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A Consistent Pricing Model for Index Options and Volatility Derivatives

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Abstract

We propose and study a flexible modeling framework for the joint dynamics of an index and a set of forward variance swap rates written on this index, allowing options on forward variance swaps and options on the underlying index to be priced consistently. Our model reproduces various empirically observed properties of variance swap dynamics and allows for jumps in volatility and returns.

An affine specification using Lévy processes as building blocks leads to analytically tractable pricing formulas for options on variance swaps as well as efficient numerical methods for pricing of European options on the underlying asset. The model has the convenient feature of decoupling the vanilla skews from spot/volatility correlations and allowing for different conditional correlations in large and small spot/volatility moves.

We show that our model can simultaneously fit prices of European options on S&P 500 across strikes and maturities as well as options on the VIX volatility index. The calibration of the model is done in two steps, first by matching VIX option prices and then by matching prices of options on the underlying.

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Contents

1	Introduction	1
1.1	Contribution	2
1.2	Outline	3
2	Variance Swaps and Forward Variances	5
2.1	Variance Swaps	5
2.2	Forward Variances	6
2.3	Options on Forward Variance Swaps	7
3	A Model for the Joint Dynamics of Variance Swaps and the Underlying Index	7
3.1	Variance Swap Dynamics	8
3.2	Dynamics of the Underlying Asset	9
3.3	Pricing of Vanilla Options	11
3.4	Examples	13
3.4.1	Normally Distributed Jumps	13
3.4.2	Exponentially Distributed Jumps	14
3.5	Impact of Jumps on the Valuation of Variance Swaps	15
4	The VIX Index	16
4.1	Description	16
4.2	VIX Futures	18
5	Implementation	19
5.1	Data	19
5.2	Calibration	19
6	Conclusion	21
A	Characteristic Functions	25
A.1	Normally Distributed Jumps	25
A.2	Exponentially Distributed Jumps	26
B	Proof of proposition 2	27
C	Tables	28

1 Introduction

Volatility indices –such as the VIX index– and derivatives written on such indices have gained popularity in markets as tools for hedging volatility risk and as market-based indicators of volatility. Variance swap contracts [19] are increasingly used by market operators to take a pure exposure to volatility or hedge the volatility exposure of options portfolios.

The existence of a liquid market for volatility derivatives such as VIX options, VIX futures and a well developed over-the-counter market for options on variance swaps, and the use of variance swaps and volatility index futures as hedging instruments for other derivatives have led to the need for a pricing framework in which volatility derivatives and derivatives on the underlying asset can be priced in a consistent manner. In order to yield derivative prices in line with their hedging costs, such models should be based on a realistic representation of the joint dynamics of the underlying asset and variance swaps written on this, while also be able to match the observed prices of the liquid derivatives –futures, calls, puts and variance swaps– used as hedging instruments.

In principle, any continuous-time model with stochastic volatility and/or jumps implies some joint dynamics for variance swaps and the underlying asset price. Broadie and Jain [8] studied the valuation of volatility derivatives in the Heston model; Carr et al. [11] study the pricing of volatility derivatives in models based on Lévy processes. However, in many commonly used models the dynamics implied for variance swaps is unrealistic [10, 5]: for example, exponential Lévy models imply a constant variance swap term structure while one-factor stochastic volatility models predict perfect correlation among movements in variance swaps at all maturities.

Also, as pointed out by Bergomi [5, 7], the joint dynamics of forward volatilities and the underlying asset is neither explicit nor tractable in most commonly used models, which makes parameter selection/calibration difficult and does not enable the user to choose a parameter to take a view on forward volatility. As a result, classical models such as the Heston model or (time changed) exponential-Lévy models are unable to match empirical properties of variance swaps and VIX options [5, 7, 6, 10, 21]. Some of these issues can be tackled using multi-factor stochastic volatility models [10, 21] but these models remain incapable of reproducing finer features of the data such as the magnitude of the VIX option skew or the different conditional correlations in large and small spot/volatility moves (see Table 1).

