

TWO PROPERTIES OF ENDOMORPHISMS OF WEYL ALGEBRAS

NIELS LAURITZEN AND JESPER FUNCH THOMSEN

ABSTRACT. We show that endomorphisms of Weyl algebras over fields of characteristic zero are flat and that birational endomorphisms are automorphisms by reducing to positive characteristic. We also give examples showing that endomorphisms of Weyl algebras are not in general flat over fields of positive characteristic.

INTRODUCTION

We prove that endomorphisms of Weyl algebras over a field of characteristic zero are flat. More precisely, let A be the n -th Weyl algebra over a field of characteristic zero and $\varphi : A \rightarrow A$ an endomorphism with $S = \varphi(A)$. We prove that $S \subset A$ is a flat ring extension in the sense of [4, Ch. 2, §2.8] i.e., A is flat both as a left and a right S -module. The endomorphism φ gives an extension of division rings $Q(S) \subset Q(A)$. We call φ birational if $Q(S) = Q(A)$ and show that birational endomorphisms of Weyl algebras over fields of characteristic zero are automorphisms. This is a non-commutative analogue (over fields of characteristic zero) of Keller's classical result that birational endomorphisms of affine spaces with invertible Jacobian are automorphisms [11]. In general, $Q(S) \subset Q(A)$ is an extension whose left- and right dimensions are bounded by $\deg(\varphi)^{2n}$ (see Proposition 4.2).

The key component in our approach is reduction to positive characteristic, where the n -th Weyl algebra is finite free over its center, which is a polynomial ring in $2n$ variables [15] with canonical Poisson bracket coming from the commutator in the lifted Weyl algebra (see [3] and §1.3 of this paper).

The Dixmier conjecture [6, §11.1] states that an endomorphism of the first Weyl algebra, i.e. $n = 1$ above, over a field of characteristic zero is in fact an automorphism. The Jacobian conjecture states that an endomorphism of affine n -space with invertible Jacobian over a field of characteristic zero is an automorphism for $n \geq 2$. The natural extension of the Dixmier conjecture to $n \geq 1$ is a non-commutative analogue¹ of the Jacobian conjecture and seems inherently intractable.

It is known over a field of arbitrary characteristic, that an endomorphism of affine n -space with invertible Jacobian is flat [2, (2.1) THEOREM][14]. We give examples of endomorphisms of the first Weyl algebra over fields of positive characteristic for which flatness fails.

A proof of flatness of endomorphisms of Weyl algebras over fields of characteristic zero was presented in [18], but it seems to contain a mistake (see §2.3 of this paper for further details), which we at present do not know how to circumvent.

1. PRELIMINARIES

Most of this section is aimed at introducing the Weyl algebra over commutative rings of prime characteristic and the reduction from zero to positive characteristic. Except for a few results, we have deliberately done this in some detail to make the paper self contained. The study of the Weyl algebra over rings of prime characteristic was initiated in [15].

¹The Dixmier conjecture for the n -th Weyl algebra implies the Jacobian conjecture for affine n -space, see [2, p. 297]. The Jacobian conjecture for affine $2n$ -space implies the Dixmier conjecture for the n -th Weyl algebra, see [3] and [17].

Throughout this paper \mathbb{N} denotes the natural numbers $\{0, 1, 2, \dots\}$ and R a commutative ring.

1.1. The Weyl algebra over a commutative ring.

Proposition 1.1. *Let S be a ring and $\partial, x \in S$ with $[\partial, x] = \partial x - x\partial = 1$. With the convention $\partial^r = x^r = 0$ for $r < 0$,*

(i)

$$\text{ad}(\partial)^i \text{ad}(x)^j (x^m \partial^n) = (-1)^j i! j! \binom{m}{i} \binom{n}{j} x^{m-i} \partial^{n-j},$$

(ii)

$$[\partial^m, x^n] = \sum_{k \geq 1} k! \binom{m}{k} \binom{n}{k} x^{n-k} \partial^{m-k}$$

for $i, j, m, n \in \mathbb{N}$.

Proof. The formula in (i) follows using that $\text{ad}(x)$ and $\text{ad}(\partial)$ are derivations of S with $\text{ad}(\partial)(x) = [\partial, x] = -\text{ad}(x)(\partial) = 1$.

The identity in (ii) goes back to [12, THEOREM XIII]. A proof may be given by induction using that $[\partial^{m+1}, x^n] = [\partial, x^n] \partial^m + \partial[\partial^m, x^n]$ and $[\partial^m, x^{n+1}] = [\partial^m, x] x^n + x[\partial^m, x^n]$ (see [6, Lemma 2.1]). \square

Definition 1.2. *The n -th Weyl algebra $A_n(R)$ over R is the free R -algebra on $x_1, \dots, x_n, \partial_1, \dots, \partial_n$ with relations*

$$(1.1.1) \quad \begin{aligned} [x_i, x_j] &= 0 \\ [\partial_i, \partial_j] &= 0 \\ [\partial_i, x_j] &= \delta_{ij} \end{aligned}$$

for $1 \leq i, j \leq n$, where δ_{ij} denotes the Kronecker delta. By abuse of notation we let x_i and ∂_i denote their canonical images in $A_n(R)$ for $i = 1, \dots, n$.

Definition 1.3. *For $v = (v_1, \dots, v_n) \in \mathbb{N}^n$ and $m \in \mathbb{N}$, the notation $v \leq m$ means that $v_i \leq m$ for every $i = 1, \dots, n$. For $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, we let $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$ in $A_n(R)$. The element $x^\alpha \partial^\beta \in A_n(R)$ is called a monomial.*

Proposition 1.4. *The Weyl algebra $A_n(R)$ is a free R -module with a basis consisting of the monomials*

$$M = \{x^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{N}^n\}.$$

Proof. See [8, §2, Lemma 3]. \square

1.2. Positive characteristic. If $\text{char}(R) > 0$, $A_n(R)$ is a finitely generated module over its center. The following result is a consequence of Proposition 1.1 and Proposition 1.4.

