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On critical cases in limit theory for stationary increments Lévy driven moving averages *

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Abstract

In this paper we present some limit theorems for power variation of stationary increments Lévy driven moving averages in the setting of critical regimes. In [5] the authors derived first and second order asymptotic results for k-th order increments of stationary increments Lévy driven moving averages. The limit theory heavily depends on the interplay between the given order of the increments, the considered power, the Blumenthal–Getoor index of the driving pure jump Lévy process L and the behaviour of the kernel function g at 0. In this work we will study the critical cases, which were not covered in the original work [5].

Key words: Power variation, limit theorems, moving averages, fractional processes, stable convergence, high frequency data.

JEL Classification: C10, C13, C14.

1 Introduction and main results

In the last decade a lot of scientific research has been devoted to limit theory for high frequency observations of stochastic processes. Power variation functionals and related statistics play a major role in the analysis of fine properties of a stochastic process, in stochastic integration theory and statistical inference. The asymptotic theory for high frequency statistics of various classes of stochastic processes has been intensively investigated in the literature. We refer e.g. to [4, 12, 13, 15] for limit theory for power variations of Itô semimartingales, to [2, 3, 9, 11, 14] for the asymptotic results in the framework

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of fractional Brownian motion and related processes, and to [8, 20] for investigations of power variation of the Rosenblatt process.

In a recent paper [5] the power variation of *stationary increments Lévy driven moving averages* has been studied. Let us recall the definitions, notations and main results of this paper. We consider an infinitely divisible process with stationary increments $(X_t)_{t\geq 0}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, given via

$$X_t = \int_{-\infty}^{t} \big\{ g(t-s) - g_0(-s) \big\}\, dL_s,$$

where $L = (L_t)_{t\in\mathbb{R}}$ is a symmetric Lévy process on $\mathbb{R}$ with $L_0 = 0$. That is, for all $u \in \mathbb{R}$, $(L_{t+u} - L_u)_{t\geq 0}$ is a Lévy process indexed by $\mathbb{R}_+$ which distribution is invariant under multiplication with -1. Furthermore, g and g_0 are deterministic functions from $\mathbb{R}$ into $\mathbb{R}$ vanishing on $(-\infty, 0)$. Throughout the paper we will need the notion of *Blumenthal–Getoor index* of L, which is defined via

$$\beta := \inf \Big\{ r \geq 0 : \int_{-1}^{1} |x|^r \, \nu(dx) < \infty \Big\} \in [0, 2], \tag{1.1}$$

where ν denotes the Lévy measure of L. It is well-known that $\sum_{s\in[0,1]} |\Delta L_s|^p$ is finite when $p > \beta$, while it is infinite for $p < \beta$. Here $\Delta L_s = L_s - L_{s-}$ where $L_{s-} = \lim_{u\uparrow s,\, u<s} L_u$.

There are some famous subclasses of stationary increments Lévy driven moving averages. When $g_0 = 0$, the process X is a moving average, and in this case X is a stationary process. If $g(s) = g_0(s) = s_+^\alpha$, X is a fractional Lévy process. In particular, when L is a β-stable Lévy process with $\beta \in (0, 2)$, X is called a linear fractional stable motion and it is self-similar with index $H = \alpha + 1/\beta$; see e.g. [18] (since in this case the stability index and the Blumenthal–Getoor index of L coincide, they are both denoted by β).

In order to describe the main results of [5] we need to introduce some notation and a set of assumptions. First of all, we consider the kth order increments $\Delta^n_{i,k} X$ of X, $k \in \mathbb{N}$, that are defined by

$$\Delta^n_{i,k} X := \sum_{j=0}^{k} (-1)^j \binom{k}{j} X_{(i-j)/n}, \qquad i \geq k.$$

For instance, we have that $\Delta^n_{i,1} X = X_{\frac{i}{n}} - X_{\frac{i-1}{n}}$ and $\Delta^n_{i,2} X = X_{\frac{i}{n}} - 2X_{\frac{i-1}{n}} + X_{\frac{i-2}{n}}$. The main functional is the power variation computed on the basis of kth order increments:

$$V(p;k)_n := \sum_{i=k}^{n} |\Delta^n_{i,k} X|^p, \qquad p > 0.$$

Now, we introduce the following set of assumptions on g, g_0 and ν:

Assumption (A): The function $g\colon \mathbb{R} \to \mathbb{R}$ satisfies

$$g(t) \sim c_0 t^\alpha \qquad \text{as } t \downarrow 0 \quad \text{for some } \alpha > 0 \text{ and } c_0 \neq 0,$$

where $g(t) \sim f(t)$ as $t \downarrow 0$ means that $\lim_{t\downarrow 0} g(t)/f(t) = 1$. For some $\theta \in (0,2]$, $\limsup_{t\to\infty} \nu(x : |x| \geq t)t^\theta < \infty$ and $g - g_0$ is a bounded function in $L^\theta(\mathbb{R}_+)$. Furthermore, g is k-times continuous differentiable on $(0,\infty)$ and there exists a $\delta > 0$ such that $|g^{(k)}(t)| \leq Kt^{\alpha-k}$ for all $t \in (0,\delta)$, $g^{(k)} \in L^\theta((\delta,\infty))$ and $|g^{(k)}|$ is decreasing on (δ,∞).

Assumption (A-log): In addition to (A) suppose that $\int_\delta^\infty |g^{(k)}(s)|^\theta \log(1/|g^{(k)}(s)|)\, ds < \infty$.

Assumption (A) ensures that the process X is well-defined, cf. [5, Section 2.4]. When L is a β-stable Lévy process, we always choose $\theta = \beta$ in assumption (A). Before we proceed with the main statements, we need some more notation. Let $h_k : \mathbb{R} \to \mathbb{R}$ be given by

$$h_k(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x-j)_+^\alpha, \qquad x \in \mathbb{R}, \tag{1.2}$$

where $y_+ = \max\{y, 0\}$ for all $y \in \mathbb{R}$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by $(L_t)_{t\geq 0}$, $(T_m)_{m\geq 1}$ be a sequence of $\mathbb{F}$-stopping times that exhaust the jumps of $(L_t)_{t\geq 0}$, that is, $\{T_m(\omega) : m \geq 1\} \cap \mathbb{R}_+ = \{t \geq 0 : \Delta L_t(\omega) \neq 0\}$ and $T_m(\omega) \neq T_n(\omega)$ for all $m \neq n$ with $T_m(\omega) < \infty$. Let $(U_m)_{m\geq 1}$ be independent and uniform $[0,1]$-distributed random variables, defined on an extension $(\Omega', \mathcal{F}', \mathbb{P}')$ of the original probability space, which are independent of $\mathcal{F}$.

The following first order limit theory for the power variation $V(p;k)_n$ has been proved in [5]. We refer to [1, 17] for the definition of $\mathcal{F}$-stable convergence in law which will be denoted $\xrightarrow{\mathcal{L}-s}$. Moreover, $\xrightarrow{\mathbb{P}}$ will denote convergence in probability.

Theorem 1.1 (First order asymptotics [5])**.** *Suppose (A) is satisfied and assume that the Blumenthal–Getoor index satisfies $\beta < 2$. We have the following three cases:*

(i) Suppose that (A-log) holds if $\theta = 1$. If $\alpha < k - 1/p$ and $p > \beta$ then the $\mathcal{F}$-stable convergence hold

$$n^{\alpha p} V(p;k)_n \xrightarrow{\mathcal{L}-s} |c_0|^p \sum_{m:\, T_m \in [0,1]} |\Delta L_{T_m}|^p V_m \quad \text{where} \quad V_m = \sum_{l=0}^{\infty} |h_k(l + U_m)|^p. \tag{1.3}$$

(ii) Suppose that L is a symmetric β-stable Lévy process with scale parameter $\sigma > 0$. If $\alpha < k - 1/\beta$ and $p < \beta$ then it holds

$$n^{-1+p(\alpha+1/\beta)} V(p;k)_n \xrightarrow{\mathbb{P}} m_p \tag{1.4}$$

where $m_p = |c_0|^p \sigma^p (\int_\mathbb{R} |h_k(x)|^\beta\, dx)^{p/\beta} \mathbb{E}[|Z|^p]$ and Z is a symmetric β-stable random variable with scale parameter 1.

