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Abstract

A task with ideal execution time L such as the execution of a computer program or the transmission of a file on a data link may fail, and the task then needs to be restarted. The task is handled by a complex system with features similar to the ones in classical reliability: failures may be mitigated by using server redundancy in parallel or k-out-of-n arrangements, standbys may be cold or warm, one or more repairmen may take care of failed components, etc. The total task time X (including restarts and pauses in failed states) is investigated with particular emphasis on the tail $\mathbb{P}(X > x)$. A general alternating Markov renewal model is proposed and an asymptotic exponential form $\mathbb{P}(X > x) \sim C\mathrm{e}^{-\gamma x}$ identified for the case of a deterministic task time $L \equiv \ell$. The rate γ is given by equating the spectral radius of a certain matrix to 1, and the asymptotic form of $\gamma = \gamma(\ell)$ as $\ell \to \infty$ is derived, leading to the asymptotics of $\mathbb{P}(X > x)$ for random task times L. A main finding is that X is always heavy-tailed if L has unbounded support. The case where the Markov renewal model is derived by lumping in a continuous-time finite Markov process with exponential holding times is given special attention, and the study includes analysis of the effect of processing rates that differ with state or time.

Keywords: Alternating renewal process, computer reliability, data transmission, failure rate, fault-tolerant computing, heavy tails, Markov renewal equation, matrix perturbation, phase-type distribution, RESTART, tail asymptotics, Perron-Frobenius theory, phase-type distribution, spectral radius, Tauberian theorem.

1 Introduction

This paper studies some systems typically identical to those of interest in Reliability Theory and Availability ([4], [5], [19]). The system consists of a large number of components and at a given time it has a state depending on the characteristics of the components (operating, under repair, in cold or hot standby, rebooting etc.). In brief, a finite state space $\mathcal{E}$ is partitioned into two subspaces, $\mathcal{D}$ and $\mathcal{U}$ (Down/Up, or Operating/Failed, etc.). Individual components may be failed and under repair also in an Up state, but the system as a whole can only operate in the Up states and is paused upon exit from there to remain in the Down states until all necessary repairs have been undertaken.

In classical reliability theory one is interested in the length of time the system is in some Up state (the longer without failure, the better). Repair time is of less importance. Availability is concerned with the fraction of time that the system is in an Up state, taking into account the multiplicity of processors (thus the availability fraction could be greater than 1). In this paper we focus on the processing of a task of length L (the *task time*) which can only finish processing after the system has been in the set $\mathcal{U}$ of Up states for a time greater than L (L may be random or constant, $L = \ell$). If during processing the system fails and goes to a Down state, the performed work is lost and when entering an Up state again, the processing has to be restarted from scratch. That is, each time the system leaves $\mathcal{U}$ and goes to $\mathcal{D}$, it must start over when reentering $\mathcal{U}$. The quantity of interest is the *total task time* X.

The study of total task times in problems of this type have a long tradition in many specific and distinct failure recovery schemes. In particular in the RESUME (also referred to as preemptive resume) scenario, if there is a processor failure while a job is being executed, after repair is implemented the job can continue where it left off. In the REPLACE situation (also referred to as preemptive repeat different), if a job fails, it is replaced by a different job from the same distribution. Here, no details concerning the previous job are necessary in order to continue. For these two schemes, see Kulkarni *et al.* [14], [15]. Further models and methods of failure recovery are in Chlebus *et al.* [9] for restartable processors and in De Prisco *et al.* [10] (stage checkpointing)

As indicated above, we are here concerned with RESTART (also referred to as preemptive repeat identical). There are many examples where this is relevant. The obvious one alluded to above involves execution of a program on some computer. If the computer fails, and the intermediate results are not saved externally (e.g., by *checkpointing*), then the job must restart from the beginning. As another example, one might wish to copy a file from a remote system using some standard protocol as FTP or HTTP. The time it takes to copy a file is proportional to its length. A transmission error immediately aborts the copy and discards the partially received data, forcing the user to restart the copy from the beginning. Yet another example would be receiving 'customer service' by telephone. Often, while dealing with a particular service agent, the connection is broken. Then the customer must redial the service center, and invariably (after waiting in a queue) ends up talking to a different agent, and having to explain everything from the beginning.

Computing expected values and transforms is usually easy in any of the models mentioned above. Also the important problem of tail behaviour, that is, the probability of long delays, has been succesfully attacked in a number of the above references. However, for the RESTART policy it resisted detailed analysis until the recent work of Sheahan *et al.* [18] and Asmussen *et at.* [2] where the tail of H was found in a variety of combinations of tail behavior of F and G. A main and surprising finding was that the tail of H is always heavy-tailed if F has unbounded support. The consequence is that delays can be very severe in the RESTART setting. For further recent work in this direction, see Jelenković *et al.* [11], [12].

2 Statement of main results

We now proceed to a more precise description of the model. Without the task being processed (equivalently, with task time $T \equiv 0$), the system is assumed to develop according to an alternating Markov renewal process with state space $\mathcal{E} = \mathcal{U} \cup \mathcal{D}$ and imbedded Markov chain at jump times $\xi_0, \xi_1, \dots$ If $\xi_0 = u \in \mathcal{U}$, the first sojourn in state ξ_0 terminates at time T_u and then state $\xi_1 \in \mathcal{D}$ is entered, and if $\xi_0 = d \in \mathcal{D}$, the first sojourn in state ξ_0 terminates at time T_d where state $\xi_1 \in \mathcal{U}$ is entered. When it is unimportant to specify whether states are in $\mathcal{U}$ or in $\mathcal{D}$, they are just denoted by $i, j \dots$; when a state in $\mathcal{E}$ is denoted by u, it is silently understood that $u \in \mathcal{U}$, and similarly for $d \in \mathcal{D}$.

The transition rules are thus specified by the set of probability measures

$$
\begin{aligned}
F_{du}(t) &= \mathbb{P}_d\big(T_d \le t,\, \xi_1 = u\big)\,, && d \in \mathcal{D},\, u \in \mathcal{U}\,,\\
F_{ud}(t) &= \mathbb{P}_d\big(T_i \le t,\, \xi_1 = d\big)\,, && u \in \mathcal{U},\, d \in \mathcal{D}\,,
\end{aligned}
$$

$F_{ij} = 0$ when i, j are either both in $\mathcal{U}$ or both in $\mathcal{D}$. In particular, the transition probabilities p_{ij} of the Markov chain ξ are given by $p_{du} = F_{du}(\infty)$, $p_{ud} = F_{ud}(\infty)$, $p_{ij} = 0$ when i, j are both in either $\mathcal{U}$ or in $\mathcal{D}$, and we have

$$
F_d(t) = \mathbb{P}_d(T_d \le t) = \sum_{u \in \mathcal{U}} F_{du}(t)\,, \quad F_u(t) = \mathbb{P}_u(T_u \le t) = \sum_{d \in \mathcal{D}} F_{ud}(t)\,.
$$

The term *alternating* comes from only transitions from $\mathcal{U}$ to $\mathcal{D}$ and vice versa being possible, not ones within $\mathcal{U}$ or $\mathcal{D}$. At the time T_u or T_d of the first state space change, the process repeats itself with new starting value ξ_1. See [1, pp. 206–7] for more detail on the basics of Markov renewal processes.

The model for how the task is handled is that processing only occurs in an up state u of the Markov renewal process and then the processing rate is $\rho_u(t)$ at time $t < T_u$ for a suitable stochastic process ρ_u. Thus, if the task length is $L \equiv \ell$ and $\xi_0 = u \in \mathcal{U}$, the task finishes processing in the first Markov renewal interval $[0, T_u)$ at time

$$
\tau_u(\ell) = \inf\Big\{ t > 0 : \int_0^t \rho_u(s)\, \mathrm{d}s = \ell \Big\}
$$

if $\tau_u(\ell) < T_u$. Otherwise, no processing takes place in the down state $d \in \mathcal{D}$ entered after T_u, and at the time where the next state ξ_2 in $\mathcal{U}$ is entered, the whole process is restarted with the same processing time ℓ but in Markov state ξ_2. No specific assumptions are made at the moment on the dependence between T_u, ρ_u, ξ_1, but to avoid trivialities, we need $\mathbb{P}\big(\tau_u(\ell) < T_u\big) > 0$ for at least one $u \in \mathcal{U}$, ensuring that the task will eventually be finished.

The main example of processing rate modeling is of course $\rho_u(t) \equiv 1$ for all u and t, and this may safely be used as basis for intuition for quite a while. We return to time-varying rates in Section 7.

Our main example of the Markov renewal set-up comes from continuous-time finite Markov processes modeling the time evolution of models in classical reliability theory ([4], [5], [19]). We present this in Section 3. However, there are others, e.g. vanilla RESTART with repair. Here we can take $\mathcal{U}, \mathcal{D}$ as one-point sets $\mathcal{U} = \{u\}$, $\mathcal{D} = \{d\}$, and $\rho_u(t) \equiv 1$. Then F_u can be seen as the distribution of either the operating time or the failure time (denoted G in [2]), F_d the distribution of the repair time, and the Markov renewal model allows for incorporating general distributions of F_u, F_d, not just exponential ones.

Our main result on the general set-up with deterministic task length is the following:

Theorem 2.1. *Consider a deterministic task length $L = \ell$ and denote by $\boldsymbol{R}(\alpha)$ the $\mathcal{E} \times \mathcal{E}$ matrix with entries*

$$
\begin{aligned}
r_{du}(\alpha) &= \mathbb{E}_d\big[\mathrm{e}^{\alpha T_d};\, \xi_1 = u\big]\,, && d \in \mathcal{D},\, u \in \mathcal{U}\,,\\
r_{ud}(\alpha) &= \mathbb{E}_u\big[\mathrm{e}^{\alpha T_u};\, \tau_u(\ell) \ge T_d\,, \xi_1 = d\big]\,, && u \in \mathcal{U}, d \in \mathcal{D}\,,
\end{aligned}
$$

all other $r_{ij}(\alpha) = 0$. Assume there exists $\gamma = \gamma(\ell)$ such that $\boldsymbol{R}(\gamma)$ is irreducible with spectral radius 1. Then $\mathbb{P}_i(X > x) \sim C_i \mathrm{e}^{-\gamma x}$ as $x \to \infty$, with $C_i = C_i(\ell)$ specified in Corollary 4.1 below.

The assumption that γ exists is automatic is essentially a condition on light tails of the T_d, for example that each T_d is gamma-like. By this we mean that the density exists and has asymptotic form

$$
bt^{a-1}\mathrm{e}^{-\delta y} \tag{2.1}
$$

We discuss this in more detail in Section 4. The case of heavy tails of some T_d is not included by Theorem 2.1 but can be treated as well. For example, we have:

Theorem 2.2. *Consider a deterministic task length $L = \ell$ and assume that each T_d has a subexponential density, more precisely that there exists a subexponential density $\varphi(y)$ such that for all d $\mathbb{P}(T_d \in \cdot)$ is absolutely continuous for large y with a density of asymptotic form $c_d \varphi(y)$. Then $\mathbb{P}_i(X > x) \sim C_i \overline{\Phi}(x)$ as $x \to \infty$, with $C_i = C_i(\ell)$ specified in (6.4) below and $\Phi(x) = \int_x^\infty \varphi(y)\,\mathrm{d}y$.*

[See (6.2) below for the definition of a subexponential density. The constants C_i are obviously not the same as in Theorem 2.1; similar slight abuse of notation occurs throughout the paper].