Proposition 1.5. *Suppose that $\text{char}(R) = m > 0$ and let*

$$C = R[x_1^m, \dots, x_n^m, \partial_1^m, \dots, \partial_n^m] \subset A_n(R).$$

Then C is a central subalgebra and $A_n(R)$ is a free module over C of rank m^{2n} with basis

$$\{x^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{N}^n, 0 \leq \alpha, \beta \leq m-1\}.$$

Example 1.6. Consider $R = \mathbb{Z}/6\mathbb{Z}$. Then $3x^2$ is a central element in $A_1(R)$, but $3x^2 \notin R[x^6, \partial^6]$.

The following result is central in this paper. Here we basically follow [1, Theorem 3.1 and Proposition 3.2] in the proof.

Theorem 1.7. *Suppose that $\text{char}(R) = p$ and let*

$$C = R[x_1^p, \dots, x_n^p, \partial_1^p, \dots, \partial_n^p] \subset A_n(R),$$

where p is a prime number.

(i) *The center of $A_n(R)$ is equal to C .*

(ii) *If $X_1, \dots, X_n, D_1, \dots, D_n \in A_n(R)$ satisfy the commutation relations for the Weyl algebra i.e.,*

$$(1.2.1) \quad \begin{aligned} [X_i, X_j] &= 0 \\ [D_i, D_j] &= 0 \\ [D_i, X_j] &= \delta_{ij} \end{aligned}$$

for $i, j = 1, \dots, n$, then

$$\{X^\alpha D^\beta \mid 0 \leq \alpha, \beta \leq p-1\}$$

is a basis for $A_n(R)$ as a module over C .

(iii) *Let $\varphi : A_n(R) \rightarrow A_n(R)$ be an R -algebra endomorphism. Then*

(a) *$\varphi(C) \subset C$.*

(b) *φ is injective/surjective if $\varphi|_C$ is injective/surjective.*

Proof. Proposition 1.5 implies that C is a central subalgebra and that $A_n(R)$ is a free module over C with basis $\{x^\alpha \partial^\beta \mid 0 \leq \alpha, \beta \leq p-1\}$. Suppose that

$$z = \sum_{0 \leq \alpha, \beta \leq p-1} \lambda_{\alpha, \beta} x^\alpha \partial^\beta$$

is an element of the center of $A_n(R)$ with $\alpha, \beta \in \mathbb{N}^n$ and $\lambda_{\alpha, \beta} \in C$. If $\lambda_{\alpha, \beta} \neq 0$ for some $(\alpha, \beta) \neq (0, 0)$, then there exists an element $D \in \{x_1, \dots, x_n, \partial_1, \dots, \partial_n\}$ with $[D, z] \neq 0$ by Proposition 1.1. This proves (i).

Let $M = \{X^\alpha D^\beta \mid 0 \leq \alpha, \beta \leq p-1\}$. Applying $\text{ad}(D_i)$ and $\text{ad}(X_i)$ successively, it follows by Proposition 1.1 that M is linearly independent over C . Suppose that R is an integral domain and let K denote the field of fractions of C . Then every element in $A_n(R)$ is a K -linear combination of elements in M . However, the formula in Proposition 1.1 applies to show that the coefficients in such a linear combination belong to C proving that M is a generating set over C .

In the proof of (ii) for general rings of characteristic p , we may assume that R is noetherian by replacing R with the \mathbb{Z} -algebra generated by the coefficients of $X_1, \dots, X_n, D_1, \dots, D_n$ in the monomial basis from Proposition 1.4. This assumption provides the existence of finitely many prime ideals $P_1, \dots, P_m \subset R$ such that

$$(1.2.2) \quad P_1 \cdots P_m = (0).$$

Let N denote the C -submodule of $A_n(R)$ generated by M . By the integral domain case, we have

$$(1.2.3) \quad A_n(R) = N + PA_n(R)$$

for $P = P_1, \dots, P_m$. By iterating (1.2.3) we get $N = A_n(R)$ using (1.2.2). This proves (ii).

For the proof of (iii), let $X_i = \varphi(x_i)$ and $D_i = \varphi(\partial_i)$ for $i = 1, \dots, n$. Then $X_i^p, D_i^p \in C$ by (ii), since $[X_i, D_j^p] = 0$ and $[D_i, X_j^p] = 0$ for $i, j = 1, \dots, n$. Therefore $\varphi(C) \subset C$ and (a) is proved. If $\varphi|_C$ is injective/surjective, then φ is injective/surjective again using (ii). This proves (b). \square

1.3. The Poisson bracket on the center. In this section we recall, following Belov-Kanel and Kontsevich [3], how the standard Poisson bracket on the center in prime characteristic is related to the commutator in the Weyl algebra.

Definition 1.8. A Poisson algebra P over R is a commutative R -algebra with an R -bilinear skew-symmetric pairing $\{\cdot, \cdot\} : P \times P \rightarrow P$ called the Poisson bracket, such that $(P, \{\cdot, \cdot\})$ is a Lie algebra over R and $\{a, \cdot\} : P \rightarrow P$ is a derivation for every $a \in P$ i.e., the Leibniz rule $\{a, bc\} = \{a, b\}c + b\{a, c\}$ holds for every $b, c \in A$. A Poisson ideal $I \subset P$ is an ideal with the property that $\{a, x\} \in I$ for every $a \in P$ and $x \in I$.

Example 1.9. In general, a Poisson bracket $\{\cdot, \cdot\}$ on a Poisson R -algebra A generated by $\xi_1, \dots, \xi_m \in A$ is uniquely determined by $\{\xi_i, \xi_j\}$ for $i, j = 1, \dots, m$.

The standard Poisson bracket on the polynomial ring $P = R[x_1, \dots, x_n, y_1, \dots, y_n]$ is uniquely determined by

$$\{x_i, x_j\} = 0$$

$$\{y_i, y_j\} = 0$$

$$\{x_i, y_j\} = \delta_{ij}$$

for $i, j = 1, \dots, n$ and given by the formula

$$(1.3.1) \quad \{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right),$$

where $f, g \in P$.

