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Edgeworth expansion for the pre-averaging estimator*

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Abstract

In this paper, we study the Edgeworth expansion for a pre-averaging estimator of quadratic variation in the framework of continuous diffusion models observed with noise. More specifically, we obtain a second order expansion for the joint density of the estimators of quadratic variation and its asymptotic variance. Our approach is based on martingale embedding, Malliavin calculus and stable central limit theorems for continuous diffusions. Moreover, we derive the density expansion for the studentized statistic, which might be applied to construct asymptotic confidence regions.

Keywords: diffusion processes, Edgeworth expansion, high frequency observations, quadratic variation, pre-averaging.

JEL Classification: C10, C13, C14

1 Introduction

In the last decade the estimation of quadratic variation of Itô semimartingales have been investigated by many researchers. Typically, this estimation problem is considered in the infill asymptotics setting, i.e. the underlying observations

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are recorded from high frequency data of continuous/discontinuous Itô semimartingales, diffusion processes corrupted by noise or related models. A recent comprehensive monograph [6] presents a detailed asymptotic analysis for estimators of quadratic variation and related objects in various frameworks.

In financial mathematics, it is nowadays widely accepted that financial data is contaminated by *microstructure noise* such as rounding errors, bid-ask bounds and misprints, when observed at ultra high frequency. This fact prevents us from using classical realised volatility estimator at such frequencies. In this work we consider a continuous SDE model corrupted by additive i.i.d. noise, i.e. the observations are

$$Y_{t_i} = X_{t_i} + \varepsilon_{t_i},$$

where X is a continuous diffusion process, ε is an i.i.d process independent of X and $t_i = i\Delta_n$ with $\Delta_n \to 0$. It is well-known that realised volatility has an explosive behaviour and more delicate methods are required to estimate the quadratic variation of the latent diffusion process X. The most famous estimation approaches in this framework are the *multiscale approach* of [14], the *realised kernel method* proposed in [2] and the *pre-averaging concept* originally introduced in [9] among others. All these estimators are consistent, asymptotically mixed normal and have the convergence rate $\Delta_n^{-1/4}$, which is known to be optimal.

Due to this relatively slow rate of convergence the quality of the mixed normal approximation is rather questionable even at high frequencies. The aim of this paper is to derive the second order Edgeworth expansion for the pre-averaging estimator to improve the mixed normal approximation of the unknown density. We remark that our work is related to a recent paper [10], which investigates the Edgeworth expansion for power variations of continuous diffusion processes in the noise-free setting. However, in the framework of continuous diffusions corrupted by additive i.i.d. noise the stochastic second order expansion of the pre-averaging statistics is more involved. Our methodology relies on martingale methods, stochastic expansion of the pre-averaging statistics and general theory of Edgeworth expansion associated with mixed normal limits studied in [13]. The latter approach is heavily using different aspects of Maliavin calculus, such as integration by parts formula and conditions for smoothness of probability laws. In a second step, we will present the Edgeworth expansion for the density of the studentized statistic, which can be potentially used to construct more precise confidence regions for the quadratic variation of the diffusion process X.

The paper is organised as follows. We describe the main setting and recall the pre-averaging approach in Section 2. Section 3 presents a second order stochastic decomposition for the pre-averaging estimator of the quadratic variation. We demonstrate the general theory of Edgeworth expansion with respect to mixed normal limits in Section 4. In Section 5 we apply the asymptotic theory to the pre-averaging estimator and present Edgeworth expansion for the studentized version of the statistic. In Section 6, we deal separately with the case of constant volatility, which does not satisfy our non-degeneracy condition.

Section 7 demonstrates an example and Section 8 collects some steps of the proof.

2 Setting

In this paper, we deal with infill asymptotics, i.e. the data is observed at equidistant grid $i\Delta_n$, $i \in \mathbb{N}$, over a finite horizon $[0,T]$ and $\Delta_n \to 0$. We also impose that $1/\Delta_n$ is a positive integer. The terminal time T is assumed to be fixed and we assume $T = 1$ without loss of generality. For simplicity, we use the notation $t_i := i\Delta_n$. On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$ (to be specified in Section 5.2), we consider a diffusion model that satisfies the stochastic differential equation

$$dX_t = b^{[1]}(X_t)dW_t + b^{[2]}(X_t)dt \tag{2.1}$$

with a bounded random variable X_0 as starting value, a standard Brownian motion W and continuous functions $b^{[1]}, b^{[2]} : \mathbb{R} \to \mathbb{R}$. We intentionally choose the notations $b^{[1]}, b^{[2]}$ to emphasize that the diffusion term $b^{[1]}$ dominates the drift term $b^{[2]}$ in asymptotic expansions throughout the paper. We are interested in estimating the integrated volatility which we denote by

$$V = \int_0^1 (b^{[1]}(X_t))^2 dt. \tag{2.2}$$

However, due to the microstructure noise effects, we are not able to observe the process X directly, but only with distortions. More specifically, we assume that the underlying observations $(Y_{t_i})_{i\geqslant 0}$ are given by

$$Y_{t_i} = X_{t_i} + \varepsilon_{t_i}, \tag{2.3}$$

where $(\varepsilon_{t_i})_{i\geqslant 0}$ is a sequence of i.i.d. random variables with

$$\mathbb{E}[\varepsilon_{t_i}] = 0 \qquad \text{and} \qquad \mathbb{E}[\varepsilon_{t_i}^2] = \omega^2 > 0, \tag{2.4}$$

and ε_{t_i} is $\mathcal{F}_{t_i}$-measurable. In addition, we assume that the processes ε and X are independent. Such additive noise models are widely used in financial mathematics, see e.g. [2, 5, 9, 15] among many others.

We require some notation to describe the pre-averaging estimator which is originally due to [5, 9]. We pick a sequence of positive integers $(k_n)_{n=1}^{\infty}$ and a positive real number θ such that

$$k_n\Delta_n^{1/2} = \theta + o(\Delta_n^{1/2}) \text{ and } d_n := \left\lfloor \frac{1}{k_n\Delta_n} \right\rfloor \in \mathbb{N}. \tag{2.5}$$

Moreover, we consider a continuous, non-negative function g on $[0,1]$ which is piecewise continuously differentiable with a piecewise Lipschitz derivative g'. This function should also satisfy

$$g(0) = g(1) = 0 \quad \text{and} \int_0^1 g^2(s)ds > 0. \tag{2.6}$$

Furthermore, we introduce the following notations associated with g:

$$h(j/k_n) = g((j+1)/k_n) - g(j/k_n),\ \psi_1^n = k_n \sum_{j=0}^{k_n-1} (h(j/k_n))^2,$$
$$\psi_2^n = \frac{1}{k_n} \sum_{j=1}^{k_n-1} (g(j/k_n))^2,\ \psi_3^n = \frac{1}{k_n} \sum_{j=1}^{k_n-1} g(j/k_n),$$
$$\psi_4^n = \frac{1}{(k_n)^2} \sum_{j=1}^{k_n-1} (g(j/k_n))^2 (j-1/2).$$

Moreover, we need the notations written below. The first four are limits of the terms $\psi_i^n, 1 \leqslant i \leqslant 4$.

$$\psi_1 = \int_0^1 [g'(s)]^2 ds,\ \psi_2 = \int_0^1 g^2(s) ds,\ \psi_3 = \int_0^1 g(s) ds,\ \psi_4 = \int_0^1 s g^2(s) ds,$$
$$\psi_5 = \int_0^1 \int_0^u g(s) ds g^2(u) du,\ \psi_6 = \int_0^1 \left(\int_0^u g(s) ds g(u) - \int_0^u (g(s) - \psi_2) ds \right)^2 du$$
$$\psi_7 = \int_0^1 \int_0^u \int_0^s [2g(s) + g(r)]^2 dr ds g^2(u) du.$$

For any process U we define the pre-averaged increment at stage $i\Delta_n$ via

$$\bar{U}_{t_i} = \sum_{j=1}^{k_n-1} g(j/k_n) \Delta_{i+j}^n U = \sum_{j=0}^{k_n-1} -h(j/k_n) U_{t_{i+j}},$$

where $\Delta_i^n U = U_{t_i} - U_{t_{i-1}}$. Finally, we are ready to introduce the pre-averaging estimator for the quadratic variation V :

$$V_n = \frac{1}{\psi_2^n} \sum_{i=0}^{d_n-1} (\bar{Y}_{t_{ik_n}})^2 - \frac{\psi_1^n d_n \Delta_n}{2\psi_2^n k_n} \sum_{i=1}^{1/\Delta_n} (\Delta_i^n Y)^2. \tag{2.7}$$

We remark that V_n is essentially the estimator proposed in [5] with the difference that we only use non-overlapping windows in this paper. This makes it easier to determine the dominating martingale M_n of the estimator, which is required in Sections 3 and 4, while computation of the martingale part of the original estimator investigated in [5] is far from being obvious. As we will see below, we need a consistent estimator of the asymptotic conditional variance associated with V_n, which is defined as

$$F_n = \frac{2\Delta_n^{-1/2}}{3(\psi_2^n)^2} \sum_{i=0}^{d_n-1} (\bar{Y}_{t_{ik_n}})^4. \tag{2.8}$$

We recall that a sequence of random variables $(Y_n)_{n\geqslant 1}$, which are defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and take values in a metric space E, is said to converge stably with the limit Y, which is defined on an extension $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathbb{P})$, if for any bounded, continuous function f and any bounded $\mathcal{F}$-measurable random variable Z it holds that

$$\mathbb{E}[f(Y_n)Z] \to \overline{\mathbb{E}}[f(Y)Z], \quad n \to \infty. \tag{2.9}$$

In short, we use the notation $Y_n \xrightarrow{d_{st}} Y$. We say that Y is mixed normal with random variance Z^2, and write $Y \sim MN(0, Z^2)$, if $Y \sim ZU$, where $U \sim N(0,1)$, $Z > 0$ and U, Z are independent.

Denoting $Z_n = \Delta_n^{-1/4}(V_n - V)$, we proceed to the first asymptotic result whose proof essentially follows from the work of [5, 9]. We provide a sketch of the proof in Section 8.

Theorem 2.1. *Under the condition $\mathbb{E}[(\varepsilon_{t_1})^8] < \infty$, we deduce that*

$$Z_n \xrightarrow{d_{st}} M \sim MN(0, C) \text{ with } C = 2\theta \int_0^1 \left((b^{[1]}(X_t))^2 + \frac{\omega^2 \psi_1}{\theta^2 \psi_2} \right)^2 dt.$$

Moreover, we obtain

$$F_n \xrightarrow{\mathbb{P}} C.$$

We note that due to

$$Z_n \xrightarrow{d_{st}} M \text{ and } F_n \xrightarrow{\mathbb{P}} C,$$

and the properties of stable convergence, the studentized statistic satisfies

$$\frac{Z_n}{\sqrt{F_n}} \xrightarrow{d} N(0,1).$$

In this paper, we will first derive an asymptotic expansion for the pair (Z_n, F_n) and then proceed to calculate the related Edgeworth expansion for the studentized statistic

$$\frac{Z_n}{\sqrt{F_n}}.$$

Example 2.2. Our prime example for g is the function

$$g(x) = x \wedge (1 - x).$$

In this case, we obtain

$$\psi_1^n = 1, \ \psi_2^n = \frac{k_n^2 + 2}{12k_n^2}, \ \psi_3^n = \frac{1}{4}, \ \psi_4^n = \frac{k_n^2 + 2}{24k_n^2} \text{ when } k_n \text{ is even}$$

and

$$\psi_1^n = \frac{k_n - 1}{k_n}, \ \psi_2^n = \frac{k_n^2 - 1}{12k_n^2}, \ \psi_3^n = \frac{k_n^2 - 1}{4k_n^2}, \ \psi_4^n = \frac{k_n^2 - 2}{24k_n^2} \text{ when } k_n \text{ is odd.}$$

In addition, we get

$$\psi_1 = 1, \ \psi_2 = \frac{1}{12}, \ \psi_3 = \frac{1}{4}, \ \psi_4 = \frac{1}{24}, \ \psi_5 = \frac{1}{96}, \ \psi_6 = \frac{143}{24192}, \ \psi_7 = \frac{1}{105}.$$

3 Stochastic decomposition of Z_n

In this section, we provide a stochastic decomposition for the bias corrected version of the random variable Z_n defined in the previous section. This second order stochastic expansion is essential for obtaining the Edgeworth expansion discussed in Sections 4 and 5. Since the first term of the estimator V_n defined in (2.7) uses the observations Y_{t_i} with $0 \leqslant i \leqslant d_n k_n$, we effectively estimate the quadratic variation of X over the interval $[0, d_n k_n \Delta_n]$. For this reason, we consider the bias corrected statistic

$$\overline{Z}_n = Z_n + \Delta_n^{-1/4} \int_{d_n k_n \Delta_n}^{1} (b^{[1]}(X_t))^2 dt. \tag{3.1}$$

Obviously, the statistic $\overline{Z}_n$ also satisfies Theorem 2.1, since the correction converges to 0 in probability. However, the bias may affect the higher order asymptotics. In the next step, we proceed with the estimation of the bias to construct a feasible statistic. We basically follow the procedure proposed in [4, Section 4]. Let p_n be a sequence of integers satisfying $p_n \to \infty$, $p_n \Delta_n \to 0$ and $p_n \sqrt{\Delta_n} \to \infty$, and set $J_n := \{1/\Delta_n - p_n + 1, \ldots, 1/\Delta_n\}$. For each $t \in [d_n k_n \Delta_n, 1]$ we define

$$(\widehat{b}^{[1]}(X_t))^2 := \frac{1}{\psi_2^n k_n \Delta_n p_n} \sum_{i+k_n \in J_n} (\bar{Y}_{t_i})^2 - \frac{\psi_1^n}{2\theta^2 \psi_2^n p_n} \sum_{i \in J_n} (\Delta_i^n Y)^2,$$

which is constant in t. It has been proved in [4] that this local estimator is consistent for $(b^{[1]}(X_t))^2$. Thus, a feasible version of the statistic $\overline{Z}_n$ is obtained via

$$Z_n^\star = Z_n + \Delta_n^{-1/4} \int_{d_n k_n \Delta_n}^{1} (\widehat{b}^{[1]}(X_t))^2 dt. \tag{3.2}$$

We remark that $1 - d_n k_n \Delta_n = O(\Delta_n^{1/2})$, which implies that

$$Z_n^\star - \overline{Z}_n = o_{\mathbb{P}}(\Delta_n^{1/4}). \tag{3.3}$$

In the next step, we will show that

$$Z_n^\star = M_n + \Delta_n^{1/4} N_n$$

where M_n and N_n are some tight sequences of random variables. Before we go into details, we need more notations. We again consider the SDE defined in (2.1). However, in this section we assume that $b^{[1]}, b^{[2]} \in C^4(\mathbb{R})$. Under this smoothness assumptions, we apply Ito's lemma and write $b^{[k]}(X_t), k = 1, 2$ in the stochastic differential form as

$$db^{[k]}(X_t) = b^{[k.1]}(X_t) dW_t + b^{[k.2]}(X_t) dt.$$

Similarly, we define the processes $b^{[k_1.k_2.k_3]}(X_t)$, $k_1, k_2, k_3 = 1, 2$. Throughout the paper, we will also use the shorthand notations

$$b_t^{[k_1,\ldots,k_d]} = b^{[k_1,\ldots,k_d]}(X_t), \; d = 1, 2, 3, \; k_1, \ldots, k_d = 1, 2. \tag{3.4}$$

The following process, which is the first order approximation of $\bar{Y}_{t_{ik_n}}$, will play an important role throughout the proofs:

$$\alpha_{t_{ik_n}} = b^{[1]}_{t_{ik_n}} \overline{W}_{t_{ik_n}} + \bar{\varepsilon}_{t_{ik_n}}. \tag{3.5}$$

Note that the quantity $\alpha_{t_{ik_n}}$ is obtained from $\bar{Y}_{t_{ik_n}}$ via freezing the volatility process at time t_{ik_n} and ignoring the drift process $b^{[2]}$. We also need to define a function g_n and a process $W(i,t)$:

$$g_n(s) = \sum_{j=1}^{k_n-1} g(j/k_n) 1_{((j-1)\Delta_n, j\Delta_n]}(s), \tag{3.6}$$

$$W(i,t) = \int_{t_{ik_n} \wedge t}^{t_{(i+1)k_n} \wedge t} g_n(u - t_{ik_n}) dW_u. \tag{3.7}$$

We note that $g_n(s)$ vanishes for $s \leqslant 0$ and $s > (k_n - 1)\Delta_n$. Moreover, we obtain the identity

$$\overline{W}_{t_{ik_n}} = W(i, t_{(i+1)k_n}).$$

The next proposition, which gives the expansion of $Z_n^\star$, is a central result of this section.