A new modeling approach, recently proposed by Bergomi [5, 7, 6] (see also related work of Bühler [10] and Gatheral [21]) is one in which, instead of modeling “instantaneous” volatility, one models directly the (forward) variance swaps for a discrete tenor of maturities. This approach, which can be seen as the analog of the LIBOR market model for volatility modeling, turns out to be quite flexible and allows to deal with the issues raised above

while retaining some tractability.

These models are based on diffusion dynamics where market variables are driven by a multidimensional Brownian motion. Recent price history across most asset classes has pointed to the importance of discontinuities in the evolution of prices; this anecdotal evidence is supplemented by an increasing body of statistical evidence for jumps in price dynamics [1, 2, 3, 17]. Volatility indices such as the VIX have exhibited, especially during the recent crisis, large fluctuations which strongly point to the existence of jumps, or spikes, in volatility. Figure 1, which depicts the daily closing levels of S&P 500 and the VIX volatility index from September 22nd, 2003 to February 27th, 2009, reveals to the simultaneity of large drops in the S&P 500 with spikes in volatility, which corresponds to the well-known “leverage effect”. Comparing the relative changes in the two series (Figure 2) reveals that, while there is already a negative correlation in small changes in the series, large changes –jumps– exhibit an even stronger negative correlation, close to -1 (see Table 1). These observations are confirmed by a recent study of Tauchen and Todorov [27], who find significant statistical evidence of simultaneous jumps of opposite sign in the VIX and the underlying index. These empirical facts need to be accounted for in a realistic model for variance swap dynamics. They are also important from a pricing perspective: Broadie and Jain [9] find that, for a wide range of models and parameter specifications, the effect of discrete sampling on the valuation of volatility derivatives is typically small while the effect of jumps can be significant. Jumps in volatility are also important in order to produce the positive “skew” of implied volatilities of VIX options [6, 21].

1.1 Contribution

Following the approach proposed by Bergomi [5, 7, 6], the present work proposes an arbitrage-free modeling framework for the joint dynamics of forward variance swap rates along with the underlying index, which

1. captures the information in index option prices by matching the index implied volatility smiles.
2. is capable of reproducing any observed term structure of variance swap rates.
3. captures the information in options on VIX futures by matching their prices/ implied volatility smiles.
4. implies a realistic joint dynamics of spot and forward implied volatilities, allowing in particular for jumps in volatility and returns.
5. allows for the spot/volatility correlation and the implied volatility skews (of vanilla options) to be parametrized independently.

6. is able to handle the term structure of vanilla skew separately from the term structure of volatility of volatility.
7. is tractable and enables efficient pricing of vanilla options, which is a key point for calibration and implementation of the model.

We address these different issues, while retaining tractability, by introducing a common jump factor which affects variance swaps and the underlying index with opposite signs. Describing these jumps in terms of a Poisson random measure leads to an analytically tractable framework, where VIX options *and* calls/puts on the underlying index can be simultaneously priced using Fourier-based methods. Our framework is in fact a class of models and allows for various specifications of the jump size distribution; we give two worked-out examples and detail their implementation.

The difference between our modeling framework and the Bergomi model [7] is mainly the ability to meet points 5), 6) and 7) above. Also, thanks to a semi-analytic representation of call option prices, our model also satisfies the tractability property 1), allowing efficient calibration to the whole implied volatility surface. This is an advantage over the Bergomi model where only the short-term implied volatility smile can be matched [5, 7].

