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The kernel of the second order Cauchy difference on semigroups

Henrik Stetkær

Abstract

Let S be a semigroup, H a 2-torsion free, abelian group and $C^2 f$ the second order Cauchy difference of a function $f : S \to H$.

Assuming that H is uniquely 2-divisible or S is generated by its squares we prove that the solutions f of $C^2 f = 0$ are the functions of the form $f(x) = j(x) + B(x, x)$, where j is a solution of the symmetrized additive Cauchy equation and B is bi-additive. Under certain conditions we prove that the terms j and B are continuous, if f is.

We relate the solutions f of $C^2 f = 0$ to Fréchet's functional equation and to polynomials of degree less than or equal to 2.

Key words: Functional equation; Whitehead; Fréchet; Cauchy difference; second order Cauchy difference
2010 Mathematics Subject Classification: 39B32, 39B52

1 Introduction

1.1 The functional equation

Let $(S, \cdot)$ be a semigroup, and $(H, +)$ an abelian group. The Cauchy difference $Cf : S \times S \to H$ of a map $f : S \to H$ is

$$Cf(x, y) := f(xy) - f(x) - f(y), \quad x, y \in S,$$

while its second order Cauchy difference $C^2 f : S \times S \times S \to H$ is

$$C^2 f(x, y, z) := f(xyz) - f(yz) - f(xz) - f(xy) + f(x) + f(y) + f(z)$$

for $x, y, z \in S$. $C^2 f$ is an iterated Cauchy difference in the sense that the following identities hold for all $x, y, z \in S$:

$$C\{Cf(x, \cdot)\}(y, z) = C\{Cf(\cdot, z)\}(x, y) = C^2 f(x, y, z). \tag{1}$$

The functional equation $C^2 f = 0$, which is equivalent to

$$f(xyz) + f(x) + f(y) + f(z) = f(xy) + f(yz) + f(xz), \; x, y, z \in S, \tag{2}$$

was called Whitehead's functional equation by Faĭziev and Sahoo [2], because it occurs in Whitehead [11, Chapter II.5]. We follow the terminology of [2] in this respect. So the set of solutions of Whitehead's functional equation is the kernel of the second order Cauchy difference C^2. The formula (2) expresses the value of f on products of 3 elements by its values on products of fewer factors. The order of the arguments x, y, z in the products yz, xz and xy on the right hand side of (2) is easy to remember: You delete x, y and z successively from the product xyz without changing the order.

The continuous, complex valued solutions of (2) on $S = \mathbb{R}$ are the polynomials of degree ≤ 2 that vanish at 0, i.e., the functions of the form $f(x) = ax + bx^2$, $x \in \mathbb{R}$, where a and b are complex constants. We shall generalize that fact.

The natural general setting of the functional equation (2) is for S to be a semigroup, because the very formulation of (2) requires only an associative composition in S, not an identity element and inverses. Thus we study in the present paper Whitehead's functional equation (2) on semigroups S, generalizing previous works in which S is a group. Similarly the range space H of the functions need not be more complicated than an abelian group. However, to make the exposition transparent and simple we impose as a blanket assumption that H is 2-torsion free.

Jensen's functional equation on groups involves the group inversion, so it does not make sense on general semigroups. However, when H is 2-torsion free, the normalized solutions $f : S \to H$ of Jensen's functional equation are the same as those of the symmetrized additive Cauchy equation

$$f(xy) + f(yx) = 2f(x) + 2f(y), \; x, y \in S, \tag{3}$$

(see [9, Section 2.4]), and (3) makes sense on any semigroup S. In the present paper we let (3) substitute Jensen's functional equation.

1.2 About earlier works

Earlier works on (2) have S as a group, not a semigroup. These works include Kannappan [4] for S a vector space and H a field of characteristic $\neq 2$, Faĭziev and Sahoo [2] for H an abelian group, Stetkær [9, Exercise 13.10] for $H = \mathbb{C}$, and finally Ng and Zhao [7] for H an abelian group.

[7] derived a number of useful basic relations for functions satisfying (2) and used them to give explicit formulas for the solutions of (2) on certain groups, including free groups and other selected groups such as symmetric groups, finite cyclic groups and the dihedral group. Li and Ng [6] and Ng [8] have also information about kernels of Cauchy differences of higher order than 2, but we do not discuss that topic.