For a random task length L, conditioning on $L = \ell$ gives

$$
\mathbb{P}_i(X > x) = \int_0^\infty \mathbb{P}_i(X > x \,|\, L = \ell)\, \mathbb{P}(L \in \mathrm{d}\ell)\,. \tag{2.2}
$$

Since now ℓ is a variable (and not constant, as previously assumed), we write $\gamma(\ell)$ rather than ℓ, $\boldsymbol{R}(\alpha,\ell)$ rather than $\boldsymbol{R}(\alpha)$ to stress the dependence on both variables. In (2.2), $\mathbb{P}_i(X > x \,|\, L = \ell)$ has an asymptotic exponential form as $x \to \infty$ by Theorem 2.1. Further, the rate $\gamma(\ell)$ goes to 0 as $\ell \to \infty$ since $\boldsymbol{R}(0,\infty)$ equals the spectral radius one mtarix $\boldsymbol{P}$. We thus obtain the following easy but notable result:

Corollary 2.3. *If the task length L has unbounded support, the distribution of the total task time X is heavy-tailed in the sense that $\mathrm{e}^{\delta x}\mathbb{P}(X > x) \to \infty$ for all $\delta > 0$.*

The key step for obtaining more precise results turns out to be to identify the asymptotics of $\gamma(\ell)$ as $\ell \to \infty$. Our result is:

Theorem 2.4. *Assume that the assumptions of Theorem 2.1 on $\boldsymbol{R}(\alpha,\ell)$ hold for all large ℓ, and that for some function $\varphi(\ell)$ it holds that*

$$k^*_{ud}(\ell) = \mathbb{P}\big(T_u > \tau_u(\ell),\, \xi_1 = d\big) \sim k_{ud}\varphi(\ell) \tag{2.3}$$

as $\ell \to \infty$ for some set of constants such that $k_{ud} > 0$ for at least one pair $u \in \mathcal{U}$, $d \in \mathcal{D}$. Then

$$\gamma(\ell) \sim \mu\,\varphi(\ell) \quad \textit{as } \ell \to \infty, \qquad \textit{where} \quad \mu = \frac{\sum_{u\in\mathcal{U},d\in\mathcal{D}} k_{ud}}{\sum_{i\in\mathcal{E}} \pi_i \mathbb{E}_i T_i} \tag{2.4}$$

and $\boldsymbol{\pi} = (\pi_i)_{i\in\mathcal{E}} = (\pi_i)_{i\in\mathcal{U}\cup\mathcal{D}}$ is the stationary distribution of the Markov chain ξ, that is, the invariant probability vector for the matrix $\boldsymbol{P} = \boldsymbol{R}(0,\infty)$.

Remark 2.5. In practice, $\varphi(\ell)$ will be found by first determining the growth rate of the $h^*_{ud}(\ell) = \mathbb{P}\big(T_u > \tau_u(\ell),\, \xi_1 = d\big)$ and taking φ as the largest one. $\Diamond$

Given Theorem 2.4, it is easy to adapt the calculations of [2] to get the tail of X, in more or less sharp forms depending on the form of the distribution of L. For example:

Corollary 2.6. *Assume that L is Gamma-like as in (2.1), that $\mathbb{P}_u(\tau_u(\ell) \to \infty) = 1$ for all u and that the assumptions of Theorem 2.4 hold with $\varphi(\ell)t = \mathrm{e}^{-q\ell}$ for some q. Then*

$$\mathbb{P}_i(X > x) \sim C\frac{\log^{a-1} x}{x^{\delta/q}}$$

with C given in (5.7) below.

The paper is organised as follows. Section 3 presents some main examples and a general class of models is described where the Markov renewal model is derived by lumping in a continuous-time finite Markov process with exponential holding times. This case provided our main motivation, but the Markov renewal conversion is necessary because of non-Markovian features of the Restart mechanism. Section 4 contains the proof of Theorem 2.1, where the main ingredient is a suitable set of Markov renewal equations. Also the form of the knowns and unknowns are specialized to the

Markov model. Section 5 contains the proof of Theorem 2.4. Perron-Frobenius theory for nonnegative matrices plays a main role (see [1, I.6] for a short introduction and [6], [17] for more extensive treatments). However, the proof is by bare-hand arguments rather than general perturbation theory. The proof of Theorem 2.2 is given in Section 6; it involves discussion of Markov renewal equations with heavy tails, a less established topic than the light tailed case. Non-constant processing rates $\rho_u(t)$ are studied in Section 7 and leads into matrix formalism and differential equations somewhat similar to the theory of fluid reward models and multivariate phase-type distributions. Finally, Section 8 sketches via an example how non-exponential distributions may be incorporated in the Markov model, and some preliminaries and technical steps are deferred to the Appendix.

The paper only contains one numerical example (a more extensive set will be presented elsewhere). Nevertheless, it should be stressed that our aim is computational. In particular, we have chosen a finite state set-up leading to explicit and numerically implementable matrix formulas rather than a general state one where one would need to impose many not easily verifiable technical conditions, operators would take the role of matrices, etc.

3 From Markov to semi-Markov models

In many examples, the Markov renewal structure may be derived from embedding into a larger Markov process $J(t)$ with state space $\mathcal{E}^* = \mathcal{U}^* \cup \mathcal{D}^*$. In a state $u \in \mathcal{U}^*$, the task is processed at rate r_u and no processing can take place in a state $d \in \mathcal{D}^*$. We then take $\mathcal{U} \subseteq \mathcal{U}^*$ as the subset of states that can be entered from $\mathcal{D}$ and $\mathcal{D} \subseteq \mathcal{D}^*$ as the subset of states that can be entered from $\mathcal{U}$. If $\xi_0 = J(0) = u \in \mathcal{U}$, we further let $T_u = \inf\{T > 0 : J(t) \in \mathcal{D}^*\}$ (and similarly for T_d) and $\rho_u(t) = r_{J(t)}$.

It should be noted that in this Markovian scheme, the F_u, F_d become phase-type (PH) and the F_{ud}, F_{du} defective PH, so that standard matrix-analytic formulas apply to rewrite the expressions in the general set-up in terms of matrix expressions (matrix inverses and matrix-exponentials). More detail is given later.

The Markovian imbedding scheme is illustrated in the following examples, where the states in $\mathcal{U}$ are dark green, the ones in $\mathcal{U}^* \setminus \mathcal{U}$ light green, and similar dark and light red coloring is used for the states in $\mathcal{D}$ and $\mathcal{D}^* \setminus \mathcal{D}$

Example 3.1. Consider a system with two exponential parallel servers, hot backup and two repairmen. The failure rate of a server is β, the repair rate is λ, and a task can be (re)started if at least one server is up. We can take $E^* = \{2, 1, 0\}$, with $J(t) = i$ meaning that i servers are up (and thus $2 - i$ down, i.e.under repair).

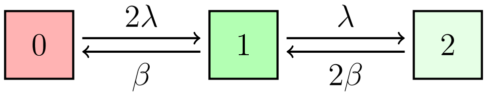

Figure 1: Parallel servers.

We have $\mathcal{U}^* = \{1, 2\}$, $\mathcal{U} = \{1\}$, $\mathcal{D}^* = \mathcal{D} = \{0\}$; note that $2 \notin \mathcal{U}$ since that even if the first service may be started in state 2, none of the following ones will be so because a failed period (a sojourn in state 0) cannot be terminated by a jump to 2. The rates are $r_1 = 1$, $r_2 = 2$ if both servers can work on the task and $r_1 = r_2 = 1$ if only one can (hot back-up).

With only one repairman, 2λ should be changed to λ, and with cold back-up, 2β should be changed to β. Everything else remains unchanged. ◊

Example 3.2. Consider the system in Example 3.1 with the modification that 2 servers have to be up before a service can start. We then take $E^* = \{2, 1+, 1-, 0\}$, with $0, 2$ having the same meaning as before, and $1-$ meaning that there is one repaired server and one under repair, $1+$ that there is one operating and one under repair. Assuming that a server that is repaired but waiting for the other to be repaired before going in to operation cannot fail, we have the transition diagram in Fig. 2.

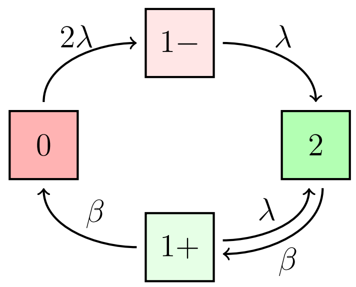

Figure 2: Start requires both servers up.

We have $\mathcal{U}^* = \{2, 1+\}$, $\mathcal{U} = \{2\}$, $\mathcal{D}^* = \{0, 1-\}$, $\mathcal{D} = \{0\}$. Again depending on the interpretation, we may take $r_{1+} = r_2 = 1$ or $r_{1+} = 1$, $r_2 = 2$. ◊

Example 3.3. Consider again the system in Example 3.1, this time modified such that the two servers a, b are heterogeneous, i.e. with failure rates β_a, β_b and repair rates λ_a, λ_b. We take $E^* = \{0, 1a, 1b, 2\}$, with 0,2 having the same meaning as before and $1a$ meaning that server a is up and server b down, and vice versa for $1b$. Cf. Figure 3(A) for the case of homogeneous rates. We have $\mathcal{U}^* = \{2, 1a, 1b\}$, $\mathcal{U} = \{1a, 1b\}$, $\mathcal{D}^* = \mathcal{D} = \{0\}$.

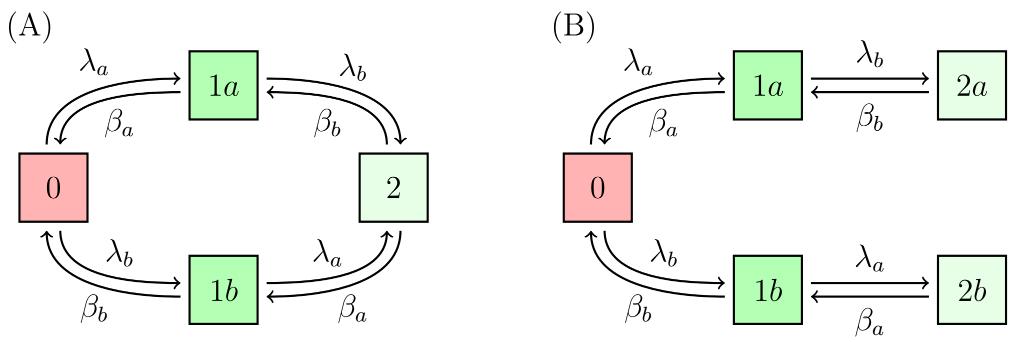

Figure 3: Heterogeneous parallel servers. (A) homogeneous service rates, (B) heterogeneous service rates.