Proposition 1.10. Assume that R is an integral domain and let φ be an R -endomorphism of $P = R[x_1, \dots, x_n, y_1, \dots, y_n]$ preserving the Poisson bracket in (1.3.1) i.e., $\{f, g\} = \{\varphi(f), \varphi(g)\}$ for every $f, g \in P$. Then $\det J(\varphi) = \pm 1$, where $J(\varphi)$ is the Jacobian matrix of φ with columns indexed by $x_1, \dots, x_n, y_1, \dots, y_n$ and rows by the coordinate functions of φ .

Proof. Let H denote the $2n \times 2n$ skew-symmetric matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

The endomorphism φ gives rise to the $2n \times 2n$ skew-symmetric matrix $H(\varphi)$ with entries $\{\varphi_i, \varphi_j\}$, where

$$\varphi_i = \varphi(x_i)$$

$$\varphi_{i+n} = \varphi(y_i)$$

for $i = 1, \dots, n$. The assumption on φ implies that $H(\varphi) = H$. Applying the determinant to the identity

$$H(\varphi) = J(\varphi)HJ(\varphi)^T$$

therefore shows that $\det J(\varphi) = \pm 1$, since $\det H = 1$. \square

Let p be a prime number and suppose that R has no p -torsion. Let π denote the canonical map $A_n(R) \rightarrow A_n(R/(p))$ and

$$C = (R/pR)[x_1^p, \dots, x_n^p, \partial_1^p, \dots, \partial_n^p].$$

For $\tilde{f}, \tilde{g} \in A_n(R)$, $[\tilde{f}, \tilde{g}] \in pA_n(R)$ if $\pi(\tilde{f}) \in C$ or $\pi(\tilde{g}) \in C$. If $\pi(\tilde{f}) = \pi(\tilde{f}_1) \in C$ and $\pi(\tilde{g}) = \pi(\tilde{g}_1) \in C$ for $\tilde{f}_1, \tilde{g}_1 \in A_n(R)$, then

$$[\tilde{f}, \tilde{g}] - [\tilde{f}_1, \tilde{g}_1] = [\tilde{f}, \tilde{g} - \tilde{g}_1] - [\tilde{g}_1, \tilde{f} - \tilde{f}_1] \in p^2A_n(R).$$

Thus for $f, g \in C$,

$$(1.3.2) \quad \{f, g\} := \pi \left(\frac{[\tilde{f}, \tilde{g}]}{p} \right) \in A_n(R/pR)$$

is independent of the choice of $\tilde{f}, \tilde{g} \in A_n(R)$ with $\pi(\tilde{f}) = f$ and $\pi(\tilde{g}) = g$.

Proposition 1.11. *The operation in (1.3.2) is the standard Poisson bracket on the center C of $A_n(R/pR)$ with*

$$(1.3.3) \quad \{x_i^p, x_j^p\} = \{\partial_i^p, \partial_j^p\} = 0 \quad \text{and} \quad \{x_i^p, \partial_j^p\} = \delta_{ij}$$

for $i, j = 1, \dots, n$.

Proof. From properties of the commutator in the Weyl algebra, $\{\cdot, \cdot\} : C \times C \rightarrow A_n(R/pR)$ is R -linear, skew-symmetric and satisfies the Leibniz rule and the “0” bracket rules in (1.3.3). Proposition 1.1(ii) and Wilson’s theorem imply that $\{x_i^p, \partial_i^p\} = 1$ for $i = 1, \dots, n$. Therefore $\{C, C\} \subset C$ and $\{\cdot, \cdot\}$ is the given standard Poisson bracket on C . \square

Proposition 1.12. *Let $\varphi : A_n(R) \rightarrow A_n(R)$ be an R -algebra endomorphism and \mathfrak{m} a maximal ideal, such that $\text{char}(R/\mathfrak{m}) = p$. Then*

$$\{\varphi_{\mathfrak{m}}(f), \varphi_{\mathfrak{m}}(g)\} = \{f, g\}$$

for every $f, g \in C_{\mathfrak{m}}$, where $\varphi_{\mathfrak{m}}$ denotes the induced endomorphism of $A_n(R/\mathfrak{m})$, $C_{\mathfrak{m}}$ the center of $A_n(R/\mathfrak{m})$ and $\{\cdot, \cdot\}$ is the Poisson bracket coming from the surjection $A_n(R/pR) \rightarrow A_n(R/\mathfrak{m})$.

Proof. With the notation above $\mathfrak{m}C$ is a Poisson ideal in C . Therefore the surjection $\gamma : A_n(R/pR) \rightarrow A_n(R/\mathfrak{m})$ induces the standard Poisson bracket on the center $C_{\mathfrak{m}}$ of $A_n(R/\mathfrak{m})$ given by $\{f, g\} := \{F, G\}$, where $\gamma(F) = f$ and $\gamma(G) = g$. Taking Theorem 1.7(iii)(a) and (1.3.2) into account, the result follows. \square

1.4. Reduction to positive characteristic. We recall some well known and useful results for reduction to positive characteristic used in this paper. The set of maximal ideals in R is denoted $\text{Specm}(R)$.

Theorem 1.13. *Suppose that R is a finitely generated integral domain over \mathbb{Z} . Then*

- (i) R/\mathfrak{m} is a finite field for every $\mathfrak{m} \in \text{Specm}(R)$ and
- (ii)

$$\bigcap_{\mathfrak{m} \in \text{Specm}(R)} \mathfrak{m} = (0).$$

- (iii) Let $f \in R$. If $\mathfrak{m} \in \text{Specm}(R_f)$, then $\mathfrak{m} \cap R \in \text{Specm}(R)$ and $R/\mathfrak{m} \cap R = R_f/\mathfrak{m}$.
- (iv) Let k denote an algebraically closed field containing R . A set of polynomials $f_1, \dots, f_m \in R[T_1, \dots, T_n]$ has a common zero in k^n if their reductions have a common zero in $(R/\mathfrak{m})^n$ for every $\mathfrak{m} \in \text{Specm}(R)$.