(iii) Suppose that $p \geq 1$. If $p = \theta$ suppose in addition that (A-log) holds. For all $\alpha > k - 1/(\beta \vee p)$ we have that

$$n^{-1+pk} V(p;k)_n \xrightarrow{\mathbb{P}} \int_0^1 |F_u|^p\, du \tag{1.5}$$

where $(F_u)_{u\in\mathbb{R}}$ *is a measurable process satisfying*

$$F_u = \int_{-\infty}^{u} g^{(k)}(u-s)\,dL_s \quad \text{a.s. for all } u\in\mathbb{R} \quad \text{and} \quad \int_0^1 |F_u|^p\,du < \infty \quad \text{a.s.}$$

We remark that Theorem 1.1 covers all possible choices of $\alpha > 0, \beta \in [0,2)$ and $p \geq 1$ except the critical cases where $p = \beta$, $\alpha = k - 1/p$ and $\alpha = k - 1/\beta$. In this paper we will show the limit theory for the critical casess $\alpha = k - 1/p$, $p > \beta$ and $\alpha = k - 1/\beta$, $p < \beta/2$. Theorem 1.2 below is the main result of our work, and to state this theorem we let $C^k(\mathbb{R}_{\geq 0})$ denote the set of continuous functions $h\colon [0,\infty) \to \mathbb{R}$ which are k-times continuous differentiable on $\mathbb{R}_{>0}$ such that $\lim_{t\downarrow 0} h^{(j)}(t)$ exists for all $j = 1,\ldots,k$, and we set

$$k_\alpha := \prod_{j=0}^{k-1}(\alpha - j).$$

Theorem 1.2 (Critical cases)**.** *Suppose (A) is satisfied and assume that the Blumenthal–Getoor index satisfies* $\beta < 2$*. Let* $f : \mathbb{R}_+ \to \mathbb{R}$ *be given by* $f(t) = g(t)t^{-\alpha}$ *for* $t > 0$ *and* $f(0) = c_0$*, and assume that* $f \in C^k(\mathbb{R}_{\geq 0})$*.*

(i) Suppose that $1/p + 1/\theta > 1$*, and for* $\theta = 1$ *suppose, in addition, that (A-log) holds. If* $\alpha = k - 1/p$ *and* $p > \beta$ *then*

$$\frac{n^{\alpha p}}{\log(n)} V(p;k)_n \xrightarrow{\mathbb{P}} |c_0 k_\alpha|^p \sum_{s\in(0,1]} |\Delta L_s|^p. \tag{1.6}$$

(ii) Suppose that L *is a symmetric* β*-stable Lévy process with scale parameter* $\sigma > 0$*. If* $\alpha = k - 1/\beta$ *and* $p < \beta/2$ *then*

$$\frac{n^{-1+p(\alpha+1/\beta)}}{(\log n)^{p/\beta}} V(p;k)_n \xrightarrow{\mathbb{P}} \tilde{m}_p, \tag{1.7}$$

where $\tilde{m}_p = |c_0 k_\alpha \sigma|^p \mathbb{E}[|Z|^p]$ *and* Z *is a symmetric* β*-stable random variable with scale parameter* 1*.*

We note the appearance of additional logarithmic rates in Theorem 1.2(i) and (ii) compared to Theorem 1.1(i) and (ii). We also remark that the mode of convergence in Theorem 1.2(i) becomes convergence in probability rather than stable convergence in Theorem 1.1(i). In [5] the authors have also shown the weak limit theory associated with Theorem 1.1(ii), which comprises a central limit theorem and a convergence towards an $(1-\alpha)\beta$-stable totally right skewed random variable. In this work we dispense with proving similar asymptotic results associated with Theorem 1.2(ii).

This paper is structured as follows. Section 2 presents some remarks about the nature and intuition of the main results. Section 3 introduces some preliminaries. We state the proof of Theorem 1.2(i) in Section 4, while the proof of Theorem 1.2(ii) is demonstrated in Section 5.

2 Background and basic ideas

In this section we explain the intuition and the methodology of the proofs of Theorems 1.1 and 1.2. For simplicity of exposition we only consider the case $k = 1$ and we set $\Delta_i^n X := \Delta_{i,1}^n X$, $h := h_1$ and $V(p)_n := V(p; 1)_n$.

In order to uncover the path properties of the process X we perform a formal differentiation with respect to time. Since $g(0) = 0$ we obtain a formal representation

$$dX_t = g(0)dL_t + \left(\int_{-\infty}^t g'(t-s)\, dL_s\right) dt = F_t\, dt.$$

We remark that the random variable F_t is not necessarily finite under assumption (A). However, under conditions of Theorem 1.1(iii), the process X is differentiable almost everywhere and $X' = F \in L^p([0,1])$, although the process F explodes at jump times of L when $\alpha < 1$ (we refer to [5, Lemma 4.3] for the proof of this statement). Thus, under the conditions of Theorem 1.1(iii), an application of the mean value theorem gives an intuitive proof of (1.5):

$$\mathbb{P}\text{-}\lim_{n\to\infty} n^{-1+p} V(p)_n = \mathbb{P}\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n |F_{\xi_i^n}|^p = \int_0^1 |F_u|^p\, du,$$

where $\xi_i^n \in ((i-1)/n, i/n)$ (see [5, Lemma 4.4] for a formal argument). This gives a sketch of the proof of the asymptotic result at (1.5).

Now, we turn our attention to the small scale behaviour of the stationary increments Lévy driven moving averages X. Recall that under conditions of Theorem 1.1(ii), $\alpha < 1 - 1/\beta$ and thus g' has an explosive behaviour at 0. Hence, we intuitively deduce the following approximation for the increments of X for a small $\Delta > 0$:

$$\begin{aligned} X_{t+\Delta} - X_t &= \int_{\mathbb{R}} \{g(t+\Delta-s) - g(t-s)\}\, dL_s \\ &\approx \int_{t+\Delta-\epsilon}^{t+\Delta} \{g(t+\Delta-s) - g(t-s)\}\, dL_s \\ &\approx c_0 \int_{t+\Delta-\epsilon}^{t+\Delta} \{(t+\Delta-s)_+^\alpha - (t-s)_+^\alpha\}\, dL_s \\ &\approx c_0 \int_{\mathbb{R}} \{(t+\Delta-s)_+^\alpha - (t-s)_+^\alpha\}\, dL_s = Y_{t+\Delta} - Y_t, \end{aligned}$$

where

$$Y_t := c_0 \int_{\mathbb{R}} \{(t-s)_+^\alpha - (-s)_+^\alpha\}\, dL_s, \tag{2.1}$$

and $\epsilon > 0$ is an arbitrary small real number with $\epsilon \gg \Delta$. In the classical terminology Y is called the *tangent process of* X. In the framework of Theorem 1.1(ii) the process Y is a symmetric fractional β-stable motion. We recall that $(Y_t)_{t\geq 0}$ has stationary increments, symmetric β-stable marginals and it is self-similar with index $H = \alpha + 1/\beta \in (1/2, 1)$, i.e.

$$(Y_{at})_{t\geq 0} \stackrel{d}{=} a^H (Y_t)_{t\geq 0}.$$

Furthermore, the symmetric fractional β-stable noise $(Y_t - Y_{t-1})_{t\geq 1}$ is mixing; see e.g. [7]. Thus, using Birkhoff's ergodic theorem we conclude that

$$\begin{aligned} n^{-1+p(\alpha+1/\beta)} V(p)_n &= \frac{1}{n}\sum_{i=1}^{n} |n^H \Delta_i^n X|^p \\ &\approx \frac{1}{n}\sum_{i=1}^{n} |n^H \Delta_i^n Y|^p \\ &\overset{d}{=} \frac{1}{n}\sum_{i=1}^{n} |Y_i - Y_{i-1}|^p \xrightarrow{\mathbb{P}} \mathbb{E}[|Y_1 - Y_0|^p]. \end{aligned}$$