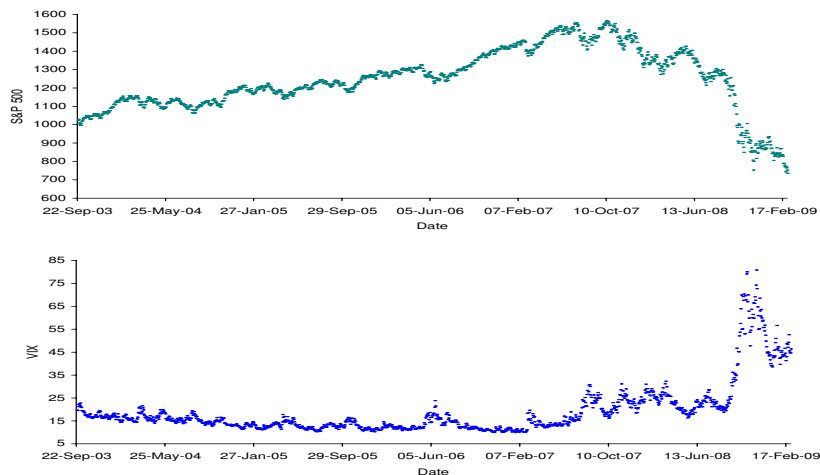


Figure 1: Time series of the VIX index (bottom) depicted together with the S&P 500 (top) covering the period from September 22nd, 2003 to February 27th, 2009.

1.2 Outline

Section 2 describes variance swap contracts, forward variance swap rates and options on variance swaps. The model is described in Section 3 and two

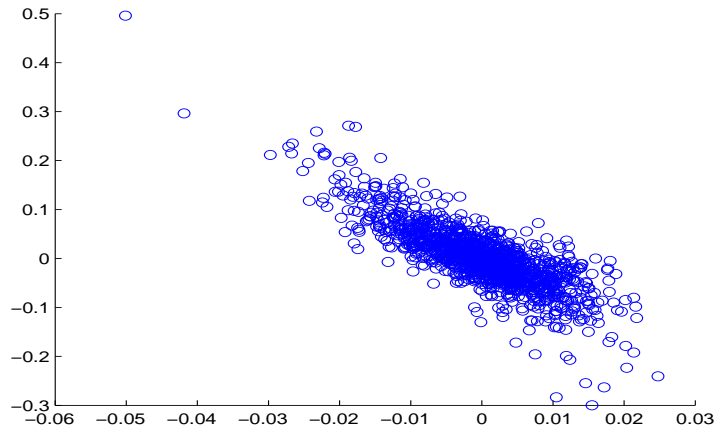


Figure 2: Daily relative changes in the VIX (vertical axis) vs daily relative changes in the S&P 500 index (horizontal axis), September 22nd, 2003 to February 27th, 2009.

Table 1: Conditional correlations between the daily returns on S&P 500 and the VIX from September 22nd, 2003 to February 27th, 2009.

Abs. Return	Unconditional	< 0.5%	> 0.5%	< 1%	> 1%	< 5%	> 5%
Correlation	-0.74	-0.45	-0.78	-0.66	-0.79	-0.76	-0.93
Observations	1368	677	691	1031	337	1348	20

model specifications are presented. Section 4 discusses the VIX index and the connection between forward variance swap rates and VIX index futures. Section 5 implements the two specifications of the model and examines their performance in jointly matching VIX options and options on the S&P 500. Section 6 concludes.

2 Variance Swaps and Forward Variances

Consider an underlying asset whose price S is modeled as a stochastic process $(S_t)_{t \geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t \geq 0}$ represents the history of the market. We assume the market is arbitrage-free and prices of traded instruments are represented as conditional expectations with respect to an equivalent pricing measure \mathbb{Q} . We shall neglect in the sequel corrections due to stochastic interest rates.

2.1 Variance Swaps

The annualized *realized variance* of a (price) process S over a time grid $t = t_0 < \dots < t_k = T$ is given by

$$RV_{t,T} = \frac{M}{k} \sum_{i=1}^k \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2, \quad (1)$$

where M is the number of trading days per year.

A variance swap (VS) with maturity T initiated at $t < T$ pays the difference between the annualized realized variance of the log-returns $RV_{t,T}$ less a strike called the variance swap rate V_t^T , determined such that the contract has zero value at the time of initiation t .