Roughly speaking [2], [9] and the present paper try to express solutions of (2) as a sum of a solution of Jensen's functional equation and a quadratic term. One reason is that solutions of Jensen's functional equation are well known, because they have been studied extensively in the literature. In contrast to [2] the present paper requires H to be 2-torsion free, and under that condition it contains most of the results of [2].

The methods of [2] and [9] do not work for semigroups, because the papers utilize the group inversion.

1.3 Our results

By Lemma 1 below any function $f : S \to H$ of the form

$$f(x) := j(x) + B(x,x), \quad x \in S, \tag{4}$$

where j is a solution of (3) and B is bi-additive, is a solution of Whitehead's functional equation (2). Our main result is the derivation of a converse: We find sufficient conditions on S and H such that any solution $f : S \to H$ of (2) decomposes as in (4). Some conditions are needed, because the decomposition (4) is not always possible (Example 8). However, it exists for many important semigroups S, for instance for connected Lie groups (Theorem 6).

The contributions of the present paper to the knowledge about solutions of (2) are the following:

1. The setting has S to be a semigroup, not necessarily a group.
2. We present sufficient conditions for the decomposition (4) to exist (Theorem 6).
3. We take continuity into account in the decomposition (4) (Theorem 7).
4. We relate the solutions of Whitehead's functional equation to polynomials of degree ≤ 2 (Proposition 10).
5. We establish a connection between Whitehead's and Fréchet's functional equations (Proposition 13).

2 Set up

We impose as blanket assumptions that $(S, \cdot)$ is a semigroup, and that $(H, +)$ is a 2-torsion free, abelian group.

We say that the abelian group H is uniquely 2-divisible, if the map $h \mapsto 2h$ is a bijection of H onto H.

We say that the semigroup S is a topological semigroup, if S is equipped with a topology such that the product map $(x,y) \mapsto xy$ from $S \times S$ to S is continuous, when $S \times S$ is given the product topology.

We say that a function f on S is *abelian* if $f(xy) = f(yx)$ and $f(xyz) = f(xzy)$ for all $x, y, z \in S$.

A map $F : S \to H$ is said to be additive, if $F(xy) = F(x) + F(y)$ for all $x, y \in S$. A map $B(\cdot,\cdot) : S \times S \to H$ is said to be bi-additive, if $B(x,\cdot) : S \to H$ and $B(\cdot,x) : S \to H$ are additive for each fixed $x \in S$.

3 Special solutions and uniqueness

Lemma 1 presents some solutions of Whitehead's equation (2).

Lemma 1. *Let $j : S \to H$ be a solution of* (3)*, and let $B : S \times S \to H$ be a bi-additive map. Then $f(x) := j(x) + B(x,x)$, $x \in S$, is a solution of Whitehead's functional equation* (2)*.*

Proof. By linearity it suffices to show that each term j and B is a solution of (2). For the B-term this is done by elementary calculations, so left is j, where we proceed as follows. That $j : S \to H$ satisfies (3) may be expressed as $Cj(\cdot, y) = -Cj(y, \cdot)$ for all $y \in S$. Using that we find from (1) that

$$C^2 j(x,y,z) = C(Cj(\cdot,z))(x,y) = -C(Cj(z,\cdot))(x,y) = -C^2 j(z,x,y),$$

so $C^2 j$ changes sign under a cyclic permutation of its variables. Thus

$$C^2 j(x,y,z) = -C^2 j(z,x,y) = C^2 j(y,z,x) = -C^2 j(x,y,z),$$

so $2C^2 j(x,y,z) = 0$. Now $C^2 j = 0$, H being 2-torsion free. □

The uniqueness of the terms in the sum in Lemma 1 is a consequence of the next lemma.

Lemma 2. *Let $f(x) = j(x) + B(x,x)$, $x \in S$, where $j : S \to H$ is a solution of* (3)*, and $B : S \times S \to H$ is bi-additive. Then $2B(x,x) = Cf(x,x)$ for all $x \in S$. In particular j and $x \mapsto B(x,x)$ are uniquely determined by f.*

Proof. From (3) we get that

$$2Cj(x,x) = 2(j(xx) - j(x) - j(x)) = 2j(xx) - 2j(x) - 2j(x) = 0,$$

so that $Cj(x,x) = 0$. Using the notation $b(x) := B(x,x)$ we find from the bi-additivity of B that

$$\begin{aligned} Cb(x,x) &= b(x^2) - 2b(x) = B(x^2,x^2) - 2B(x,x) = 4B(x,x) - 2B(x,x) \\ &= 2B(x,x). \end{aligned}$$

Finally, we obtain from $f = j + b$ that

$$Cf(x,x) = Cj(x,x) + Cb(x,x) = 0 + 2B(x,x) = 2B(x,x).$$

This shows that $x \mapsto B(x,x)$ is uniquely determined from f. Hence so is $j(x) = f(x) - B(x,x)$. □

4 Existence of the decomposition

Our main result, Theorem 6, says that the converse of Lemma 1 holds under certain extra hypotheses on S or H.