Noting that $Y_1 - Y_0$ is a symmetric β-stable random variable with scale parameter

$$\sigma_Y = \sigma |c_0| \|h\|_{L^\beta(\mathbb{R})},$$

we conclude that $m_p = \mathbb{E}[|Y_1 - Y_0|^p]$. This method sketches the proof of the convergence at (1.4); see [5, Section 4.2] for more details. However, in the critical case $\alpha = 1 - 1/\beta$ of Theorem 1.2(ii), which corresponds to $H = 1$, the previous idea does not work. In particular, the process Y is not well-defined in this framework since $h \notin L^\beta(\mathbb{R})$. In fact, we will show the convergence (1.7) more directly, by proving that

$$\frac{n^{-1+p(\alpha+1/\beta)}}{(\log n)^{p/\beta}} \mathbb{E}[V(p)_n] \to \tilde{m}_p \qquad \text{and} \qquad \frac{n^{-2+2p(\alpha+1/\beta)}}{(\log n)^{2p/\beta}} \operatorname{var}(V(p)_n) \to 0.$$

In contrast to Theorem 1.1(ii), where we only require that $p < \beta$ to ensure that $m_p < \infty$, in Theorem 1.2(ii) we need to assume the stronger condition $p < \beta/2$ to be able to compute $\operatorname{var}(V(p)_n)$. At the moment we do not know how to show (1.7) under the mere condition $p < \beta$.

At this stage we need to better understand the fine scale behaviour of the process X in order to describe the intuition behind the non-standard result of Theorem 1.1(i). For simplicity of exposition we will discuss the symmetric fractional β-stable motion Y defined at (2.1), although the stability of the driving Lévy process L does not matter for our arguments. Instead, the fact that β is the Blumenthal-Getoor index of L is more important.

As it follows by [19, Theorem 3.4], process $(Y_t)_{t\in[0,1]}$ has Hölder index α (recall $\alpha > 0$). This is in strong contrast to the framework of fractional Brownian motion, which corresponds to the case $\beta = 2$, that has Hölder index $H = \alpha + 1/2$. Thus, in terms of Hölder continuity, the fractional Brownian motion has much smoother sample paths than the linear fractional sample motion. We recall that the Hölder index of a stochastic process is the largest index θ such that the sample paths are $(\theta - \epsilon)$-Hölder continuous a.s. for all $\epsilon > 0$. Now, we will discuss the asymptotic distribution of the scaled increments $n^\alpha \Delta_i^n Y$. Although the process Y has infinitely many jumps on finite intervals, we assume for simplicity of exposition that $T \in [(j-1)/n, j/n)$ is the only jump time of L within the interval $[0,1]$. As above we consider the approximation

$$\Delta_i^n Y \approx A_i^n + B_i^n$$
$$:= c_0 \left(\int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\frac{i}{n} - s\right)^{\alpha} dL_s + \int_0^{\frac{i-1}{n}} \left\{ \left(\frac{i}{n} - s\right)^{\alpha} - \left(\frac{i-1}{n} - s\right)^{\alpha} \right\} dL_s \right)$$

Since $T \in [(j-1)/n, j/n)$ is the only jump time of L, we observe that $A_i^n = 0$ for all $i \neq j$ and $B_i^n = 0$ for all $i < j$. More precisely, we deduce that

$$\Delta_{j+l}^n Y \approx \begin{cases} c_0 \Delta Y_T \left(\frac{j}{n} - T\right)^{\alpha} & l = 0 \\ c_0 \Delta Y_T \left(\left(\frac{j+l}{n} - T\right)^{\alpha} - \left(\frac{j+l-1}{n} - T\right)^{\alpha} \right) & l \geq 1 \end{cases}$$

Now, we use the following result, which is essentially due to Tukey [21] (see also [10] and [5, Lemma 4.1]): Let Z be a random variable with an absolutely continuous distribution and let $\{x\} := x - \lfloor x \rfloor \in [0,1)$ denote the fractional part of $x \in \mathbb{R}$. Then it holds that

$$\{nZ\} \xrightarrow{\mathcal{L}-s} U \sim \mathcal{U}([0,1]),$$

where U is defined on the extended probability space and U is independent of $\mathcal{F}$. Since $\{nT\} = j - nT$ we conclude the stable convergence

$$n^{\alpha} \Delta_{j+l}^n Y \xrightarrow{\mathcal{L}-s} c_0 \Delta Y_T \left((l+U)_+^{\alpha} - (l-1+U)_+^{\alpha} \right), \qquad l \geq 0.$$

Thus, we obtain the result of (1.3) for one jump time:

$$n^{\alpha p} V(Y,p)_n \approx \sum_{i=j}^{n} |n^{\alpha}(A_i^n + B_i^n)|^p \xrightarrow{\mathcal{L}-s} |c_0 \Delta Y_T|^p \sum_{l=0}^{\infty} \left| (l+U)_+^{\alpha} - (l-1+U)_+^{\alpha} \right|^p, \tag{2.2}$$

which gives an intuitive proof of Theorem 1.1(i). A formal proof of the stable convergence at (1.3) requires a decomposition of the driving jump measure associated with L into big and small jumps, and a certain time separation between the big jumps; we refer to [5, Section 4.1] for a detailed exposition. We remark that the conditions $\alpha \in (0, 1-1/p)$ and $p > \beta$ of Theorem 1.1(i) seem to be sharp. Indeed, since $|h(x)| \leq K x^{\alpha-1}$ for large x, we obtain from (1.3) that

$$\sup_{m \geq 1} V_m < \infty$$

when $\alpha \in (0, 1-1/p)$. On the other hand $\sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^p < \infty$ for $p > \beta$, which follows from the definition of the Blumenthal–Getoor index at (1.1). In particular, the quadratic variation case $p = 2$ always falls under Theorem 1.1(i) whenever $\alpha \in (0, 1/2)$.

In the critical case $\alpha = 1 - 1/p$ of Theorem 1.2(i) the above argument fails due to the fact that $\sum_{l=0}^{\infty} \left| (l+U)_+^{\alpha} - (l-1+U)_+^{\alpha} \right|^p = \infty$. However, applying mean value theorem and the same argument as in (2.2), we deduce that

$$\frac{n^{\alpha p}}{\log(n)} V(Y,p)_n \approx \frac{|c_0 \alpha \Delta Y_T|^p}{\log(n)} \sum_{l=1}^{n-j} \frac{1}{l} \xrightarrow{\mathbb{P}} |c_0 \alpha \Delta Y_T|^p,$$

when $T \in (0,1)$. This give an intuition behind the convergence at (1.6).

3 Preliminaries

For all $p > 0$ and all measurable functions $f:\mathbb{R}\to\mathbb{R}$ we let $\|f\|_{L^p(\mathbb{R})} = (\int_{\mathbb{R}} |f(x)|^p\,dx)^{1/p}$ which defines a (semi) norm for all $p \geq 1$. Throughout the following sections all positive constants will be denoted by K, although they may change from line to line. Also the notation might change from subsection to subsection, but the meaning will be clear from the context. Throughout all the next sections we assume, without loss of generality, that $c_0 = \delta = \sigma = 1$. Recall that $g(t) = g_0(t) = 0$ for all $t < 0$ by assumption. When k and p are fixed and we want to stress that the power variation is built from a process Y we will sometimes write $V(Y)_n = \sum_{i=k}^n |\Delta_{i,k}^n Y|^p$ to simplify the notation. For all $n, i \in \mathbb{N}$ set

$$\begin{aligned} g_{i,n}(x) &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} g\big((i-j)/n - x\big), \qquad (3.1)\\ h_{i,n}(x) &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \big((i-j)/n - x\big)_+^{\alpha},\\ g_n(x) &= n^{\alpha} g(x/n), \qquad x \in \mathbb{R}. \end{aligned}$$

In addition, for each function $\phi\colon \mathbb{R}\to\mathbb{R}$ define $D^k\phi\colon\mathbb{R}\to\mathbb{R}$ by

$$D^k\phi(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \phi(x-j), \qquad x\in\mathbb{R}.$$

In this notation the function h_k, defined in (1.2), is given by $h_k = D^k\phi$ with $\phi : x \mapsto x_+^{\alpha}$. We note that if ϕ is k-times continuous differentiable on $(0,\infty)$ then for all $x > k$ we have

$$D^k\psi(x) = \int_{x-k}^{x-k+1} \Big(\int_{t_k}^{t_k+1} \cdots \Big(\int_{t_2}^{t_2+1} \psi^{(k)}(t_1)\,dt_1 \Big) \cdots dt_{k-1} \Big) dt_k,$$

which by the mean-value theorem implies that

$$D^k\psi(x) = \psi^{(k)}(y) \qquad \text{for some } y \in [x-k, x]. \qquad (3.2)$$

Furthermore, by Lemma 3.1 in [5] we have the following estimates on $g_{i,n}$.