For any semimartingale S , as $\sup_{i=1, \dots, k} |t_i - t_{i-1}| \rightarrow 0$ the realized variance converges to the quadratic variation of the log price:

$$\sum_{i=1}^k \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 \xrightarrow{\mathbb{Q}} [\log S]_T - [\log S]_t.$$

Approximating the realized variance by the quadratic variation of the log returns is justified when the sampling frequency is daily, as is the case in most variance swap contracts, to warrant replacing the realized variance by its continuous counterpart [9]. The fixed leg paid in a variance swap is then approximated by

$$V_t^T = \frac{1}{T-t} \mathbb{E}([\log S]_T - [\log S]_t \mid \mathcal{F}_t), \quad (2)$$

and we refer to this quantity as the (spot) variance swap (VS) rate prevailing at date t for the maturity T . An example of a variance swap term structure is given in Figure 3.

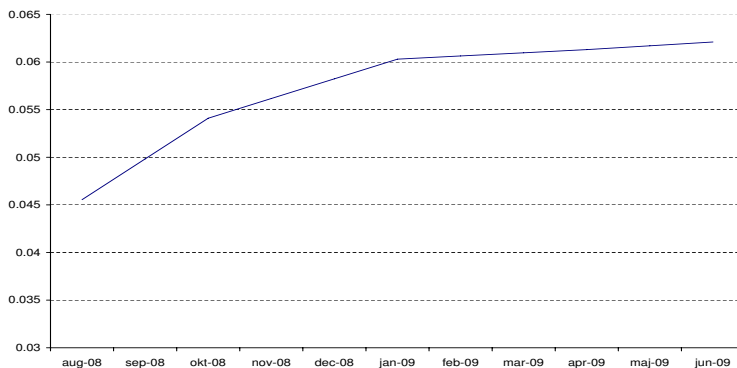


Figure 3: The term structure of 1 month forward variance swaps for the S&P 500 on August 20th, 2008.

2.2 Forward Variances

The forward variance, quoted at date t for the period $[T_1, T_2]$ is the strike that sets the value of a forward variance swap running from T_1 to T_2 to zero at time t ; it is given by

$$V_t^{T_1, T_2} = \frac{1}{T_2 - T_1} \mathbb{E}([\log S]_{T_2} - [\log S]_{T_1} \mid \mathcal{F}_t) \quad (3)$$

$$= \frac{(T_2 - t) V_t^{T_2} - (T_1 - t) V_t^{T_1}}{T_2 - T_1}, \quad (4)$$

where $t < T_1 < T_2$. The last equality follows easily by substitution of (2). In particular, as noted in [5, 6] forward variances have the martingale property under the pricing measure.

Choosing $t < s < T_1 < T_2$ it follows by substitution of (3) and the use of the law of iterated expectations that

$$\mathbb{E}(V_s^{T_1, T_2} \mid \mathcal{F}_t) = V_t^{T_1, T_2} \quad (5)$$

which shows that forward variances are martingales under the pricing measure \mathbb{Q} .

Assume that a set of settlement dates is given

$$T_0 < T_1 < \dots < T_n$$

known as the tenor structure and that the interval between two tenor dates is fixed, $\tau_i = T_{i+1} - T_i = \tau$ (equal to 30 days if we use the tenor structure of the VIX futures). We define the *forward variance* over the time interval $[T_i, T_{i+1}]$ as

$$V_t^i = V_t^{T_i, T_{i+1}}. \quad (6)$$

2.3 Options on Forward Variance Swaps

A call option with strike K and maturity T_1 on a forward variance swap for the period $[T_1, T_2]$ gives the holder the option to enter at date T_1 into a variance swap running from T_1 to T_2 with some predetermined strike K . Hence, the value at time T_1 is

$$\begin{aligned} \pi(T_1, T_1, T_2, K) &= \left(e^{-\int_{T_1}^{T_2} r_s ds} \mathbb{E} (RV_{T_1, T_2} - K \mid \mathcal{F}_{T_1}) \right)^+ \\ &= e^{-\int_{T_1}^{T_2} r_s ds} \left(V_{T_1}^{T_1, T_2} - K \right)^+, \end{aligned} \quad (7)$$

which gives us the time t value

$$\pi(t, T_1, T_2, K) = e^{-\int_t^{T_2} r_s ds} \mathbb{E} \left(\left(V_{T_1}^{T_1, T_2} - K \right)^+ \mid \mathcal{F}_t \right). \quad (8)$$

From this expression it is clear that having tractable dynamics for the forward variance swap rates enable the pricing of options on variance swaps.