Non-preemptive service and heterogeneous service rates $r_a \neq r_b$ for the servers can be handled by splitting state 2 into two states $2a$, $2b$, cf. Figure 3(B). Then $r_{2a} = r_{1a} = r_a$, $r_{2b} = r_{1b} = r_b$. $\Diamond$

4 Theorem 2.1: proof and amendments

Proof of Theorem 2.1. For $i \in \mathcal{E}$, let $Z_i(x) = \mathbb{P}_i(X > x)$. If $i = d \in \mathcal{D}$, then service starts at the time T_d of exit from d, so that the whole of T_d contributes to the total task time X and for $X > x$ to occur, only delay $(x - T_d)^+$ needs to be accumulated after T_d. Conditioning on $y = T_d$, considering the cases $y > x$ and $y \leq x$ separately, and partitioning according to the possible values $u = \xi_1 \in \mathcal{U}$ gives

$$Z_d(x) = z_d(x) + \sum_{u \in \mathcal{U}} \int_0^x Z_u(x-y)\, \widetilde{F}_{du}(\mathrm{d}y)$$

where $z_d(x) = \mathbb{P}_d(T_d > x)$ and $\widetilde{F}_{du} = F_{du}$. If $i = u \in \mathcal{U}$, then the task will be terminated before $y = T_u$ if $y > \tau_u(\ell)$ so that $X > x$ if and only if $\tau_u(\ell) > x$. This gives

$$Z_u(x) = z_u(x) + \sum_{d \in \mathcal{D}} \int_0^x Z_d(x-y)\, \mathbb{P}\big(T_u \in \mathrm{d}y,\, \tau_u(\ell) > y,\, \xi_1 = d\big)$$

where $z_u(x) = \mathbb{P}\big(T_u > \tau_u(\ell) > x\big)$. Recalling that transitions within $\mathcal{U}$ or within $\mathcal{D}$ need not be taken into account and letting

$$\widetilde{F}_{ud}(t) = \mathbb{P}\big(T_u \leq t,\, \tau_u(\ell) > T_u,\, \xi_1 = d\big)\,,$$

this can be summarised as the set

$$Z_i(x) = z_i(x) + \sum_{j \in \mathcal{E}} \int_0^x Z_j(x-y)\, \widetilde{F}_{ij}(\mathrm{d}y)\,,\ i \in \mathcal{E}, \tag{4.1}$$

of Markov renewal equations where $\widetilde{F}_{ij} \equiv 0$ if ij are both in $\mathcal{U}$ or both in $\mathcal{D}$. That the $Z_i(x)$ decay exponentially at rate γ then then follows immediately from assumption on $\boldsymbol{R}(\alpha)$ via the generalisation of the key renewal theorem stated in Lemma A.1. In more detail, that result also gives the form of the C_i which we next state separately. $\square$

Corollary 4.1. *Let $\boldsymbol{\nu} = (\nu_i)_{i \in \mathcal{E}}$, $\boldsymbol{h} = (h_i)_{i \in \mathcal{E}}$ be the left, resp. right, eigenvectors of $\boldsymbol{R}(\gamma)$ corresponding to the eigenvalue γ. Then $C_i = h_i C_1 / C_2$ where*

$$C_1 = \frac{1}{\gamma} \sum_{u \in \mathcal{U}} \nu_u \int_0^\infty \mathbb{E}\big[(\mathrm{e}^{\gamma \tau_u(\ell)} - 1)\mathbf{1}\{\tau_u(\ell) \leq y\} \,\big|\, T_u = y\big]\, F_u(\mathrm{d}y) \tag{4.2}$$

$$+ \frac{1}{\gamma} \sum_{d \in \mathcal{D}} \nu_d \int_0^\infty [\mathrm{e}^{\gamma y} - 1]\, F_d(\mathrm{d}y) \tag{4.3}$$

$$C_2 = \sum_{u \in \mathcal{U}, d \in \mathcal{D}} \nu_u h_d \int_0^\infty y \mathrm{e}^{\gamma y}\, \mathbb{P}\big(\tau_u(\ell) \geq y \,\big|\, T_u = y\big)\, F_{ud}(\mathrm{d}y) \tag{4.4}$$

$$+ \sum_{d \in \mathcal{D}, u \in \mathcal{U}} \nu_h h_u \int_0^\infty y \mathrm{e}^{\gamma y}\, F_{du}(\mathrm{d}y) \tag{4.5}$$

Proof. For $d \in \mathcal{D}$, integration by parts gives that

$$\gamma \int_0^\infty \mathrm{e}^{\gamma x} z_d(x)\,\mathrm{d}x = \gamma \int_0^\infty \mathrm{e}^{\gamma x} \mathbb{P}(T_d > x)\,\mathrm{d}x$$

reduces to the integral in (4.3). For $u \in \mathcal{U}$,

$$\begin{aligned}
\gamma \int_0^\infty \mathrm{e}^{\gamma x} z_u(x)\,\mathrm{d}x &= \gamma \int_0^\infty \mathrm{e}^{\gamma x} \mathbb{P}(T_u \geq \tau_u(\ell) > x)\,\mathrm{d}x \\
&= \int_0^\infty H_u(\mathrm{d}y) \int_0^\infty \gamma \mathrm{e}^{\gamma x} \mathbb{P}\big(\tau_u(\ell) \in (x, y] \,\big|\, T_u = y\big)\,\mathrm{d}x \\
&= \int_0^\infty H_u(\mathrm{d}y) \int_0^\infty \gamma \mathrm{e}^{\gamma x} \mathbb{P}\big(\tau_u(\ell) \mathbf{1}\{\tau_u(\ell) \leq y\} > x \,\big|\, T_u = y\big)\,\mathrm{d}x \\
&= \int_0^\infty H_u(\mathrm{d}y) \mathbb{E}\big[\mathrm{e}^{\gamma \tau_u(\ell) \mathbf{1}\{\tau_u(\ell) \leq y\}} - 1 \,\big|\, T_u = y\big].
\end{aligned}$$

Considering the cases $\tau_u(\ell) \leq y$ and $\tau_u(\ell) > y$ separately gives the integral in (4.2). Inserting in (A1.1) gives the expression for C_1. The one for C_2 follows from (A1.2) by similar manipulations. □

Remark 4.2. In the simplest case $\rho_u(t) \equiv 1$, the integral in (4.2) reduces to

$$(\mathrm{e}^{\gamma \ell} - 1) \int_\ell^\infty F_u(\mathrm{d}y) = (\mathrm{e}^{\gamma \ell} - 1) \mathbb{P}(T_u > \ell)$$

and the one in (4.4) to

$$\int_0^\ell y \mathrm{e}^{\gamma y} F_{du}(\mathrm{d}y) = \mathbb{E}\big[T_u \mathrm{e}^{\gamma T_u};\, T_u \leq \ell, \xi_1 = d\big]\,. \qquad \lozenge$$

Remark 4.3. The expressions for γ are implicit even for simple RESTART ([2]), so in the present generality, numerical evaluation seems inevitable. This has two steps. The first is computing the elements of $\boldsymbol{R}(\alpha)$. How difficult this is depends on the specific model parameters (but see Section 7 below). The next step is then evaluating eigenvalues of $\boldsymbol{R}(\alpha)$ and finding the roots of the equation $1 = \mathrm{spr}(\boldsymbol{R}(\alpha))$ which can be done using standard software.

The dimension of the matrices can be reduced from $\mathcal{E} \times \mathcal{E}$ to $\mathcal{U} \times \mathcal{U}$ or $\mathcal{D} \times \mathcal{D}$ by noting that $[\mathrm{spr}(\boldsymbol{R}(\alpha))]^2 = \mathrm{spr}(\boldsymbol{R}(\alpha)^2)$, and hence the equation determining γ is (in obvious block notation)

$$1 = \mathrm{spr}\big(\boldsymbol{R}_{\mathcal{U}\mathcal{D}}(\gamma) \boldsymbol{R}_{\mathcal{D}\mathcal{U}}(\gamma)\big) = \mathrm{spr}\big(\boldsymbol{R}_{\mathcal{D}\mathcal{U}}(\gamma) \boldsymbol{R}_{\mathcal{U}\mathcal{D}}(\gamma)\big)\,. \qquad \lozenge$$

We proceed to discuss Theorem 2.1 in the setting of the Markov model of Section 3. In the following, we will use the partitioning

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*} & \boldsymbol{Q}_{\mathcal{U}^*\mathcal{D}^*} \\ \boldsymbol{Q}_{\mathcal{D}^*\mathcal{U}^*} & \boldsymbol{Q}_{\mathcal{D}^*\mathcal{D}^*} \end{pmatrix}$$

of the generator $\boldsymbol{Q}$ of the Markov process J on $\mathcal{E}^* = \mathcal{U}^* \cup \mathcal{D}^*$. Recall the definition of a phase-type (PH) distribution and its parameters commonly denoted as rate matrix, initial vector and exit vector given in, e.g., [1, III.4]. Let further $\boldsymbol{e}_u$ denote the $\mathcal{U}^*$-column vector with 1 at entry $u \in \mathcal{U}^*$ and 0 otherwise (and similarly for $\boldsymbol{e}_d$), and $\boldsymbol{e}$ the column vector with 1 at all entries (of dimension $\mathcal{U}^*$ or $\mathcal{D}^*$ depending on the context).

Proposition 4.4. *In the Markov model,*

- F_{ud} *is defective PH with rate matrix* $\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*}$*, initial vector* $\boldsymbol{e}_u^\mathsf{T}$ *and exit vector* $\boldsymbol{Q}_{\mathcal{U}^*\mathcal{D}^*}\boldsymbol{e}_d$*;*
- F_{du} *is defective PH with rate matrix* $\boldsymbol{Q}_{\mathcal{D}^*\mathcal{D}^*}$*, initial vector* $\boldsymbol{e}_d^\mathsf{T}$ *and exit vector* $\boldsymbol{Q}_{\mathcal{D}^*\mathcal{U}^*}\boldsymbol{e}_u$*;*
- F_u *is PH with rate matrix* $\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*}$*, initial vector* $\boldsymbol{e}_u^\mathsf{T}$ *and exit vector* $\boldsymbol{Q}_{\mathcal{U}^*\mathcal{D}^*}\boldsymbol{e}$*;*
- F_d *is PH with rate matrix* $\boldsymbol{Q}_{\mathcal{D}^*\mathcal{D}^*}$*, initial vector* $\boldsymbol{e}_d^\mathsf{T}$ *and exit vector* $\boldsymbol{Q}_{\mathcal{D}^*\mathcal{U}^*}\boldsymbol{e}$*.*

Remark 4.5. The implication of Proposition 4.4 is that standard matrix-analytic machinery becomes available to rewrite many expressions considered so far in terms of matrices (facilitating computer implementation). For example,

$$z_d(x) = \mathbb{P}(T_d > x) = \boldsymbol{e}_d^\mathsf{T}\mathrm{e}^{\boldsymbol{Q}_{\mathcal{D}^*\mathcal{D}^*}x}\boldsymbol{e}\,, \quad r_{du}(\alpha) = \boldsymbol{e}_d^\mathsf{T}\big(-\alpha\boldsymbol{I} - \boldsymbol{Q}_{\mathcal{D}^*\mathcal{D}^*}\big)^{-1}\boldsymbol{Q}_{\mathcal{D}^*\mathcal{U}^*}\boldsymbol{e}_u\,.$$