Lemma 3.1. *Assume that g satisfies condition (A). Then we obtain the following estimates*

$$\begin{aligned} |g_{i,n}(x)| &\leq K(i/n - x)^{\alpha}, \qquad x \in [(i-k)/n, i/n], \qquad (3.3)\\ |g_{i,n}(x)| &\leq K n^{-k} ((i-k)/n - x)^{\alpha-k}, \qquad x \in (i/n - 1, (i-k)/n), \qquad (3.4)\\ |g_{i,n}(x)| &\leq K n^{-k} \Big(\mathbb{1}_{[(i-k)/n-1, i/n-1]}(x) + g^{(k)}((i-k)/n - x) \mathbb{1}_{(-\infty,(i-k)/n-1)}(x) \Big), \qquad (3.5)\\ &\qquad x \in (-\infty, i/n - 1]. \end{aligned}$$

The same estimates trivially hold for the function $h_{i,n}$.

4 Proof of Theorem 1.2(i)

In this section we prove the assertions of Theorem 1.2(i). The proof will be divided into the following three steps. First, in Step (i), we prove Theorem 1.2(i) in case L is a compound Poisson process. Next, in Step (ii), an approximation lemma is derived which is used in Step (iii) together with the result of Step (i) to obtain the general result of Theorem 1.2(i). The proof relies heavily on a crucial decomposition derived in [5], see (4.3) below.

Step (i) (compound Poisson case): Suppose that $L = (L_t)_{t\in\mathbb{R}}$ is a compound Poisson process and let $0 \leq T_1 < T_2 < \ldots$ denote the jump times of the Lévy process $(L_t)_{t\geq 0}$ chosen in increasing order. Consider a fixed $\epsilon > 0$ and let $n \in \mathbb{N}$ satisfy $\varepsilon > n^{-1}$. We define

$$\Omega_\varepsilon := \Big\{\omega \in \Omega : \text{for all } j \geq 1 \text{ with } T_j(\omega) \in [0,1] \text{ we have } |T_{j+1}(\omega) - T_j(\omega)| > \varepsilon/2 \\ \text{and } \Delta L_s(\omega) = 0 \text{ for all } s \in [-\epsilon, 0]\Big\}.$$

Notice that $\Omega_\varepsilon \uparrow \Omega$ as $\varepsilon \downarrow 0$. Now, we decompose for $i = k, \ldots, n$

$$\Delta^n_{i,k} X = M_{i,n,\varepsilon} + R_{i,n,\varepsilon}, \tag{4.1}$$

where

$$M_{i,n,\varepsilon} = \int_{\frac{i}{n}-\frac{\varepsilon}{2}}^{\frac{i}{n}} g_{i,n}(s)\,dL_s, \qquad R_{i,n,\varepsilon} = \int_{-\infty}^{\frac{i}{n}-\frac{\varepsilon}{2}} g_{i,n}(s)\,dL_s,$$

and the function $g_{i,n}$ is introduced in (3.1). The term $M_{i,n,\varepsilon}$ represents the dominating quantity, while $R_{i,n,\varepsilon}$ turns out to be negligible.

The dominating term: In the following we will show that almost surely on Ω_ϵ we have

$$\frac{n^{\alpha p}}{\log(n)} \sum_{i=k}^{n} |M_{i,n,\varepsilon}|^p \to |k_\alpha|^p \sum_{s\in(0,1]} |\Delta L_s|^p \qquad \text{as } n \to \infty. \tag{4.2}$$

To show (4.2) the following representation is crucial. For each m let $i_m = i_m(\omega, n)$ denote the (random) index satisfying $T_m \in ((i_m - 1)/n, i_m/n]$. On Ω_ε we have that

$$\frac{n^{\alpha p}}{\log(n)} \sum_{i=k}^{n} |M_{i,n,\varepsilon}|^p = V_{n,\varepsilon} \qquad \text{with} \tag{4.3}$$

$$V_{n,\varepsilon} = \frac{n^{\alpha p}}{\log(n)} \sum_{m\colon T_m\in(0,1]} |\Delta L_{T_m}|^p \left(\sum_{l=0}^{[\varepsilon n/2]+v_m} |g_{i_m+l,n}(T_m)|^p\right)$$

where the random indexes $v_m = v_m(\omega, n, \epsilon)$ are given by $v_m = 0$ if $([\epsilon n/2] + i_m)/n - \epsilon/2 < T_m$ and $v_m = -1$ else, see (4.4) in [5]. We proceed by showing that for each fixed m we have almost surely

$$\frac{n^{\alpha p}}{\log(n)} \sum_{l=0}^{[\varepsilon n/2]+v_m} |g_{i_m+l,n}(T_m)|^p \to |k_\alpha|^p \qquad \text{as } n \to \infty. \tag{4.4}$$

To this aim we start by noticing that

$$\frac{n^{\alpha p}}{\log(n)} \sum_{l=0}^{k} |g_{i_m+l,n}(T_m)|^p \to 0 \qquad \text{as } n \to \infty \tag{4.5}$$

which follows by the estimate $|g(x)| \le K|x|^\alpha$ for all $x \in (0,1)$. Next we will show that

$$\frac{1}{\log(n)} \sum_{l=k+1}^{[\varepsilon n/2]+v_m} |n^\alpha g_{i_m+l,n}(T_m) - h_k(l + \{nT_m\})|^p \to 0 \qquad \text{as } n \to \infty. \tag{4.6}$$

We let $f(t) = g(t)t^{-\alpha}$ for $t > 0$ and $f(0) = 1$. Then $g(s) = s^\alpha f(s)$ for $s \ge 0$ and we find that

$$\eta_n(s) := n^\alpha g(s/n) - s^\alpha = n^\alpha (s/n)^\alpha \{f(s/n) - f(0)\} = n^\alpha \psi_1(s/n)\psi_2(s/n)$$

where $\psi_1(s) = s^\alpha$ and $\psi_2(s) = f(s) - f(0)$ for $s \ge 0$. We start by noticing that

$$n^\alpha g_{i_m+l,n}(T_m) - h_k(l + \{nT_m\}) = D^k \eta_n(l + \{nT_m\}).$$

For all $s > k$ there exists, cf. (3.2), a $\xi_s^n \in [s-k, s]$ such that

$$(D^k \eta_n)(s) = \eta_n^{(k)}(\xi_s^n) = n^{\alpha-k} \sum_{j=0}^{k} \binom{k}{j} \psi_1^{(j)}(\xi_s^n/n)\psi_2^{(k-j)}(\xi_s^n/n) \tag{4.7}$$

where the the last equality follows by the product rule. Applying (4.7) on $s = l + \{nT_m\}$ we obtain $\zeta_l^n \in [l-k, l+1]$ such that

$$|(D^k \eta_n)(l + \{nT_m\})| \le K \sum_{j=0}^{k} n^{j-k} |\xi_l^n|^{\alpha-j} \psi_2^{(k-j)}(\xi_l^n/n)$$

where we have use that $\psi_1^{(j)}(t) = \alpha(\alpha-1)\cdots(\alpha-j+1)t^{\alpha-j}$ for $t > 0$. Hence,

$$\frac{1}{\log(n)} \sum_{l=k+1}^{[\varepsilon n/2]+v_m} |n^\alpha g_{i_m+l,n}(T_m) - h_k(l + \{nT_m\})|^p \le K \sum_{j=0}^{k} a_{j,n} \tag{4.8}$$

where for $j = 0, \ldots, k-1$ we have

$$a_{j,n} = \frac{n^{(j-k)p}}{\log(n)} \sum_{l=k+1}^{[\varepsilon n/2]+v_m} |\xi_l^n|^{(\alpha-j)p} \le K \frac{n^{(j-k)p}}{\log(n)} n^{(\alpha-j)p+1} = K n^{(\alpha-k)p+1} \frac{1}{\log(n)} \to 0$$

as $n \to \infty$, where we have use that $\psi_2^{(k-j)}$ is bounded on $(0,1]$ in the first inequality and that $(\alpha - k)p = -1$ to conclude the convergence to zero. For $j = k$ we have

$$a_{k,n} \le \frac{n^{-p}}{\log(n)} \sum_{l=k+1}^{[\varepsilon n/2]+v_m} |\xi_l^n|^{p-1} \le K \frac{1}{\log(n)} \to 0 \qquad \text{as } n \to \infty$$

where we have used that $(\alpha-k)p=-1$ and $|\psi_2(x)|\le Kx$ for all $x\in(0,1]$. This completes the proof of (4.6).