Proposition 3.1. *We obtain*

$$Z_n^\star = M_n + \Delta_n^{1/4} N_n,$$

where

$$M_n = \frac{\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} \left(\alpha^2_{t_{ik_n}} - \mathbb{E}[\alpha^2_{t_{ik_n}} | \mathcal{F}_{t_{ik_n}}] \right) \text{ and } N_n = \sum_{k=1}^{6} N_{n,k} + R_n$$

with

$$\begin{aligned}
N_{n,1} =& \frac{2\Delta_n^{-1/2}}{\psi_2^n} \sum_{i=0}^{d_n-1} b^{[1]}_{t_{ik_n}} \int_{t_{ik_n}}^{t_{(i+1)k_n}} \nu_i^n(u) dW_u \\
N_{n,2} =& \frac{2\Delta_n^{-1/2}}{\psi_2^n} \sum_{i=0}^{d_n-1} b^{[1]}_{t_{ik_n}} b^{[2.1]}_{t_{ik_n}} \int_{t_{ik_n}}^{t_{(i+1)k_n}} \int_{t_{ik_n}}^{u} dW_s W(i,u) g_n(u - t_{ik_n}) du, \\
N_{n,3} =& \frac{\Delta_n^{3/2} k_n^2 (\psi_3^n)^2}{\psi_2^n} \sum_{i=0}^{d_n-1} (b^{[2]}_{t_{ik_n}})^2, \\
N_{n,4} =& \frac{\Delta_n^{3/2} k_n^2 (2\psi_4^n - \psi_2^n)}{2\psi_2^n} \sum_{i=0}^{d_n-1} 2 b^{[1]}_{t_{ik_n}} b^{[1.2]}_{t_{ik_n}} + (b^{[1.1]}_{t_{ik_n}})^2 \\
N_{n,5} =& \frac{2\Delta_n^{-1/2}}{\psi_2^n} \sum_{i=0}^{d_n-1} \bar{\varepsilon}_{t_{ik_n}} \left(\psi_3^n k_n \Delta_n b^{[2]}_{t_{ik_n}} + b^{[1.1]}_{t_{ik_n}} \int_{t_{ik_n}}^{t_{(i+1)k_n}} \int_{t_{ik_n}}^{u} dW_s dW(i,u) \right), \\
N_{n,6} =& \frac{\Delta_n^{-1/2} \psi_1^n d_n}{\psi_2^n k_n} \left[\omega^2 - \frac{\Delta_n}{2} \sum_{i=1}^{1/\Delta_n} (\Delta \varepsilon_i^n)^2 \right], \\
R_n =& o_{\mathbb{P}}(1)
\end{aligned}$$

where

$$\nu_i^n(u) = b^{[1.1]}_{t_{ik_n}} \left(\int_{t_{ik_n}}^u \left[2(W_s - W_{t_{ik_n}}) g_n(s - t_{ik_n}) + W(i,s) \right] dW_s \right) g_n(u - t_{ik_n})$$
$$+ b^{[1.1]}_{t_{ik_n}} \left(\int_{t_{ik_n}}^u g_n(s - t_{ik_n}) ds g_n(u - t_{ik_n}) + \int_u^{t_{(i+1)k_n}} (g_n^2(s - t_{ik_n}) - \psi_2^n) ds \right)$$
$$+ \psi_3^n k_n \Delta_n g_n(u - t_{ik_n}) b^{[2]}_{t_{ik_n}}.$$

Proof. See Section 8. □

The meaning and the asymptotic behaviour of the quantity (M_n, N_n) will be explained in Section 5.

4 Asymptotic expansion theory with respect to mixed normal limit

In this section, we briefly summarize the main elements of the martingale expansion for a mixed normal limit, which was developed in [13]. Suppose that, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$, we have a auxiliary random variable Z_n satisfying

$$Z_n = M_n + r_n N_n,$$

where N_n is a tight sequence of random variables and (r_n) is a sequence of positive numbers satisfying $r_n \to 0$ (in our framework $r_n = \Delta_n^{1/4}$). Note that we had this type of decomposition in the previous section. In addition, we assume that M_n is a terminal value of some continuous $(\mathcal{F}_t)$-martingale $(M_t^n)_{t \in [0,1]}$ with $M_0^n = 0$. We assume that M_n (and hence Z_n) converges stably in law to a mixed normal limit M :

$$M_n \xrightarrow{d_{st}} M \sim MN(0, C).$$

Here, M is defined on an extension $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathbb{P})$. Let F_n be a reference random variable, which is general here but will be F_n of (2.8) in Section 5. We are interested in the asymptotic expansion of (Z_n, F_n).

Let $(C_t^n)_{t \in [0,1]}$ denote the quadratic variation process of M^n and $(M_t)_{t \in [0,1]}$, defined on an extenstion $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathbb{P})$, be a process satisfying $M = M_1$. For $C_n := C_1^n$ and F_n, we denote

$$\widehat{C}_n = r_n^{-1}(C_n - C) \text{ and } \widehat{F}_n = r_n^{-1}(F_n - F),$$

respectively, where C and F are some random variables. We impose the following crucial assumption where [B1](i) involves a functional stable convergence.

[B1] **(i)** $(M_\cdot^n, N_n, \widehat{C}_n, \widehat{F}_n) \xrightarrow{d_{st}} (M_\cdot, N, \widehat{C}, \widehat{F})$,

(ii) $M_t \sim MN(0, C_t)$ for each $t \in [0,1]$.

All information concerning the Edgeworth expansion for (Z_n, F_n) is contained in two random symbols $\underline{\sigma}$ and $\overline{\sigma}$, which are introduced in the next subsection.

4.1 The random symbols $\underline{\sigma}$ and $\overline{\sigma}$

We start with the random symbol $\underline{\sigma}$. Let $\widetilde{\mathcal{F}} = \mathcal{F} \vee \sigma(M)$. We assume that there exists random variables $\widetilde{C}(z),\ \widetilde{N}(z),\ \widetilde{F}(z)$ such that

$$\widetilde{C}(M) = \mathbb{E}[\hat{C}|\widetilde{\mathcal{F}}], \quad \widetilde{N}(M) = \mathbb{E}[N|\widetilde{\mathcal{F}}], \quad \widetilde{F}(M) = \mathbb{E}[\hat{F}|\widetilde{\mathcal{F}}].$$

Then, we define the adaptive (classical) random symbol $\underline{\sigma}$ by

$$\underline{\sigma}(z, \mathrm{i}u, \mathrm{i}v) = \frac{(\mathrm{i}u)^2}{2}\widetilde{C}(z) + \mathrm{i}u\widetilde{N}(z) + \mathrm{i}v\widetilde{F}(z). \tag{4.1}$$

The anticipative random symbol $\overline{\sigma}$ is more involved and only given implicitly. For this purpose, we define

$$\Phi_n(u, v) = \mathbb{E}[\exp(-u^2C/2 + \mathrm{i}vF)(e_1^n(u) - 1)\psi_n] \tag{4.2}$$

where $e_t^n(u) = \exp(\mathrm{i}uM_t^n + u^2C_t^n/2)$ and ψ_n is a truncation functional that takes values in $[0, 1]$ satisfying at least (i) $\mathbb{P}[\psi_n = 1] = 1 - o(r_n^{1+\kappa})$ as $n \to \infty$ for some positive constant κ, and (ii) $C_n - C$ is bounded whenever $\psi_n = 1$. We observe that $(e_t^n(u))_{t\in[0,1]}$ is an exponential martingale, that is integrable under the truncation by ψ_n. Computations of $\Phi_n(u, v)$ can be done as if $\psi_n = 1$ in practice since the effect of the truncation is asymptotically negligible.

For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$, [1] let $|\alpha| = \alpha_1 + \alpha_2$. For a function of two variables, we use the following differential operator notations:

$$d^\alpha = d_{x_1}^{\alpha_1} d_{x_2}^{\alpha_2} \text{ and } \partial^\alpha = \mathrm{i}^{-|\alpha|} d^\alpha.$$

Set $\Phi_n^\alpha(u, v) = \partial^\alpha \Phi_n(u, v)$. We suppose that the limit

$$\Phi^\alpha(u, v) = \lim_{n\to\infty} r_n^{-1} \Phi_n^\alpha(u, v)$$

exists and has the form

$$\Phi^\alpha(u, v) = \partial^\alpha \mathbb{E}[\exp(-u^2C/2 + \mathrm{i}vF)\overline{\sigma}(\mathrm{i}u, \mathrm{i}v)] \tag{4.3}$$

for every $\alpha \in \mathbb{Z}_+^2$, where $\overline{\sigma}$ is given by

$$\overline{\sigma} = \sum_j \overline{c}_j (\mathrm{i}u)^{m_j} (\mathrm{i}v)^{n_j} \text{ (finite sum)} \tag{4.4}$$

where $n_j, m_j \in \mathbb{N}$ and $\overline{c}_j$ are random variables. See [13] for details of the random symbols.

[1] $\mathbb{N}_0 = \mathbb{Z}_+ = \{0, 1, 2, ...\}$.

Remark 4.1. We note that the random symbol $\underline{\sigma}$ dates back to [11] which deals with a martingale expansion associated with a normal limit. On the other hand, the random symbol $\overline{\sigma}$ first appeared in [13] and is due to the mixed normality of the limit. Indeed, if C is deterministic, we may pretend $\psi_n = 1$ by a suitable stopping argument and obtain $\Psi_n(u, v) = 0$ due to the martingale property of $e_t^n(u)$. That means $\overline{\sigma} = 0$.

4.2 The asymptotic expansion of (Z_n, F_n)

We define the full random symbol σ by

$$\sigma = \underline{\sigma} + \overline{\sigma}.$$

We recall from (4.1) and (4.4) that $\underline{\sigma}$ and $\overline{\sigma}$ are finite polynomials in $(\mathrm{i}u, \mathrm{i}v)$ with random coefficients. Hence, σ admits the decomposition

$$\sigma(z, \mathrm{i}u, \mathrm{i}v) = \sum_j c_j(z)(\mathrm{i}u)^{m_j}(\mathrm{i}v)^{n_j} \text{ (finite sum)}$$

for some $n_j, m_j \in \mathbb{N}$. We set the approximated density of (Z_n, F_n) as

$$\begin{aligned} p_n(z, x) =& \mathbb{E}[\phi(z; 0, C)|F = x]p^F(x) \\ &+ r_n \sum_j (-d_z)^{m_j}(-d_x)^{n_j} \left(\mathbb{E}[c_j(z)\phi(z; 0, C)|F = x]p^F(x)\right), \end{aligned} \tag{4.5}$$

where $\phi(\cdot; 0, b^2)$ and p^F denote the densities of $N(0, b^2)$ and F, respectively. We note that p^F exists due to the condition that will be imposed later. For $K, \gamma > 0$, let

$$\mathcal{E}(K, \gamma) = \left\{h : \mathbb{R}^2 \to \mathbb{R} | h \text{ is measurable and } |h(z, x)| \leqslant K(1 + |z| + |x|)^\gamma\right\}.$$

For $h \in \mathcal{E}(K, \gamma)$, we denote

$$\Delta_n(h) = \left|\mathbb{E}[h(Z_n, F_n)] - \int h(z, x)p_n(z, x)dzdx\right|.$$

We are now at the stage to recall a basic result. We need elements of Malliavin calculus to state it; see e.g. the book [8] for the main concepts. In what follows, we will only treat one-dimensional functionals M^n and F_n, and a two-dimensional Gaussian process as the input process, for simplicity of notation. However, it is sufficient for the purpose of this paper.

For $\mathbb{H} = L^2([-1, 1] \times \{1, 2\}, dt \times \nu)$, ν being the counting measure, let $w = (w(\mathsf{h}))_{\mathsf{h} \in \mathbb{H}}$ be a Gaussian process associated with the Hilbert space $\mathbb{H}$. That is, w is a family of centered Gaussian random variables such that $\mathbb{E}[w(\mathsf{h})w(\mathsf{g})] = \int_{[-1,1]\times\{1,2\}} \mathsf{h}\mathsf{g}\, dx d\nu$ for $\mathsf{h}, \mathsf{g} \in \mathbb{H}$. The Malliavin derivative is denoted by D, while its dual, also called divergence operator, is denoted by $\delta = D^*$. For a separable Hilbert space E, the Sobolev spaces $\mathbb{D}_{s,p}(\mathsf{E})$ of E-valued random variables are well defined, where s is the index of differentiability and p is the index of integrability. We simply write $\mathbb{D}_{s,p}$ for $\mathbb{D}_{s,p}(\mathbb{R})$. Let $\mathbb{D}_{s,\infty}(E) = \bigcap_{p>1} \mathbb{D}_{s,p}(E)$.

For a multivariate functional $U = (U_1, \ldots, U_d)$ the Malliavin covariance matrix of U is given by $\sigma_U := (\langle DU_i, DU_j \rangle_{\mathbb{H}})_{1\leqslant i,j\leqslant d}$. We also set $\Delta_U := \det \sigma_U$.

Besides [B1], we will consider the following conditions. We note that the $\mathbb{R}$-valued functional ξ_n appearing below is used to construct a truncation functional ψ_n.

$[\mathbf{B2}]_\ell$ **(i)** $F \in \mathbb{D}_{\ell,\infty}$ and $C \in \mathbb{D}_{\ell,\infty}$.

(ii) $M_n \in \mathbb{D}_{\ell+1,\infty}$, $F_n \in \mathbb{D}_{\ell+1,\infty}$, $C_n \in \mathbb{D}_{\ell,\infty}$, $N_n \in \mathbb{D}_{\ell+1,\infty}$ and $\xi_n \in \mathbb{D}_{\ell,\infty}$. Moreover, for every $p > 1$,

$$\sup_{n\in\mathbb{N}} \Big\{ \|M_n\|_{\ell+1,p} + \|\hat{C}_n\|_{\ell+1,p} + \|\hat{F}_n\|_{\ell+1,p} + \|N_n\|_{\ell+1,p} + \|\xi_n\|_{\ell,p} \Big\} < \infty$$

$[\mathbf{B3}]$ **(i)** $\lim_{n\to\infty} \mathbb{P}[|\xi_n| \leqslant 1/2] = 1$.

(ii) $|C_n - C| > r_n^{1-a}$ implies $|\xi_n| \geqslant 1$, where $a \in (0, 1/3)$ is a constant.

