrm{d}s \right)^2 \right] < \infty,$$

and $g(s) = f(\|s\|) - f(\|s - h_0\|)$.

Above $\partial \Gamma_n$ denotes the boundary of Γ_n as a subset of $\mathbb{Z}^2$. The proof of Theorem 5.1 is almost identical to the proof of Theorem 2.1, but with $\mathbb{Z}^d$ replaced by $\mathbb{Z}^2$ as the index set for the observations.

Acknowledgements

A   Cumulant formulas

For two variables Y_1, Y_2 on the form (2.4),

$$Y_1 = \int_{\mathbb{R}^d} f_1(s) \, M(\mathrm{d}s) \qquad Y_2 = \int_{\mathbb{R}^d} f_2(s) \, M(\mathrm{d}s) \,,$$

with kernels f_1, f_2 satisfying (2.3), the joint cumulant function is given as

$$\log \mathbb{E}\big(e^{i(\lambda_1 Y_1 + \lambda_2 Y_2)}\big) = \int_{\mathbb{R}^d} \log \mathbb{E}\big(e^{i(\lambda_1 f_1(s) + \lambda_2 f_2(s))M'}\big) \, \mathrm{d}s \,,$$

see [6] for details. Assuming that $E|M'|^k < \infty$ and differentiating k times with respect to λ_1 gives

$$\kappa_k(Y_1) = \kappa_k(M') \int_{\mathbb{R}^d} f_1(s)^k \, \mathrm{d}s \,,$$

where $\kappa_k(X)$ denotes the kth cumulant of the variable X. Similarly, if $E(M')^2 < \infty$ we find

$$\operatorname{Cov}(Y_1, Y_2) = \operatorname{Var}(M') \int_{\mathbb{R}^d} f_1(s) f_2(s) \, \mathrm{d}s \,.$$

If furthermore $\mathbb{E}Y_1 = \mathbb{E}Y_2 = 0$ and $E(M')^4 < \infty$, it holds

$$\mathbb{E}\big((Y_1^2 - \mathbb{E}Y_1^2)(Y_2^2 - \mathbb{E}Y_2^2)\big)$$
$$= \kappa_4(M') \Big(\int_{\mathbb{R}^d} f_1(s)^2 f_2(s)^2 \, \mathrm{d}s \Big) + 2 \operatorname{Var}(M')^2 \Big(\int_{\mathbb{R}^d} f_1(s) f_2(s) \, \mathrm{d}s \Big)^2 .$$

B Corollary to Ottaviani's inequality

In this Appendix $(X_n)_{n \geq 1}$ will denote a sequence of random variables. Let

$$S_n = X_1 + \cdots + X_n \quad \text{and} \quad M_n = \max_{1 \leq k \leq n} |S_k| \, .$$

We will use the following version of Ottaviani's inequality, see [7, Section 6.24].

Theorem B.1. *Let $(X_n)_{n \geq 1}$ be a sequence of independent random variables. Then,*

$$\mathbb{P}(M_n > x + y) \cdot \min_{1 \leq k \leq n} \mathbb{P}(|S_n - S_k| \leq y) \leq \mathbb{P}(|S_n| > x)$$

for all $x, y \in \mathbb{R}$.

Corollary B.2. Let $(X_n)_{n \geq 1}$ be a sequence of independent random variables. Then for all $p \geq 1$, $\sup_{n \geq 1} \mathbb{E}|S_n|^p < \infty$ if and only if $\mathbb{E}M^p < \infty$, where $M = \sup_{n \geq 1} M_n$.

Note that the corollary can be shown for $0 < p < 1$ as well. However, in the present paper we only need the result for $p = 2, 4$.

Proof of the corollary. First notice that it is easy to show directly that $\mathbb{E}M^p < \infty$ implies $\sup_{n \geq 1} \mathbb{E}|S_n|^p < \infty$. The opposite statement follows from monotone convergence, if we can show that for all $n \geq 1$

$$\mathbb{E}M_n^p \leq C \max_{1 \leq k \leq n} \mathbb{E}|S_k|^p \tag{B.1}$$

for a finite constant C that only depends on p. By assumption, $\sup_{n \geq 1} \mathbb{E}|S_n|^p < \infty$ and therefore,

$$m = \max_{1 \leq k \leq n} \mathbb{E}|S_k|^p < \infty \, .$$

Let $\tau = (2^{p+1}m)^{1/p}$. Then, by Markov's inequality, we have for $1 \leq k \leq n$

$$\mathbb{P}(|S_n - S_k| > \tau) \leq \frac{\mathbb{E}(|S_n - S_k|^p)}{\tau^p} \leq 2^p \frac{\mathbb{E}|S_n|^p + \mathbb{E}|S_k|^p}{\tau^p} \leq \frac{1}{2},$$

where the inequality $|x + y|^p \leq 2^p(|x|^p + |y|^p)$ has been applied. It follows that $\min_{1 \leq k \leq n} \mathbb{P}(|S_n - S_k| \leq \tau) \geq \frac{1}{2}$, so by Theorem B.1,

$$\mathbb{P}(M_n > x) = \mathbb{P}(M_n > (x - \tau) + \tau) \leq 2\mathbb{P}(|S_n| + \tau > x).$$

Integrating both sides with respect to $px^{p-1}\, \mathrm{d}x$ gives

$$\mathbb{E}M_n^p \leq 2\mathbb{E}\big((|S_n| + \tau)^p\big) \leq 2^{p+1}\mathbb{E}(|S_n|^p + \tau^p) \leq 2^{p+1}(1 + 2^{p+1})m \, . \qquad \square$$

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