Proof. The first three results follow from the fact that R is a Jacobson ring (see [5, CHAPTER V, §3.4]). Notice that the identity $R/\mathfrak{m} \cap R = R_f/\mathfrak{m}$ in (iii) is a consequence of $f \notin \mathfrak{m} \cap R$. Assume in (iv) that f_1, \dots, f_m do not have a common zero in k^n . Then Hilbert’s Nullstellensatz implies that

$$\lambda_1 f_1 + \dots + \lambda_m f_m = r$$

for $\lambda_1, \dots, \lambda_m \in R[T_1, \dots, T_n]$ and $r \in R \setminus \{0\}$. By (i), there exists $\mathfrak{m} \in \text{Spec}(R)$, such that $r \notin \mathfrak{m}$. This shows that f_1, \dots, f_m cannot have a common zero in $(R/\mathfrak{m})^n$ contradicting our assumption. \square

2. FLATNESS

Let $A\text{-Mod}$ denote the category of left A -modules and $\text{Mod-}A$ the category of right A -modules, where A is a ring.

2.1. Flat ring homomorphisms. A ring homomorphism $\varphi : S \rightarrow T$ endows T with the left S -module structure $s.t = \varphi(s)t$ and the right S -module structure $t.s = t\varphi(s)$, where $s \in S$ and $t \in T$. We call φ *left flat* if $M \mapsto M \otimes_S T$ is an exact functor from $\text{Mod-}S$ to $\text{Mod-}T$, *right flat* if $M \mapsto T \otimes_S M$ is an exact functor $S\text{-Mod} \rightarrow T\text{-Mod}$ and *flat* if it is both left and right flat.

Lemma 2.1. *Suppose that R has prime characteristic and let φ be an injective endomorphism of $A_n(R)$. Then φ is right/left flat if and only if its restriction $\varphi|_C$ to the center $C \subset A_n(R)$ is flat.*

Proof. Let $S = \varphi(A_n(R))$ and $C_S = \varphi(C)$. Then the product map $C \otimes_{C_S} S \rightarrow A_n(R)$ is an isomorphism by Theorem 1.7 (ii) (see also [16, §3.3, Corollary 2]).

Assume that the restriction $\varphi|_C$ is a flat ring homomorphism of commutative rings. For a left S -module M , the natural isomorphism

$$A_n(R) \otimes_S M \cong C \otimes_{C_S} M$$

of abelian groups therefore shows that φ is right flat. The ‘‘opposite’’ product map $S \otimes_{C_S} C \rightarrow A_n(R)$ similarly shows that φ is left flat if $\varphi|_C$ is flat.

Suppose that φ is left (or right) flat i.e., $A_n(R)$ is flat as a left (or right) module over the subring S . Then $A_n(R)$ is a flat C_S -module, since $C_S \subset S \subset A_n(R)$ and S is a free C_S -module. This implies that C is a flat C_S -module as the second step of the extension $C_S \subset C \subset A_n(R)$ is free and therefore faithfully flat. \square

Notice that an injective endomorphism as in Lemma 2.1 is right flat if and only if it is left flat.

2.2. Failure of flatness in positive characteristic. Let k be a field of positive characteristic $p > 0$. Consider the endomorphism $\varphi : A_1(k) \rightarrow A_1(k)$ given by

$$\begin{aligned}\varphi(x) &= x \\ \varphi(\partial) &= \partial + x^{p-1}\partial^p.\end{aligned}$$

In this section we will prove that φ is not a flat ring endomorphism by showing that the restriction of φ to the center of $A_1(k)$ fails to be flat as an endomorphism of commutative rings.

The computation of the restriction to the center can be quite difficult potentially involving complicated p -th powers in the Weyl algebra. For the endomorphism φ , a classical formula from Jacobson’s book [10, p. 187] helps greatly in an otherwise complicated computation: suppose that A is a ring of prime characteristic p and $a, b \in A$. Then

$$(2.2.1) \quad (a + b)^p = a^p + b^p + \sum_{i=1}^{p-1} s_i(a, b),$$

where $s_i(a, b)$ is the coefficient of t^{i-1} in

$$D^{p-1}(a),$$

where $D : A[t] \rightarrow A[t]$ is the derivation $\text{ad}(ta + b)$ and t is a central indeterminate.

Lemma 2.2. *The formula*

$$(\partial + x^{p-1}\partial^p)^p = (x^p)^{p-1}(\partial^p)^p$$

holds for $x, \partial \in A_1(k)$.

Proof. This is a consequence of Jacobson's formula (2.2.1) with $a = \partial$ and $b = x^{p-1}\partial^p$ using $(p-1)! = -1$ (Wilson's theorem). \square

Consider now the restriction of φ to the center in terms of the ring homomorphism $f : k[u, v] \rightarrow k[u, v]$ given by

$$\begin{aligned} f(u) &= u \\ f(v) &= u^{p-1}v^p. \end{aligned}$$

This ring homomorphism is injective and therefore φ is injective by Theorem 1.7. From [13, Theorem 7.4], we have for a flat ring homomorphism $A \rightarrow B$ that

$$IB \cap JB = (I \cap J)B$$

where I and J are ideals in A . Now let $A = k[u, u^{p-1}v^p]$, $B = k[u, v]$, $I = (u^{p-1})$ and $J = (u^{p-1}v^p)$.

Since A is isomorphic to a polynomial ring in the variables u and $u^{p-1}v^p$,

$$I \cap J = (u^{2(p-1)}v^p) \subset A.$$

But $u^{p-1}v^p \in IB \cap JB$ and $u^{p-1}v^p \notin (u^{2(p-1)}v^p)B$. This proves that f is not a flat ring homomorphism and therefore φ is not flat by Lemma 2.1.

2.3. Flatness in characteristic zero. In [18, Theorem 5.1] Tsuchimoto claims a proof of flatness of an endomorphism of $A_n(K)$ with K a field of characteristic zero. At a crucial point he uses that $A_n(R)$ has no "albert holes" [18, Corollary 4.2], where R is a Dedekind domain. However, in the proof of Corollary 4.2, he applies the wrong statement that $M \mapsto \text{Gr}(M)$ commutes with $-\otimes R/I$ for a filtered A -module M with R commutative and A an almost commutative R -algebra. Also, [18, Proposition 5.6(4)] seems not to be obtained in [16] or [17] as stated in [18]. It is not clear to us how to repair these shortcomings in a straightforward manner.

In this section we present our proof of flatness of endomorphisms of Weyl algebras over fields of characteristic zero.