Similarly, with constant processing rate $\rho_u(t) \equiv 1$, we have $\tau_u(\ell) = \ell$ and so

$$\begin{aligned} r_{ud}(\alpha) = \mathbb{E}\big[\mathrm{e}^{\alpha T_u};\, T_u < \ell,\, \xi_1 = d\big] &= \int_0^\ell \mathrm{e}^{\alpha y}\boldsymbol{e}_u^\mathsf{T}\mathrm{e}^{\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*}y}\boldsymbol{Q}_{\mathcal{U}^*\mathcal{D}^*}\boldsymbol{e}_d\,\mathrm{d}y \\ &= \boldsymbol{e}_u^\mathsf{T}\big(-\alpha\boldsymbol{I} - \boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*}\big)^{-1}\big(\mathrm{e}^{\alpha\ell}\mathrm{e}^{\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*}\ell} - \boldsymbol{I}\big)\boldsymbol{Q}_{\mathcal{D}^*\mathcal{U}^*}\boldsymbol{e}_u \end{aligned}$$

We return in Section 7 to the question of how to evaluate such quantities in more general settings than $\rho_u(t) \equiv 1$. ♢

We also have

Proposition 4.6. *In the Markov model* γ *always exists and is unique.*

Proof. From general results on phase-type distributions based on Perron-Frobenius theory, the tail of F_{du} has asymptotic form $c_{du}x^{n_{ud}}\mathrm{e}^{-\alpha^* x}$ for suitable constants α^*, n_{ud}, c_{du} (if $\boldsymbol{Q}_{\mathcal{D}\mathcal{D}}$ is irreducible, $-\alpha^*$ is the eigenvalue of maximal real part and all $n_{ud} = 0$). Thus the $r_{du}(\alpha)$ are defined for $\alpha < \alpha^*$ and some have limit ∞ as $\alpha \uparrow \alpha^*$. Further, letting $\rho^* = \max_{u\in\mathcal{U}^*}\rho_u$, $r_{ud}(\alpha)$ can be bounded by $\mathbb{E}[\mathrm{e}^{\alpha T_u};\, T_u \le \ell/\rho^*]$ which is finite for all α. Thus $\mathrm{spr}\big(\boldsymbol{R}(\alpha)\big)$ is defined for all $\alpha \in [0, \alpha^*)$ and by dominated convergence, it is continuous in that domain. The limit at $\alpha = 0$ is < 0 because $\boldsymbol{R}(0)$ is a proper subtransition matrix (some row sums are < 1 because of the condition $T_u < \tau_u(\ell)$ imposed), and the limit as $\alpha \uparrow \alpha^*$ is ∞ by Lemma A.2. Also that lemma gives that $\mathrm{spr}\big(\boldsymbol{R}(\alpha)\big)$ is strictly increasing in α. Putting these facts together completes the proof. □

5 Theorem 2.4: proof and amendments

Let $\boldsymbol{h}(\ell)$ be the right Perron-Frobenus eigenvector of $\boldsymbol{R}\big(\gamma(\ell), \ell\big)$ corresponding to the eigenvalue 1 and normalized by $\boldsymbol{\pi}\boldsymbol{h}(\ell) = 1$. Since $\gamma(\ell) \to 0$ and hence $\boldsymbol{R}\big(\gamma(\ell), \ell\big) \to \boldsymbol{P}$ as $\ell \to \infty$, one expects the following lemma to hold in view of $\boldsymbol{P}\boldsymbol{e} = \boldsymbol{e}$, $\boldsymbol{\pi}\boldsymbol{e} = 1$; we include the proof since it is short.

Lemma 5.1. $\boldsymbol{h}(\ell) \to \boldsymbol{e}$ *as* $\ell \to \infty$.

Proof. Assume that $\boldsymbol{h}(\ell_n) \to \boldsymbol{e}$ fails for some sequence $\{\ell_n\}$. The assumption $\boldsymbol{\pi}\boldsymbol{h}(\ell) = 1$ and $\pi_1 > 0$ for all i ensures that the sequence $\{\boldsymbol{h}(\ell_n)\}$ is relatively compact, so if passing to a subsequence if necessary we may assume that $\boldsymbol{h}(\ell_n) \to \boldsymbol{e}+\boldsymbol{f}$ with $\boldsymbol{f} \neq 0$. From $\boldsymbol{R}\bigl(\gamma(\ell),\ell\bigr)\boldsymbol{h}(\ell_n) = \boldsymbol{h}(\ell_n)$ and $\boldsymbol{R}\bigl(\gamma(\ell),\ell\bigr) \to \boldsymbol{P}$ we then get $\boldsymbol{P}(\boldsymbol{e}+\boldsymbol{f}) = \boldsymbol{e}+\boldsymbol{f}$. Since $\boldsymbol{P}$ is a transition matrix, we have $\boldsymbol{P}\boldsymbol{e} = \boldsymbol{e}$ and so $\boldsymbol{f}$ is an eigenvector of $\boldsymbol{P}$ corresponding to the eigenvalue 1. Thus $\boldsymbol{f} = c\boldsymbol{e}$ for some c by the Perron-Frobenius theorem, and

$$1 = \boldsymbol{\pi}\boldsymbol{h}(\ell_n) \to \boldsymbol{\pi}(\boldsymbol{e}+\boldsymbol{f}) = 1 + c\boldsymbol{\pi}\boldsymbol{e} = 1 + c$$

then gives $c = 0$, a contradiction. □

Proof of Theorem 2.4. Let $\boldsymbol{K}^*(\ell)$ denote the $\mathcal{E}\times\mathcal{E}$ matrix with udth element $k^*_{ud}(\ell)$ and the $\mathcal{U}\times\mathcal{U}$-, $\mathcal{D}\times\mathcal{U}$- and $\mathcal{D}\times\mathcal{D}$ blocks identically 0, Let further $\boldsymbol{A}(\alpha,\ell)$ be the matrix with elements

$$a_{ud}(\alpha,\ell) = \int_0^\ell (\mathrm{e}^{\alpha y} - 1 - \alpha y)\, H_{ud}(\mathrm{d}y)\,, \quad a_{du}(\alpha,\ell) = \int_0^\infty (\mathrm{e}^{\alpha y} - 1 - \alpha y)\, H_{ud}(\mathrm{d}y)$$

for $u \in \mathcal{U}, d \in \mathcal{D}$ and all other $a_{ij}(\alpha,\ell) = 0$, let $\boldsymbol{M}^*(\ell)$ be the matrix with elements

$$m^*_{ud}(\ell) = \int_0^\ell H_{ud}(\mathrm{d}y)\,, \quad m^*_{du}(\ell) = \int_0^\infty y\, H_{ud}(\mathrm{d}y)$$

for $u \in \mathcal{U}, d \in \mathcal{D}$ and all other $m_{ij}(\alpha,\ell;1) = 0$, and let $\boldsymbol{M} = \lim_{\ell\to\infty}\boldsymbol{M}(\ell)$, i.e. $\boldsymbol{M}$ is the matrix with elements

$$m_{ud} = \int_0^\infty y\, H_{ud}(\mathrm{d}y) = \mathbb{E}[T_u;\, \xi_1 = d]\,, \quad m_{du} = \int_0^\infty y\, H_{ud}(\mathrm{d}y) = \mathbb{E}[T_d;\, \xi_1 = u]$$

and all other $m_{ij} = 0$. We then have the identity

$$\boldsymbol{A}(\alpha,\ell) = \boldsymbol{R}(\alpha,\ell) - \boldsymbol{P} + \boldsymbol{K}^*(\ell) - \alpha\boldsymbol{M}^*(\ell)\,. \tag{5.1}$$

Write for convenience $\boldsymbol{h}(\ell) = \boldsymbol{e} + \gamma(\ell)\boldsymbol{n}(\ell)$. Taking $\alpha = \gamma(\ell)$ in (5.1) and multipying by $\boldsymbol{h}(\ell)$ to the right, we obtain

$$\begin{aligned}\mathrm{O}\bigl(\gamma(\ell)^2\bigr) &= \boldsymbol{e} + \gamma(\ell)\boldsymbol{n}(\ell) - \boldsymbol{P}\boldsymbol{e} - \gamma(\ell)\boldsymbol{P}\boldsymbol{n}(\ell)\\ &\quad + \boldsymbol{K}^*(\ell)\boldsymbol{e} + \gamma(\ell)\boldsymbol{K}^*(\ell)\boldsymbol{n}(\ell) - \gamma(\ell)\boldsymbol{M}^*(\ell)\boldsymbol{e} - \gamma(\ell)^2\boldsymbol{M}^*(\ell)\boldsymbol{n}(\ell)\end{aligned}$$

Noting that $\boldsymbol{P}\boldsymbol{e} = \boldsymbol{e}$ and $\boldsymbol{\pi}\boldsymbol{P} = \boldsymbol{\pi}$, that $\gamma(\ell)\boldsymbol{n}(\ell) \to \boldsymbol{0}$ by Lemma 5.1 and that $\boldsymbol{\pi}\boldsymbol{n}(\ell) = \boldsymbol{0}$ because of $\boldsymbol{\pi}\boldsymbol{h}(\ell) = 1$, it follows by multiplication by $\boldsymbol{\pi}$ to the left that

$$\begin{aligned}\mathrm{o}\bigl(\gamma(\ell)\bigr) &= \boldsymbol{\pi}\boldsymbol{K}^*(\ell)\boldsymbol{e} + \gamma(\ell)\boldsymbol{\pi}\boldsymbol{K}^*(\ell)\boldsymbol{e} - \gamma(\ell)\boldsymbol{\pi}\boldsymbol{M}^*(\ell)\boldsymbol{e}\,,\\ \mathrm{o}(1) &= \frac{1}{\gamma(\ell)}\boldsymbol{\pi}\boldsymbol{K}^*(\ell)\boldsymbol{e} + \boldsymbol{\pi}\boldsymbol{K}^*(\ell)\boldsymbol{n}(\ell) - \boldsymbol{\pi}\boldsymbol{M}^*(\ell)\boldsymbol{e}\end{aligned}$$

Since $\boldsymbol{\pi}\boldsymbol{M}^*(\ell)\boldsymbol{e} \to \boldsymbol{\pi}\boldsymbol{M}\boldsymbol{e}$ and $\boldsymbol{K}^*(\ell) = \mathrm{o}(1)$ by (2.3) , this gives

$$\boldsymbol{\pi}\boldsymbol{M}\boldsymbol{e} = \lim_{\ell\to\infty}\Bigl[\frac{\varphi(\ell)}{\gamma(\ell)}\frac{\boldsymbol{\pi}\boldsymbol{K}^*(\ell)\boldsymbol{e}}{\varphi(\ell)} + \mathrm{o}(1)\Bigr] = \lim_{\ell\to\infty}\frac{\varphi(\ell)}{\gamma(\ell)}\boldsymbol{\pi}\boldsymbol{K}\boldsymbol{e}$$

where $\boldsymbol{K} = \lim \boldsymbol{K}^*(\ell)/\varphi(\ell)$ is given by (2.3). Hence

$$\lim_{\ell\to\infty}\frac{\gamma(\ell)}{\varphi(\ell)} = \frac{\boldsymbol{\pi}\boldsymbol{K}(\ell)\boldsymbol{e}}{\boldsymbol{\pi}\boldsymbol{M}\boldsymbol{e}} = \mu\,.$$ □