3 A Model for the Joint Dynamics of Variance Swaps and the Underlying Index

Our goal is to construct a model which

- allows for an arbitrary initial variance swap term structure.
- allows to specify directly the dynamics of the observable forward variance swaps V_t^i on a discrete tenor of maturities $(T_i, i = 1..n)$.
- allows for flexible modeling of the variance swap curve: for example, it should be able to accommodate the fact that variance swaps at the long end of the maturity curve have a lower variability than those at the short end.
- allows for jumps in volatility [27] and in the underlying asset [1, 2, 3, 17].
- allows for a flexible specification of the spot price dynamics.
- is analytically tractable i.e. leads to efficient numerical methods for pricing/calibration of calls/puts both on the underlying and volatility derivatives such as options on forward variance swaps and options on forward volatility.

In order to achieve these goals, we first specify the dynamics of (a discrete tenor of) forward variance swaps (Section 3.1) using an affine specification

which allows Fourier-based pricing of European-type volatility derivatives. Once the dynamics of forward variance swaps has been fixed, we specify a jump-diffusion dynamics for the underlying asset which is compatible with the variance swap dynamics (Section 3.2). Presence of a jump component as well as a diffusion component in the underlying asset allows us to satisfy this compatibility condition while simultaneously matching values/implied volatilities of options on the underlying asset (Section 3.3).

3.1 Variance Swap Dynamics

Given that the forward variance swap rate is a (positive) martingale under the pricing measure, we model it as

$$\begin{aligned} V_t^i &= V_0^i e^{X_t^i} \\ &= V_0^i \exp \left\{ \int_0^t \mu_s^i ds + \int_0^t \omega e^{-k_1(T_i-s)} dZ_s + \int_0^t \int_{\mathbb{R}} e^{-k_2(T_i-s)} x J(dx ds) \right\}, \end{aligned} \quad (9)$$

where $J(dxdt)$ is a Poisson random measure with compensator $\nu(dx)dt$ and Z a Wiener process independent of J . The martingale condition imposes

$$\mu_t^i = -\frac{1}{2}\omega^2 e^{-2k_1(T_i-t)} - \int_{\mathbb{R}} \left(\exp \left\{ e^{-k_2(T_i-t)} x \right\} - 1 \right) \nu(dx) . \quad (10)$$

For $t > T_i$ we let $V_t^i = V_{T_i}^i$. This specification allows for jumps and the exponential functions inside the integrals allow to control the term structure of volatility of volatility. For example, if k_1 is large and k_2 small, the diffusion Z results mostly in fluctuations at the short end of the curve, while the jumps impact the entire variance swap curve. Correlated Brownian factors can be added if more flexibility in the variance swap curve dynamics is desired, e.g. correlation between movements in the short and the long end of the curve. Likewise, functional forms other than exponentials can be used in (9), although the exponential specification is quite flexible as we will observe in the examples.

Expressions such as (8) can be evaluated using Fourier-based methods [13, 24, 18] given the characteristic function of $X_{T_i}^i$, which in this case has a simple form:¹

$$\begin{aligned} \mathbb{E} \left[e^{iuX_{T_i}^i} \right] &= \exp \left\{ -\frac{1}{2}u^2 \int_0^{T_i} \omega^2 e^{-2k_1(T_i-s)} ds + iu \int_0^{T_i} \mu_s^i ds \right. \\ &\quad \left. + \int_0^{T_i} \int_{\mathbb{R}} \left(\exp \left\{ iue^{-k_2(T_i-s)} x \right\} - 1 \right) \nu(dx) dt \right\} . \end{aligned} \quad (11)$$

¹We hope that the reader will not confuse the complex number i with the index in the forward variance swap rate V_t^i .