2.3.1. Good filtrations. The *degree* of a monomial $x^\alpha \partial^\beta \in A_n(R)$ is defined as $\deg(x^\alpha \partial^\beta) := |\alpha| + |\beta|$, where $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $|\beta| = \beta_1 + \dots + \beta_n$ for $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$.

The increasing sequence $B = B_0 \subset B_1 \subset \dots$ of finite rank free R -submodules given by

$$B_m = \text{Span}_R \left\{ x^\alpha \partial^\beta \mid \deg(x^\alpha \partial^\beta) \leq m \right\} \subset A_n(R)$$

is a filtration of $A_n(R)$ (called *the Bernstein filtration*) i.e., $\bigcup_{i=0}^{\infty} B_i = A_n(R)$ and $B_i B_j \subset B_{i+j}$ for $i, j \in \mathbb{N}$. Furthermore,

$$\text{Gr}_B(A_n(R)) = B_0 \oplus B_1/B_0 \oplus \dots$$

is the commutative polynomial ring over R in the $2n$ variables $[x_1], \dots, [x_n], [\partial_1], \dots, [\partial_n] \in B_1/B_0$ (see [8, §2]).

Let M be a left module over $A_n(R)$. A *good filtration* of M is an increasing sequence $0 = M_{-1} \subset M_0 \subset M_1 \subset \dots$ of finitely generated R -submodules of M with $\bigcup_i M_i = M$ and $B_i M_j \subset M_{i+j}$ for $i, j \in \mathbb{N}$, such that the graded module

$$\text{Gr}(M) = M_0 \oplus M_1/M_0 \oplus \dots$$

is finitely generated over $\text{Gr}_B(A_n(R))$. If M is finitely generated by $m_1, \dots, m_r \in M$, then $M_i = B_i m_1 + \dots + B_i m_r$, $i = 0, 1, 2, \dots$, is a good filtration of M .

2.3.2. *Grothendieck generic freeness.* The Grothendieck generic freeness lemma [9, Exposé IV, Lemme 6.7] is a very important tool in our proof of flatness. We need it in the slightly strengthened version presented in [7, Theorem 14.4].

Lemma 2.3. *Let A be a noetherian integral domain, B a finitely generated A -algebra and M a finitely generated B -module. Then there exists $f \in A$, such that M_f is a free A_f -module. If in addition, B is positively graded, with A acting in degree zero, and if M is a graded B -module, then f may be chosen so that each graded component of M_f is free over A_f .*

Proposition 2.4. *Let R denote a noetherian integral domain and M a finitely generated $A_n(R)$ -module. Then there exists a nonzero element $f \in R$, such that M_f is a free R_f -module.*

Proof. As M is a finitely generated $A_n(R)$ -module, it has a good filtration $\{M_i\}_{i \in \mathbb{N}}$. The associated graded module $\text{Gr}(M)$ is a finitely generated module over a polynomial ring with coefficients in R , and thus, by Lemma 2.3, there exists a nonzero $f \in R$ such that the R -modules $(M_i/M_{i-1})_f$ are free, for $i \in \mathbb{N}$. By choosing a basis for each $(M_i/M_{i-1})_f$, $i \in \mathbb{N}$, and lifting the collection of these elements to M_f we obtain a basis of M_f . In particular, M_f is free. \square

Lemma 2.5. *Let R denote a finitely generated integral domain over \mathbb{Z} , M a finitely generated $A_n(R)$ -module and let*

$$N = \bigcap_{\mathfrak{m} \in \text{Specm}(R)} \mathfrak{m}M.$$

Then there exists $f \in R$ with $N_f = 0$.

Proof. Applying Proposition 2.4 to the finitely generated module M/N , we may find a nonzero $g \in R$, such that $(M/N)_g$ is a free R_g -module. Similarly we may find a nonzero $h \in R$, such that N_h is a free R_h -module, since M is left noetherian as a finitely generated left module over $A_n(R)$. Therefore N_f and $(M/N)_f$ are free as modules over R_f , where $f = gh$. Now consider the short exact sequence

$$0 \rightarrow N_f \rightarrow M_f \rightarrow (M/N)_f \rightarrow 0.$$

and fix a maximal ideal \mathfrak{m} in R_f . As $(M/N)_f$ is free over R_f , we obtain an induced injective map

$$(2.3.1) \quad N_f \otimes_{R_f} R_f/\mathfrak{m} \rightarrow M_f \otimes_{R_f} R_f/\mathfrak{m}.$$

By (iii) in Theorem 1.13, the field R_f/\mathfrak{m} is isomorphic to R/\mathfrak{m}_R , where $\mathfrak{m}_R = \mathfrak{m} \cap R$. In particular, the map (2.3.1) is identified with the map

$$N/\mathfrak{m}_R N = N \otimes_R R/\mathfrak{m}_R \rightarrow M \otimes_R R/\mathfrak{m}_R = M/\mathfrak{m}_R M,$$

which is zero by the definition of N . Therefore

$$N_f \otimes_{R_f} R_f/\mathfrak{m} = 0$$

and $N_f = 0$ by the freeness of N_f . \square

We now prove flatness in characteristic zero by reducing to positive characteristic.

2.3.3. *Flatness.*

Theorem 2.6. *Let K be a field of characteristic zero. Then an endomorphism $\varphi : A_n(K) \rightarrow A_n(K)$ is flat.*

Proof. Let $S = \varphi(A_n(K))$. We will prove that $A_n(K)$ is flat as a left S -module (φ is injective, since $A_n(K)$ is a simple ring). The proof that $A_n(K)$ is flat as a right S -module is similar and is left to the reader. It suffices to prove that the multiplication map

$$(2.3.2) \quad I \otimes_S A_n(K) \rightarrow A_n(K)$$

is injective for every finitely generated right ideal I in S . Let M denote the right $A_n(K)$ -module $I \otimes_S A_n(K)$, and assume that $m \in M$ maps to zero under (2.3.2). We will prove that m is zero.

