A note on Wilson's functional equation

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Abstract

Let S be a semigroup, and $\mathbb{F}$ a field of characteristic $\neq 2$. If the pair $f, g : S \to \mathbb{F}$ is a solution of Wilson's μ-functional equation such that $f \neq 0$, then g satisfies d'Alembert's μ-functional equation.

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Let S be a semigroup with an involution $x \mapsto x^*$, and $\mathbb{F}$ a field of characteristic $\neq 2$ with unit element 1. Finally $\mu : S \to \mathbb{F}$ is a multiplicative function (meaning $\mu(xy) = \mu(x)\mu(y)$ for all $x, y \in S$) such that $\mu(xx^*) = 1$ for all $x \in S$. The classic example is $\mu = 1$.

Wilson's μ-functional equation is

$$f(xy) + \mu(y)f(xy^*) = 2f(x)g(y), \ \forall x, y \in S, \tag{1}$$

where $f, g : S \to \mathbb{F}$ are unknown functions to be determined, while d'Alembert's μ-functional equation is the special case of (1) having $f = g$, i.e.,

$$g(xy) + \mu(y)g(xy^*) = 2g(x)g(y), \ \forall x, y \in S, \tag{2}$$

so here $g : S \to \mathbb{F}$ is the unknown function to be determined.

Theorem 1. *If the pair $f, g : S \to \mathbb{F}$ satisfies* (1) *and $f \neq 0$, then g satisfies* (2).

The theorem reveals a close relation between (1) and (2). This is interesting, because (2) has been studied extensively, so the theorem reduces the work on (1) to a situation in which f is the only unknown. The theorem developed from a series of papers of more and more generality, dating at least back to Kaczmarz [3] from 1924. See Stetkær [4] for a brief account. The set up of the most recent paper in the series ([1] by Bouikhalene and Elqorachi) has S as a monoid. Our contribution here is to extend their result to semigroups. In our opinion that is the natural framework for the theorem. A novel feature of our proof is to involve translates of solutions.

Proof. We define the left translate of a function $\phi : S \to \mathbb{F}$ by an element $x \in S$ as the function $\{L'(x)\phi\}(y) := \phi(xy)$, $y \in S$. Note that if (f, g) is a solution of (1), then so is $(L'(x)f, g)$ for any $x \in S$.

Let g be fixed. When a solution $f \neq 0$ of (1) is odd, i.e., $\mu(x)f(x^*) = -f(x)$ for all $x \in S$, then proofs in the literature give that g satisfies (2) (see [1, Proposition 2.1], or [2, Proposition 1] for the classic case). So we may assume that 0 is the only odd solution of (1).

By our hypothesis (1) has a solution $f \neq 0$. As noted above left translates of solutions of (1) are solutions, so $F := \alpha L'(x)f + \beta L'(y)f$ is a solution for any $\alpha, \beta \in \mathbb{F}$ and $x, y \in S$. By an easy computation F is odd, if $\alpha f(x) + \beta f(y) = 0$. In particular if $(\alpha, \beta) = (f(y), -f(x))$. Hence $f(y)L'(x)f - f(x)L'(y)f$ is an odd solution of (1). By our assumption it vanishes, so

$$f(y)f(xz) = f(x)f(yz) \quad \text{for all } x, y, z \in S.$$

Since $f \neq 0$, there exists an $x_0 \in S$ such that $f(x_0) \neq 0$. Taking $y = x_0$ we find that $f(xz) = f(x)\chi(z)$, where $\chi(z) := f(x_0 z)/f(x_0)$. Using that $f(x_0) \neq 0$ we get from $f(xz) = f(x)\chi(z)$ that $\chi : S \to \mathbb{C}$ is multiplicative. That f satisfies (1) may now be written $f(x)\chi(y) + f(x)\mu(y)\chi(y^*) = 2f(x)g(y)$, from which we infer that

$$g(y) = \frac{\chi(y) + \mu(y)\chi(y^*)}{2} \quad \text{for all } y \in S.$$

A direct verification shows that g is a solution of (2). □

References

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