Set $\phi(t)=t_+^\alpha$ for all $t\in\mathbb{R}$. Next we will show that

$$\frac{1}{\log(n)}\sum_{l=k+1}^{[\varepsilon n/2]+v_m}|h_k(l+\{nT_m\})-\phi^{(k)}(l)|^p\to 0 \qquad \text{as } n\to\infty. \tag{4.9}$$

We have that $h_k=D^k\phi$, and hence by (3.2) we deduce the estimate

$$\begin{aligned}|h_k(l+\{nT_m\})-\phi^{(k)}(l)| &\le \sup_{s\in[l-k,l+1]}|\phi^{(k)}(s)-\phi^{(k)}(l)|\\ &\le K\sup_{s\in[l-k,l+1]}|\phi^{(k+1)}(s)|\le K(l-k)^{\alpha-k-1},\end{aligned} \tag{4.10}$$

which shows that

$$\begin{aligned}&\frac{1}{\log(n)}\sum_{l=k+1}^{[\varepsilon n/2]+v_m}|h_k(l+\{nT_m\})-\phi^{(k)}(l)|^p\\ &\quad\le K\frac{1}{\log(n)}\sum_{l=k+1}^{[\varepsilon n/2]+v_m}(l-k)^{(\alpha-k)p-p}\le Kn^{-p}\frac{1}{\log(n)}\to 0 \qquad \text{as } n\to\infty,\end{aligned}$$

and completes the proof of (4.9). Furthermore, we have

$$\frac{1}{\log(n)}\sum_{l=k+1}^{[\varepsilon n/2]+v_m}|\phi^{(k)}(l)|^p=|k_\alpha|^p\frac{1}{\log(n)}\sum_{l=k+1}^{[\varepsilon n/2]+v_m}l^{-1}\to|k_\alpha|^p \qquad \text{as } n\to\infty. \tag{4.11}$$

By applying the inequality

$$\Big|\Big(\sum_{r=1}^{l}|a_r|^p\Big)^{1/p}-\Big(\sum_{r=1}^{l}|b_r|^p\Big)^{1/p}\Big|\le\Big(\sum_{r=1}^{q}|a_r-b_r|^p\Big)^{1/p} \tag{4.12}$$

for $p\ge 1$ (follows by Minkowski's inequality), and the inequality

$$\Big|\sum_{r=1}^{l}|a_r|^p-\sum_{r=1}^{l}|b_r|^p\Big|\le\sum_{r=1}^{l}|a_r-b_r|^p \tag{4.13}$$

for $p\in(0,1)$ (follows by sub additivity), we deduce (4.4) from (4.5), (4.6), (4.9) and (4.11).

The rest term: In the following we will show that

$$\frac{n^{\alpha p}}{\log(n)}\sum_{i=k}^{n}|R_{i,n,\epsilon}|^p\xrightarrow{\mathbb{P}}0 \qquad \text{as } n\to\infty. \tag{4.14}$$

The fact that the random variables in (4.14) are usually not integrable makes the proof of (4.14) considerable more complicated, however, we can rely on some estimates already derived in [5] to show this part. Since $\alpha = k - 1/p$ we have the simple estimate

$$\frac{n^{\alpha p}}{\log(n)} \sum_{i=k}^{n} |R_{i,n,\epsilon}|^p \le \frac{1}{\log(n)} \Big(\max_{i=k,\ldots,n} |n^k R_{i,n,\epsilon}| \Big)^p. \tag{4.15}$$

We will divide the proof of (4.14) into the two special cases $\theta \in (0,1]$ and $\theta \in (1,2]$ which need separate treatments. Suppose first that $\theta \in (0,1]$. To show (4.14) it suffices, according to (4.15), to show that

$$\sup_{n\in\mathbb{N},\, i=k,\ldots,n} n^k |R_{i,n,\epsilon}| < \infty \qquad \text{a.s.}$$

The proof of this follows by (4.11) of [5] (which also holds under the assumption $\alpha = k - 1/p$). Next we assume that $\theta \in (1,2]$. According to (4.15) of [5] (which also holds under the assumption $\alpha = k - 1/p$) we have that

$$U := \sup_{n\in\mathbb{N},\, i=k,\ldots,n} \frac{n^k |R_{i,n,\epsilon}|}{(\log n)^{1/q}} < \infty \qquad \text{a.s.}$$

where $q > 1$ denotes the conjugated number to $\theta > 1$ determined by $1/\theta + 1/q = 1$. By (4.15) we have

$$\frac{n^{\alpha p}}{\log(n)} \sum_{i=k}^{n} |R_{i,n,\epsilon}|^p \le \Big[(\log n)^{1/q - 1/p} U \Big]^p$$

which shows (4.14) since $1/q - 1/p = 1 - 1/\theta - 1/p < 0$, the latter follows by the assumption $1/p + 1/\theta > 1$. This completes the proof of (4.14) in general.

End of the proof of Step (i): By the decomposition (4.1) together with (4.2) and (4.14) we deduce from the two inequalities (4.12) and (4.13) that for all $\epsilon > 0$

$$\frac{n^{\alpha p}}{\log(n)} V(X)_n \xrightarrow{\mathbb{P}} |k_\alpha|^p \sum_{s\in(0,1]} |\Delta L_s|^p \qquad \text{on } \Omega_\epsilon \text{ as } n \to \infty. \tag{4.16}$$

Since $\Omega_\epsilon \uparrow \Omega$ as $\epsilon \downarrow 0$, (4.16) implies

$$\frac{n^{\alpha p}}{\log(n)} V(X)_n \xrightarrow{\mathbb{P}} |k_\alpha|^p \sum_{s\in(0,1]} |\Delta L_s|^p \qquad \text{as } n \to \infty$$

which completes the proof of Step (i).

Step (ii) (an approximation): To prove Theorem 1.2(i) in the general case we need the following approximation result. Consider a general symmetric Lévy process $L = (L_t)_{t\in\mathbb{R}}$ as in Theorem 1.2(i) and let N be the corresponding Poisson random measure $N(A) := \sharp\{t \in \mathbb{R} : (t, \Delta L_t) \in A\}$ for all measurable $A \subseteq \mathbb{R} \times (\mathbb{R} \setminus \{0\})$. By our assumptions (in particular, by symmetry), the process $X(j)$ given by

$$X_t(j) = \int_{(-\infty,t]\times[-\frac{1}{j},\frac{1}{j}]} \big\{ (g(t-s) - g_0(-s))x \big\} \, N(ds, dx) \tag{4.17}$$

is well-defined. The following estimate on the processes $X(j)$ given in the below lemma will be crucial. The overall idea in proving the lemma are similar to that of Lemma 4.2 in [5], however, different estimates are needed due to different assumptions and statements.

