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Abstract

We present a spatio-temporal modelling framework for stochastic fields that obey exact symmetry in space and time, i.e. the field amplitude considered as a stochastic process in time at a fixed position in space is identical, as a stochastic process, to the field amplitude considered as a stochastic process in space at a fixed time point. The stochastic fields are given in explicit form and include intermittency as a basic ingredient. A causal version is discussed with respect to turbulence modelling and in relation to Taylor's Frozen Flow Hypothesis.

Taylor's Frozen Field Hypothesis (TFFH) [14] originates from turbulence studies and was formulated as a relation between second order moments of spatial and temporal velocity increments in homogeneous and stationary turbulent flows. The basic assumption is that of the existence of a steady mean flow V that is much larger than fluctuating contributions. The Hypothesis then states that second order moments of timewise velocity increments on a time scale t correspond to second order moments of spatial increments on a spatial scale Vt (in direction of the mean flow) and vice versa. The application of TFFH is a standard tool in turbulence studies where it is used to interpret timeseries as spatial recordings, and, in many cases, without restriction to second order statistics. In particular, TFFH is widely used to estimate spatial structure functions from hot-wire anemometry, see e.g. [8].

The basic idea of converting spatial scales to temporal scales has also been applied to a great variety of other phenomena in natural sciences ranging from rain field measurements and modelling [10] to the interpretation of galactic turbulence [5]. The basic criterion for the applicability of TFFH is the existence of a steady advection velocity that carries the observable of interest through spatial scales without relevant distortion. There are however situations where the converting quantity between space and time is not the mean velocity (although present) [15] and there are observations in atmospheric sciences where TFFH holds but no clear advection velocity V can be identified and/or the fluctuations are not small compared to V [9]. This raises the question whether space-time similarity in the spirit of Taylor can be caused by other mechanism than a steady mean flow, as was pointed out in [6].

In this letter we discuss a class of stochastic fields that show space-time symmetry without referring to a constant advection velocity. The spatio-temporal stochastic modelling framework we present is explicit (not defined in terms of solutions of stochastic differential equations and not defined implicitely as, for instance, in [6]), displays intermittency and is intrinsically symmetric, as a stochastic field, with respect to space-time conversion. Moreover, the modelling framework is mathematically tractable and can be formulated to obey causality which provides the possibility to specify the ingredients of the model for a wide range of applications.

We define the modelling framework for the spatio-temporal field amplitude $Y_t(x)$ in the simplest case as

$$Y_t(x) = \mu + \int_{\mathbb{R}^2} G(t-s, x-\xi)\sigma_s(\xi)W(\mathrm{d}s, \mathrm{d}\xi) \tag{1}$$

where t denotes time, x denotes position in one-dimensional space, μ is the mean amplitude $\mu = \mathrm{E}\{Y_t(x)\}$ ($\mathrm{E}\{\ \}$ denotes the expectation), G is a deterministic kernel (suitable for the integral to exist) and W is two-dimensional white noise, i.e. an independently scattered normal random measure. The stochastic field σ is assumed to be independent of W, stationary in time and homogeneous in its spatial variable which implies stationarity and homogeneity for Y. The field σ allows the distribution of Y and that of of its increments to be strongly non-Gaussian with intermittent amplitudes, as has been discussed in [2, 3]

In the following we specify the model ingredients G and σ such that Y behaves for fixed t as a process in x stochastically identical to Y for a fixed x as a process of t, i.e $Y_t(0)$ is stochastically identical (as a stochastic process) to $Y_0(ct)$, where c is a constant. Without loss of generality we will assume that $c = 1$ (any other choice corresponds to a linear change of time).

The symmetry between space and time can easily be achieved by assuming a symmetric kernel

$$G(t, x) = G(x, t) \tag{2}$$

and by the requirement that $\sigma_t(0)$ and $\sigma_0(ct)$ are stochastically identical processes. While the former requirement is simply a restriction of the class of deterministic kernels to functions symmetric in their arguments, the latter is discussed below in more details.

In the turbulence context the model (1) (with some extra term accounting for skewness) represents the main component of the velocity field and σ^2 plays the role of the turbulent energy dissipation [2, 3, 4]. To account for intermittency and scaling, a stochastic intermittency field for σ is employed in [1, 13, 11, 12, 7]. This stochastic intermittency field can, for instance, be realized as a continuous cascade process in space and time.