(iii) For any $p > 1$, $\limsup_{n\to\infty} \mathbb{E}\big[1_{\{|\xi_n|\leqslant 1\}} \Delta_{(M_n,F)}^{-p}\big] < \infty$ and $C^{-1} \in \bigcap_{p>1} L^p$.

$[\mathbf{B4}]_{\ell,\mathfrak{m},\mathfrak{n}}$ **(i)** $\underline{\sigma}$ is a random symbol of the form

$$\underline{\sigma}(z, \mathtt{i}u, \mathtt{i}v) \quad = \quad \sum_j b_j z^{k_j} (\mathtt{i}u)^{m_j} (\mathtt{i}v)^{n_j} \qquad (b_j \in \mathbb{D}_{4,\infty},\ m_j \leqslant 2,\ n_j \leqslant 1)$$

(ii) There exists a random symbol $\overline{\sigma}$ having a representation

$$\overline{\sigma}(\mathtt{i}u, \mathtt{i}v) \quad = \quad \sum_j c_j (\mathtt{i}u)^{m_j} (\mathtt{i}v)^{n_j} \qquad (c_j \in \mathbb{D}_{\ell,\infty},\ m_j \leqslant \mathfrak{m},\ n_j \leqslant \mathfrak{n})$$

and (4.3) holds for every $\alpha \in \mathbb{Z}_+^2$.

In assumption [B5], the term Φ_n (see (4.2)) involves the truncation functional ψ_n which is described below. Suppose that $\psi : \mathbb{R} \to [0, 1]$ in $C^\infty(\mathbb{R})$ satisfies $\psi(x) = 1$ if $|x| \leqslant 1/2$ and $\psi(x) = 0$ if $|x| \geqslant 1$. Let $Q_n = (M_n, F)$ and $R_n = (N_n, \hat{F}_n)$ and define a random matrix R'_n by

$$R'_n \quad = \quad \sigma_{Q_n}^{-1}\big(r_n \langle DQ_n, DR_n \rangle_{\mathbb{H}} + r_n \langle DR_n, QR_n \rangle_{\mathbb{H}} + r_n^2 \langle DR_n, DR_n \rangle_{\mathbb{H}}\big).$$

Then $\sigma_{(Z_n,C_n)} = \sigma_{Q_n}(I_2 + R'_n)$. Let

$$\xi'_n \quad = \quad r_n^{-1} |R'_n|^2.$$

Then, the truncation functional ψ_n is composed by

$$\psi_n \quad = \quad \psi(\xi_n)\psi(\xi'_n). \tag{4.6}$$

The functional ξ_n will be set more concretely in Section 8.5 for our application.

[**B5**] For every $\alpha \in \mathbb{Z}_+^2$ and some $\varepsilon = \varepsilon(\alpha) \in (0,1)$,

$$\limsup_{n\to\infty} \sup_{(u,v)\in\Lambda_n^0(2,q)} r_n^{-1}|(u,v)|^{3-\varepsilon}|\Phi_n^\alpha(u,v)| \quad < \quad \infty,$$

where $\Lambda_n^0(2,q) = \{(u,v);\ |(u,v)| \leqslant r_n^{-q}\}$ and $q = (1-a)/2$.

The following customizes Theorem 1 of [13].

Theorem 4.2. *Let* $\mathfrak{n} = \max_j n_j$ *and* $\ell = \max(5, 2[(\mathfrak{n}+3)/2])$. *Let* K, $\gamma \in (0,\infty)$ *and* $\kappa \in (0,1)$ *be arbitrary numbers. Suppose that Conditions* $[B1]$, $[B2]_\ell$, $[B3]$, $[B4]_{\ell,\mathfrak{m},\mathfrak{n}}$ *and* $[B5]$ *are satisfied. Then for some constant* $K_1 = K_1(K,\gamma,\kappa)$,

$$\sup_{h\in\mathcal{E}(K,\gamma)} \Delta_n(h) \quad \leqslant \quad K_1\mathbb{P}\big[|\xi_n| > 1/2\big]^\kappa + o(r_n). \tag{4.7}$$

In other words, p_n is the second order Edgeworth expansion of the distribution of the pair (Z_n, F_n), if the event truncated by ξ_n is sufficiently small.

See [13] for details of the above theorem and other information, and also arXiv 1210.3680v3 for updates. The Malliavin calculus is used to derive the asymptotic expansion formula p_n. Further, one needs the non-degeneracy of the Malliavin covariance since the problem of validity of the asymptotic expansion is deeply related with the regularity of the distribution of the underlying functional. There is a counterexample, even in the classical expansion, if [B3] (iii) is not satisfied. Condition [B5] is also a requirement for the non-degeneracy of the same kind but regarding the correction term corresponding to the anticipative random symbol.

Remark 4.3. Under a stronger assumption that $\mathbb{P}[|\xi_n| \leqslant 1/2] = 1 - O(r_n^\kappa)$ for any $\kappa > 0$, in place of [B3] (i), we can simply use $\psi_n = \psi(\xi_n)$ without ξ_n' apparently, and remove the first term on the right-hand side of (4.7). This makes presentation of the result slightly simpler though the deeper truncation (4.6) is re-constructed in the proof. In this case, Condition [B5] may also become stronger since the truncation reduces.

5 Main results: Asymptotic expansion for the pre-averaging estimator

In this section, we utilise the results from the general theory for a mixed normal limit and obtain the Edgeworth expansion for the pre-averaging estimator.

5.1 Assumptions

We will consider F_n defined in (2.8) as a consistent estimator of C. We denote by C_b^∞ the set of smooth functions on $\mathbb{R}$ whose all derivatives of positive orders are bounded. Let

$$a(x) \quad = \quad 2\theta\left((b^{[1]}(x))^2 + \frac{\omega^2\psi_1}{\theta^2\psi_2}\right)^2.$$

We are assuming that supp$\mathcal{L}\{X_0\}$ is compact and that ω is positive. We impose the following condition on the processes $b^{[1]}$ and $b^{[2]}$.

[V] **(i)** $b^{[1]}, b^{[2]} \in C_b^\infty$ and $b^{[1]}(x) \neq 0$ for $x \in \mathrm{supp}\, \mathcal{L}\{X_0\}$.

(ii) $\sum_{k=1}^\infty |d_x^k a(X_0)| > 0$.

Remark 5.1. If $b^{[1]}$ is nonnegative, Condition [V] (ii) can be replaced by

(ii′) $\sum_{k=1}^\infty |d_x^k b^{[1]}(X_0)| > 0$.

Remark 5.2. By assumption, supp $\mathcal{L}\{X_0\}$ is compact. We do not assume uniform ellipticity of the diffusion on the whole domain of $b^{[1]}$. The microstructure noise serves as a smoother of the distribution of M_n. On the other hand, we need regularity of the distribution of C defined in Theorem 2.1. Practically this would be satisfied once C is random. If C is deterministic, the problem of asymptotic expansion becomes a classical one that is tractable by [11]. We refer to Section 6 for the exposition of this setting.

5.2 Stable limit theorem

We have seen in the previous section that the random coefficients of $\underline{\sigma}$ solely depend on the limit of the stable convergence found in the condition **[B1]**. Hence, we need to compute $M, N, \widehat{C}, \widehat{F}$. In this section, we assume that

$$\varepsilon_{t_i} = \omega \Delta_n^{-1/2}(B_{t_i} - B_{t_{i-1}}) \qquad (i \geqslant 0) \tag{5.1}$$

where for the Gaussian process $w = (w(\mathsf{h}))_{\mathsf{h}\in\mathbb{H}}$, $B_t = w(\mathbb{1}_{[-1,t]\times\{2\}})$ for $t \in [-1, 1]$, as well as $W_t = w(\mathbb{1}_{[0,t]\times\{1\}})$ for $t \in [0, 1]$. We assume that w is independent of X_0. The Malliavin derivative in the directions of W and B are denoted by $D^{(1)}$ and $D^{(2)}$, respectively.

The particular Gaussian framework of the model (5.1) is imposed to be able to use Malliavin calculus. Our results can be directly extended to a more general setting

$$\varepsilon_{t_i} = \Delta_n^{-1/2} \int_{t_{i-1}}^{t_i} \omega_s dB_s,$$

where ω is adapted to the filtration $\mathcal{G}_t = \sigma(X_0, (W_s)_{s\leqslant t})$, under mild assumptions on the stochastic process $(\omega_t)_{t\geqslant 0}$ (cf. [5]). However, we dispense with the detailed exposition of this case.

Similarly to $g_n(s)$ and $W(i, t)$, we define

$$h_n(s) = \sum_{j=0}^{k_n-1} h(j/k_n) 1_{(t_{j-1},t_j]}(s),$$

$$\varepsilon(i, t) = -\omega \Delta_n^{-1/2} \int_{t_{ik_n-1}\wedge t}^{t_{(i+1)k_n-1}\wedge t} h_n(u - t_{ik_n}) dB_u.$$

We remark that

$$\bar{\varepsilon}_{t_{ik_n}} = \varepsilon(i, t_{(i+1)k_n}) = \varepsilon(i, t_{(i+1)k_n - 1}).$$

Let

$$\alpha(i,u) \quad = \quad b^{[1]}_{t_{ik_n}} W(i,u) + \varepsilon(i,u)$$

and remark that $\alpha(i,u)$, conditionally on $\mathcal{F}_{t_{ik_n}}$, is distributed as

$$N\left(0, \int_{t_{ik_n - 1}}^{u} [g_n(u - t_{ik_n})^2 (b^{[1]}_{t_{ik_n}})^2 + \omega^2 \Delta_n^{-1} h_n(u - t_{ik_n})^2] du\right).$$

We will consider the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,1]}$, each $\mathcal{F}_t$ being generated by X_0, $\{W_s\}_{s\in[0,t]}$ and $\{B_s\}_{s\in[-1,t]}$. Itô's lemma implies that M_n is the terminal value of the $\mathbf{F}$-continuous martingale

$$M^n_t = \frac{2\Delta_n^{-1/4}}{\psi^n_2} \sum_{i=0}^{d_n - 1} \int_{t_{ik_n}\wedge t}^{t_{(i+1)k_n}\wedge t} \alpha(i,u) d\alpha(i,u). \tag{5.2}$$

According to the expression (5.2), we observe that the quadratic variation of M^n satisfies

$$C_n = \langle M^n \rangle_1 \quad = \quad \frac{4\Delta_n^{-1/2}}{(\psi^n_2)^2} \sum_{i=0}^{d_n - 1} \int_{t_{ik_n}}^{t_{(i+1)k_n}} \alpha^2(i,u) [d\alpha(i,u)]^2,$$

where

$$[d\alpha(i,u)]^2 = [(b^{[1]}_{t_{ik_n}})^2 g_n(u - t_{ik_n})^2 + \omega^2 \Delta_n^{-1} h_n(u - t_{ik_n})^2] du. \tag{5.3}$$

Before going to the stable limit theorem, we observe that some of terms included in N_n converge in probability. We want to separate them from others.

Lemma 5.3. *We obtain the convergence in probability*

$$N_{n,k} \xrightarrow{\mathbb{P}} N_k, \ k = 2, 3, 4$$

where the quantities $N_{n,k}$ are defined in Proposition 3.1 and

$$\begin{aligned}
N_2 =& \frac{\theta(\psi_3)^2}{\psi_2} \int_0^1 b^{[1]}_u b^{[2.1]}_u du, \\
N_3 =& \frac{\theta(\psi_3)^2}{\psi_2} \int_0^1 (b^{[2]}_u)^2 du, \\
N_4 =& \frac{\theta(2\psi_4 - \psi_2)}{2\psi_2} \int_0^1 2b^{[1]}_u b^{[1.2]}_u + (b^{[1.1]}_u)^2 du,
\end{aligned}$$

After recalling the following notation used in the previous section

$$\widehat{C}_n = \Delta_n^{-1/4}(C_n - C),\ \widehat{F}_n = \Delta_n^{-1/4}(F_n - C),$$

we are ready to state our stable convergence result.

Theorem 5.4. *We obtain the stable convergence*

$$\left(M_n, N_n, \widehat{C}_n, \widehat{F}_n\right) \xrightarrow{d_{st}} (M, N, \widehat{C}, \widehat{F}) \sim MN\left(\mu, \int_0^1 \Sigma_s ds\right),$$

where $\mu_1 = \mu_3 = \mu_4 = \Sigma_s^{12} = \Sigma_s^{23} = \Sigma_s^{24} = 0$,

$$\mu_2 = \int_0^1 \mu_2(b_s^{[1]}, b_s^{[2]}, b_s^{[1.1]})dW_s + \sum_{k=2}^4 N_k$$

and

$$\Sigma_s^{11} = 2\theta\left((b_s^{[1]})^2 + \frac{\omega^2\psi_1}{\theta^2\psi_2}\right)^2, \quad \Sigma_s^{22} = \sigma_2(b_s^{[1]}, b_s^{[2]}, b_s^{[1.1]}) - \left[\mu_2(b_s^{[1]}, b_s^{[2]}, b_s^{[1.1]})\right]^2$$

$$\Sigma_s^{33} = \frac{16\theta^3}{3}\left((b_s^{[1]})^2 + \frac{\omega^2\psi_1}{\theta^2\psi_2}\right)^4, \quad \Sigma_s^{44} = \frac{128}{3}\theta^3\left((b_s^{[1]})^2 + \frac{\omega^2\psi_1}{\theta^2\psi_2}\right)^4$$

$$\Sigma_s^{13} = \frac{8\theta^2}{3}\left((b_s^{[1]})^2 + \frac{\omega^2\psi_1}{\theta^2\psi_2}\right)^3, \quad \Sigma_s^{14} = 8\theta^2\left((b_s^{[1]})^2 + \frac{\omega^2\psi_1}{\theta^2\psi_2}\right)^3$$

$$\Sigma_s^{34} = \frac{44\theta^2}{3}\left((b_s^{[1]})^2 + \frac{\omega^2\psi_1}{\theta^2\psi_2}\right)^3,$$

with

$$\mu_2(x,y,z) = \frac{\theta x}{\psi_2}\left([(\psi_3)^2 + 2\psi_4 - \psi_2]z + 2(\psi_3)^2 y\right)$$

$$\sigma_2(x,y,z) = \frac{4\theta^2 x^2}{\psi_2^2}\left[(\psi_7 + \psi_6)z^2 + (4\psi_5 - \psi_2)\psi_3 yz + \psi_3^2\psi_2 y^2\right] + \frac{4\omega^2\psi_1}{\psi_2^2}\left[\psi_3^2 y^2 + \psi_4 z^2\right] + \frac{3\omega^4\psi_1^2}{\theta^4\psi_2^2}.$$

Proof. See Section 8. □

5.3 Computation of $\underline{\sigma}$ and $\overline{\sigma}$

It is now straightword to compute $\underline{\sigma}$. Indeed, the mixed normality of the limit in Theorem 5.4 and (4.1) imply

$$\underline{\sigma}(z, \mathrm{i}u, \mathrm{i}v) = (\mathrm{i}u)^2\mathcal{H}_1(z) + \mathrm{i}u\mathcal{H}_2 + \mathrm{i}v\mathcal{H}_3(z) \tag{5.4}$$

where

$$\mathcal{H}_1(z) = z\frac{\int_0^1 \Sigma_s^{13}ds}{2\int_0^1 \Sigma_s^{11}ds}, \quad \mathcal{H}_2 = \mu_2, \quad \mathcal{H}_3(z) = z\frac{\int_0^1 \Sigma_s^{14}ds}{\int_0^1 \Sigma_s^{11}ds}.$$

Now, we pass to the calculation of $\overline{\sigma}$. The anticipative random symbol $\overline{\sigma}$ in (4.3) is characterized by

$$\Phi^{\alpha}(u,v) = \partial^{\alpha}\mathbb{E}[\exp((-u^2/2+\mathrm{i}v)C)\overline{\sigma}(\mathrm{i}u,\mathrm{i}v)] \tag{5.5}$$

in the present situation. Using techniques from Malliavin calculus, we obtain the following result.