Lemma 4.1. *Suppose that* $\alpha = k - 1/p$ *and* $p > \beta$. *Then*

$$\lim_{j\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(\frac{n^{\alpha p}}{\log(n)} V(X(j))_n > \epsilon\Big) = 0 \qquad \text{for all } \epsilon > 0.$$

Proof. By Markov's inequality and the stationary increments of $X(j)$ we have that

$$\mathbb{P}\Big(\frac{n^{\alpha p}}{\log(n)} V(X(j))_n > \epsilon\Big) \leq \epsilon^{-1} \frac{n^{\alpha p}}{\log(n)} \sum_{i=k}^{n} \mathbb{E}[|\Delta_{i,k}^n X(j)|^p] \leq \epsilon^{-1} \frac{n^{\alpha p+1}}{\log(n)} \mathbb{E}[|\Delta_{k,k}^n X(j)|^p].$$

Hence it is enough to show that

$$\lim_{j\to\infty} \limsup_{n\to\infty} \mathbb{E}[|Y_{n,j}|^p] = 0 \qquad \text{with} \quad Y_{n,j} := a_n \Delta_{k,k}^n X(j), \quad a_n := \frac{n^{\alpha+1/p}}{(\log n)^{1/p}}. \tag{4.18}$$

To show (4.18) it sufficers to show

$$\lim_{j\to\infty} \limsup_{n\to\infty} \xi_{n,j} = 0 \qquad \text{where} \qquad \xi_{n,j} = \int_{|x|\leq 1/j} \chi_n(x)\, \nu(dx) \quad \text{and}$$

$$\chi_n(x) = \int_{-\infty}^{k/n} \Big(|a_n g_{k,n}(s)x|^p \mathbb{1}_{\{|a_n g_{k,n}(s)x|\geq 1\}} + |a_n g_{k,n}(s)x|^2 \mathbb{1}_{\{|a_n g_{k,n}(s)x|\leq 1\}}\Big)\, ds,$$

which follows from the representation

$$Y_{n,j} = \int_{(-\infty,k/n]\times[-\frac{1}{j},\frac{1}{j}]} \big(a_n g_{k,n}(s)x\big)\, N(ds,dx),$$

and by [16, Theorem 3.3 and the remarks above it]. Suppose for the moment that there exists a finite constant $K > 0$ such that

$$\chi_n(x) \leq K(|x|^p + x^2) \qquad \text{for all } x \in [-1,1]. \tag{4.19}$$

Then,

$$\limsup_{j\to\infty} \big\{ \limsup_{n\to\infty} \xi_{n,j} \big\} \leq K \limsup_{j\to\infty} \int_{|x|\leq 1/j} (|x|^p + x^2)\, \nu(dx) = 0$$

since $p > \beta$. Hence it suffices to show the estimate (4.19), which we will do in the following.

Let $\Phi_p \colon \mathbb{R} \to \mathbb{R}_+$ denote the function $\Phi_p(y) = |y|^2 \mathbb{1}_{\{|y|\leq 1\}} + |y|^p \mathbb{1}_{\{|y|>1\}}$. We split χ_n into the following three terms which need different treatments

$$\begin{aligned}\chi_n(x) &= \int_{-k/n}^{k/n} \Phi_p\Big(a_n g_{k,n}(s)x\Big)\, ds + \int_{-1}^{-k/n} \Phi_p\Big(a_n g_{k,n}(s)x\Big)\, ds \\ &\quad + \int_{-\infty}^{-1} \Phi_p\Big(a_n g_{k,n}(s)x\Big)\, ds \\ &=: I_{1,n}(x) + I_{2,n}(x) + I_{3,n}(x).\end{aligned}$$

Estimation of $I_{1,n}$: By (3.3) of Lemma 3.1 we have that

$$|g_{k,n}(s)| \le K(k/n - s)^{\alpha}, \qquad s \in [-k/n, k/n]. \tag{4.20}$$

Since Φ_p is increasing on $\mathbb{R}_+$, (4.20) and the estimate $a_n \le n^{\alpha+1/p}$ implies that

$$I_{1,n}(x) \le K \int_0^{2k/n} \Phi_p\Big(x n^{\alpha+1/p} s^{\alpha}\Big)\, ds.$$

Hence using (4.26) and (4.27) from [5] (which also holds under our assumptions) we obtain the estimate $I_{1,n}(x) \le K(|x|^p + x^2)$.

Estimation of $I_{2,n}$: By (3.4) of Lemma 3.1 it holds that

$$|g_{k,n}(s)| \le K n^{-k} |s|^{\alpha-k}, \qquad s \in (-1, -k/n). \tag{4.21}$$

Again, due to the fact that Φ_p is increasing on $\mathbb{R}_+$, (4.21) implies that

$$I_{2,n}(x) \le K \int_{k/n}^{1} \Phi_p(x(\log n)^{-1/p} s^{-1/p})\, ds \tag{4.22}$$

where we have used the assumption $\alpha = k - 1/p$. For $p \neq 2$ we have

$$\begin{aligned} &\int_{k/n}^{1} |x s^{-1/p}|^2 \mathbb{1}_{\{|xs^{-1/p}| \le 1\}}\, ds \\ &\quad \le K\Big(x^2 (n^{-1})^{-2/p+1} \mathbb{1}_{\{|x|^p \le n^{-1}\}} + x^2 (|x|^p)^{-2/p+1} \mathbb{1}_{\{|x|^p > n^{-1}\}}\Big) \qquad (4.23) \\ &\quad \le K\Big(x^2 + |x|^p\Big), \qquad (4.24) \end{aligned}$$

where the first term in (4.23) is estimated less than or equal to Kx^2 for for $p > 2$, and less than or equal to Kx^p for $p < 2$. For $p = 2$ we have

$$\begin{aligned} &\int_{k/n}^{1} |x(\log n)^{-1/p} s^{-1/p}|^2 \mathbb{1}_{\{|x(\log n)^{-1/p} s^{-1/p}| \le 1\}}\, ds \\ &\qquad \le x^2 (\log n)^{-1} \int_{k/n}^{1} s^{-1}\, ds \le K x^2. \qquad (4.25) \end{aligned}$$

Moreover,

$$\begin{aligned} &\int_{k/n}^{1} |x(\log n)^{-1/p} s^{-1/p}|^p \mathbb{1}_{\{|x n^{\alpha+1/p-k} s^{\alpha-k}| > 1\}}\, ds \\ &\qquad \le K|x|^p (\log n)^{-1} \int_{k/n}^{1} s^{-1}\, ds \le K|x|^p. \qquad (4.26) \end{aligned}$$

By (4.22), (4.24), (4.25) and (4.26) we obtain the estimate $I_{2,n}(x) \le K(|x|^p + x^2)$.

Estimation of $I_{3,n}$: For $s < -1$ we have $|g_{k,n}(s)| \le K n^{-k} |g^{(k)}(-k/n - s)|$ by (3.5) of Lemma 3.1, and hence

$$I_{3,n}(x) \le K \int_1^{\infty} \Phi_p\big(x g^{(k)}(s)\big)\, ds. \tag{4.27}$$

We have that
$$\int_1^\infty |xg^{(k)}(s)|^2 \mathbb{1}_{\{|xg^{(k)}(s)|\le 1\}}\, ds \le x^2 \int_1^\infty |g^{(k)}(s)|^2\, ds. \tag{4.28}$$
Since $|g^{(k)}|$ is decreasing on $(1,\infty)$ and $g^{(k)} \in L^\theta((1,\infty))$ for some $\theta \le 2$, the integral on the right-hand side of (4.28) is finite. For $x \in [-1,1]$ we have

$$\int_1^\infty |xg^{(k)}(s)|^p \mathbb{1}_{\{|xg^{(k)}(s)|>1\}}\, ds \le |x|^p \int_1^\infty |g^{(k)}(s)|^p \mathbb{1}_{\{|g^{(k)}(s)|>1\}}\, ds. \tag{4.29}$$

From our assumptions the integral on the right-hand side of (4.29) is finite. By (4.27), (4.28) and (4.29) we have that $I_{n,3}(x) \le K(|x|^p + x^2)$ for all $x \in [-1,1]$, which completes the proof of (4.19) and therefore also the proof of the lemma. □

Step (iii): The general case. In the following we will prove Theorem 1.2(i) in the general case by combining the above Steps (i) and (ii).