Assume that I is generated as a right ideal in S by elements $\varphi(a_i)$, for $i = 1, 2, \dots, m$, with $a_i \in A_n(K)$. We may then write

$$m = \sum_{i=1}^m \varphi(a_i) \otimes b_i,$$

for certain elements $b_i \in A_n(K)$. Now fix a finitely generated \mathbb{Z} -subalgebra R of K , such that all the elements $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_m$ and $\varphi(x_i), \varphi(\partial_i)$, for $i = 1, 2, \dots, n$, are contained in $A_n(R) \subseteq A_n(K)$. Then there exists an induced endomorphism

$$\varphi_R : A_n(R) \rightarrow A_n(R)$$

whose base change to K equals φ . We let S_R denote the image of φ_R , and let I_R denote right ideal in S_R generated by $\varphi(a_1), \varphi(a_2), \dots, \varphi(a_m) \in S_R$. Finally we let M_R denote $I_R \otimes_{S_R} A_n(R)$ and let m_R denote the element $\sum_i \varphi(a_i) \otimes b_i$ in M_R . The base change of S_R, I_R and M_R to K then equals S, I and M respectively, and the multiplication map $M_R \rightarrow A_n(R)$ will base change to (2.3.2). Moreover, m equals $m_R \otimes 1$ in $M_R \otimes_R K = M$.

It suffices to prove that m_R is zero in some localization $(M_R)_f$, for $f \in R$. By Lemma 2.5 this will follow if m_R is zero modulo every maximal ideal \mathfrak{m} of R . So fix a maximal ideal \mathfrak{m} of R , and let \bar{m} denote the image of m_R in $M_R \otimes_R R/\mathfrak{m}$. Consider also the induced morphism $\bar{\varphi} : A_n(R/\mathfrak{m}) \rightarrow A_n(R/\mathfrak{m})$ with image $S_{R/\mathfrak{m}}$. Here R/\mathfrak{m} is a field of positive characteristic by (i) of Theorem 1.13.

By Proposition 1.12, the induced map $\bar{\varphi}|_C : C \rightarrow C$ on the center C of $A_n(R/\mathfrak{m})$ preserves the canonical Poisson bracket. Therefore $\det(\bar{\varphi}_C) = \pm 1$ by Proposition 1.10 and φ_C is flat by [2, (2.1) THEOREM][14]. By Lemma 2.1, this shows that $A_n(R/\mathfrak{m})$ is flat as a left module over $S_{R/\mathfrak{m}}$. Letting $I_{R/\mathfrak{m}}$ denote the right ideal in $S_{R/\mathfrak{m}}$ generated by the image of I_R in $S_{R/\mathfrak{m}}$, it follows that the multiplication map

$$(2.3.3) \quad M_R \otimes_R R/\mathfrak{m} = I_{R/\mathfrak{m}} \otimes_{S_{R/\mathfrak{m}}} A_n(R/\mathfrak{m}) \rightarrow A_n(R/\mathfrak{m}),$$

is injective. But \bar{m} maps to zero under (2.3.3) as $m_R \in M_R$ maps to zero in $A_n(R)$ under the multiplication map $M_R \rightarrow A_n(R)$. We conclude that \bar{m} is zero as claimed. \square

3. AUTOMORPHISMS AND POLYNOMIAL EQUATIONS

Let $f \in A_n(R) \setminus \{0\}$. The degree ($\deg f$) of f is defined as the maximum of the degrees of the monomials occurring with non-zero coefficient in the monomial expansion of f from Proposition 1.4. Notice that $\deg(fg) = \deg(f) + \deg(g)$ if $f, g \in R \setminus \{0\}$ and R is an integral domain.

Definition 3.1. Let φ be an endomorphism of $A_n(R)$. Then the degree of φ is defined as

$$\deg \varphi = \max\{\deg \varphi(x_1), \deg \varphi(\partial_1), \dots, \deg \varphi(x_n), \deg \varphi(\partial_n)\}.$$

The following result comes from [17, Proposition 4.2].

Lemma 3.2. Let k be a field of prime characteristic p and φ an automorphism of $A_n(k)$. Then

$$\deg(\varphi^{-1}) \leq \deg(\varphi)^{2n-1}.$$

Proof. If φ is an automorphism, then the induced endomorphism $\varphi|_C$ (see (iii) of Theorem 1.7) of the center $C \subset A_n(k)$ is an automorphism. The bound on degrees is therefore a consequence of [2, (1.4) COROLLARY], since

$$\deg \varphi = \max\{\deg \varphi(x_1)^p, \deg \varphi(\partial_1)^p, \dots, \deg \varphi(x_n)^p, \deg \varphi(\partial_n)^p\}/p.$$

□

Lemma 3.3. *Let $K \subset L$ be a field extension with $\text{char}(K) = 0$. If φ is an automorphism of $A_n(L)$ and $\varphi(x_i), \varphi(\partial_i) \in A_n(K)$, then $\varphi^{-1}(x_i), \varphi^{-1}(\partial_i) \in A_n(K)$ for $i = 1, \dots, n$.*

Proof. We may write

$$(3.0.4) \quad x_i = \sum_{\alpha, \beta \in \mathbb{N}^n} \lambda_{\alpha\beta}^i \varphi(x)^\alpha \varphi(\partial)^\beta \quad \text{and} \quad \partial_i = \sum_{\alpha, \beta \in \mathbb{N}^n} \mu_{\alpha\beta}^i \varphi(x)^\alpha \varphi(\partial)^\beta$$

for $i = 1, \dots, n$ and $\lambda_{\alpha\beta}^i, \mu_{\alpha\beta}^i \in L$. Applying $\text{ad}(\varphi(x_i))$ and $\text{ad}(\varphi(\partial_j))$ to both sides of the two identities in (3.0.4) gives $\lambda_{\alpha\beta}^i, \mu_{\alpha\beta}^i \in K$ by use of Proposition 1.1. □

Proposition 3.4. *Let K denote a field with $\text{char}(K) = 0$ and $R \subset K$ a finitely generated \mathbb{Z} -subalgebra. If φ is an endomorphism of $A_n(R)$, such that the induced endomorphism of $A_n(R/\mathfrak{m})$ is an automorphism for every $\mathfrak{m} \in \text{Specm}(R)$, then φ is an automorphism of $A_n(K)$.*