Proposition 5.5. *We obtain the identity*

$$\overline{\sigma}(\mathrm{i}u,\mathrm{i}v) = \mathrm{i}u\left(-\frac{u^2}{2}+\mathrm{i}v\right)^2\mathcal{H}_4 + \mathrm{i}u\left(-\frac{u^2}{2}+\mathrm{i}v\right)\mathcal{H}_5, \tag{5.6}$$

where $c(x) = \left[(b^{[1]})^2(x) + \frac{\omega^2\psi_1}{\theta^2\psi_2}\right]^2$ *and*

$$\begin{aligned}
\mathcal{H}_4 =& \frac{4\theta^3\psi_3^2}{\psi_2}\int_0^1 (b^{[1]})^2(X_t)\left(\int_t^1 c'(X_r)D_t^{(1)}X_r dr\right)^2 dt,\\
\mathcal{H}_5 =& \frac{2\theta^2\psi_3^2}{\psi_2}\int_0^1 (b^{[1]})^2(X_t)\left(\int_t^1 \left[c''(X_r)(D_t^{(1)}X_r)^2 + c'(X_r)D_t^{(1)}D_t^{(1)}X_r\right]dr\right)dt.
\end{aligned}$$

Proof. See Section 8. □

5.4 The asymptotic expansion of the pre-averaging estimator

In view of (5.4) and (5.6), we observe that the full random symbol $\sigma = \underline{\sigma} + \overline{\sigma}$ is given by

$$\sigma(z,\mathrm{i}u,\mathrm{i}v) = \sum_{j=1}^{8} c_j(z)(\mathrm{i}u)^{m_j}(\mathrm{i}v)^{n_j} \tag{5.7}$$

where

$$\begin{aligned}
&m_1 = 1,\ \ n_1 = 0,\ \ c_1(z) = \mathcal{H}_2, && m_2 = 0,\ \ n_2 = 1,\ \ c_2(z) = \mathcal{H}_3(z),\\
&m_3 = 2,\ \ n_3 = 0,\ \ c_3(z) = \mathcal{H}_1(z), && m_4 = 1,\ \ n_4 = 1,\ \ c_4(z) = \mathcal{H}_5,\\
&m_5 = 3,\ \ n_5 = 0,\ \ c_5(z) = \frac{1}{2}\mathcal{H}_5, && m_6 = 1,\ \ n_6 = 2,\ \ c_6(z) = \mathcal{H}_4,\\
&m_7 = 3,\ \ n_7 = 1,\ \ c_7(z) = \mathcal{H}_4, && m_8 = 5,\ \ n_8 = 0,\ \ c_8(z) = \frac{1}{4}\mathcal{H}_4.
\end{aligned}$$

We continue as in Section 4.2 and define the density $p_n(z,x)$ by

$$\begin{aligned}
p_n(z,x) =& \phi(z;0,x)p^C(x) \\
&+ \Delta_n^{1/4}\sum_{j=1}^{8}(-d_z)^{m_j}(-d_x)^{n_j}\left(\phi(z;0,x)p^C(x)\mathbb{E}[c_j(z)|C=x]\right),
\end{aligned} \tag{5.8}$$

according to (4.5). For $h \in \mathcal{E}(K,\gamma)$, we also recall the notation

$$\Delta_n(h) = \left| \mathbb{E}[h(Z_n^\star, F_n)] - \int h(z,x) p_n(z,x) dz dx \right|.$$

The following theorem is the main result of this article.

Theorem 5.6. *Suppose that Condition $[V]$ is satisfied. Let $K > 0, \gamma > 0$. Then*

$$\sup_{h \in \mathcal{E}(K,\gamma)} \Delta_n(h) = o(\Delta_n^{1/4})$$

Proof. See Section 8. □

This theorem is not the end of the story. From the point of view of statistical applications, the Edgeworth expansion associated to the studentized statistic $Z_n^\star/\sqrt{F_n}$ is more interesting. Since the representation of σ in (5.7) is the same as in [10, Section 6], we easily obtain the second order Edgeworth expansion of $Z_n^\star/\sqrt{F_n}$.

Corollary 5.7. *Suppose that Condition $[V]$ is fulfilled. We define the random variables $\widetilde{\mathcal{H}}_1$ and $\widetilde{\mathcal{H}}_3$ via the identity $\mathcal{H}_k(z) = z\widetilde{\mathcal{H}}_k$, $k = 1, 3$. Then the second order Edgeworth expansion of $Z_n^\star/\sqrt{F_n}$ is given by*

$$\begin{aligned} p^{Z_n^\star/\sqrt{F_n}}(y) &= \phi(y;0,1) + \Delta_n^{1/4}\phi(y;0,1)\Big[y\Big(\mathbb{E}[\mathcal{H}_2 C^{-1/2}] - \frac{1}{2}\mathbb{E}[\mathcal{H}_5 C^{-3/2}] \\ &+ \frac{3}{4}\mathbb{E}[\mathcal{H}_4 C^{-5/2}] + \mathbb{E}[\widetilde{\mathcal{H}}_3 C^{-1/2}] - 3\mathbb{E}[\widetilde{\mathcal{H}}_1 C^{-1/2}]\Big) \\ &+ y^3 \Big(\mathbb{E}[\widetilde{\mathcal{H}}_1 C^{-1/2}] - \frac{1}{2}\mathbb{E}[\widetilde{\mathcal{H}}_3 C^{-1/2}]\Big)\Big] \end{aligned}$$

Note that the polynomial involved in the second order term is odd of order 3. However, in general it is not connected to the third order Hermite polynomial, which appears in the classical Edgeworth expansion in the framework of i.i.d. observations, see e.g. Theorem 2.5 in [3].

6 The case of constant volatility

The main focus of this paper was to investigate asymptotic expansions when the estimated object C is random, as seen in the previous sections. We remark however that Condition $[V]$(ii) is not satisfied when $b^{[1]}(x) = b$ for all x. In particular, the asymptotic expansion of Corollary 5.7 can not be directly applied to the case of constant volatility.

For the sake of completeness, we thus present the second order Edgeworth expansion in the setting $b^{[1]}(x) = b$ identically, which relies on an earlier work [11]. This article studies asymptotic expansions associated with a classical central limit theorem and it does not require Condition $[V]$(ii). We note that the expression for the asymptotic density $p^{Z_n^\star/\sqrt{F_n}}$ simplifies quite a bit in the case of constant volatility. In particular, in view of Remark 4.1, we obtain that $\overline{\sigma} = 0$. Hence, $\mathcal{H}_4 = \mathcal{H}_5 = 0$. The following version of Corollary 5.7 is a consequence of [11, Theorem 1].

Theorem 6.1. *Suppose that $b^{[1]}(x) = b$ identically and Condition $[V]$(i) holds. Then the second order Edgeworth expansion of $Z_n^\star/\sqrt{F_n}$ is given by*

$$p^{Z_n^\star/\sqrt{F_n}}(y) = \phi(y;0,1) \\ + \Delta_n^{1/4}\phi(y;0,1)C^{-1/2}\Big[y\Big(\mathbb{E}[\mathcal{H}_2] + \widetilde{\mathcal{H}}_3 - 3\widetilde{\mathcal{H}}_1\Big) + y^3\Big(\widetilde{\mathcal{H}}_1 - \frac{1}{2}\widetilde{\mathcal{H}}_3\Big)\Big].$$

Proof. First, we notice that Condition $[V]$(i) implies the assumptions of [11, Theorem 1]. The asymptotic expansion of [11, Theorem 1] has not been obtained for the pair $(Z_n^\star, F_n)$, hence we require a stochastic expansion for the studentized statistic $Z_n^\star/\sqrt{F_n}$ directly. Denoting $r_n = \Delta_n^{1/4}$, the Taylor expansion yields

$$\frac{Z_n^\star}{\sqrt{F_n}} = (M_n + r_n N_n)\left(\frac{1}{C^{1/2}} - \frac{(F_n - C)}{2C^{3/2}} + o_{\mathbb{P}}(r_n)\right) \\ =: \mathcal{M}_n + r_n\mathcal{N}_n,$$

where, recalling the notation $\widehat{F}_n = r_n^{-1}(F_n - C)$, we have

$$\mathcal{M}_n = \frac{M_n}{C^{1/2}} \quad \text{and} \quad \mathcal{N}_n = \frac{N_n}{C^{1/2}} - \frac{M_n\widehat{F}_n}{2C^{3/2}} + o_{\mathbb{P}}(1).$$

Let us denote $\mathcal{C}_n := C_n/C$. Applying Theorem 5.4, we deduce the joint stable convergence

$$(\mathcal{M}_n, \mathcal{N}_n, r_n^{-1}(\mathcal{C}_n - 1)) \xrightarrow{d_{st}} \left(\frac{M}{C^{1/2}}, \frac{N}{C^{1/2}} - \frac{M\widehat{F}}{2C^{3/2}}, \frac{\widehat{C}}{C}\right) =: (\mathcal{M}, \eta, \xi),$$

where ξ and η follow the notation from Theorem 1 of [11]. Recalling equations (4.1) and (5.4), we observe the identities

$$\mathbb{E}[\eta|\mathcal{M} = z] = \frac{\mathbb{E}[\mathcal{H}_2]}{C^{1/2}} - \frac{z^2\widetilde{\mathcal{H}}_3}{2C^{1/2}} \quad \text{and} \quad \mathbb{E}[\xi|\mathcal{M} = z] = \frac{2z\widetilde{\mathcal{H}}_1}{C^{1/2}}.$$

The second order Edgeworth expansion of [11, Theorem 1] implies the formula

$$p^{Z_n^\star/\sqrt{F_n}}(y) = \phi(y;0,1) + \frac{1}{2}r_n\partial_y^2(\mathbb{E}[\xi|\mathcal{M} = y]\phi(y;0,1)) - r_n\partial_y(\mathbb{E}[\eta|\mathcal{M} = y].$$

A straightforward calculation implies the desired asymptotic expansion. □

7 Example

Example 7.1. Let $a > 0$ and $\sigma > 0$. We consider the Black-Scholes model to illustrate the computations of the previous sections:

$$dX_t = aX_t dt + \sigma X_t dW_t.$$

In this framework we have that

$$b_t^{[1]} = \sigma X_t, \quad b_t^{[1.1]} = \sigma^2 X_t, \quad b_t^{[1.2]} = a\sigma X_t,$$
$$b_t^{[2]} = a X_t, \quad b_t^{[2.1]} = a\sigma X_t, \quad b_t^{[2.2]} = a^2 X_t.$$

Then, we immediately obtain that

$$C = 2\theta \int_0^1 \left(\sigma^2 X_t^2 + \frac{\omega^2\psi_1}{\theta^2\psi_2}\right)^2 dt, \; \widetilde{\mathcal{H}}_1 = \frac{2\theta}{3} \frac{\int_0^1 \left(\sigma^2 X_t^2 + \frac{\omega^2\psi_1}{\theta^2\psi_2}\right)^3 dt}{\int_0^1 \left(\sigma^2 X_t^2 + \frac{\omega^2\psi_1}{\theta^2\psi_2}\right)^2 dt}, \; \widetilde{\mathcal{H}}_3 = 6\widetilde{\mathcal{H}}_1,$$

$$\mathcal{H}_2 = \frac{\theta\sigma[(\psi_3^2 + 2\psi_4 - \psi_2)\sigma^2 + 2\psi_3^2 a]}{\psi_2} \int_0^1 X_t^2 dW_t + \frac{\theta[2\psi_3^2(\sigma^2 a + a^2) + (2\psi_4 - \psi_2)(2\sigma^2 a + \sigma^4)]}{2\psi_2} \int_0^1 X_t^2 dt.$$

We observe that $D_s^{(1)} X_t$ satisfies, for $t \geqslant s$,

$$D_s^{(1)} X_t = a D_s^{(1)} X_t dt + \sigma D_s^{(1)} X_t dW_t, \quad D_s^{(1)} X_s = \sigma X_s.$$

This easily implies

$$D_s^{(1)} X_t = \sigma X_s \exp\left[(a - \sigma^2/2)(t-s) + \sigma(W_t - W_s)\right] = \sigma X_t.$$

Hence, we deduce that $c(x) = \left[\sigma^2 x^2 + \frac{\omega^2\psi_1}{\theta^2\psi_2}\right]^2$ and

$$\mathcal{H}_4 = \frac{4\theta^3\psi_3^2\sigma^4}{\psi_2} \int_0^1 X_t^2 \left(\int_t^1 c'(X_r) X_r dr\right)^2 dt,$$
$$\mathcal{H}_5 = \frac{2\theta^2\psi_3^2\sigma^4}{\psi_2} \int_0^1 X_t^2 \left(\int_t^1 \left[c''(X_r) X_r^2 + c'(X_r) X_r\right] dr\right) dt.$$

Using the above quantities we may obtain the second order Edgeworth expansion of $Z_n^\star/\sqrt{F_n}$ using Corollary 5.7.

8 Proofs

8.1 Sketch of the proof of Theorem 2.1

Without loss of generality, we suppose that the processes $b^{[1]}$ and $b^{[2]}$ are bounded. This is done following a standard localization procedure, see [1] for details. The first order approximation of $\bar{Y}_{t_{ik_n}}$ is given by the process $\alpha_{t_{ik_n}} = b_{t_{ik_n}}^{[1]} \overline{W}_{t_{ik_n}} + \bar{\varepsilon}_{t_{ik_n}}$ (see also the statement of Proposition 3.1). Hence, we obtain that the dominating term in Z_n is

$$M_n = \frac{\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n - 1} \left\{\alpha_{t_{ik_n}}^2 - \mathbb{E}[\alpha_{t_{ik_n}}^2 | \mathcal{F}_{t_{ik_n}}]\right\}.$$

Using the notation

$$\beta_{t_{ik_n}} = \frac{\Delta_n^{-1/4}}{\psi_2^n}\left(\alpha_{t_{ik_n}}^2 - \mathbb{E}[\alpha_{t_{ik_n}}^2|\mathcal{F}_{t_{ik_n}}]\right),$$

we observe that $\beta_{t_{ik_n}}$ is $\mathcal{F}_{t_{(i+1)k_n}}$-measurable and $\mathbb{E}[\beta_{t_{ik_n}}|\mathcal{F}_{t_{ik_n}}] = 0$ holds. Moreover, we get

$$\sum_{i=0}^{d_n-1}\mathbb{E}[\left(\beta_{t_{ik_n}}\right)^2|\mathcal{F}_{t_{ik_n}}] = \frac{2\Delta_n^{-1/2}}{(\psi_2^n)^2}\sum_{i=0}^{d_n-1}\left((b_{t_{ik_n}}^{[1]})^2\psi_2^n k_n\Delta_n + \frac{\psi_1^n\omega^2}{k_n}\right)^2 \xrightarrow{\mathbb{P}} C.$$

Therefore, the first claim follows from Theorem IX.7.28 of [7]. As for the second claim, we again observe that the first order approximation of $\bar{Y}_{t_{ik_n}}$ is given by the process $\alpha_{t_{ik_n}}$. Moreover, we see that the main term in F_n satisfies

$$\frac{2\Delta_n^{-1/2}}{3(\psi_2^n)^2}\sum_{i=0}^{d_n-1}\mathbb{E}[\left(\alpha_{t_{ik_n}}\right)^4|\mathcal{F}_{t_{ik_n}}] = 2\Delta_n^{-1/2}\sum_{i=0}^{d_n-1}\left((b_{t_{ik_n}}^{[1]})^2 k_n\Delta_n + \frac{\psi_1^n\omega^2}{\psi_2^n k_n}\right)^2 \xrightarrow{\mathbb{P}} C.$$