Corollary 5.2. *Consider the Markov model with $\rho_u(t) \equiv 1$, assume that $\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*}$ is irreducible, let $-\delta$ be the eigenvalue with maximal real part and $\boldsymbol{h}_{\mathcal{U}^*}, \boldsymbol{\nu}_{\mathcal{U}^*}$ the corresponding right and left eigenvectors normalized by $\boldsymbol{\nu}_{\mathcal{U}^*}\boldsymbol{h}_{\mathcal{U}^*} = 1$. Then $\gamma(\ell) \sim \mu \mathrm{e}^{-\delta\ell}$ as $\ell \to \infty$, where*

$$\mu = \frac{(\boldsymbol{\pi}_{\mathcal{U}^*}\boldsymbol{h}_{\mathcal{U}^*}) \cdot (\boldsymbol{\nu}_{\mathcal{U}^*}\boldsymbol{e})}{\boldsymbol{\pi}_{\mathcal{U}^*}(-\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*})^{-1}\boldsymbol{e} + \boldsymbol{\pi}_{\mathcal{D}^*}(-\boldsymbol{Q}_{\mathcal{D}^*\mathcal{D}^*})^{-1}\boldsymbol{e}}$$

with $\boldsymbol{\pi}^* = (\boldsymbol{\pi}_{\mathcal{U}^*}\ \boldsymbol{\pi}_{\mathcal{D}^*})$ the stationary distribution of $\boldsymbol{Q}$.

Proof. By Perron-Frobenius theory, $-\delta$ is a simple eigenvalue and

$$\begin{aligned} k^*_{ud}(\ell) &= \mathbb{P}\big(T_u > \tau_u(\ell),\, \xi_1 = d\big) = \mathbb{P}(T_u > \ell,\, \xi_1 = d) \\ &= \boldsymbol{e}_u \exp\{\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*}\ell\}\boldsymbol{e}_d \sim \boldsymbol{e}_u(\boldsymbol{h}_{\mathcal{U}^*}\boldsymbol{\nu}_{\mathcal{U}^*})\boldsymbol{e}_d \cdot \mathrm{e}^{-\delta\ell} \end{aligned}$$

Thus we may take $\varphi(\ell) = \mathrm{e}^{-\delta\ell}$ and the expression for μ then easily comes out. □

Remark 5.3. The assumption of $\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*}$ being irreducible holds in all the examples we present. For an example where it fails, assume

$$\mathcal{U}^* = \{1,2\}\,, \quad \mathcal{D}^* = \{3\}\,, \quad \boldsymbol{Q} = \begin{pmatrix} -a & a & 0 \\ 0 & -a & a \\ b & c & -b-c \end{pmatrix}$$

with at least $a, b > 0$. Here the eigenvalue $-a$ of $\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*}$ is not simple and $\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*}$ is reducible (1 cannot be reached from 2). Nevertheless, the assumptions of Theorem 2.4 hold since $\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*}$ is the rate matrix of an Erlang$(2, a)$ distribution and so

$$\begin{aligned} \mathbb{P}_1(T_1 > \ell,\, \xi_1 = 3) &= \mathbb{P}_1(T_1 > \ell) = a\ell\mathrm{e}^{-a\ell} + \mathrm{e}^{-a\ell}\,, \\ \mathbb{P}_2(T_2 > \ell,\, \xi_1 = 3) &= \mathbb{P}_2(T_2 > \ell) = \mathrm{e}^{-a\ell}\,. \end{aligned}$$

I.e., we may take $\varphi(\ell) = \ell\mathrm{e}^{-a\ell}$ and get $k^*_{13} = a$, $k^*_{23} = 0$. In general, dealing with reducibility and eigenvalues that are not simple lead into the Jordan canonical form; we omit the details. ♢

Proof of Corollary 5.2. We use once more formula (2.2) for a random task length L, stating that

$$\mathbb{P}_i(X > x) = \int_0^\infty \mathbb{P}_i(X > x \,|\, L = \ell)\, \mathbb{P}(L \in \mathrm{d}\ell)\,. \tag{5.2}$$

It is shown in [2] that here it is permissible to insert the approximations in Theorem 2.4, 2.4, leading to

$$\mathbb{P}_i(X > x) \sim \int_{\ell_0}^\infty h_i C_1/C_2 \exp\{-\mu\mathrm{e}^{-q\ell}\}\, a\mathrm{e}^{-\delta\ell}\, \mathrm{d}\ell\,. \tag{5.3}$$

Here h_i, C_1, C_2 depend on ℓ. Now $h_i \to 1$ by Lemma 5.1 and similarly $\nu_i \to \pi_i$ as $\ell \to \infty$. It then follows from the assumption $\mathbb{P}_u(\tau(\ell) \to \infty) = 1$ that C_1, C_2 have limits

$$
\begin{aligned}
D_1 &= \sum_{u\in\mathcal{U}} \pi_u \cdot 1 \cdot 0 + \sum_{d\in\mathcal{D}} \pi_d \mathbb{E} T_d = \sum_{d\in\mathcal{D}} \pi_d \mathbb{E} T_d && (5.4)\\
D_2 &= \sum_{u\in\mathcal{U}, d\in\mathcal{D}} \pi_u \cdot 1 \int_0^\infty y \cdot 1 \cdot 1\, F_{ud}(\mathrm{d}y) + \sum_{d\in\mathcal{D}, u\in\mathcal{U}} \pi_d \cdot 1 \int_0^\infty y \cdot 1\, F_{du}(\mathrm{d}y) && (5.5)\\
&= \sum_{i\in\mathcal{E}} \pi_i \mathbb{E}_i T_i && (5.6)
\end{aligned}
$$

The asymptotics of the integral in (5.3) is determined in Lemma A.3, and the result follows with

$$
C = \frac{\Gamma(\delta/q) D_1}{\mu^{\delta/q} q^a D_2}\,. \tag{5.7}
$$

□

Remark 5.4. Given Theorem 2.1, the asymptotics of $\mathbb{P}(X > x)$ can easily be found in a variety of combinations of the forms of $\varphi(\ell)$ and the distribution of L. To this end, simply insert in the integral estimates of [2] (note that in many cases only logarithmic asymptotics comes out). We omit the lengthy statement of all of the available results, but remark that the case where $\varphi(\ell)$ and the distribution of L are somewhat alike is particularly nice. More precisely, with the assumption

$$
\mathbb{P}(L \in \mathrm{d}\ell) = \mu\varphi'(\ell)\big(\mu\varphi(\ell)\big)^{\beta-1} L_0\big(\varphi(\ell)\big)
$$

on the density of L where $\beta > 0$ and L_0 is slowly varying at 0, a Tauberian argument gives

$$
\mathbb{P}(X > x) \sim \frac{\Gamma(\beta) D_1}{\mu^\beta D_2} \frac{L_0(1/x)}{x^\beta}\,. \qquad \Diamond
$$

Example 5.5. The asymptotic parameter, γ, depends heavily on the type of system under examination. But it is still of interest to get some idea of how it behaves. A detailed study of this set of problems will be presented elsewhere, but we consider here the simplest system, a single server (ON state) that has an exponential failure rate of β, and a single repairman (DOWN state) with exponential repair rate, λ.

In this case, the equation $\mathrm{spr}\big(\boldsymbol{R}(\alpha)\big) = 1$ reduces to the root finding formula,

$$
f(\alpha = \gamma) = \frac{\lambda}{(\lambda - \alpha)} \frac{\beta[1 - \mathrm{e}^{-\ell(\beta-\alpha)}]}{(\beta - \alpha)} - 1 = 0.
$$

From Figure 4 we see that γ depends heavily on both ℓ and λ. The property that $\gamma(0|\lambda, \beta) = \lambda$ is an artifact of the fact that the repair time distribution is exponential(λ), and for $\delta = \ell\beta \ll 1$ the probability that the system will fail one more time is of order δ. One expects in general that the behavior of γ near $\ell = 0$ should be dominated by the asymptotic failure distribution, namely $\pi_d \exp(\boldsymbol{Q}_{\mathcal{D}\mathcal{D}}\, t)\, e$. For large ℓ, the exponential decay of γ given by Theorem 2.4 is confirmed by the figure at least for the simple system examples given. ◊

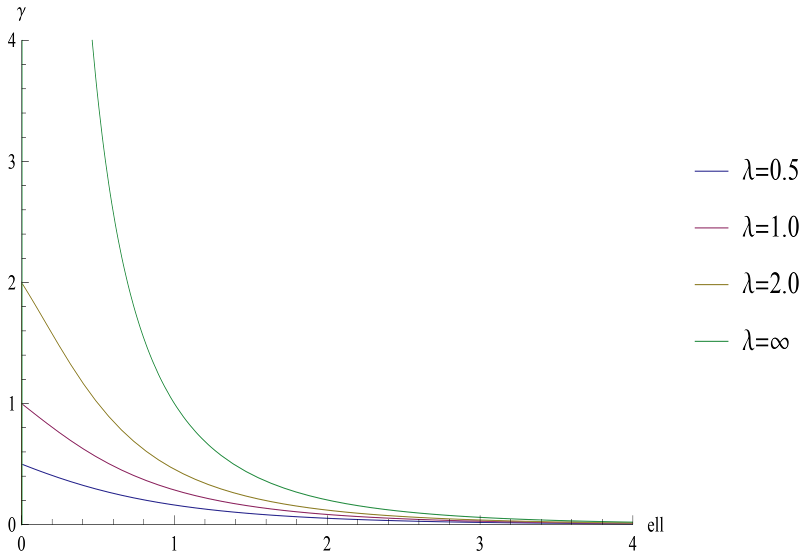

Figure 4: $\gamma(\ell \,|\, \lambda, \beta)$ as a function of ℓ for $\lambda \in \{0.5,\, 1.0,\, 2.0,\, \infty\}$ and $\beta = 1$, for the simplest system of one UP and one DOWN state. For this system, $\gamma(0|\lambda,\, \beta) = \lambda$.