Proof of Theorem 1.2(i). For each $j \in \mathbb{N}$ let $\hat{L}(j)$ be the Lévy process given by

$$\hat{L}_t(j) - \hat{L}_u(j) = \sum_{u\in(s,t]} \Delta L_u \mathbb{1}_{\{|\Delta L_u|>\frac{1}{j}\}}, \qquad s<t,$$

and set
$$\hat{X}_t(j) = \int_{-\infty}^t \big(g(t-s) - g_0(-s)\big)\, d\hat{L}_s(j).$$

Since $\hat{L}(j)$ is a compound Poisson process, Step (i) shows that

$$\frac{n^{\alpha p}}{\log(n)} V(\hat{X}(j))_n \xrightarrow{\mathbb{P}} Z_j := |k_\alpha|^p \sum_{s\in(0,1]} |\Delta \hat{L}_s(j)|^p \qquad \text{as } n\to\infty. \tag{4.30}$$

By monotone convergence we have as $j\to\infty$,

$$Z_j = |k_\alpha|^p \sum_{s\in(0,1]} |\Delta L_s|^p \mathbb{1}_{\{|\Delta L_s|>\frac{1}{j}|\}} \xrightarrow{\text{a.s.}} |k_\alpha|^p \sum_{s\in(0,1]} |\Delta L_s|^p =: Z. \tag{4.31}$$

Suppose first that $p \ge 1$ and decompose

$$\Big(\frac{n^{\alpha p}}{\log(n)} V(X)_n\Big)^{1/p} = \Big(\frac{n^{\alpha p}}{\log(n)} V(\hat{X}(j))_n\Big)^{1/p} + \Big[\Big(\frac{n^{\alpha p}}{\log(n)} V(X)_n\Big)^{1/p} - \Big(\frac{n^{\alpha p}}{\log(n)} V(\hat{X}(j))_n\Big)^{1/p}\Big]$$
$$=: Y_{n,j} + U_{n,j}.$$

Equations (4.30) and (4.31) show

$$Y_{n,j} \xrightarrow[n\to\infty]{\mathbb{P}} Z_j^{1/p} \qquad \text{and} \qquad Z_j^{1/p} \xrightarrow[j\to\infty]{\mathbb{P}} Z^{1/p}.$$

Note that $X - \hat{X}(j) = X(j)$, where $X(j)$ is defined in (4.17). For all $\epsilon > 0$ we have by the inequality (4.12) that

$$\limsup_{j\to\infty}\limsup_{n\to\infty}\mathbb{P}\big(|U_{n,j}| > \epsilon\big) \le \limsup_{j\to\infty}\limsup_{n\to\infty}\mathbb{P}\Big(\frac{n^{\alpha p}}{\log(n)}V(X(j))_n > \epsilon^p\Big) = 0$$

where the last equality follows by Lemma 4.1. Hence, we deduce that

$$\Big(\frac{n^{\alpha p}}{\log(n)}V(X)_n\Big)^{1/p} \xrightarrow{\mathbb{P}} Z^{1/p}$$

which completes the proof of Theorem 1.2(i) when $p \ge 1$. For $p < 1$, Theorem 1.2(i) follows by (4.30), (4.31), the inequality (4.13) and Lemma 4.1. $\square$

5 Proof of Theorem 1.2(ii)

As we mentioned in Section 2 we will prove Theorem 1.2(ii) by showing that

$$\frac{n^{-1+p(\alpha+1/\beta)}}{(\log n)^{p/\beta}}\mathbb{E}[V(p;k)_n] \to \tilde{m}_p, \tag{5.1}$$

$$\frac{n^{-2+2p(\alpha+1/\beta)}}{(\log n)^{2p/\beta}}\operatorname{var}(V(p;k)_n) \to 0, \tag{5.2}$$

which we will perform in two steps.

(i): Due to stationarity of the increments of X we deduce the identity

$$\frac{n^{-1+p(\alpha+1/\beta)}}{(\log n)^{p/\beta}}\mathbb{E}[V(p;k)_n] = \mathbb{E}\left[\Big|\frac{n^{\alpha+1/\beta}}{(\log n)^{1/\beta}}\Delta^n_{k,k}X\Big|^p\right].$$

We remark that the random variable $\Delta^n_{k,k}X$ is symmetric β-stable with scale parameter $\|g_{k,n}\|_{L^\beta(\mathbb{R})}$. Hence, it suffices to prove that

$$\frac{n^{\alpha+1/\beta}}{(\log n)^{1/\beta}}\|g_{k,n}\|_{L^\beta(\mathbb{R})} \to |k_\alpha| \tag{5.3}$$

to show (5.1), cf. the scaling properties of β-stable random variables or alternatively use [18, Proposition 1.2.17]. By substitution we obtain the identity

$$\frac{n^{\alpha\beta+1}}{\log(n)}\int_{\mathbb{R}}|g_{k,n}(x)|^\beta\,dx = \frac{1}{\log(n)}\int_{\mathbb{R}}|D^k g_n(s)|^\beta\,ds,$$

where the function g_n has been defined at (3.1). Applying (3.2) and recalling that the function $g^{(k)} \in L^\beta((1,\infty))$ is decreasing on $(1,\infty)$ and $\alpha = k - 1/\beta$, we find that

$$\frac{1}{\log(n)}\int_{n+k}^{\infty}|D^k g_n(s)|^\beta\,ds \le \frac{K}{\log(n)}\int_1^\infty |g^{(k)}(x)|^\beta\,dx \to 0.$$

In addition we have

$$\frac{1}{\log(n)}\int_0^{k+1}|D^k g_n(s)|^\beta\,ds \le K\frac{1}{\log(n)} \to 0.$$

Hence to show (5.3) it is enough to show that as $n\to\infty$

$$\frac{1}{\log(n)}\int_{k+1}^{n+k}|D^k g_n(s)-h_k(s)|^\beta\,ds \to 0 \tag{5.4}$$

and

$$\frac{1}{\log(n)}\int_{k+1}^{n+k}|h_k(s)|^\beta\,ds \to |k_\alpha|^\beta. \tag{5.5}$$

Indeed, this follows by applying the integral versions of the inequalities (4.12) and (4.13).

To prove (5.4) we argue as in the proof of (4.6). The estimate (4.7) implies (as in proof of (4.8)) that

$$\frac{1}{\log(n)}\int_{k+1}^{n+k}|D^k g_n(s)-h_k(s)|^\beta\,ds \le K\sum_{j=0}^{k} a_{j,n}$$

where for $j=0,\dots,k-1$ we have

$$a_{j,n} := \frac{n^{(j-k)\beta}}{\log(n)}\int_{k+1}^{n+k}|\max\{s,s-k\}|^{(\alpha-j)\beta}\,ds \le n^{(\alpha-k)\beta+1}\frac{1}{\log(n)} \to 0$$

as $n\to\infty$, where the convergence to zero follows by the assumption $(\alpha-k)\beta=-1$. For $j=k$ we have

$$a_{k,n} := \frac{n^{-\beta}}{\log(n)}\int_{k+1}^{n+k}|\max\{s,s-k\}|^{\beta-1}\,ds \le K\frac{1}{\log(n)} \to\ 0,$$

which completes the proof of (5.4). To show (5.5) we have, as in (4.10), that for all $s>k$

$$|h_k(s)-\phi^{(k)}(s)| \le K(s-k)^{\alpha-k-1} \tag{5.6}$$

where $\phi(t)=t^\alpha$ for all $t\ge 0$. Eq. (5.6) implies that

$$\frac{1}{\log(n)}\int_{k+1}^{n+k}|h_k(s)-\phi^{(k)}(s)|^\beta\,ds \to 0 \qquad \text{as } n\to\infty \tag{5.7}$$

since $(\alpha-k)\beta=-1$. Moreover,

$$\frac{1}{\log(n)}\int_{k+1}^{n+k}|\phi^{(k)}(s)|^\beta\,ds = |k_\alpha|^\beta\frac{1}{\log(n)}\int_{k+1}^{n+k}s^{-1}\,ds \to |k_\alpha|^\beta. \tag{5.8}$$

By (5.7) and (5.8) we deduce (5.5), which completes the proof of (5.1).