□

8.2 Proofs of Proposition 3.1 and Theorem 5.4

Proof of Proposition 3.1. Proceeding as the in previous section, we apply a localization procedure and suppose that all processes of the form $b^{[k_1\dots k_m]}$, $k_i = 1,2$, are bounded. We will apply the following version of Burkholder's inequality several times. For any process U as in (2.1) with bounded drift and diffusion terms and any $p \geqslant 0$, we have

$$\mathbb{E}[|U_t - U_s|^p] \leqslant C_p|t-s|^{p/2}. \tag{8.1}$$

Hence, we may apply this result for the following terms: $b^{[1]}$, $b^{[2]}$, $b^{[1.1]}$, $b^{[1.2]}$, $b^{[2.1]}$ and $b^{[2.2]}$. We expand and denote

$$\begin{aligned}\Delta_n^{-1/4}V_n = &\frac{\Delta_n^{-1/4}}{\psi_2^n}\sum_{i=0}^{d_n-1}\left(\bar{X}_{t_{ik_n}}\right)^2 + \frac{2\Delta_n^{-1/4}}{\psi_2^n}\sum_{i=0}^{d_n-1}\bar{X}_{t_{ik_n}}\bar{\varepsilon}_{t_{ik_n}}\\ &+ \frac{\Delta_n^{-1/4}}{\psi_2^n}\sum_{i=0}^{d_n-1}\left[\bar{\varepsilon}_{t_{ik_n}}^2 - \frac{\psi_1^n\omega^2}{k_n}\right] + \frac{\Delta_n^{-1/4}\psi_1^n d_n}{\psi_2^n k_n}\left[\omega^2 - \frac{\Delta_n}{2}\sum_{i=1}^{1/\Delta_n}(\Delta Y_i^n)^2\right]\\ &=: R_n^{(1)} + R_n^{(2)} + R_n^{(3)} + \Delta_n^{1/4}N_{n,6} + o_{\mathbb{P}}(\Delta_n^{1/4}).\end{aligned} \tag{8.2}$$

Let's look at the term $R_n^{[2]}$ first. Due to

$$\bar{X}_{t_{ik_n}} = b_{t_{ik_n}}^{[1]}\overline{W}_{t_{ik_n}} + \int_{t_{ik_n}}^{t_{(i+1)k_n}}\left[(b_u^{[1]} - b_{t_{ik_n}}^{[1]})dW(i,u) + g_n(u - t_{ik_n})b_u^{[2]}du\right],$$

we obtain

$$
\begin{aligned}
R_n^{(2)} =& \frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} \left\{ b_{t_{ik_n}}^{[1]} \overline{W}_{t_{ik_n}} \bar{\varepsilon}_{t_{ik_n}} + \int_{t_{ik_n}}^{t_{(i+1)k_n}} (b_u^{[1]} - b_{t_{ik_n}}^{[1]}) dW(i,u) \times \bar{\varepsilon}_{t_{ik_n}} \right\} \\
&+ \frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} \int_{t_{ik_n}}^{t_{(i+1)k_n}} g_n(u - t_{ik_n}) b_u^{[2]} du \times \bar{\varepsilon}_{t_{ik_n}} \\
=& \frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} b_{t_{ik_n}}^{[1]} \overline{W}_{t_{ik_n}} \bar{\varepsilon}_{t_{ik_n}} \\
&+ \frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} \left(b_{t_{ik_n}}^{[1.1]} \int_{t_{ik_n}}^{t_{(i+1)k_n}} \int_{t_{ik_n}}^{u} dW_s dW(i,u) + b_{t_{ik_n}}^{[2]} \psi_3^n k_n \Delta_n \right) \bar{\varepsilon}_{t_{ik_n}} \\
&+ o_{\mathbb{P}}(\Delta_n^{1/4}) \\
=:& R_n^{(2.1)} + \Delta_n^{1/4} N_{n,5} + o_{\mathbb{P}}(\Delta_n^{1/4}). \qquad (8.3)
\end{aligned}
$$

In above computation, $o_{\mathbb{P}}(\Delta_n^{1/4})$-error terms were obtained by applying (8.1) to the processes $b^{[1.1]}$ and $b^{[2]}$. Let's provide some details. (8.1) applied to the process $b^{[2]}$ implies

$$
\mathbb{E}[|b_u^{[2]} - b_{t_{ik_n}}^{[2]}|^p] \leqslant C_p \Delta_n^{p/4}.
$$

for each $0 \leqslant i \leqslant d_n - 1$ and $t_{ik_n} \leqslant u \leqslant t_{(i+1)k_n}$. Then, we obtain

$$
\int_{t_{ik_n}}^{t_{(i+1)k_n}} g_n(u - t_{ik_n})(b_u^{[2]} - b_{t_{ik_n}}^{[2]}) du \times \bar{\varepsilon}_{t_{ik_n}} = O_{\mathbb{P}}(\Delta_n) \text{ and}
$$

$$
\frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} \int_{t_{ik_n}}^{t_{(i+1)k_n}} g_n(u - t_{ik_n})(b_u^{[2]} - b_{t_{ik_n}}^{[2]}) du \times \bar{\varepsilon}_{t_{ik_n}} = O_{\mathbb{P}}(\Delta_n^{1/2}),
$$

where we used the independece of X and ε, and the i.i.d assumption on ε for the second result above.

Now, we pass to the expansion of $R_n^{(1)}$. Analogous to $W(i,t)$ defined in (3.6), we define a new process

$$
X(i,t) = \int_{t_{ik_n}}^{t} g_n(u - t_{ik_n}) b_u^{[1]} dW_u + \int_{t_{ik_n}}^{t} g_n(u - t_{ik_n}) b_u^{[2]} du.
$$

We remark that

$$
\bar{X}_{t_{ik_n}} = X(i, t_{(i+1)k_n}).
$$

Now, Ito's lemma yields

$$
\begin{aligned}
(\bar{X}_{t_{ik_n}})^2 =& 2\int_{t_{ik_n}}^{t_{(i+1)k_n}} X(i,u)(b_u^{[1]} dW(i,u) + g_n(u - t_{ik_n}) b_u^{[2]} du) \\
&+ \int_{t_{ik_n}}^{t_{(i+1)k_n}} (g_n(u - t_{ik_n}))^2 (b_u^{[1]})^2 du.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
R_n^{(1)} =& \frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} \int_{t_{ik_n}}^{t_{(i+1)k_n}} X(i,u) b_u^{[1]} dW(i,u) \\
&+ \frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} \int_{t_{ik_n}}^{t_{(i+1)k_n}} X(i,u) g_n(u - t_{ik_n}) b_u^{[2]} du \\
&+ \frac{\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} \int_{t_{ik_n}}^{t_{(i+1)k_n}} (g_n(u - ik_n\Delta_n))^2 (b_u^{[1]})^2 du \\
=:& R_n^{(1.1)} + R_n^{(1.2)} + R_n^{(1.3)}.
\end{aligned}
$$

Here, $R_n^{(1.1)}$ is decomposed as

$$
\begin{aligned}
R_n^{(1.1)} &= \frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} \int_{t_{ik_n}}^{t_{(i+1)k_n}} \int_{t_{ik_n}}^{u} b_s^{[1]} dW(i,s) b_u^{[1]} dW(i,u) \\
&+ \frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} \int_{t_{ik_n}}^{t_{(i+1)k_n}} \int_{t_{ik_n}}^{u} g_n(s - t_{ik_n}) b_s^{[2]} ds \ b_u^{[1]} dW(i,u) \\
&= \frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} (b_{t_{ik_n}}^{[1]})^2 \int_{t_{ik_n}}^{t_{(i+1)k_n}} W(i,u) dW(i,u) \\
&+ \frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} b_{t_{ik_n}}^{[1]} b_{t_{ik_n}}^{[1.1]} \int_{t_{ik_n}}^{t_{(i+1)k_n}} \int_{t_{ik_n}}^{u} (W_s - W_{t_{ik_n}}) dW(i,s) dW(i,u) \\
&+ \frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} b_{t_{ik_n}}^{[1]} b_{t_{ik_n}}^{[2]} \int_{t_{ik_n}}^{t_{(i+1)k_n}} \int_{t_{ik_n}}^{u} g_n(s - t_{ik_n}) ds dW(i,u) \\
&+ \frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} b_{t_{ik_n}}^{[1]} b_{t_{ik_n}}^{[1.1]} \int_{t_{ik_n}}^{t_{(i+1)k_n}} \int_{t_{ik_n}}^{u} dW_s W(i,u) dW(i,u) + o_{\mathbb{P}}(\Delta_n^{1/4}) \\
&=: R_n^{(1.1.1)} + R_n^{(1.1.2)} + R_n^{(1.1.3)} + R_n^{(1.1.4)} + o_{\mathbb{P}}(\Delta_n^{1/4}). \qquad (8.4)
\end{aligned}
$$

Using (8.2), (8.3) and (8.4), we immediately observe that

$$
\begin{aligned}
R_n^{(1.1.1)} =& \frac{\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} (b_{t_{ik_n}}^{[1]} \overline{W}_{t_{ik_n}})^2 - \mathbb{E}[(b_{t_{ik_n}}^{[1]} \overline{W}_{t_{ik_n}})^2 | \mathcal{F}_{t_{ik_n}}] \text{ and} \\
M_n =& R_n^{(1.1.1)} + R_n^{(2.1)} + R_n^{(3)}.
\end{aligned}
$$

Regarding the term $R_n^{(1.2)}$, we proceed similarly and obtain

$$
\begin{aligned}
R_n^{(1.2)} &= \frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} b^{[1]}_{t_{ik_n}} \int_{t_{ik_n}}^{t_{(i+1)k_n}} W(i,u) g_n(u-t_{ik_n}) b_u^{[2]} du \\
&+ \frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} (b^{[2]}_{t_{ik_n}})^2 \int_{t_{ik_n}}^{t_{(i+1)k_n}} \int_{t_{ik_n}}^{u} g_n(s-t_{ik_n}) ds \; g_n(u-t_{ik_n}) du + o_{\mathbb{P}}(\Delta_n^{1/4}) \\
&= \frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} b^{[1]}_{t_{ik_n}} b^{[2]}_{t_{ik_n}} \int_{t_{ik_n}}^{t_{(i+1)k_n}} W(i,u) g_n(u-t_{ik_n}) du \\
&+ \frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} b^{[1]}_{t_{ik_n}} b^{[2.1]}_{t_{ik_n}} \int_{t_{ik_n}}^{t_{(i+1)k_n}} \int_{t_{ik_n}}^{u} dW_s W(i,u) g_n(u-t_{ik_n}) du \\
&+ \frac{\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} (b^{[2]}_{t_{ik_n}})^2 \left(\int_{t_{ik_n}}^{t_{(i+1)k_n}} g_n(u-t_{ik_n}) du \right)^2 + o_{\mathbb{P}}(\Delta_n^{1/4}) \\
&=: R_n^{(1.2.1)} + \Delta_n^{1/4} N_{n,2} + \Delta_n^{1/4} N_{n,3} + o_{\mathbb{P}}(\Delta_n^{1/4}). \qquad (8.5)
\end{aligned}
$$

In view of (8.4) and (8.5), we note that Ito's product rule yields

$$
\begin{aligned}
R_n^{(1.1.3)} + R_n^{(1.2.1)} &= \frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} b^{[1]}_{t_{ik_n}} b^{[2]}_{t_{ik_n}} \overline{W}_{t_{ik_n}} \int_{t_{ik_n}}^{t_{(i+1)k_n}} g_n(u-t_{ik_n}) du \\
&= \frac{2\psi_3^n k_n \Delta_n^{3/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} b^{[1]}_{t_{ik_n}} b^{[2]}_{t_{ik_n}} \overline{W}_{t_{ik_n}} =: \Delta_n^{1/4} N_{n,1}^{(1)}. \qquad (8.6)
\end{aligned}
$$

Finally, we look at $R_n^{(1.3)}$ and V. For both terms we use

$$
(b_u^{[1]})^2 = (b^{[1]}_{t_{ik_n}})^2 + 2\int_{t_{ik_n}}^{u} b_s^{[1]} \left(b_s^{[1.1]} dW_s + b_s^{[1.2]} ds \right) + \int_{t_{ik_n}}^{u} (b_s^{[1.1]})^2 ds.
$$

Then, recalling the definition of $Z_n^\star$ in (3.2) and the estimate (3.3), we get

$$
\begin{aligned}
& R_n^{(1.3)} - \Delta_n^{-1/4} \int_0^{d_n k_n \Delta_n} (b_u^{[1.1]})^2 du \\
&= \frac{2\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} b^{[1]}_{t_{ik_n}} b^{[1.1]}_{t_{ik_n}} \int_{t_{ik_n}}^{t_{(i+1)k_n}} (g_n^2(u-t_{ik_n}) - \psi_2^n)(W_u - W_{t_{ik_n}}) du \\
&+ \frac{\Delta_n^{-1/4}}{\psi_2^n} \sum_{i=0}^{d_n-1} \left[2b^{[1]}_{t_{ik_n}} b^{[1.2]}_{t_{ik_n}} + (b^{[1.1]}_{t_{ik_n}})^2 \right] \int_{t_{ik_n}}^{t_{(i+1)k_n}} ((g_n(u-t_{ik_n}))^2 - \psi_2^n) u du \\
&=: \Delta_n^{1/4} (N_{n,1}^{(2)} + N_{n,4}). \qquad (8.7)
\end{aligned}
$$

Using (8.4), (8.6), (8.7) and Ito's formula, we obtain

$$
R_n^{(1.1.2)} + R_n^{(1.1.4)} + \Delta_n^{1/4} \left(N_{n,1}^{(1)} + N_{n,1}^{(2)} \right) = \Delta_n^{1/4} N_{n,1}.
$$