6 Theorem 2.2: proof and amendments

For Theorem 2.2, one needs a Markov renewal version of the key renewal theorem with defective heavy tails. This appears, however, not to be in the literature. One may note in this connection that already for an ordinary renewal equation

$$Z(x) = z(x) + \int_0^\infty Z(x-y)\, F(\mathrm{d}y) \tag{6.1}$$

(with only one unknown function Z), such a result is relatively new and was only given fairly recently in Asmussen, Foss & Korshunov [3]. The details of this analysis are quite technical and one needs to go somewhat beyond the ordinary subexponential setting by imposing conditions not only on the tail of F but also on the local behavior. This involves the definition of a distribution F to have a subexponential density, namely that there exists $\widehat{x} < \infty$ such that F has a density $f(x)$ on $[\widehat{x} < \infty)$ and that

$$f^{*2}(x) = 2\int_0^{\widehat{x}} f(x-y)\, F(\mathrm{d}y) + \int_{\widehat{x}}^{x-\widehat{x}} f(x-y) f(y)\, \mathrm{d}y \sim 2f(x)\,, \quad x \to \infty. \tag{6.2}$$

A complete and rigorous treatment of the relevant version of the Markov key renewal theorem will be presented elsewhere, with Lemma 6.1 stated below without proof being the main extension of [3], but we present here the basic intuition. If $\theta = \int_0^\infty F(\mathrm{d}x) < 1$ in the simple case of (6.1), the idea in [3] is to use the convolution structure of the renewal equation to view $Z(x)$ as the density of the sum of a geometric sum of r.v.'s with distribution F/θ and a r.v W with density $z(x)/\int z$. Three cases arise according to the balance between the heaviness of z and the tail of F. The one corresponding to the present case is $z(x)$ being heavier than the density f of F, and here the contribution from the geometric sum vanishes, giving $Z(x) \sim z(x)/\int z$.

In the setting of Theorem 2.2, the representation in [1], Prop. 4.4 p. 209, of the solution $\big(Z_i(x)\big)_{i\in\mathcal{E}}$ to the set of Markov renewal equations (4.1) can be written as

$$Z_i(x) = \sum_{n=0}^{\infty}\sum_{j\in\mathcal{E}}\int_0^x z_j(x-y)\mathbb{P}_i\big(\xi_n(\ell)=j,\, S_n(\ell)\in \mathrm{d}y\big) \tag{6.3}$$

with the following notational conventions: N is defined as the Markov renewal epoch at which the task is processed, Δ is some extra absorbing state Δ, and $\xi(\ell)$ is a Markov chain with state space $\mathcal{E}\cup\{\Delta\}$ such that $\xi_n(\ell)=\xi_n$ for $n<N$ and $\xi_n(\ell)=\Delta$ for $n\ge N$. Further, $S_n(\ell)=T_0^*+\cdots+T_n^*$ for $n<N$, $S_n(\ell)=\infty$ for $n\ge N$ where given the Markov chain $\xi(\ell)$, the T_k^* are independent with T_k^* distributed as T_i on the event $\xi_k(\ell)=i$ (the definition of $S_n(\ell)$ for $n\ge N$ is redundant but one may take, e.g., $S_n(\ell)=\infty$. We need the following extension of Propositions 7, 8 of [3]:

Lemma 6.1. *Let $G_1,\ldots,G_A$ be a finite set of distributions such that each G_a admits a subexponential density $g_a(x)$ with $g_a(x)\sim c_a\varphi(x)$ for some subexponential density $\varphi(x)$. Then for any $n_1,\ldots,n_k$*

$$\big(g_1^{n_1}*\cdots*g_A^{n_A}\big)(x)\sim (n_1c_1+\cdots+c_An_A)\varphi(x)\,,$$

and for any $\epsilon>0$, there exists C_ϵ such that

$$\big(g_1^{n_1}*\cdots*g_A^{n_A}\big)(x)\le C_\epsilon(1+\epsilon)^{n_1+\cdots+n_A}\varphi(x)\,,$$

Now the transition probabilities $p_{ij}(\ell)$ of $\xi(\ell)$ are given by $p_{\Delta\Delta}(\ell)(\ell)=1$, $p_{du}(\ell)=p_{du}$,

$$p_{ud}(\ell)=\mathbb{P}_u\big(T_d<\tau_u(\ell),\xi_1=d\big)\,,\quad p_{u\Delta}=1-\sum_{d\in\mathcal{D}}p_{ud}(\ell)=\mathbb{P}_u\big(T_d\ge\tau_u(\ell)\big)$$

all other $p_{ij}(\ell)=0$. Since Δ is absorbing, $\mathrm{spr}\big(\boldsymbol{P}(\ell)\big)<1$ and so Perron-Frobenius theory gives the existence of $b<1$ such that the n-step transition probabilities $p_{ij}^n(\ell)$ decay at rate at most b^n. For $j\ne\Delta$ one has $\mathbb{P}_i\big(\xi_n(\ell)=j\big)=p_{ij}^n(\ell)$ and letting W_d be a r.v. with density $z_j^*(y)=z_j(y)/\overline{z}_j$ where $\overline{z}_j=\int_0^\infty z_j(y)\,\mathrm{d}y$, we can rewrite (6.3) as

$$Z_i(x)=\sum_{n=0}^{\infty}\sum_{j\in\mathcal{E}}p_{ij}^n(\ell)\overline{z}_j\mathbb{E}\big[(g_{n,j}*z_j^*)(x)\big]$$

where $g_{n,j}$ is the conditional density of $S_n(\ell)$ given $\xi_0=i,\xi_1,\ldots,\xi_{n-1}$. By the first part of Lemma 6.1, $g_{n,j}(x)\sim c_n\varphi(x)$ for some (random) c_n. Further, the $z_d(x)$ are heavier than $\varphi(x)$ (that $\Phi(x)/\varphi(x)\to\infty$ is a standard estimate in the heavy-tailed area), and so $(g_{n,j}*z_j^*)(x)\sim z_j^*(x)$ by Proposition 7 of [3]. Choosing ϵ with $(1+\epsilon)b<1$, we can then use the second part of Lemma 6.1 and a dominated convergence argument to conclude that

$$Z_i(x)\sim\sum_{n=0}^{\infty}\sum_{j\in\mathcal{E}}p_{ij}^n(\ell)\overline{z}_j\,z_j^*(x)\sim\sum_{n=0}^{\infty}\sum_{j\in\mathcal{E}}p_{ij}^n(\ell)c_d\Phi(x)\,.$$

This completes the proof of Theorem 2.2, with the expression

$$C_i = \boldsymbol{e}_i\big(\boldsymbol{I} - \boldsymbol{P}(\ell)\big)^{-1}\boldsymbol{c} \tag{6.4}$$

for the C_i where $\boldsymbol{c}$ is the vector with j-th entry c_d for $i = d \in \mathcal{D}$ and 0 for $i = u \in \mathcal{U}$. □

Example 6.2. Consider as in Section 2 vanilla RESTART with repair, $\mathcal{U} = \{u\}$, $\mathcal{D} = \{d\}$, and $\rho_u(t) \equiv 1$. Then F_u is the distribution of the failure time (denoted G so far), F_d the distribution of the repair time, and with heavy-tailed F_d, Theorem 2.2 immediately gives that the total task time X has tail proportional to $\overline{F}(x)$.

A particular case of a heavy-tailed F_d could arise if repair means rebooting of the processor, such that the ideal time for rebooting is B but that rebooting may fail at Poisson(β) times and then itself needs to be restarted. This means that given $B = b$, T_d is the RESTART total task time with exponential(β) failure and task length b, so that by [2] $\overline{F}_d \sim C(b)\mathrm{e}^{-\eta(b)x}$, with the values of $C(b), \eta(b)$ given there. For a random B with unbounded suport, T_d becomes heavy-tailed. For example, for a Gamma-like B we have that (again by reference to [2])

$$\overline{F}_d(t) \sim c_{\alpha,\delta}\frac{\log^{\alpha-1} x}{x^{\eta_{\alpha,\delta}}}$$

with $c_{\alpha,\delta}, \eta_{\alpha,\delta}$ specified in [2]. Thus F_d is regularly varying and Theorem 2.2 applies. □

7 Time-varying processing rates

As is clear from the formulas presented so far, a main problem for numerical implementation beyond the simplest case $\rho_u(t) \equiv \rho$ is evaluation of quantities like

$$g_{ud}(\ell) = \mathbb{P}\big(T_u > \tau_u(\ell),\, \xi_1 = d\big) = \mathbb{P}\Big(\int_0^{T_u} \rho_u(t)\,\mathrm{d}t > \ell\,, \xi_1 = d\Big)\,.$$

Example 7.1. A simple example is obtained by letting the server speed depend on time. E.g. the server slowing down with time could be modeled by the service rate being proportional to $t^{-\alpha}$ with $0 < \alpha < 1$, and then $\tau_u(\ell)$ is proportional to $t^{1-\alpha}$. □

Another tractable and more interesting case is what we call the *independent Markov rate model* where the server speed is determined by an independent external environment. . This is defined in terms of a set of Markov process $I = \{I(t)\}_{t\geq 0}$ with state space $\{1, \ldots, m\}$ and a set of rates $r_1, \ldots, r_m$, such that $\rho_u(t) = r_{I(t)}$. The basic assumption is that I is independent of the Markov renewal process. We denote the rate matrix by $\boldsymbol{A}$ and the initial distribution by $\boldsymbol{\alpha}$. A natural choice of $\boldsymbol{\alpha}$ is the stationary distribution for I, and dependence of the rates on u can be obtained by taking I complicated enough. Write $\boldsymbol{\Delta_r}$ for the diagonal matrix with the r_i on the diagonal.

Proposition 7.2. *Consider the independent Markov model with all $r_i > 0$ and assume that T_u is independent of ξ_1 and exponential(δ). Then*

$$g_{ud}(\ell) = p_{ud}\, \boldsymbol{\alpha} \exp\{\boldsymbol{B}(\delta)\ell\}\boldsymbol{e}$$

where $\boldsymbol{B}(\delta) = \boldsymbol{\Delta}_{\boldsymbol{r}}^{-1}\boldsymbol{A} - \dfrac{\delta}{2}\big(\boldsymbol{\Delta}_{\boldsymbol{r}}^{-1}\boldsymbol{e}\boldsymbol{e}^{\mathsf{T}} + \boldsymbol{e}\boldsymbol{e}^{\mathsf{T}}\boldsymbol{\Delta}_{\boldsymbol{r}}^{-1}\big)$.

Proof. Fix u, d and define $\boldsymbol{f}(x)$ as the row vector with elements

$$f_j(x) = \mathbb{P}_{\boldsymbol{\alpha}}\Big(\int_0^{T_u} \rho_u(t)\,\mathrm{d}t > x,\ I\big(\tau_u(x)\big) = j\Big)\,.$$

We shall identify $f_j(x+h)$ up to $\mathrm{o}(h)$ terms and thereby derive a differential equation for $\boldsymbol{f}(x)$, giving $\boldsymbol{f}(x) = \boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{B}(\delta)x}$ from which the result follows by letting $x = \ell$, summing over j (corresponding to multiplication by $\boldsymbol{e}$) and using the assumed independence of ξ_1 to extract the transition probability p_{ud}.