We start by Lemmas 3 and 4, in which we derive some properties of the solutions of (2).

Lemma 3. *If $f : S \to H$ is a solution of* (2)*, then Cf is a bi-additive.*

Proof. This immediate from the characterization (1) of the second order Caucy difference, because a map $F : S \to H$ is additive, when $CF = 0$. □

Lemma 4 is inspired by the formulas of Lemma 2.

Lemma 4. *Let $f : S \to H$ be a solution of* (2)*. Then the map*

$$J(x) := 2f(x) - Cf(x,x) \text{ for } x \in S,$$

satisfies (3)*.*

Proof. For any $x, y \in S$ we find, using the bi-additivity of Cf, that

$$\begin{aligned} J(xy) + J(yx) &= 2f(xy) - Cf(xy,xy) + 2f(yx) - Cf(yx,yx) \\ &= 2f(xy) + 2f(yx) - Cf(x,x) - Cf(x,y) - Cf(y,x) - Cf(y,y) \\ &\quad - Cf(y,y) - Cf(y,x) - Cf(x,y) - Cf(x,x) \\ &= 2f(xy) + 2f(yx) - 2Cf(x,x) - 2Cf(y,y) - 2Cf(x,y) - 2Cf(y,x) \\ &= 2f(xy) + 2f(yx) - 4f(x) + 2J(x) - 4f(y) + 2J(y) \\ &\quad - 2f(xy) + 2f(x) + 2f(y) - 2f(yx) + 2f(x) + 2f(y) \\ &= 2J(x) + 2J(y). \end{aligned}$$

□

Many important groups and semigroups are generated by their squares. For instance all connected Lie groups, like $SL(2,\mathbb{R})$ and the Heisenberg group (see [9, Section A.5.2]). The monoid of complex 2×2 matrices under matrix multiplication is generated by its squares (for a proof see [1, p. 192]), and it is not a group.

Lemma 5. *If S is generated by its squares, and $B(\cdot,\cdot) : S \times S \to H$ is bi-additive, then $B(x,y) \in 2H$ for all $x, y \in S$.*

Proof. Let $x, y \in S$. By assumption x has the form $x = x_1^2 x_2^2 \cdots x_n^2$ for some $x_1, x_2, \ldots, x_n \in S$. Now, by the additivity in the first coordinate,

$$B(x,y) = B(x_1^2 x_2^2 \cdots x_n^2, y) = \sum_{i=1}^{n} 2B(x_i,y) = 2\sum_{i=1}^{n} B(x_i,y) \in 2H. \quad \square$$

Theorem 6. *Assume that H is uniquely 2-divisible or/and that S is generated by its squares.*

If $f : S \to H$ is a solution of Whitehead's functional equation (2)*, then there exist a solution j of* (3) *and a bi-additive map $B : S \times S \to H$, such that $f(x) = j(x) + B(x, x)$ for all $x \in S$.*

Proof. If H is uniquely 2-divisible then obviously $Cf(x, y) \in 2H$. If S is generated by its squares, we get the same thing from Lemmas 3 and 5. In any case $B(x, y) := Cf(x, y)/2$ makes sense. B is bi-additive according to Lemma 3 (and the blanket assumption that H is 2-torsion free). Then we see from Lemma 4 that $j(x) := f(x) - B(x, x)$ exists and satisfies (3). Finally $f(x) = j(x) + B(x, x)$. □

The term $B(\cdot, \cdot)$ in the decomposition $f(x) = j(x)+B(x, x)$ in Theorem 6 is abelian, B being bi-additive, so f is abelian if and only if the term j is abelian. Jensen's functional equation, which is equivalent to (3), has on some groups non-abelian solutions (for an example see [9, Example 12.4]). Hence so does Whitehead's functional equation.