This finishes the proof of Proposition 3.1. □

Proof of Theorem 5.4. We write

$$M_n = \sum_{i=0}^{d_n-1} \chi_{i,1}^n, \quad N_n = \sum_{k=2}^{4} N_{n,k} + \sum_{i=0}^{d_n-1} \chi_{i,2}^n,$$
$$\widehat{C}_n = K_n + \sum_{i=0}^{d_n-1} \chi_{i,3}^n, \quad \widehat{F}_n = L_n + \sum_{i=0}^{d_n-1} \chi_{i,4}^n,$$

where

$$\chi_{i,1}^n = \frac{\Delta_n^{-1/4}}{\psi_2^n} \left(\alpha_{t_{ik_n}}^2 - \mathbb{E}[\alpha_{t_{ik_n}}^2 | \mathcal{F}_{t_{ik_n}}]\right),$$
$$\chi_{i,2}^n = \frac{2\Delta_n^{-1/2}}{\psi_2^n} b_{t_{ik_n}}^{[1]} \int_{t_{ik_n}}^{t_{(i+1)k_n}} \nu_i^n(u) dW_u +$$
$$\frac{2\Delta_n^{-1/2}}{\psi_2^n} \bar{\varepsilon}_{t_{ik_n}} \left(\psi_3^n k_n \Delta_n b_{t_{ik_n}}^{[2]} + b_{t_{ik_n}}^{[1.1]} \int_{t_{ik_n}}^{t_{(i+1)k_n}} \int_{t_{ik_n}}^{u} dW_s dW(i,u)\right),$$
$$+ \frac{\Delta_n^{-1/2} \psi_1^n}{2\psi_2^n (k_n)^2} \sum_{j=1}^{k_n} \left[2\omega^2 - (\Delta \varepsilon_{ik_n+j}^n)^2\right],$$
$$\chi_{i,3}^n = \frac{4\Delta_n^{-3/4}}{(\psi_2^n)^2} \int_{t_{ik_n-1}}^{t_{(i+1)k_n-1}} \left(\alpha^2(i,u) - \mathbb{E}[\alpha^2(i,u) | \mathcal{F}_{t_{ik_n}}]\right) [d\alpha(i,u)]^2,$$
$$\chi_{i,4}^n = \frac{2\Delta_n^{-3/4}}{3(\psi_2^n)^2} \left(\alpha_{t_{ik_n}}^4 - \mathbb{E}[\alpha_{t_{ik_n}}^4 | \mathcal{F}_{t_{ik_n}}]\right),$$
$$K_n = \Delta_n^{-1/4} \left(\frac{4\Delta_n^{-1/2}}{(\psi_2^n)^2} \sum_{i=0}^{d_n-1} \int_{t_{ik_n-1}}^{t_{(i+1)k_n-1}} \mathbb{E}[\alpha^2(i,u) | \mathcal{F}_{t_{ik_n}}][d\alpha(i,u)]^2 - C\right),$$
$$L_n = \frac{2\Delta_n^{-3/4}}{3(\psi_2^n)^2} \sum_{i=0}^{d_n-1} \left((\bar{Y}_{t_{ik_n}})^4 - \alpha_{t_{ik_n}}^4\right)$$
$$+ \Delta_n^{-1/4} \left(\frac{2\Delta_n^{-1/2}}{3(\psi_2^n)^2} \sum_{i=0}^{d_n-1} \mathbb{E}[\alpha_{t_{ik_n}}^4 | \mathcal{F}_{t_{ik_n}}] - C\right),$$

and the quantity $[d\alpha(i,u)]^2$ has been introduced in (5.3). We observe that $K_n \xrightarrow{\mathbb{P}} 0$ and $L_n \xrightarrow{\mathbb{P}} 0$. We recall that $\alpha_{t_{ik_n}}$, conditionally on $\mathcal{F}_{t_{ik_n}}$, is distributed as

$$N\left(0, \psi_2^n k_n \Delta_n (b_{t_{ik_n}}^{[1]})^2 + \frac{\psi_1^n \omega^2}{k_n}\right).$$

Then, for $\chi_i^n = (\chi_{i,1}^n, \chi_{i,2}^n, \chi_{i,3}^n, \chi_{i,4}^n)$ we obtain

$$\sum_{i=0}^{d_n-1} \mathbb{E}[\chi_{i,k}^n \chi_{i,l}^n | \mathcal{F}_{t_{ik_n}}] \xrightarrow{\mathbb{P}} \int_0^1 \Sigma_s^{kl} ds,$$
$$\sum_{i=0}^{d_n-1} \mathbb{E}[\chi_{i,k}^n (W_{t_{(i+1)k_n}} - W_{t_{ik_n}}) | \mathcal{F}_{t_{ik_n}}] \xrightarrow{\mathbb{P}} \int_0^1 \mu_s^k ds$$
$$\sum_{i=0}^{d_n-1} \mathbb{E}[\chi_{i,k}^n (B_{t_{(i+1)k_n}} - B_{t_{ik_n}}) | \mathcal{F}_{t_{ik_n}}] \xrightarrow{\mathbb{P}} 0$$

for $1 \leqslant k, l \leqslant 4$, where the stochastic processes Σ^{kl} and μ^k are introduced in Theorem 5.4. For any $\delta > 0$ and $1 \leqslant k \leqslant 4$, we observe that

$$\sum_{i=0}^{d_n-1} \mathbb{E}[|\chi_{i,k}^n|^2 \mathbb{1}_{\{|\chi_{i,k}^n| > \delta\}} | \mathcal{F}_{t_{ik_n}}] \leqslant \delta^{-2} \sum_{i=0}^{d_n-1} \mathbb{E}[|\chi_{i,k}^n|^4 | \mathcal{F}_{t_{ik_n}}] \leqslant C_\epsilon k_n \Delta_n \to 0.$$

Now, let Q be a bounded continuous martingale with $\langle W, Q \rangle = \langle B, Q \rangle = 0$. A standard argument (see e.g. [5]) shows that

$$\mathbb{E}[\chi_{i,k}^n (Q_{t_{(i+1)k_n}} - Q_{t_{ik_n}}) | \mathcal{F}_{t_{ik_n}}] = 0.$$

This implies

$$\sum_{i=0}^{d_n-1} \mathbb{E}[\chi_{i,k}^n (Q_{t_{(i+1)k_n}} - Q_{t_{ik_n}}) | \mathcal{F}_{t_{ik_n}}] \to 0.$$

Then, we are done using Theorem IX.7.28 of [7]. □

8.3 Nondegeneracy of C

For

$$C = 2\theta \int_0^1 \left((b_u^{[1]})^2 + \frac{\omega^2 \psi_1}{\theta^2 \psi_2} \right)^2 du,$$

we get

$$D_r^{(1)} C = 8\theta \int_r^1 \left((b_u^{[1]})^2 + \frac{\omega^2 \psi_1}{\theta^2 \psi_2} \right) b_u^{[1]} (b_u^{[1]})' D_r^{(1)} X_u du$$

and $D_r^{(2)} C = 0$ for $r \in [0, 1]$. We write

$$\sigma_{22}(t) = \int_0^t \left(8\theta \int_r^1 \left((b_u^{[1]})^2 + \frac{\omega^2 \psi_1}{\theta^2 \psi_2} \right) b_u^{[1]} (b_u^{[1]})' D_r^{(1)} X_u du \right)^2 dr.$$

Then C's Malliavin covariance $\sigma_C = \sigma_{22}(1)$, i.e. $\|DC\|_{\mathbb{H}}^2 = \sigma_{22}(1)$.

We will work with the two-dimensional stochastic process $\bar{X}_t = (X_t^{(1)}, X_t^{(2)})$ defined by the stochastic integral equation

$$\bar{X}_t = \bar{X}_0 + \int_0^t V_1(\bar{X}_s) \circ dW_s + \int_0^t V_0(\bar{X}_s) ds$$

for $t \in [0,1]$, where $\circ$ denotes the Stratonovich integral. The coefficients are given by

$$V_1(x) = \begin{bmatrix} b^{[1]}(x^{(1)}) \\ 0 \end{bmatrix}, \qquad V_0(x) = \begin{bmatrix} \tilde{b}^{[2]}(x^{(1)}) \\ a(x^{(1)}) \end{bmatrix}$$

for $x = (x^{(1)}, x^{(2)})$, $\tilde{b}^{[2]} = b^{[2]} - 2^{-1} b^{[1]} (b^{[1]})'$ and

$$a(x^{(1)}) = 2\theta \left((b^{[1]}(x^{(1)}))^2 + \frac{\omega^2 \psi_1}{\theta^2 \psi_2} \right)^2$$

already defined. Condition [V] implies the Hörmander condition

$$\mathrm{Lie}[V_0; V_1](x^{(1)}, 0) = \mathbb{R}^2 \qquad (\forall x^{(1)} \in \mathrm{supp}\mathcal{L}\{X_0\})$$

where $\mathrm{Lie}[V_0; V_1]$, the Lie algebra generated by V_1 and V_0, is defined in the following way. The Lie bracket of V and W is given by

$$[V, W](x) = \mathcal{D}V(x)W(x) - \mathcal{D}W(x)V(x),$$

where $\mathcal{D}V(x)$ is the derivative of V at x. Then, we define the Lie algebra by $\mathrm{Lie}[V_0; V_1] = \mathrm{span}\left(\bigcup_{j=0}^{\infty} \Sigma_j \right)$, where $\Sigma_0 = \{V_1\}$ and $\Sigma_j = \{[V, V_i];\ V \in \Sigma_{j-1},\ i = 0, 1\}$ $(j \geqslant 1)$. It is then possible to deduce that

$$\sigma_{22}(t)^{-1} \in \bigcap_{p>1} L^p \tag{8.8}$$

for every $t \in (0, 1]$. For details, we refer the reader to [10, 12, 13].

8.4 Setting s_n and a local nondegeneracy of (M_t^n, C)

The Malliavin covariance matrix of (M_t^n, C) is denoted by

$$\sigma_{(M_t^n, C)} = \begin{bmatrix} \sigma_{11}(n,t) & \sigma_{12}(n,t) \\ \sigma_{21}(n,t) & \sigma_{22}(1) \end{bmatrix}.$$

Recall that

$$\begin{aligned} \alpha(i,t) &= b^{[1]}_{t_{ik_n}} W(i,t) + \varepsilon(i,t), \\ W(i,t) &= \int_{t_{ik_n} \wedge t}^{t_{(i+1)k_n} \wedge t} g_n(u - t_{ik_n}) dW_u, \\ \varepsilon(i,t) &= \omega \Delta_n^{-1/2} \int_{t_{ik_n - 1} \wedge t}^{t_{(i+1)k_n - 1} \wedge t} -h_n(u - t_{ik_n}) dB_u. \end{aligned}$$

Denoting by $\langle U \rangle$ the quadratic variation process of U, we conclude that

$$\begin{aligned} \langle \alpha(i, \cdot) \rangle_t &= (b^{[1]}_{t_{ik_n}})^2 \int_{t_{ik_n} \wedge t}^{t_{(i+1)k_n} \wedge t} g_n(u - t_{ik_n})^2 du \\ &\quad + \omega^2 \Delta_n^{-1} \int_{t_{ik_n - 1} \wedge t}^{t_{(i+1)k_n - 1} \wedge t} h_n(u - t_{ik_n})^2 du. \end{aligned}$$

and

$$\begin{aligned} D_r^{(1)}W(i,t) &= g_n(r-t_{ik_n})\mathbb{1}_{(t_{ik_n}\wedge t, t_{(i+1)k_n}\wedge t]}(r) \\ &= g_n(r-t_{ik_n})\mathbb{1}_{(t_{ik_n}, t_{(i+1)k_n}]}(r)\mathbb{1}_{\{r\leqslant t\}}. \end{aligned}$$

Obviously,

$$\mathbb{E}[\alpha^2_{t_{ik_n}}|\mathcal{F}_{t_{ik_n}}] = (b^{[1]}_{t_{ik_n}})^2\psi_2^n k_n\Delta_n + \omega^2\Delta_n^{-1}\int_{t_{ik_n-1}}^{t_{(i+1)k_n-1}} h_n(u-t_{ik_n})^2du.$$

Since

$$\begin{aligned} M_t^n &= \frac{2\Delta_n^{-1/4}}{\psi_2^n}\sum_{i=0}^{d_n-1}\int_{t_{ik_n}\wedge t}^{t_{(i+1)k_n}\wedge t}\alpha(i,u)d\alpha(i,u). \\ &= \frac{\Delta_n^{-1/4}}{\psi_2^n}\sum_{i=0}^{d_n-1}\big\{\alpha(i,t)^2-\langle\alpha(i,\cdot)\rangle_t\big\}, \end{aligned}$$

it follows that

$$\begin{aligned} &D_r^{(1)}M_t^n = \frac{\Delta_n^{-1/4}}{\psi_2^n} \\ &\times\sum_{i=0}^{d_n-1}\Bigg\{2\alpha(i,t)\Big((b^{[1]})'_{t_{ik_n}}D_r^{(1)}X_{t_{ik_n}}W(i,t)\mathbb{1}_{(0,t_{ik_n}]}(r)+b^{[1]}_{t_{ik_n}}D_r^{(1)}W(i,t)\Big) \\ &-2b^{[1]}_{t_{ik_n}}(b^{[1]})'_{t_{ik_n}}D_r^{(1)}X_{t_{ik_n}}\int_{t_{ik_n}\wedge t}^{t_{(i+1)k_n}\wedge t}g_n(u-t_{ik_n})^2du\mathbb{1}_{(0,t_{ik_n}]}(r)\Bigg\} \\ &=\frac{\Delta_n^{-1/4}}{\psi_2^n}\sum_{i=0}^{d_n-1}\Bigg[\sum_{l=i+1}^{d_n-1}2(b^{[1]})'_{t_{lk_n}}D_r^{(1)}X_{t_{lk_n}}\Bigg\{\alpha(l,t)W(l,t) \\ &-b^{[1]}_{t_{lk_n}}\int_{t_{lk_n}\wedge t}^{t_{(l+1)k_n}\wedge t}g_n(u-t_{lk_n})^2du\Bigg\}+2\alpha(i,t)b^{[1]}_{t_{ik_n}}G_i(r)\mathbb{1}_{\{r\leqslant t\}}\Bigg]\mathbb{1}_{I_i}(r). \end{aligned}$$

and

$$\begin{aligned} D_r^{(2)}M_t^n &= \frac{-\Delta_n^{-1/4}}{\psi_2^n}\sum_{i=0}^{d_n-1}2\alpha(i,t)\omega\Delta_n^{-1/2}h_n(r-t_{ik_n})\mathbb{1}_{(t_{ik_n-1}\wedge t, t_{(i+1)k_n-1}\wedge t]}(r) \\ &= \frac{\Delta_n^{-1/4}}{\psi_2^n}\sum_{i=0}^{d_n-1}2\alpha(i,t)\omega\Delta_n^{-1/2}H_i(r)\mathbb{1}_{\{r\leqslant t\}}\times\mathbb{1}_{(t_{ik_n-1}, t_{(i+1)k_n-1}]}(r). \end{aligned}$$

where $I_i=(t_{ik_n}, t_{(i+1)k_n}]$, $I_{i,j}=(t_{ik_n+j-1}, t_{ik_n+j}]$, and we use the notations $G_i(r)=\sum_{j=1}^{k_n-1}g(j/k_n)\mathbb{1}_{I_{i,j}}(r)$ and $H_i(r)=\sum_{j=0}^{k_n-1}-h(j/k_n)\mathbb{1}_{I_{i,j}}(r)$. Hence, the

Malliavin covariance matrix of M_t^n is expressed by

$$\begin{aligned}
&\sigma_{M_t^n} \equiv \sigma_{11}(n,t)\\
&= \frac{1}{(\psi_2^n)^2}\sum_{i=0}^{d_n-1}\int_{t_{ik_n}}^{t_{(i+1)k_n}}\Bigg[2\Delta_n^{-1/4}\alpha(i,t)b^{[1]}_{t_{ik_n}}G_i(r)\mathbb{1}_{\{r\leqslant t\}}+\Delta_n^{1/4}\sum_{l=i+1}^{d_n-1}2(b^{[1]}_{t_{lk_n}})'\\
&\times D_rX_{t_{lk_n}}\Delta_n^{-1/2}\big[\alpha(l,t)W(l,t)-b^{[1]}_{t_{lk_n}}\int_{t_{lk_n}\wedge t}^{t_{(l+1)k_n}\wedge t}g_n(u-t_{lk_n})^2du\big]\Bigg]^2dr\\
&+\frac{1}{(\psi_2^n)^2}\sum_{i=0}^{d_n-1}\int_{t_{ik_n-1}\wedge t}^{t_{(i+1)k_n-1}\wedge t}[2\Delta_n^{-1/4}\alpha(i,t)\omega\Delta_n^{-1/2}H_i(r)]^2\,dr.
\end{aligned}$$