For $\int_0^{T_u}\rho_u(t)\,\mathrm{d}t > x + h$ to occur, we must have $\tau_u(x) < T_u$. The contribution to $f_j(x+h)$ from the event $I\big(\tau_u(x+h)\big) = I\big(\tau_u(x)\big) = j$ is therefore

$$\begin{aligned} &f_j(x)\big(1 + a_{jj}h/r_j\big)\mathbb{P}\big(T_u > \tau_u(x) + h/r_j \,|\, T_u > \tau_u(x)\big) + \mathrm{o}(h) \\ &\quad = f_j(x)\big(1 + a_{jj}h/r_j - \delta h/r_j\big) + \mathrm{o}(1)\,, \end{aligned}$$

where the $\mathrm{o}(h)$ terms takes care of, e.g., the possibility of jumps out of j and back in the time interval $\big(\tau_u(x), \tau_u(x+h)\big]$. Consider next the contribution from the event $I\big(\tau_u(x)\big) = k$, $I\big(\tau_u(x+h)\big) = j$ and a single jump from k to j in the time interval $\big(\tau_u(x), \tau_u(x+h)\big]$. This jump must occur before $\tau_u(x) + h/r_k$, and its position is then approximately at $\tau_u(x) + Uh/r_k$ where U is an independent uniform$(0,1)$ r.v. Passage to $x+h$ must therefore occur at time $\tau_u(x) + Uh/r_k + (1-U)h/r_j$, and the probability that T_u survives from $\tau_u(x)$ to there is approximately

$$\mathbb{E}\big\{-\delta[Uh/r_k + (1-U)h/r_j]\big\} = 1 - h\frac{\delta}{2}(1/r_k + 1/r_j)\,.$$

The asked for contribution is thus

$$\sum_{k\neq j} f_k(x)\Big(a_{kj}h/r_k - h\frac{\delta}{2}(1/r_k + 1/r_j)\Big)\,. \tag{7.1}$$

Alltogether,

$$\begin{aligned} f_j(x+h) &= f_j(x) + \sum_{k=1}^{m} f_k(x)\Big(a_{kj}h/r_k - h\frac{\delta}{2}(1/r_k + 1/r_j)\Big) + \mathrm{o}(h)\,, \\ f_j'(x) &= \sum_{k=1}^{m} f_k(x)\Big(a_{kj}/r_k - \frac{\delta}{2}(1/r_k + 1/r_j)\Big)\,. \end{aligned}$$

In matrix notation, this means $\boldsymbol{f}'(x) = \boldsymbol{f}(x)\boldsymbol{B}(\delta)$ which together with $\boldsymbol{f}(0) = \boldsymbol{\alpha}$ gives the desired conclusions. □

Remark 7.3. The assumption of T_u being exponential is easily generalized to a PH distribution at the expense of the differential equations and the form of $\boldsymbol{B}$ become somewhat more complicated. We omit the details. $\square$

We next turn to the Markov model of Section 3, where the rate generating process is internal rather than external. That is, the role of I is taking by the J-process of Section 3.

Proposition 7.4. *Consider the Markov model with $r_v > 0$ for all $v \in \mathcal{U}$. Then*

$$g_{ud}(\ell) = \boldsymbol{e}_u^{\mathsf{T}} \exp\bigl\{\boldsymbol{\Delta}_{\boldsymbol{r}}^{-1} \boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*} \ell\bigr\} (-\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*})^{-1} \boldsymbol{Q}_{\mathcal{U}^*\mathcal{D}^*} \boldsymbol{e}_d \,.$$

Proof. Define $\boldsymbol{f}(x)$ as the row vector with elements

$$f_v(x) = \mathbb{P}_u\Bigl(\int_0^{T_u} r_{J(t)}\,\mathrm{d}t > x,\ J\bigl(\tau_u(x-)\bigr) = v\Bigr), \quad v \in \mathcal{U}\,.$$

Then $\boldsymbol{f}'(x) = \boldsymbol{f}(x)\boldsymbol{\Delta}_{\boldsymbol{r}}^{-1}\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*}$ and thus $\boldsymbol{f}(x) = \boldsymbol{e}_u^{\mathsf{T}} \exp\bigl\{\boldsymbol{\Delta}_{\boldsymbol{r}}^{-1}\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*} x\bigr\}$. Indeed, the only difference from the proof of Proposition 7.2 is that the $\delta/2$ term in (7.1) does not enter because the possibility $T_u \in \bigl(\tau_u(x), \tau_u(x+h)\bigr]$ is taken care of by the rows of $\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*}$ not summing to 0 (what is missing is the rates of entering $\mathcal{D}^*$). Now just note that

$$\begin{aligned} &\mathbb{P}_u\Bigl(\int_0^{T_u} \rho_u(t)\,\mathrm{d}t > \ell,\, \xi_1 = d \,\Big|\, J\bigl(\tau_u(\ell-)\bigr) = v\Bigr) \\ &\quad = \int_{\tau_u(\ell)}^{\infty} \boldsymbol{e}_v^{\mathsf{T}} \exp\{\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*}\bigl(t - \tau_u(\ell)\bigr)\}\boldsymbol{Q}_{\mathcal{U}^*\mathcal{D}^*}\boldsymbol{e}_d\,\mathrm{d}t = \boldsymbol{e}_v^{\mathsf{T}}(-\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*})^{-1}\boldsymbol{Q}_{\mathcal{U}^*\mathcal{D}^*}\boldsymbol{e}_d \end{aligned}$$

$\square$ $\square$

Proposition 7.5. *Consider the Markov model with $r_v > 0$ for all $v \in \mathcal{U}$. Then*

$$r_{ud}(\alpha, \ell) = \boldsymbol{e}_u^{\mathsf{T}}\Bigl(\boldsymbol{I} - \exp\bigl\{\boldsymbol{\Delta}_{\boldsymbol{r}}^{-1}\bigl(\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*} + \alpha\boldsymbol{I}\bigr)\ell\bigr\}\Bigr)(-\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*} - \alpha\boldsymbol{I})^{-1}\boldsymbol{Q}_{\mathcal{U}^*\mathcal{D}^*}\boldsymbol{e}_d\,.$$

Proof. We have

$$\begin{aligned} r_{ud}(\alpha, \ell) &= \mathbb{E}[\mathrm{e}^{\alpha T_u};\, \xi_1 = d] - \mathbb{E}\bigl[\mathrm{e}^{\alpha T_u};\, \int_0^{T_u} r_{J(t)}\,\mathrm{d}t > \ell,\, \xi_1 = d\bigr] \\ &= \boldsymbol{e}_u^{\mathsf{T}}(-\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*} - \alpha\boldsymbol{I})^{-1}\boldsymbol{Q}_{\mathcal{U}^*\mathcal{D}^*}\boldsymbol{e}_d - \mathbb{E}\bigl[\mathrm{e}^{\alpha T_u};\, \int_0^{T_u} r_{J(t)}\,\mathrm{d}t > \ell,\, \xi_1 = d\bigr]\,. \end{aligned}$$

The idea is now to note the identity

$$\mathrm{e}^{\alpha T_u} = \exp\bigl\{\alpha\tau_u(\ell)\bigr\} \cdot \exp\bigl\{\alpha\bigl(T_u - \tau_u(\ell)\bigr)\bigr\}$$

(valid on the set $\bigl\{T_u > \tau_u(\ell)\bigr\}$), take care of the first factor by the differential equation approach and the second by the conditioning argument in the proof of Proposition 7.4.

So, define this time $\boldsymbol{f}(x)$ as the row vector with elements

$$f_v(x) = \mathbb{E}_u\bigl[\exp\{\alpha\tau_u(x)\};\ \int_0^{T_u} r_{J(t)}\,\mathrm{d}t > x,\ J\bigl(\tau_u(x-)\bigr) = v\bigr]\,, \quad v \in \mathcal{U}\,.$$

Now

$$\exp\bigl\{\alpha\tau_u(x+h)\bigr\} = \exp\bigl\{\alpha\tau_u(x)\bigr\}\cdot\exp\{\alpha h/r_k\} \approx \exp\bigl\{\alpha\tau_u(x)\bigr\}\cdot(1+\alpha h/r_k)$$

on the set $\bigl\{T_u > \tau_u(x+h),\ J\bigl(\tau_u(x)\bigr) = k\bigr\}$. This gives

$$\boldsymbol{f}'(x) = \boldsymbol{f}(x)\boldsymbol{\Delta}_{\boldsymbol{r}}^{-1}(\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*} + \alpha\boldsymbol{I})\,, \quad \boldsymbol{f}(x) = \boldsymbol{e}_u^{\mathsf{T}}\exp\bigl\{\boldsymbol{\Delta}_{\boldsymbol{r}}^{-1}(\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*} + \alpha\boldsymbol{I})x\bigr\}\,.$$

Further,

$$\begin{aligned}
&\mathbb{E}_u\bigl[\exp\bigl\{\alpha\bigl(T_u - \tau_u(\ell)\bigr)\bigr\},\ \int_0^{T_u}\rho_u(t)\,\mathrm{d}t > \ell,\ \xi_1 = d\,\big|\,J\bigl(\tau_u(\ell-)\bigr) = v\bigr] \\
&\quad = \int_{\tau_u(\ell)}^{\infty}\exp\bigl\{\alpha\bigl(t-\tau_u(\ell)\bigr)\bigr\}\boldsymbol{e}_v^{\mathsf{T}}\exp\bigl\{\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*}\bigl(t-\tau_u(\ell)\bigr)\bigr\}\boldsymbol{Q}_{\mathcal{U}^*\mathcal{D}^*}\boldsymbol{e}_d\,\mathrm{d}t \\
&\quad = \boldsymbol{e}_v^{\mathsf{T}}(-\boldsymbol{Q}_{\mathcal{U}^*\mathcal{U}^*} - \alpha\boldsymbol{I})^{-1}\boldsymbol{Q}_{\mathcal{U}^*\mathcal{D}^*}\boldsymbol{e}_d\,.
\end{aligned}$$

The rest of the proof is easy manipulations. □

Remark 7.6. In Propositions 7.2–7.5, the assumption of strictly positive rates can be dispensed with by working with a reduced state space formed only by the states having rates > 0. Consider for example Proposition 7.2 and assume $r_i > 0$ for $i = 1,\ldots,n$, $r_i = 0$ for $i = n+1,\ldots,m$ with $1 \le n < m$. The $f_i(x)$ then only need to be defined for $i = 1,\ldots,n$. Dividing $\boldsymbol{A}$ as

$$\boldsymbol{A} = \begin{pmatrix}\boldsymbol{A}_{11} & \boldsymbol{A}_{12}\\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22}\end{pmatrix}\,,$$

where $\boldsymbol{A}_{11}$ is $n\times n$, $\boldsymbol{A}_{12}$ $n\times(m-n)$ etc., the rate of starting an excursion in $\{n+1,\ldots,m\}$ from i and ending it by a jump to j is

$$\sum_{k=n+1}^{n} a_{jk}\int_0^{\infty}\boldsymbol{e}_k\mathrm{e}^{\boldsymbol{A}_{22}t}\boldsymbol{e}_j\,\mathrm{d}t = a_{jk}\boldsymbol{e}_k(-\boldsymbol{A}_{22})^{-1}\boldsymbol{e}_j\,,$$

and so Proposition 7.2 holds if $\boldsymbol{B}(\delta)$ is modified to

$$\boldsymbol{\Delta}_{\boldsymbol{r};11}^{-1}\bigl(\boldsymbol{A}_{11} - \boldsymbol{A}_{11}(\boldsymbol{A}_{22})^{-1}\boldsymbol{A}_{21}\bigr) - \frac{\delta}{2}\bigl(\boldsymbol{\Delta}_{\boldsymbol{r};11}^{-1}\boldsymbol{e}\boldsymbol{e}^{\mathsf{T}} + \boldsymbol{e}\boldsymbol{e}^{\mathsf{T}}\boldsymbol{\Delta}_{\boldsymbol{r};11}^{-1}\bigr)\,,$$

in obvious block notation for $\boldsymbol{\Delta}_{\boldsymbol{r};11}^{-1}$.