5 About continuous solutions

The treatments of Whitehead's functional equation (2) in [6] and [7] work for free groups and quotients of such groups, and continuity is not an issue. But continuity is important in many examples from geometry and physics. In those examples the functions usually assume values in $\mathbb{R}$ or $\mathbb{C}$, not in $\mathbb{Z}$ or $\mathbb{Q}$.

Theorem 7 shows that the decomposition (4) under some conditions on H respects continuity: Each term in the decomposition of a continuous solution of Whitehead's functional equation is continuous. These conditions are satisfied for $H = \mathbb{R}$ and $H = \mathbb{C}$. The continuous, complex-valued solutions of Jensen's functional equation and the continuous, complex-valued, bi-additive maps are known on many groups.

Theorem 7. *Let S be a topological semigroup. Assume furthermore that either*

(a) H is a topological vector space over $\mathbb{R}$ or $\mathbb{C}$, or

(b) H is a Hausdorff, locally compact, σ-compact topological group, which is uniquely 2-divisible.

If $f : S \to H$ is a continuous solution of Whitehead's functional equation (2)*, then the components j and B in the decomposition $f(x) = j(x)+B(x, x)$ from Theorem 6 are continuous.*

Proof. It suffices to prove that the map

$$(x,y) \mapsto B(x,y) = \tfrac{1}{2}Cf(x,y) = \tfrac{1}{2}\{f(xy) - f(x) - f(y)\}$$

of $S \times S$ into S is continuous. Since $(x,y) \mapsto f(xy) - f(x) - f(y)$ is continuous, it remains to show that the map $h \mapsto h/2$ from H to H is continuous.

(a) This is trivially satisfied by the very definition of a topological vector space.

(b) The homomorphism $h \mapsto h + h$ is continuous, H being a topological group. By the Open Mapping Theorem for groups (see, e.g., [3, (5.29) Theorem]) the inverse mapping $h \mapsto h/2$ is continuous as well. □

6 Two examples

Example 8 reveals that decomposition as in Theorem 6 is not always possible, so the hypotheses of the theorem can not be deleted. The obstacle is that $H = \mathbb{Z}$ is not 2-divisible. In Example 9 we get around the obstruction, even though H is the same as in Example 8, because the S is 2-divisible.

Example 8. Let $S = H = \mathbb{Z}$. The map $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) := n(n-1)/2$, $n \in \mathbb{Z}$, is a solution of Whitehead's functional equation (2) by [7, Theorem 3.1]. Its putative decomposition (4) must by the uniqueness of the decomposition (Lemma 2), be

$$f(n) = \frac{n(n-1)}{2} = -\frac{n}{2} + \frac{n^2}{2}, \quad n \in \mathbb{Z}.$$

But the right hand side has the fraction 1/2 in each term, and 1/2 is not in $H = \mathbb{Z}$. Thus the decomposition in Theorem 6 is not possible here.

Example 9. Let $C_n = \langle a \rangle$ be a cyclic group of odd order $n = 2q - 1$, where $q \in \{2, 3, \dots\}$.

Since $a^{2q-1} = e$, we have $a = a^{2q}$, so C_n is generated by its squares. Thus Theorem 6 applies. To exhibit the details we read from [7, Theorem 5.1 and Remark 5.2] that the solutions $f : C_n \to H$ of (2) are the maps of the form

$$f(a^p) = ph_0 + \frac{p(p-1)}{2}\, h_1 \quad \text{for } p \in \mathbb{Z},$$

where $h_0, h_1 \in H$ satisfy $nh_0 = nh_1 = 0$. Since $n = 2q - 1$ we get from $nh_1 = 0$ that $h_1 = 2qh_1$, and so further that

$$f(a^p) = ph_0 + \frac{p(p-1)}{2}\, 2qh_1 = p(h_0 - qh_1) + p^2 qh_1 \quad \text{for } p \in \mathbb{Z},$$

which is the splitting from Theorem 6 of the solution f.

7 A connection to polynomials of degree two

In this section we relate the kernel of C^2 to polynomials of degree at most 2.

The difference operator Δ_x, $x \in S$, maps the function $f : S \to H$ to the function $(\Delta_x f)(y) := f(yx) - f(y)$, $y \in S$. If $S = M$ is an abelian monoid and multiplication by 2 is a bijection of H, then the solutions $f : M \to H$ of $\Delta_x \Delta_y \Delta_z f = 0$ are the polynomials on M of degree at most 2 (see Székelyhidi [10, Theorem 9.1] for a derivation). Roughly spoken Proposition 10 says that the kernel of C^2 consists of polynomials of degree at most 2, that vanish at the neutral element e of the monoid, but we must emphasize that the "polynomials" here are placed in a non-abelian setting.