The cross Malliavin covariance $\langle M_t^n, C\rangle_{\mathbb{H}}$ of (M_t^n, C) is given by

$$\begin{aligned}
&\sigma_{12}(t)=\frac{1}{\psi_2^n}\sum_{i=0}^{d_n-1}\int_{t_{ik_n}}^{t_{(i+1)k_n}}\Bigg\{\Bigg[2\Delta_n^{-1/4}\alpha(i,t)b^{[1]}_{t_{ik_n}}G_i(r)\mathbb{1}_{\{r\leqslant t\}}+2\Delta_n^{1/4}\times\\
&\sum_{l=i+1}^{p}(b^{[1]})'_{t_{lk_n}}D_r^{(1)}X_{t_{lk_n}}\Delta_n^{-1/2}\big[\alpha(l,t)W(l,t)-b^{[1]}_{t_{lk_n}}\int_{t_{lk_n}\wedge t}^{t_{(l+1)k_n}\wedge t}g_n^2(u-t_{lk_n})du\big]\Bigg]\\
&\times\left[8\theta\int_r^1\left((b_u^{[1]})^2+\frac{\omega^2\psi_1}{\theta^2\psi_2}\right)b_u^{[1]}(b_u^{[1]})'D_r^{(1)}X_udu\right]\Bigg\}dr.
\end{aligned}$$

Now, let

$$\sigma(n,t) \quad = \quad \begin{bmatrix}\sigma_{11}(n,t) & \sigma_{12}(n,t)\\ \sigma_{12}(n,t) & \sigma_{22}(t)\end{bmatrix}.$$

Then, we modify $\sigma(n,t)$ for $t=t_{(p+1)k_n}$ by

$$\tilde{\sigma}(n,t) \quad = \quad \begin{bmatrix}\tilde{\sigma}_{11}(n,t) & \tilde{\sigma}_{12}(n,t)\\ \tilde{\sigma}_{12}(n,t) & \sigma_{22}(t)\end{bmatrix} \qquad (t=t_{(p+1)k_n})$$

with

$$\begin{aligned}
&\tilde{\sigma}_{11}(n,t)=\frac{1}{(\psi_2^n)^2}\sum_{i=0}^{p}\int_{t_{ik_n}}^{t_{(i+1)k_n}}\left[2\Delta_n^{-1/4}\alpha_{t_{ik_n}}b^{[1]}_{t_{ik_n}}G_i(r)\right]^2dr\\
&+\frac{1}{(\psi_2^n)^2}\sum_{i=0}^{p}\int_{t_{ik_n-1}}^{t_{(i+1)k_n-1}}[2\Delta_n^{-1/4}\alpha_{t_{ik_n}}\omega\Delta_n^{-1/2}H_i(r)]^2\,dr+\frac{1}{(\psi_2^n)^2}\sum_{i=0}^{p}\int_{t_{ik_n}}^{t_{(i+1)k_n}}\\
&\times\left[\Delta_n^{1/4}\sum_{l=i+1}^{p}2(b^{[1]})'_{t_{lk_n}}D_r^{(1)}X_{t_{lk_n}}\Delta_n^{-1/2}\big[\alpha_{t_{lk_n}}\overline{W}_{t_{lk_n}}-b^{[1]}_{t_{lk_n}}\psi_2^nk_n\Delta_n\big]\right]^2dr.
\end{aligned}$$

and

$$
\begin{aligned}
\tilde{\sigma}_{12}(n,t) &= \frac{1}{\psi_2^n}\sum_{i=0}^{p}\int_{t_{ik_n}}^{t_{(i+1)k_n}}\Bigg\{\Bigg[\Delta_n^{1/4}\sum_{l=i+1}^{p}2(b^{[1]})'_{t_{lk_n}}D_r^{(1)}X_{t_{lk_n}} \\
&\quad\times\Delta_n^{-1/2}\big[\alpha_{t_{lk_n}}\overline{W}_{t_{lk_n}}-b^{[1]}_{t_{lk_n}}\int_{t_{lk_n}\wedge t}^{t_{(l+1)k_n}\wedge t}g_n(u-t_{lk_n})^2du\big]\Bigg] \\
&\quad\times\Bigg[8\theta\int_r^1\left((b_u^{[1]})^2+\frac{\omega^2\psi_1}{\theta^2\psi_2}\right)b_u^{[1]}(b_u^{[1]})'D_r^{(1)}X_udu\Bigg]\Bigg\}dr,
\end{aligned}
$$

evaluated at $t=t_{(p+1)k_n}$, respectively. Let

$$
J_{n,l}^{(1)} = \Delta_n^{-1/4}\overline{W}_{t_{lk_n}} = \Delta_n^{-1/4}\int_{t_{lk_n}}^{t_{(l+1)k_n}}g_n(s-lk_n\Delta_n)dW_s
$$

and

$$
J_{n,l}^{(2)} = \Delta_n^{-1/4}\overline{\varepsilon}_{t_{lk_n}} = -\omega\Delta_n^{-3/4}\int_{t_{lk_n}}^{t_{(l+1)k_n}}h_n(s-lk_n\Delta_n)dB_s.
$$

By definition,

$$
\alpha_{t_{lk_n}} = \Delta_n^{1/4}\big(b^{[1]}_{t_{lk_n}}J_{n,l}^{(1)}+J_{n,l}^{(2)}\big)
$$

and

$$
\begin{aligned}
\alpha_{t_{lk_n}}\overline{W}_{t_{lk_n}}-b^{[1]}_{t_{lk_n}}\psi_2^nk_n\Delta_n &= b^{[1]}_{t_{lk_n}}\big(\overline{W}^2_{t_{lk_n}}-\psi_2^nk_n\Delta_n\big)+\overline{\varepsilon}_{t_{lk_n}}\overline{W}_{t_{lk_n}} \\
&= \Delta_n^{1/2}J_{n,l}^{(3)}+\Delta_n^{1/2}J_{n,l}^{(4)}+\Delta_n^{1/2}J_{n,l}^{(5)}
\end{aligned}
$$

where

$$
\begin{aligned}
J_{n,l}^{(3)} &= \Delta_n^{-1/2}b^{[1]}_{t_{lk_n}}\int_{t_{lk_n}}^{t_{(l+1)k_n}}dW_s2g_n(s-lk_n\Delta_n)\int_{t_{lk_n}}^{s}dW_ug_n(u-lk_n\Delta_n)dW_u \\
J_{n,l}^{(4)} &= \Delta_n^{-1/2}\int_{t_{lk_n}}^{t_{(l+1)k_n}}dW_sg_n(s-lk_n\Delta_n)\int_{t_{lk_n}}^{s}dB_u(-\omega)\Delta_n^{-1/2}h_n(u-t_{lk_n}) \\
J_{n,l}^{(5)} &= \Delta_n^{-1/2}\int_{t_{lk_n}}^{t_{(l+1)k_n}}dB_s(-\omega)\Delta_n^{-1/2}h_n(s-t_{lk_n})\int_{t_{lk_n}}^{s}dW_ug_n(u-lk_n\Delta_n)
\end{aligned}
$$

We claim that for every $q>1$,

$$
\sup_p\Big\|\Delta_n^{1/2}\sum_{i=0}^{p}\mathcal{C}(i,p)\Big\|_q = O(\Delta_n^{1/4}) \tag{8.9}
$$

as $n\to\infty$ for

$$
\begin{aligned}
&\mathcal{C}(i,p)=\Delta_n^{-1/4}\alpha_{t_{ik_n}}b^{[1]}_{t_{ik_n}}\Delta_n^{1/4} \\
&\times\sum_{l=i+1}^{p}(b^{[1]}_{t_{lk_n}})'\left(\Delta_n^{-1/2}\int_{I_i}D_r^{(1)}X_{t_{lk_n}}dr\right)\Delta_n^{-1/2}[\alpha_{t_{lk_n}}\overline{W}_{t_{lk_n}}-b^{[1]}_{t_{lk_n}}\psi_2^nk_n\Delta_n)].
\end{aligned}
$$

We have

$$
\begin{aligned}
\mathcal{C}(i,p) &= \left((b^{[1]}_{t_{ik_n}})^2 J^{(1)}_{n,l} + b^{[1]}_{t_{ik_n}} J^{(2)}_{n,l}\right) \\
&\quad \times \Delta_n^{1/4} \sum_{l=i+1}^{p} (b^{[1]}_{t_{lk_n}})' \left(\Delta_n^{-1/2} \int_{I_i} D^{(1)}_r X_{t_{lk_n}} dr\right) \left(J^{(3)}_{n,l} + J^{(4)}_{n,l} + J^{(5)}_{n,l}\right).
\end{aligned}
$$

Then the estimate (8.9) follows from Lemma 5 of [13]. By using (8.9), we see

$$
\begin{aligned}
\sigma_{M^n_{t_{(p+1)k_n}}} &= \frac{1}{(\psi^n_2)^2} \sum_{i=0}^{p} \int_{t_{ik_n}}^{t_{(i+1)k_n}} \Bigg\{ \left[2\Delta_n^{-1/4} \alpha_{t_{ik_n}} b^{[1]}_{t_{ik_n}} G_i(r)\right]^2 \\
&\quad + \left[\Delta_n^{1/4} \sum_{l=i+1}^{p} (b^{[1]}_{t_{lk_n}})' D^{(1)}_r X_{t_{lk_n}} \Delta_n^{-1/2} [2\alpha_{t_{lk_n}} \overline{W}_{t_{lk_n}} - 2b^{[1]}_{t_{lk_n}} \psi^n_2 k_n \Delta_n)]\right]^2 \\
&\quad + \left[2\Delta_n^{-1/4} \alpha_{t_{ik_n}} \omega \Delta_n^{-1/2} H_i(r)\right]^2 \Bigg\} dr + O_{L^q}(\Delta_n^{1/4}) \\
&= \hat{\sigma}_{11}(n, t_{(p+1)k_n}) + O_{L^q}(\Delta_n^{1/4})
\end{aligned}
$$

as $n \to \infty$ uniformly in p, for every $q > 1$. That is,

$$
\sup_{\substack{(t,p):\, t = t_{(p+1)k_n}, \\ p = 0,\dots,d_n-1}} \left\|\sigma_{11}(n,t) - \tilde{\sigma}_{11}(n,t)\right\|_q = O(\Delta_n^{1/4})
$$

for every $q > 1$. In a similar fashion, we also obtain

$$
\sup_{\substack{(t,p):\, t = t_{(p+1)k_n}, \\ p = 0,\dots,d_n-1}} \left\|\sigma_{12}(n,t) - \tilde{\sigma}_{12}(n,t)\right\|_q = O(\Delta_n^{1/4})
$$

as $n \to \infty$ for every $q > 1$. In this way, we obtain the following result.

Lemma 8.1.

$$
\sup_{\substack{(t,p):\, t = t_{(p+1)k_n}, \\ p = 0,\dots,d_n-1}} \left\|\sigma(n,t) - \tilde{\sigma}(n,t)\right\|_q = O(\Delta_n^{1/4})
$$

as $n \to \infty$ for every $q > 1$.

Define $m_n(p)$ by

$$
\begin{aligned}
m_n(p) &= \frac{1}{(\psi^n_2)^2} \sum_{i=0}^{p} \int_{t_{ik_n}}^{t_{(i+1)k_n}} \left[2\Delta_n^{-1/4} \alpha_{t_{ik_n}} b^{[1]}_{t_{ik_n}} G_i(r)\right]^2 dr \\
&\quad + \frac{1}{(\psi^n_2)^2} \sum_{i=0}^{p} \int_{t_{ik_n - 1}}^{t_{(i+1)k_n - 1}} [2\Delta_n^{-1/4} \alpha_{t_{ik_n}} \omega \Delta_n^{-1/2} H_i(r)]^2 \, dr.
\end{aligned}
$$

By the Cauchy-Schwarz inequality, we see

$$
\begin{aligned}
\det \tilde{\sigma}\left(n, t_{(p+1)k_n}\right) &= \tilde{\sigma}_{11}(n, t_{(p+1)k_n}) \sigma_{22}(t_{(p+1)k_n}) - \tilde{\sigma}_{12}(n, t_{(p+1)k_n})^2 \\
&\geqslant m_n(p)\, \sigma_{22}(t_{(p+1)k_n}). \qquad (8.10)
\end{aligned}
$$

Let $u_n = \lfloor d_n/2 \rfloor k_n \Delta_n$ and let $p_n = \lfloor d_n/2 \rfloor - 1$. Set $\hat{m}_n = m_n(p_n)$. Define s_n by

$$s_n \quad = \quad \frac{1}{2}\left[\hat{m}_n \sigma_{22}(u_n) + \psi\left(\frac{\hat{m}_n}{2c_1}\right)\right],$$

where $\psi : \mathbb{R} \to [0,1]$ in $C^\infty(\mathbb{R})$ and satisfies $\psi(x) = 1$ if $|x| \leqslant 1/2$ and $\psi(x) = 0$ if $|x| \geqslant 1$. Then, we observe that $s_n \geqslant 1/2$ if $\hat{m}_n \leqslant c_1$, and $s_n \geqslant 2^{-1}\hat{m}_n\sigma_{22}(u_n)$ otherwise.

Lemma 8.2. *For each $q > 1$,*

$$\limsup_{n\to\infty} \mathbb{E}[s_n^{-q}] \quad < \quad \infty.$$

Proof. The result easily follows from (8.8). □

Let

$$m_n^\dagger \quad = \quad \frac{4}{\psi_2^2}\sum_{i=0}^{p_n} \Delta_n^{-1/2}\big([b^{[1]}_{t_{ik_n}} \overline{W}_{t_{ik_n}}]^2 + [\overline{\varepsilon}_{t_{ik_n}}]^2\big)\bigg((b^{[1]}_{t_{ik_n}})^2 \psi_2^n k_n \Delta_n + \frac{\omega^2 \psi_1^n}{k_n}\bigg).$$

Then, for every $q > 1$ the orthogonality between $\overline{W}_{t_{ik_n}}$ and $\overline{\varepsilon}_{t_{ik_n}}$ gives

$$\big\|\hat{m}_n - m_n^\dagger\big\|_q \quad = \quad O(\Delta_n^{1/4})$$

Let $\Pi^n = \{t_{ik_n}; i = 0, ..., d_n - 1\}$.

Lemma 8.3. *For sufficiently small positive number c_1,*

$$\sup_{t\geqslant 1/2} \mathbb{P}\big[\det \sigma_{(M_t^n, C)} < s_n\big] \quad = \quad O(\Delta_n^\xi)$$

where ξ is arbitrary positive number.