To give an explicit example of a stochastic field σ with spatio-temporal symmetry we adopt the set-up of stochastic intermittency fields and define

$$\sigma_t(x) = \exp\{L(A_t(x))\} \tag{3}$$

where $A_t(x) = (x, t) + A_0$ is a translational invariant region in space-time attached to each point (x, t) and where L is a homogeneous Lévy basis [1], i.e. an independently scattered infinitely divisible random measure with distributions independent of position in space-time. Such a Lévy basis associates a random number to each

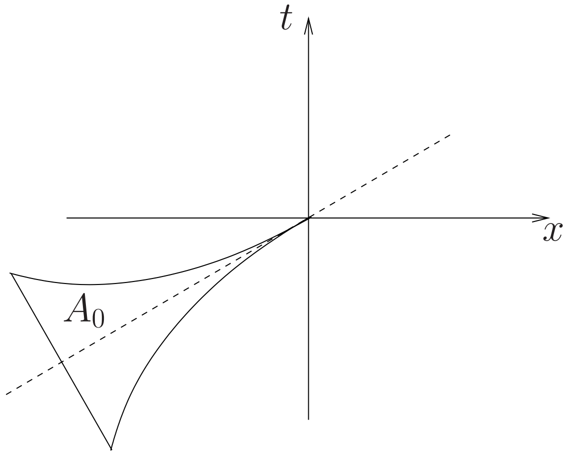

Figure 1: Illustration of the set A_0 symmetric around the identity line $x = t$.

set in $\mathbb{R}^2$ and the dependence structure of σ at different space-time locations only depends on L of the overlap of the associated sets A. Such a set-up easily accounts for scaling relations corresponding to a geometrical property of A_0 [13, 11, 12, 7].

We now specify the set A_0 to be symmetric with respect to the identity line $x = t$ (see Figure 1) which implies that

$$A_0(0) \cap A_0(x) = A_0(0) \cap A_t(0) \tag{4}$$

for all $x = t$ and consequently the processes $\sigma_t(0)$ and $\sigma_0(x)$ are stochastically identical implying that $Y_t(0)$ and $Y_0(t)$ are also stochastically identical.

It is important to note that no requirement about $\mu = \mathrm{E}\{Y_t(x)\}$ enters the model, in fact μ can be any value including zero. Another important point is that no assumptions about the fluctuations being small compared to the mean amplitude are used.

Causality, in the sense that present field amplitudes only depend on past innovations, can easily be incorporated by the requirement $G(t, x) = 0$ for $t \leq 0$ (implying that $G(t, x) = 0$ for $x \leq 0$) and $A_t(x) \subset [-\infty, t] \times \mathbb{R}$.

In case that σ is an arbitrary stationary and homogeneous stochastic field with $\epsilon = \mathrm{E}\{\sigma_t(x)^2\}$ finite, we arrive at the original statement of TFFH, namely that second order moments of spatial increments are equal to second order moments of temporal increments. To see this we note that the spatial second order moments can be expressed as

$$\mathrm{E}\{(Y_0(x) - Y_0(0))^2\} = \epsilon \int_{\mathbb{R}^2} (G(-s, x - \xi) - G(-s, -\xi))^2 \mathrm{d}s\mathrm{d}\xi \tag{5}$$

and the temporal second order moments as

$$\mathrm{E}\{(Y_t(0) - Y_0(0))^2\} = \epsilon \int_{\mathbb{R}^2} (G(t - s, -\xi) - G(-s, -\xi))^2 \mathrm{d}s\mathrm{d}\xi. \tag{6}$$

These two expressions give the same results if we assume (2) and set $x = t$.

The above modelling framework can straightforwardly be extended to k spatial dimensions

$$Y_t(x) = \mu + \int_{\mathbb{R}^{k+1}} G(t - s, x - \xi)\sigma_s(\xi) W(\mathrm{d}s, \mathrm{d}\xi) \tag{7}$$

where W is white noise in $k+1$ dimensions. We define the intermittency field in the same way as above (3) where now A is a subset of $\mathbb{R}^{k+1}$. Let $x = (x_1, x_2, \ldots, x_k)$ denote a vector in $\mathbb{R}^k$ and let $\bar{x} = (0, x_2, \ldots, x_k)$. We specify the set A_0 to be symmetric with respect to the identity plane $x_1 = t$ which implies that

$$A_0(x) \cap A_0(\bar{x}) = A_0(\bar{x}) \cap A_t(\bar{x}) \tag{8}$$

for all $x_1 = t$ and consequently the processes $\sigma_0(x)$ and $\sigma_t(\bar{x})$ are stochastically identical. The assumption

$$G(t, x_1, x_2, \ldots, x_k) = G(x_1, t, x_2, \ldots, x_n) \tag{9}$$

then implys that $Y_t(\bar{x})$ and $Y_0(x)$ are also stochastically identical. We therefore have that $Y_t(x)$ as a process in time t at fixed x is stochastically identical, as a process, to $Y_t(x))$ considered as a process of x_1 for fixed t and fixed $\bar{x}$.

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