If in the Markov model $\mathcal{D}$ consists of a single state, Proposition 7.4 gives the tail probability of $\int_0^{T_u} r_{J(t)}\,\mathrm{d}t$. This is a 'fluid reward', and with the extension sketched in Remark 7.6, Proposition 7.4 reduces to an well-known expression for the marginal distributions in the Kulkarni class of multivariate PH distributions, [13]. □

8 Phase-type modeling

Standard Markov models assume exponential distributions of failure times, repair times etc. This may seem intrinsically inherent in the Markovian set-up because of the memoryless property of the exponential distribution, but there is is fact a simple approach going far beyond this by using PH distributions. We shall not go into the general formulation but only consider a basic examples, the *Erlang distribution* $E_p = E_p(\lambda)$ with $p = 1, 2, \ldots$ stages defined as the sum of p i.i.d. exponential(μ) r.v.'s (the stages). The density is $\lambda^p t^{p-1} \mathrm{e}^{-\lambda t}/\Gamma(p)$ where $\Gamma(p) = (p-1)!$. The Erlang is therefore just a Gamma with integer shape parameter. However, the important fact is the probabilistic interpretation in terms of the stages which we model as a Markovian movement between stages $1, 2$ and the final one, completion, with transition rate λ for each of the two transitions. In some example the stages may have a physical interpretation; e.g., stage 1 may be the real repair and stage 2 checking or warm-up. Usually, the Erlang(p) distribution is, however, mainly used in a descriptive way to allow for including non-exponential distributions in a Markovian set-up.

We will consider an elaborate example of the use of the Erlang distribution which from the general view of the paper also illustrates how identical Markov processes may lead to different Markov renewal schemes. We look at a system with 3 identical components, each with exponential(λ) operating times, and two repairmen with $E_p(\mu)$ repair times. A Markov state of the whole system is specified with the number $i = 0, 1, 2, 3$ of failed components and the stage(s) in which the servers are currently operating. In Fig. 6, state 2:12 indicates that 2 components are failed and one server working in stage 1, the other in stage 2. In state 1:2 only one component is failed, i.e. under repair, and the repairman working on it is in stage 2. In state 3:11, all 3 components are failed, one waiting for repair and the servers working on the other two are both in stage 1; etc. The system may be parallel as in Figure 5, in which case $\mathcal{U}^* = \{$0,1:1,1:2,2:11,2:12,2:22$\}$ or, e.g., 2-out-of-3 as in Figure 6 so that $\mathcal{U}^* = \{$0,1:1,1:2$\}$.

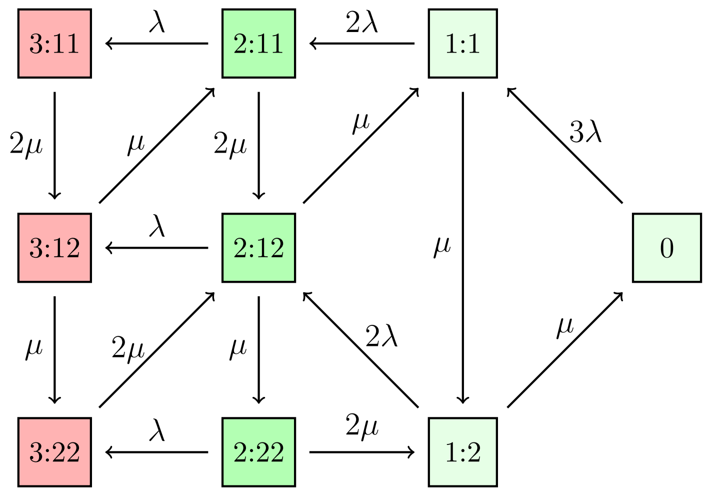

Figure 5: E_2 repair times, parallel.

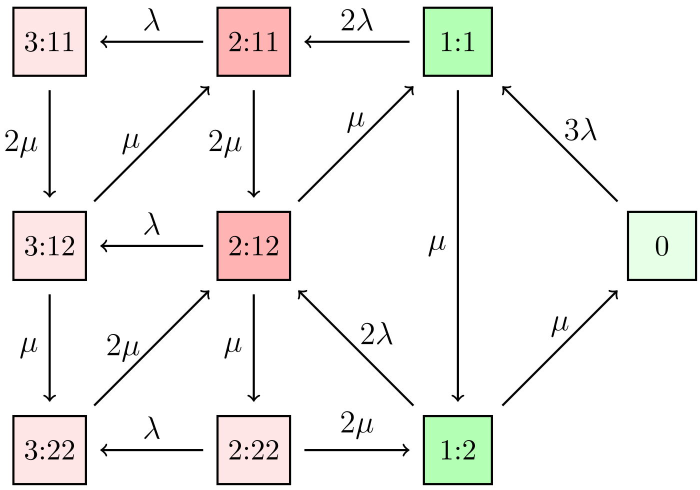

Figure 6: E_2 repair times, 2-out-of-3.

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A Some lemmas

The following result is given in [1, Th. VII.4.6]:

Lemma A.1. *Consider a Markov renewal equation*

$$Z_i(x) = z_i(x) + \sum_{j\in\mathcal{E}} \int_0^x Z_j(x-y)\,\widetilde{F}_{ij}(\mathrm{d}y)\,,\ i\in\mathcal{E},$$

with $\mathcal{E}$ finite and the matrix $\big(\|\widetilde{F}_{ij}\|\big)$ irreducible. Let $r_{ij} = \int_0^\infty \mathrm{e}^{\gamma u}\,\widetilde{F}_{ij}(\mathrm{d}u)$, suppose that for some real γ the matrix $\boldsymbol{R} = (r_{ij})$ has spectral radius 1, and choose $\boldsymbol{\nu}$, $\boldsymbol{h}$ with $\boldsymbol{\nu A} = \boldsymbol{\nu}$, $\boldsymbol{Ah} = \boldsymbol{h}$, $\nu_i > 0$, $h_i > 0$, $i \in \mathcal{E}$. Then $Z_i(x) \sim h_i C_1/C_2$ where

$$C_1 = \sum_{j=1}^{p} \nu_j \int_0^\infty \mathrm{e}^{\gamma x} z_j(x)\,\mathrm{d}x\,, \tag{A1.1}$$

$$C_2 = \sum_{k,j=1}^{p} \nu_k h_j \int_0^\infty x\mathrm{e}^{\gamma x}\,\widetilde{F}_{kj}(\mathrm{d}x)\ . \tag{A1.2}$$

The following lemma in unsurprising given the Perron-Frobenius theory, but included here in this precise form for the sake of easy reference:

Lemma A.2. *Let for $k = 1,2$ $\boldsymbol{A}(k) = \big(a_{ij}(k)\big)$ be irreducible non-negative matrices and let $\gamma(k) = \mathrm{spr}\big(\boldsymbol{A}(k)\big)$. Assume that $\boldsymbol{A}(1) \geq \boldsymbol{A}(2)$ and that $a_{i'j'}(1) > a_{i'j'}(2)+b > 0$ for at least one pair $i'j'$ and some $b > 0$. Then $\gamma(1) \geq c(b)\gamma(2)$ for some constant $c(b) > 1$ depending only on $\boldsymbol{A}(2)$ and satisfying $c(b) \to \infty$ as $b \to \infty$.*

Proof. Irreducibility of $\boldsymbol{A}(2)$ ensures that for some n and all ij there exists a path $i_0 i_1 \cdots i_{n-1} i_n$ with $i_0 = i, i_n = j$, all $a_{i_{k-1}i_k}(2) > 0$ and $i_{k-1}i_k = i'j'$ for some k'. Hence

$$a_{ij}(1) > a_{ij}(2) + b \prod_{k=1,\,k\neq k'}^{n} a_{i_{k-1}i_k}(2) > a_{ij}(2)\big(1 + bc_1(i,j)\big)$$

for some $c_1(i,j) > 0$. Thus $\boldsymbol{A}(1)^n \geq c(b)^n \boldsymbol{A}(2)^n$ where $c(b) = \big(1 + \inf_{i,j} c_1(i,j)\big)^n$. But with $\boldsymbol{\pi}(1)$ the positive left (row) eigenvector of $\boldsymbol{A}(1)$ corresponding to $\gamma(1)$ and $\boldsymbol{h}(1)$ the positive right (column) eigenvector of $\boldsymbol{A}(2)$ corresponding to $\gamma(2)$, we have

$$\boldsymbol{\pi}(2)\boldsymbol{A}(1)^{nm}\boldsymbol{h}(1) \sim \gamma(1)^{nm}\boldsymbol{\pi}(2)\boldsymbol{h}(1)\,, \quad \boldsymbol{\pi}(2)\boldsymbol{A}(2)^{nm}\boldsymbol{h}(1) \sim \gamma(2)^{nm}\boldsymbol{\pi}(2)\boldsymbol{h}(1)$$

which in view of $\boldsymbol{A}(1)^n \geq c(b)^n \boldsymbol{A}(2)^n$ and $\boldsymbol{\pi}(2)\boldsymbol{h}(1) > 0$ is only possible if $\gamma(1) \geq c(b)\gamma(2)$. □

Lemma A.3. *Let* $I = \displaystyle\int_0^\infty \exp\{-\mu \mathrm{e}^{-q\ell} x\}\, \ell^{a-1} \mathrm{e}^{-\delta\ell}\, \mathrm{d}\ell$. *Then* $I \sim \dfrac{\Gamma(\delta/q)}{\mu^{\delta/q} q^a} \dfrac{\log^{a-1} x}{x^{\delta/q}}$ *as* $x \to \infty$.

Proof. Using the substitution $y = \mu \mathrm{e}^{-q\ell} x$, we can rewrite I as

$$\int_0^{\mu x} \mathrm{e}^{-y} \Big(\frac{\log x + \log\mu - \log y}{q}\Big)^{a-1} \Big(\frac{y}{\mu x}\Big)^{\delta/q} \frac{\mathrm{d}y}{qy} = \frac{1}{\mu^{\delta/q} q^a} \frac{\log x}{x^{\delta/q}} I_1$$

where

$$I_1 = \int_0^\infty \omega(x,y)^{\alpha-1} y^{\delta/q-1} \mathrm{e}^{-y}\, \mathrm{d}y\,, \quad \omega(x,y) = 1 + \log\mu/\log x - \log y/\log x\,.$$

Now for $x \geq \mathrm{e}$ and $\varepsilon > 0$, we have $\big|\omega(x,y)\big| \leq \omega^*(y)$ where $\omega^*(y) = d\max(y^\varepsilon, y^{-\varepsilon})$ for some $d = d_\varepsilon$. Choosing ε such that $\varepsilon|\alpha - 1| < \alpha$, $\omega^*(y)^{\alpha-1} y^{\delta/q-1} \mathrm{e}^{-y}$ is integrable, and since $\omega(x,y) \to 1$ as $x \to \infty$ with y fixed, the dominated convergence theorem implies that $I_1 \to \Gamma(\delta/q)$, concluding the proof. □