Proof. With the help of (8.10), Lemmas 8.1 and 8.2, we have

$$\begin{aligned}
&\sup_{t\geqslant 1/2} \mathbb{P}\big[\det \sigma_{(M_t^n, C)} < s_n\big] \leqslant \sup_{t\geqslant 1/2} \mathbb{P}\big[\det \sigma(n,t) < s_n\big] \\
\leqslant\ & \sup_{t\in\Pi^n; t\geqslant 1/2} \mathbb{P}\big[\det \sigma(n,t) < 1.5 s_n\big] \\
&+ \sup_{s,t; |s-t|\leqslant k_n\Delta_n} \mathbb{P}\big[|\det \sigma(n,t) - \det \sigma(n,s)| > 0.5 s_n\big] \\
\leqslant\ & \sup_{t\in\Pi^n; t\geqslant 1/2} \mathbb{P}\big[\det \tilde{\sigma}(n,t) < 2 s_n\big] \\
&+ \sup_{t\in\Pi^n; t\geqslant 1/2} \mathbb{P}\big[|\tilde{\sigma}(n,t) - \sigma(n,t)| > 0.5 s_n\big] + O(\Delta_n^\xi) \\
\leqslant\ & \sup_{p\geqslant p_n} \mathbb{P}\big[m_n(p)\, \sigma_{22}(t_{(p+1)k_n}) < 2 s_n\big] \\
&+ \sup_{t\in\Pi^n; t\geqslant 1/2} \mathbb{P}\big[|\tilde{\sigma}(n,t) - \sigma(n,t)| > 0.5 s_n\big] + O(\Delta_n^\xi) \\
\leqslant\ & \mathbb{P}\big[m_n(p_n)\, \sigma_{22}(u_n) < 2 s_n\big] + O(\Delta_n^\xi) \\
\leqslant\ & \mathbb{P}\big[\hat{m}_n \leqslant c_1\big] + O(\Delta_n^\xi) \\
\leqslant\ & \mathbb{P}\big[m_n^\dagger \leqslant c_1\big] + O(\Delta_n^\xi) = O(\Delta_n^\xi)
\end{aligned}$$

where ξ can be any positive number, if we chose a sufficiently small positive number c_1. □

8.5 Composition of ξ_n

We again consider $\psi : \mathbb{R} \to [0,1]$ in $C^\infty(\mathbb{R})$ that satisfies $\psi(x) = 1$ if $|x| \leqslant 1/2$ and $\psi(x) = 0$ if $|x| \geqslant 1$. For $r_n = \Delta_n^{1/4}$, let

$$\xi_n'' = r_n^{-c^*/2}(C_n - C) + 2[1 + 4\Delta_{(M_n,C)}s_n^{-1}]^{-1} + r_n^{c_1^*/2}C^2,$$

where $c_1^* > 0$, and $c^* \in (2q, 1)$ for $q = (1-a)/2$ for a fixed $a \in (0, 1/3)$. Let

$$\tilde{\xi}_n = \int_{[0,1]^2} \left(\frac{r_n^{-q/2}|C_t^n - C_t - C_s^n + C_s|}{|t-s|^{3/8}} \right)^8 dtds,$$

where $C_t^n = \langle M^n \rangle_t$. We compose ξ_n as $\xi_n = \xi_n'' + \tilde{\xi}_n$.

Tracing the derivation of the stochastic expansion, we can see the expansion holds in $\mathbb{D}_{s,p}$ sense and Condition [B2]$_\ell$ holds for arbitrary $\ell \in \mathbb{N}$. It is easy to verify [B3] (i) by estimating $\mathbb{P}\big[2[1 + 4\Delta_{(M_n,C)}s_n^{-1}]^{-1} > 2/5\big]$ with the aid of Lemma 8.3. Condition [B3] (ii) is immediate by definition. Condition [B3] (iii) follows from Lemma 8.2.

8.6 Estimate of Φ_n^α

We shall verify Condition [B5] to prove the validity of the asymptotic expansion. We know

$$\Phi_n^\alpha(u,v) = \mathtt{i}^{-|\alpha|}d_{(u,v)}^\alpha \mathbb{E}\left[\int_0^1 e_t^n(u)d(\mathtt{i}uM_t^n)\Psi(u,v)\psi_n\right]$$

with $\Psi(u,v) = \exp((-u^2) + iv)C)$. The FGH-decomposition ([13]) will be used:

$$e_t^n(u,v)\Psi(u,v) = \mathbb{F}_t^n(u,v)\mathbb{G}_t^n(u)\mathbb{H}_t^n(u)$$

with

$$\begin{aligned}\mathbb{F}_t^n(u,v) &= \exp\left(\mathtt{i}uM_t^n + \mathtt{i}vC\right), \\ \mathbb{G}_t^n(u) &= \exp\left(-\frac{1}{2}u^2(C - C_t)\right), \\ \mathbb{H}_t^n(u) &= \exp\left(\frac{1}{2}u^2(C_t^n - C_t)\right).\end{aligned}$$

Applying the duality twice, we obtain the representation

$$\Delta_n^{-1/4}\Phi_n^0(u,v) = \frac{2\Delta_n^{-1/2}}{\psi_2^n}\sum_{i=0}^{d_n-1}\int_0^1\int_0^1 dtds\,\mathcal{E}_i^n(u,v)_{t,s}$$

where

$$\mathcal{E}_i^n(u,v)_{t,s} \quad = \quad \mathbb{E}\big[\mathbb{F}_t^n(u,v)\mathbb{G}_t^n(u)\mathbb{H}_t^n(u)\Xi_i^n(u,v)_{t,s}\big].$$

The functional $\Xi_{t,s}^n(u,v)$ is given by

$$\begin{aligned}
&\Xi_i^n(u,v)_{t,s} \\
= \ & \mathtt{i}u\big(e_t^n(u)\Psi(u,v)\big)^{-1} \\
&\times\Big\{g_n(s-t_{ik_n})g_n(t-t_{ik_n})b_{t_{ik_n}}^{[1]}D_s^{(1)}\Big(e_t^n(u)b_{t_{ik_n}}^{[1]}D_t^{(1)}(\Psi(u,v)\psi_n)\Big) \\
+ \ & g_n(s-t_{ik_n})\Delta_n^{-1/2}h_n(t-t_{ik_n})b_{t_{ik_n}}^{[1]}(-\omega)D_s^{(1)}\Big(e_t^n(u)\Psi(u,v)D_t^{(2)}\psi_n\Big) \\
+ \ & \Delta_n^{-1/2}h_n(s-t_{ik_n})g_n(t-t_{ik_n})b_{t_{ik_n}}^{[1]}(-\omega)D_s^{(2)}\Big(e_t^n(u)b_{t_{ik_n}}^{[1]}D_t^{(1)}(\Psi(u,v)\psi_n)\Big) \\
+ \ & \Delta_n^{-1/2}h_n(s-t_{ik_n})\Delta_n^{-1/2}h_n(t-t_{ik_n})(-\omega)^2D_s^{(2)}\Big(e_t^n(u)\Psi(u,v)D_t^{(2)}\psi_n\Big)\Big\}
\end{aligned}$$

After all, $\Xi_i^n(u,v)_{t,s}$ is a polynomial of densities of $O(1)$ that are stable in L^p sense. We note that the functions $t \mapsto \Delta_n^{-1/2}h_n(t-t_{ik_n})$ are stable as $n \to \infty$.

By the integration-by-parts formula applied at most 8 times, following the (a)-(h) procedure (pages 911-912 of [13]), we can obtain the estimate

$$\limsup_{n\to\infty} \sup_{i=0,\dots,d_n-1} \sup_{t,s\in[0,1]} \sup_{(u,v)\in\Lambda_n^0(2,q)} |(u,v)|^3|\mathcal{E}_i^n(u,v)_{t,s}| \quad < \quad \infty.$$

Indeed, for $t < 1/2$, we take advantage of the decay of $\mathbb{G}_t^n(u)$ in u and the nondegeneracy of C for v, i.e., (8.8). For $t \geqslant 1/2$, we can use nondegeneracy of (M_t^n, C) for (u,v), i.e., Lemmas 8.3 and 8.2. Estimation of $\Phi_n^\alpha(u,v)$ is similarly done. In this way, Condition [B5] has been verified.

8.7 Proof of Proposition 5.5 and Theorem 5.6

Since $e_t^n(u)$ is an exponential martingale and C is bounded on the event $\{\psi_n > 0\}$, we have

$$\Phi_n(u,v) = \mathbb{E}\left[\int_0^1 e_t^n(u)d(\mathrm{i}uM_t^n)\Psi(u,v)\psi_n\right],$$

where

$$\Psi(u,v) = \exp\big((-u^2/2 + \mathrm{i}v)C\big).$$

We decompose $\Delta_n^{-1/4}\Phi_n(u,v)$ as

$$\Delta_n^{-1/4}\Phi_n(u,v) = \check{U}_n(u,v) + \hat{U}_n(u,v),$$

where

$$\check{U}_n(u,v) = \Delta_n^{-\frac{1}{4}} \sum_{i=0}^{d_n-1} \mathbb{E}\left[\Psi(u,v)\int_{t_{ik_n}}^{t_{(i+1)k_n}} e^n_{t_{ik_n-1}}(u)d(\mathrm{i}uM_t^n)\psi_n\right],$$
$$\hat{U}_n(u,v) = \Delta_n^{-\frac{1}{4}} \sum_{i=0}^{d_n-1} \mathbb{E}\left[\Psi(u,v)\int_{t_{ik_n}}^{t_{(i+1)k_n}} (e^n_t(u) - e^n_{t_{ik_n-1}}(u))d(\mathrm{i}uM_t^n)\psi_n\right].$$

We will prove that $\check{U}_n(u,v)$ is the main term and $\hat{U}_n(u,v)$ is negligible.

Let us first look at the term $\check{U}_n(u,v)$. We observe that

$$\check{U}_n(u,v) = \frac{2\Delta_n^{-1/2}}{\psi_2^n} \sum_{i=0}^{d_n-1} \int_0^1\int_0^1 dtds\, \check{\mathcal{E}}_i^n(u,v)_{t,s}$$

where $\check{\mathcal{E}}_i^n(u,v)_{t,s}$ has a representation similar to $\mathcal{E}_i^n(u,v)_{t,s}$ as

$$\check{\mathcal{E}}_i^n(u,v)_{t,s} = \mathbb{E}\big[e^n_{t_{ik_n}-1}(u)\Psi(u,v)\check{\Xi}_i^n(u,v)_{t,s}\big]$$

with

$$\begin{aligned}
&\check{\Xi}_i^n(u,v)_{t,s} = \mathtt{i}u\big(e^n_{t_{ik_n}-1}(u)\Psi(u,v)\big)^{-1}\\
&\times\Bigg\{g_n(s-t_{ik_n})g_n(t-t_{ik_n})b^{[1]}_{t_{ik_n}} e^n_{t_{ik_n}-1}(u)b^{[1]}_{t_{ik_n}} D_s^{(1)}\Big(D_t^{(1)}(\Psi(u,v)\psi_n)\Big)\\
&+ g_n(s-t_{ik_n})\Delta_n^{-1/2}h_n(t-t_{ik_n})b^{[1]}_{t_{ik_n}}(-\omega)e^n_{t_{ik_n}-1}(u)D_s^{(1)}\Big(\Psi(u,v)D_t^{(2)}\psi_n\Big)\\
&+ \Delta_n^{-1/2}h_n(s-t_{ik_n})g_n(t-t_{ik_n})b^{[1]}_{t_{ik_n}}(-\omega)e^n_{t_{ik_n}-1}(u)b^{[1]}_{t_{ik_n}} D_t^{(1)}\Big(\Psi(u,v)D_s^{(2)}\psi_n\Big)\\
&+ \Delta_n^{-1/2}h_n(s-t_{ik_n})\Delta_n^{-1/2}h_n(t-t_{ik_n})(-\omega)^2 e^n_{t_{ik_n}-1}(u)\Psi(u,v)D_s^{(2)}D_t^{(2)}\psi_n\Bigg\}.
\end{aligned}$$

For derivation of the above equality, the bounds of the supports of g_n and h_n were used. The terms stemming from the last three terms in {...} are negligible since $\mathbb{P}[|\xi_n|>1] = O(\Delta_n^L)$ for arbitrary $L>0$. By the same reason applied to the first term there, we have

$$\check{U}_n(u,v) = \frac{2\Delta_n^{-1/2}}{\psi_2^n} \sum_{i=0}^{d_n-1} \int_0^1\int_0^1 dtds\, \dot{\mathcal{E}}_i^n(u,v)_{t,s} + o(1)$$

where

$$\dot{\mathcal{E}}_i^n(u,v)_{t,s} = \mathbb{E}\big[e^n_{t_{ik_n}-1}(u)\Psi(u,v)\dot{\Xi}_i^n(u,v)_{t,s}\big]$$

and

$$\begin{aligned}
&\dot{\Xi}_i^n(u,v)_{t,s}\\
=\;& \mathtt{i}u\big(e^n_{t_{ik_n}-1}(u)\Psi(u,v)\big)^{-1}\\
&\times g_n(s-t_{ik_n})g_n(t-t_{ik_n})(b^{[1]}_{t_{ik_n}})^2 e^n_{t_{ik_n}-1}(u)\psi_n D_s^{(1)}D_t^{(1)}\Psi(u,v)\\
=\;& \mathtt{i}u\, g_n(s-t_{ik_n})g_n(t-t_{ik_n})(b^{[1]}_{t_{ik_n}})^2\psi_n\big\{l^2(D_s^{(1)}C)(D_t^{(1)}C) + lD_s^{(1)}D_t^{(1)}C\big\}
\end{aligned}$$

where $l = -\frac{u^2}{2} + \mathrm{i}v$. Therefore

$$\check{U}_n(u,v) = \frac{2\Delta_n^{-1/2}}{\psi_2^n} \sum_{i=0}^{d_n-1} \int\int dtds\, \mathbb{E}\Big[e^n_{t_{ik_n}}(u)\Psi(u,v)\mathtt{i}u\, g_n(s-t_{ik_n})g_n(t-t_{ik_n})$$

$$\times (b^{[1]}_{t_{ik_n}})^2 \psi_n \{ l^2 (D^{(1)}_s C)(D^{(1)}_t C) + l D^{(1)}_s D^{(1)}_t C \} \Big] \to$$

$$\frac{\theta\psi_3^2}{\psi_2} \int_0^1 \mathbb{E}\Big[\mathtt{i}u\, \exp\big(\mathtt{i}uM_t + C\frac{u^2}{2}\big)\Psi(u,v)(b^{[1]}_t)^2\{l^2(D^{(1)}_t C)^2 + lD^{(1)}_t D^{(1)}_t C\}\Big]\, dt$$

$$= \mathbb{E}\Big[\Psi(u,v)\, \frac{\theta\psi_3^2}{\psi_2} \int_0^1 \mathtt{i}u\, (b^{[1]}_t)^2\{l^2(D^{(1)}_t C)^2 + lD^{(1)}_t D^{(1)}_t C\}\, dt\Big]$$

$$= \mathbb{E}\Big[\Psi(u,v)\, \frac{\theta\psi_3^2}{\psi_2} \mathtt{i}u\, \{4\theta^2 l^2 \mathcal{C}_2 + 2\theta l(\mathcal{C}_3 + \mathcal{C}_4)\}\Big]$$

where $D^{(1)}_t D^{(1)}_t C = \lim_{s\uparrow t} D^{(1)}_s D^{(1)}_t C$ and

$$\begin{aligned} \mathcal{C}_2 &= \int_0^1 (b^{[1]})^2(X_t) \left(\int_t^1 c'(X_r) D^{(1)}_t X_r dr\right)^2 dt, \\ \mathcal{C}_3 &= \int_0^1 (b^{[1]})^2(X_t) \left(\int_t^1 c''(X_r)(D^{(1)}_t X_r)^2 dr\right) dt, \\ \mathcal{C}_4 &= \int_0^1 (b^{[1]})^2(X_t) \left(\int_t^1 c'(X_r) D^{(1)}_t D^{(1)}_t X_r dr\right) dt \end{aligned}$$

for $c(x) = \Big[(b^{[1]})^2(x) + \frac{\omega^2\psi_1}{\theta^2\psi_2}\Big]^2$. Convergence $\hat{U}_n(u,v) \to 0$ is easy to show. Moreover, it is possible to specify the limit Φ^α in a similar way to verify (4.3) for $F = C$. Hence, we obtain

$$\overline{\sigma}(\mathrm{i}u, \mathrm{i}v) = \frac{\theta\psi_3^2}{\psi_2} \mathrm{i}u[4\theta^2 \mathcal{C}_2 l^2 + 2\theta(\mathcal{C}_3 + \mathcal{C}_4)l].$$

Thus, Proposition 5.5 and hence Condition $[\mathrm{B4}]_{\ell,2,1}$ has been verified, which concludes the proof of Theorem 5.6. □

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