(ii): In order to prove the convergence at (5.2) we need to recall some technical results from [5]. First, we set

$$Z_i^n := |n^{\alpha+1/\beta}\Delta_{i,k}^n X|^p - \mathbb{E}[|n^{\alpha+1/\beta}\Delta_{i,k}^n X|^p]$$

and observe that with

$$\theta_n(l) := \mathrm{cov}(Z_k^n, Z_{k+l}^n)$$

we have that

$$\begin{aligned}\frac{n^{-2+2p(\alpha+1/\beta)}}{(\log n)^{2p/\beta}}\mathrm{var}(V(p;k)_n) &= \frac{1}{(\log n)^{2p/\beta}}\Big(\frac{n-k}{n^2}\theta_n(0) + \frac{2}{n^2}\sum_{l=1}^{n-k}(n-k-l+1)\theta_n(l)\Big)\\ &\le \frac{2}{(\log n)^{2p/\beta}n}\sum_{l=0}^{n-k}|\theta_n(l)|.\end{aligned}$$

For all $n \ge 1$ we set $u_n(x) = D^k g_n(x)$ for $x \in \mathbb{R}$. Then functions u_n are continuous functions from $\mathbb{R}$ into $\mathbb{R}$ and by Lemma 3.1 they satisfying the inequality

$$\begin{aligned}|u_n(x)| \le K\Big(&|x|^\alpha \mathbb{1}_{[0,k+1]}(x) + |x|^{\alpha-k}\mathbb{1}_{[k+1,n)}(x)\\ &+ n^{\alpha-k}(\mathbb{1}_{[n,n+k]}(x) + v((x-k)/n)\mathbb{1}_{(n+k,\infty)}(x))\Big),\end{aligned} \tag{5.9}$$

where $v \in L^\beta((1,\infty)) \cap C((1,\infty))$ is a fixed decreasing function which does not depend on n. In fact, we can choose $v = g^{(k)}$. For all $q \in [0,\beta/2)$ we set

$$I_{l,n,q} := \int_0^\infty |u_n(x+l)|^{\beta-q}|u_n(x)|^q\,dx.$$

Based on the inequality (5.9) we will show for all $q \in [0,\beta/2)$ and all $n \ge 1$ and $l = 0,\dots,n$ we have the estimate the estimate

$$I_{l,n,q} \le K\Big(\log(n) - \log(l \vee 1) + 1\Big). \tag{5.10}$$

To show (5.10) we observe that

$$\begin{aligned}&\int_{k+1}^n |u_n(x+l)|^{\beta-q}|u_n(x)|^q \mathbb{1}_{\{x+l<n\}}\,dx\\ &\quad\le Kl^{(\alpha-k)\beta}\int_{k+1}^n \Big(\frac{x}{l}+1\Big)^{(\alpha-k)(\beta-q)}\Big(\frac{x}{l}\Big)^{(\alpha-k)q}\,dx\\ &\quad\le K\int_{(k+1)/l}^{n/l}(y+1)^{-1+q/\beta)}y^{-q/\beta}\,dy \le K\big(\log(n)-\log(l)\big),\end{aligned}$$

since $\alpha = k - 1/\beta$. On the other hand, we have that

$$\int_{k+1}^n |u_n(x+l)|^{\beta-q}|u_n(x)|^q \mathbb{1}_{\{x+l>n\}}\,dx \le Kn^{(\alpha-k)(\beta-q)}\int_k^n x^{(\alpha-k)q}\,dx \le K.$$

We trivially obtain

$$\int_0^{k+1} |u_n(x+l)|^{\beta-q}|u_n(x)|^q\,dx \le K l^{-1+q/\beta} \le K.$$

Finally, for the last term we deduce that

$$\int_n^{\infty} |u_n(x+l)|^{\beta-q}|u_n(x)|^q\,dx \le K n^{(\alpha-k)\beta}\left(1+\int_n^{\infty} v(x/n)^{\beta}\,dx\right) \le K,$$

where we used that the function v is decreasing on $(1,\infty)$ and $v \in L^{\beta}((1,\infty))$, which completes the proof of (5.10).

By Cauchy–Schwarz inequality we have that

$$\begin{aligned} |\theta_n(l)| &\le \|Z_k^n\|_{L^2}\|Z_{k+l}^n\|_{L^2} = \mathbb{E}[|Z_k^n|^2] \\ &\le K\Big(\int_0^{\infty} |u_n(x)|^{\beta}\,dx\Big)^{2p/\beta} = \Big(I_{0,n,0}\Big)^{2p/\beta} \le K(\log n)^{2p/\beta} \end{aligned} \tag{5.11}$$

where the last inequality follows by (5.10) used on $l=0$. In the following we will show that $\theta_n(l)$ are bounded for l suitable close to n. To be more precise we claim that for each fixed $r\in(0,1)$ there exists a constant K_r only depending on r such that

$$|\theta_n(l)| \le K_r \qquad \text{for all } n\ge 1,\ l=[rn],\dots,n. \tag{5.12}$$

In the following we will show (5.12). By arguing as in the second equation after (5.46) in [5] we obtain the representation

$$\begin{aligned} \theta_n(l) &= a_p^{-2}\int_{\mathbb{R}^2} \frac{1}{|s_1s_2|^{1+p}}\psi_l^n(s_1,s_2)\,ds_1\,ds_2, \qquad \text{where} \\ \psi_l^n(s_1,s_2) &= \exp\left(-\int_{\mathbb{R}} |s_1u_n(x)-s_2u_n(x+l)|^{\beta}\,dx\right) \\ &\quad - \exp\left(-\int_{\mathbb{R}} |s_1u_n(x)|^{\beta}+|s_2u_n(x+l)|^{\beta}\,dx\right) \end{aligned}$$

and $a_p=\int_{\mathbb{R}}(1-\exp(iu))|u|^{-1-p}\,du\in\mathbb{R}_+$. We decompose $\theta_n(l)$ as

$$\theta_n(l)=\theta_n(l)_1+\theta_n(l)_2$$

where

$$\begin{aligned} \theta_n(l)_1 &= a_p^{-2}\int_{\mathbb{R}^2\setminus[-1,1]^2} \frac{1}{|s_1s_2|^{1+p}}\psi_l^n(s_1,s_2)\,ds_1\,ds_2, \\ \theta_n(l)_2 &= a_p^{-2}\int_{[-1,1]^2} \frac{1}{|s_1s_2|^{1+p}}\psi_l^n(s_1,s_2)\,ds_1\,ds_2. \end{aligned}$$

Part (b) of the proof of [5, Lemma 6.3], see (6.15), (6.17), (6.18) and (6.20) in [5], shows the inequality

$$|\theta_n(l)_2| \le K\Big(I_{l,n,p}+I_{l,n,0}\Big). \tag{5.13}$$

Combining (5.10) and (5.13) yields that for all $n \geq 1$ and all $l = [rn], \dots, n$ we have

$$|\theta_n(l)_2| \leq K\Big(\log(n) - \log(l) + 1\Big) \leq K\Big(\log(n) - \log(rn) + 1\Big) = K\Big(|\log(r)| + 1\Big).$$

To estimate $\theta_n(l)_1$ we use that $|\psi_l^n(s_1, s_2)| \leq 1$ which implies that

$$|\theta_n(l)_1| \leq a_p^{-2} \int_{\mathbb{R}^2 \setminus [-1,1]^2} \frac{1}{|s_1 s_2|^{1+p}}\, ds_1\, ds_2 \leq K < \infty,$$

and completes the proof of (5.12). By using the estimate (5.11) for all $l = 0, \dots, [rn]$ and the estimate (5.12) for all $l = [rn] + 1, \dots, n$ we have that

$$\begin{aligned}\frac{1}{\log(n)^{2p/\beta} n} \sum_{l=0}^{n-k} |\theta_n(l)| &= \frac{1}{n} \sum_{l=0}^{[nr]} \frac{|\theta_n(l)|}{(\log n)^{2p/\beta}} + \frac{1}{n} \sum_{l=[nr]+1}^{n-k} \frac{|\theta_n(l)|}{(\log n)^{2p/\beta}} \\ &\leq Kr + \frac{K_r}{(\log n)^{2p/\beta}}. \end{aligned} \tag{5.14}$$

Hence, we deduce (5.2) by letting first $n \to \infty$ in (5.14) and then $r \to 0$. This completes the proof of Theorem 1.2(ii). □

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