Proposition 10. *If M is a monoid with neutral element e, then*

$$\ker C^2 = \{f : M \to H \mid \Delta_x \Delta_y \Delta_z f = 0 \text{ for all } x, y, z \in M, \text{ and } f(e) = 0\}.$$

Proof. Note that $f(e) = 0$ for all $f \in \ker C^2$ (take $x = y = z = e$ in (2)). The proposition now follows easily from the identity

$$(\Delta_x \Delta_y \Delta_z f)(a) = C^2 f(ax, y, z) - C^2 f(a, y, z), \ x, y, z, a \in M, \tag{5}$$

and its special case of $a = e$ that amounts to

$$(\Delta_x \Delta_y \Delta_z f)(e) = C^2 f(x, y, z) - f(e), \ x, y, z \in M.$$

The identity (5) can be derived from the definitions of $\Delta_x f$ and $C^2 f$ by elementary computations. ☐

Remark 11. The proof of Proposition 10 does not use that H is 2-torsion free.

Remark 12. A special instance of Proposition 10 with $M = \mathbb{Z}$ and $H = \mathbb{Q}$ occurs as [8, Lemma 2.1(ii) for $m = 2$].

8 A connection to Fréchet's functional equation

In this section we relate Fréchet's and Whitehead's functional equations.

For functions $f : S \to H$ Fréchet's functional equation

$$f(xyz) + f(x) + f(y) + f(z) = f(xy) + f(yz) + f(zx), \ x, y, z \in S, \tag{6}$$

differs from Whitehead's (2) only in the very last term, which is $f(xz)$ in (2), but $f(zx)$ in (6). The two equations coincide on abelian semigroups S.

Fréchet's functional equation should not be confused with Fréchet's equation $\Delta_{x_1} \Delta_{x_2} \ldots \Delta_{x_{n+1}} f = 0$ that is used to characterize polynomials f of degree at most n. We met Fréchet's equation with $n = 2$ in Section 7.

Proposition 13 relates the solutions of (6) to those of (2). The hypothesis that the center $Z(S)$ of S is non-empty holds in particular if S is a group or just a monoid, but it can of course hold, even when S has no neutral element as shown by the example $S = (\,]0,\infty[,+)$.

Proposition 13. *If $Z(S) \neq \emptyset$, then the solutions $f : S \to H$ of Fréchet's functional equation* (6) *are the abelian solutions of Whitehead's functional equation* (2)*.*

Proof. We shall only prove that all solutions of (6) are abelian, because that is the only non-trivial statement of Proposition 13. So let $f : S \to H$ be a solution of (6). By assumption there is a $y_0 \in Z(S)$. Interchanging $y = y_0$ and z on the left hand side of (6) does not change it, because $y_0 \in Z(S)$. So from the right hand side we get for all $x, z \in S$ that

$$f(xy_0) + f(y_0z) + f(zx) = f(xz) + f(zy_0) + f(y_0x),$$

which, since $y_0 \in Z(S)$, reduces to $f(zx) = f(xz)$ for all $x, z \in S$, i.e., f is central. With this in mind it is clear from (6) that $f(xyz) = f(xzy)$ for all $x, y, z \in S$, so f is abelian. □

Remark 14. The proof of Proposition 13 does not use that H is 2-torsion free.

Thus we can apply our results for Whitehead's functional equation to Fréchet's functional equation, when $Z(S) \neq \emptyset$. Since the abelian solutions of (3) are the additive maps (under our blanket assumption that H is 2-torsion free), we get the following corollary of Theorem 6.

Corollary 15. *Assume that H is uniquely 2-divisible or/and that S is generated by its squares. Assume furthermore that $Z(S) \neq \emptyset$.*

If $f : S \to H$ is a solution of Fréchet's functional equation (6)*, then there exist an additive map $A : S \to H$ and a bi-additive map $B : S \times S \to H$, such that $f(x) = A(x) + B(x,x)$ for all $x \in S$.*

A version of Corollary 15 can be found in [2, Theorem 3.3]. For more information about (6) you may consult Kannappan's monograph [5, Section 5.1].

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