Proof. A potential inverse to $\varphi \in A_n(K)$ may be viewed as the solution to a set of polynomial equations with coefficients in R as follows. We are looking for elements

$$(3.0.5) \quad \begin{aligned} q_i &= \sum_{\alpha, \beta \in \mathbb{N}^n} \lambda_{\alpha\beta}^i x^\alpha \partial^\beta && \text{[candidate for } \varphi^{-1}(x_i)\text{]} \\ p_i &= \sum_{\alpha, \beta \in \mathbb{N}^n} \mu_{\alpha\beta}^i x^\alpha \partial^\beta && \text{[candidate for } \varphi^{-1}(\partial_i)\text{]}, \end{aligned}$$

for $i = 1, \dots, n$ in $A_n(K)$, such that

$$(3.0.6) \quad \begin{aligned} [p_i, p_j] &= 0 \\ [q_i, q_j] &= 0 \\ [p_i, q_j] &= \delta_{ij} \\ x_i &= \sum_{\alpha, \beta \in \mathbb{N}^n} \lambda_{\alpha\beta}^i \varphi(x)^\alpha \varphi(\partial)^\beta \\ \partial_i &= \sum_{\alpha, \beta \in \mathbb{N}^n} \mu_{\alpha\beta}^i \varphi(x)^\alpha \varphi(\partial)^\beta \end{aligned}$$

for $i, j = 1, \dots, n$. Using Proposition 1.4, the equations in (3.0.6) may be considered as a system of polynomial equations with coefficients in R in the finitely many variables $\lambda_{\alpha\beta}^i, \mu_{\alpha\beta}^i$ for $i = 1, \dots, n$ and $\alpha, \beta \in \mathbb{N}^n$ with $|\alpha| + |\beta| \leq \deg(\varphi)^{2n-1}$. By assumption and Lemma 3.2, this system of polynomial equations has a solution in R/\mathfrak{m} for every $\mathfrak{m} \in \text{Specm}(R)$. By (iv) in Theorem 1.13, the polynomial system therefore has a solution $\lambda_{\alpha\beta}^i, \mu_{\alpha\beta}^i \in \overline{K}$ i.e., φ is an automorphism in $A_n(\overline{K})$ and thus an automorphism in $A_n(K)$ by Lemma 3.3. □

4. BIRATIONAL ENDOMORPHISMS

Let A be an Ore domain i.e.,

$$sA \cap tA \neq (0) \quad \text{and} \quad As \cap At \neq (0)$$

for every $s, t \in S = A \setminus \{0\}$. Then A embeds in a division ring $Q(A)$, such that

$$(1) \quad Q(A) = \{s^{-1}a \mid a \in A, s \in S\}.$$

- (2) Any homomorphism $f : A \rightarrow T$, such that f maps S to invertible elements in T factors through $Q(A)$. [*universal property*]

The division ring $Q(A)$ is uniquely determined up to isomorphism.

An injective ring homomorphism $\varphi : A \rightarrow B$ between Ore domains A and B induces a natural injection $Q(A) \subset Q(B)$. We call φ *birational* if $Q(A) = Q(B)$. The Weyl algebra over a noetherian integral domain is an Ore domain, since it is a left and right noetherian domain.

Lemma 4.1. *Let R denote an integral domain of prime characteristic p , C the center of $A_n(R)$ and K the fraction field of R . Then the multiplication map*

$$(4.0.7) \quad A_n(R) \otimes_C K \rightarrow Q(A_n(R)),$$

is an isomorphism of rings. In particular, the center of $Q(A_n(R))$ equals K . Moreover, an injective endomorphism $\varphi : A_n(R) \rightarrow A_n(R)$ is birational if and only if the induced polynomial map $\varphi|_C : C \rightarrow C$ is birational.

Proof. By Theorem 1.7, $A_n(R)$ is a C -algebra and $A_n(R) \otimes_C K$ is a finite dimensional K -algebra containing $A_n(R)$ as a subring. Since $A_n(R)$ is a domain, it follows that $A_n(R) \otimes_C K$ is a division ring [15, p. A227]. Thus by the universal property of $Q(A_n(R))$, the map (4.0.7) has an inverse and must be an isomorphism. The claim about the center of $Q(A_n(R))$ follows, since the center of $A_n(R)$ is C .

Consider an injective endomorphism $\varphi : A_n(R) \rightarrow A_n(R)$. By (iii) of Theorem 1.7 we have a commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & K \\ \downarrow & & \downarrow \\ Q(A_n(R)) & \xrightarrow{\varphi} & Q(A_n(R)) \end{array}$$

of extensions of division rings, where the two vertical extensions are of degree p^{2n} . It follows that the top horizontal extension is of degree one if and only if the lower horizontal extension is of degree one. This is equivalent to the final claim. \square

If K is a field of characteristic zero, recall that an endomorphism of $A_n(K)$ is injective, since $A_n(K)$ is a simple ring. It follows that an endomorphism of $A_n(K)$ induces an endomorphism of $Q(A_n(K))$ which is finite in the following sense.

Proposition 4.2. *Let K denote a field of characteristic zero and let φ denote an endomorphism of $A = A_n(K)$. Let $S = \varphi(A)$ denote the image of φ . Then the dimension of $Q(A)$ as a (left or right) module over $Q(S)$ is less than or equal to $(\deg \varphi)^{2n}$.*

Proof. We will prove the bound for the left dimension (the proof for the right dimension is similar). Let $e_1, \dots, e_r \in Q(A)$ be linearly independent over $Q(S)$.

By clearing denominators we may assume that e_1, e_2, \dots, e_r are elements in A . Choose $D \in \mathbb{N}$, such that $\deg e_i \leq D$ for $i = 1, \dots, r$. For $j \in \mathbb{N}$, let B_j denote the Bernstein filtration of A and define

$$M_j = \varphi(B_j)e_1 + \varphi(B_j)e_2 + \dots + \varphi(B_j)e_r.$$

Then $M_j \subseteq B_{d+j+D}$, where $d = \deg \varphi$. By the linear independence of e_1, e_2, \dots, e_r ,

$$\dim_K(M_j) = r \dim_K(B_j).$$

This leads to the inequality

$$r \dim_K(B_j) \leq \dim_K(B_{d+j+D}),$$

for $j \in \mathbb{N}$. As

$$\dim_K(B_j) = \frac{1}{(2n)!} j^{2n} + \text{lower degree terms in } j,$$

we conclude that

$$\frac{r}{(2n)!} \leq \frac{d^{2n}}{(2n)!},$$

which gives $r \leq d^{2n}$ as claimed. \square

Theorem 4.3. *Let K denote a field of characteristic zero and let φ be an endomorphism of $A_n(K)$. If φ is birational, then φ is an automorphism.*

Proof. The birationality of φ implies the existence of $a_i, b_i, c_i, d_i \in A_n(K)$, for $i = 1, \dots, n$, such that

$$(4.0.8) \quad \begin{aligned} \varphi(a_i) &= x_i \varphi(b_i) \\ \varphi(c_i) &= \partial_i \varphi(d_i) \\ \varphi(b_i) &\neq 0 \\ \varphi(d_i) &\neq 0. \end{aligned}$$

Let T denote a finitely generated \mathbb{Z} -subalgebra of K , such that all the coefficients of $\varphi(x_i), \varphi(\partial_i), a_i, b_i, c_i$ and d_i in the monomial K -basis of $A_n(K)$ (see Proposition 1.4), are contained in T . We define $f \in T$ to be the product of all the non-zero coefficients occurring in the expansions of b_i and d_i , for $i = 1, 2, \dots, n$, in the monomial T -basis of $A_n(T)$. Let $R = T[1/f] \subset K$.

For $\mathfrak{m} \in \text{Specm}(R)$, we let $C_{\mathfrak{m}}$ denote the center of $A_n(R/\mathfrak{m})$ and $\varphi_{\mathfrak{m}}$ the induced endomorphism of $A_n(R/\mathfrak{m})$. Notice that $\varphi_{\mathfrak{m}}$ is injective by (iii) of Theorem 1.7, since $\varphi_{\mathfrak{m}}|_{C_{\mathfrak{m}}}$ is injective as $\det J(\varphi_{\mathfrak{m}}|_{C_{\mathfrak{m}}}) = \pm 1$ by Proposition 1.10. Since the relations (4.0.8) are preserved for $\varphi_{\mathfrak{m}}$, it follows that $\varphi_{\mathfrak{m}}|_{C_{\mathfrak{m}}}$ is birational and by Lemma 4.1, that $\varphi_{\mathfrak{m}}|_{C_{\mathfrak{m}}}$ is a birational endomorphism of $C_{\mathfrak{m}}$. Therefore $\varphi_{\mathfrak{m}}|_{C_{\mathfrak{m}}}$ is an automorphism by [2, (2.1) THEOREM] and $\varphi_{\mathfrak{m}}$ is an automorphism by (iii) of Theorem 1.7.

Now Proposition 3.4 applies to show that φ is an automorphism of $A_n(K)$. \square

Remark 4.4. We end this paper with two natural questions for an endomorphism φ of $A_n(K)$, where K is a field of characteristic zero. Let $A = A_n(K)$ and $S = \varphi(A) \subset A$.

- (i) Do the left- and right dimensions of $Q(A)$ over $Q(S)$ agree?
- (ii) Is $S = A$ if A is a finitely generated S -module?

The first question could perhaps be answered affirmatively by reducing to positive characteristic (where the left and right dimensions do agree). The last question is inspired by the simply connectedness of \mathbb{C}^n in the commutative case [2, (2.1) THEOREM, (e)].

REFERENCES

1. P. Adjamagbo and A. van den Essen, *A proof of the equivalence of the Dixmier, Jacobian and Poisson conjectures*, Acta Math. Vietnam. **32** (2007), no. 2-3, 205–214.
2. H. Bass, E. H. Connell, and D. Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 2, 287–330.
3. A. Belov-Kanel and M. Kontsevich, *The Jacobian conjecture is stably equivalent to the Dixmier conjecture*, Mosc. Math. J. **7** (2007), no. 2, 209–218, 349.
4. J.-E. Björk, *Rings of Differential Operators*, North-Holland Mathematical Library, vol. 21, North-Holland Publishing Co., Amsterdam-New York, 1979.
5. N. Bourbaki, *Commutative Algebra, Chapters 1–7*, Elements of Mathematics, Springer Verlag, 1989.
6. J. Dixmier, *Sur les algèbres de Weyl*, Bull. Soc. Math. France **96** (1968), 209–242.
7. D. Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995.

8. I. M. Gelfand and A. A. Kirillov, *Sur les corps liés aux algèbres enveloppantes des algèbres de Lie*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 31, 5–19.
9. A. Grothendieck, *Revêtements étales et groupe fondamental. Fasc. I: Exposés 1 à 5*, Séminaire de Géométrie Algébrique, vol. 1960/61, Institut des Hautes Études Scientifiques, Paris, 1963.
10. N. Jacobson, *Lie algebras*, Dover Publications, Inc., New York, 1979, Republication of the 1962 original.
11. O.-H. Keller, *Ganze Cremona-Transformationen*, Monatsh. Math. Phys. **47** (1939), 229–306.
12. D. E. Littlewood, *On the classification of algebras*, Proc. London Math. Soc. (2) **35** (1933), 200–240.
13. H. Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid.
14. M. Miyanishi, L. Robbiano, and S. Sui Sheng Wang, *A brief proof of Jacobian hypothesis implies flatness*, Proc. Amer. Math. Soc. **109** (1990), no. 2, 327–330.
15. P. Revoy, *Algèbres de Weyl en caractéristique p* , C. R. Acad. Sci. Paris Sér. A-B **276** (1973), A225–A228.
16. Y. Tsuchimoto, *Preliminaries on Dixmier conjecture*, Mem. Fac. Sci. Kochi Univ. Ser. A Math. **24** (2003), 43–59.
17. ———, *Endomorphisms of Weyl algebra and p -curvatures*, Osaka J. Math. **42** (2005), no. 2, 435–452.
18. ———, *Non commutative algebraic spaces of finite type over dedekind domains*, J. Math. Kyoto Univ. **46** (2006), 553–582.