3.2 Dynamics of the Underlying Asset

Once the dynamics of forward variance swaps V_t^i for a discrete set of maturities $T_i, i = 1..n$ has been specified, this imposes some constraints on the (risk neutral) dynamics of the underlying asset $(S_t)_{t \geq 0}$. S should be such that

1. the discounted spot price $\hat{S}_t = \exp(-\int_0^t (r_s - q_s) ds) S_t$ is a (positive) martingale, where q_t is the dividend yield.
2. the dynamics of the spot price is compatible with the specification of V_t^i , which puts the following constraint on the quadratic variation process $[\ln S]$ of the log-price:

$$E[[\ln S]_{T_{i+1}} - [\ln S]_{T_i} | \mathcal{F}_t] = V_t^i \quad (12)$$

To these constraints we add a third requirement, namely that:

3. the model values of calls/puts on S match the observed prices across strikes and maturities.

Typically we need at least two distinct parameters/degrees of freedom in the dynamics of the underlying asset in order to accommodate points 2) and 3) above.

The Bergomi models [5] and [7] propose to achieve this by introducing a random "local volatility" function which is reset at each tenor date and chosen such at time T_i to match the observed value of $V_{T_i}^i$. This procedure guarantees coherence between the variance swaps and the underlying asset dynamics but leads to a loss of tractability: even vanilla call options need to be priced by Monte Carlo simulation for maturities $T > T_1$.

We adopt here a different approach which allows a greater tractability while simultaneously allowing for jumps in the volatility and the price. In fact, as we will see, introducing jumps *is* the key to tractability.

The underlying asset is driven by

- a Brownian motion W , correlated with the diffusion component Z driving the variance swaps: $\langle W, Z \rangle_t = \rho t$.
- a jump component, which is driven by the *same* Poisson random measure J which drives jumps in the variance. However, we allow different jump amplitudes in the underlying and the forward variance.

Presence of a jump component as well as a diffusion component in the underlying asset allows us to satisfy this compatibility condition while simultaneously matching values/implied volatilities of options on the underlying asset (Section 3.3).

For each interval $[T_i, T_{i+1})$ we introduce a stochastic diffusion coefficient σ_i and a function $u_i(x, V^i)$ which expresses the size of the jump in the

underlying asset in terms of the jump size x in forward variance level V^i and V^i itself. We will give in section 3.4 some simple and flexible specifications for this function u_i but most developments below hold for arbitrary choice of u_i .

We will present here the detailed computations in the (typically useful) case where $T = T_m$, for $m = 1, \dots, n$, but the analysis can be easily generalized to any maturity. The dynamics of the underlying asset is then specified as

$$S_{T_m} = S_0 \exp \left\{ \int_0^{T_m} (r_s - q_s) ds + \sum_{i=0}^{m-1} \mu_i (T_{i+1} - T_i) + \sigma_i (W_{T_{i+1}} - W_{T_i}) + \sum_{i=0}^{m-1} \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}} u_i(x, V_{T_i}^i) J(dx ds) \right\}, \quad (13)$$

where $\mu_i = -\frac{1}{2}\sigma_i^2 - \int_{\mathbb{R}} \left(e^{u_i(x, V_{T_i}^i)} - 1 \right) \nu(dx)$ and the σ_i s are stochastic and revealed at time T_i by relation (16) below to match the known variance swap value $V_{T_i}^i$ at this time point. The drift term μ_i is also stochastic and \mathcal{F}_{T_i} -measurable. The random jump measure J in the index dynamics is the same as that in the VS dynamics, so the index and the variance swaps jump simultaneously. u_i is a deterministic function of x and $V_{T_i}^i$ chosen to match the observed volatility smiles implied by prices of options on the spot. W is correlated with Z with a correlation ρ .

As far as pricing of vanilla instruments is concerned, the model can be viewed simply as a flexible (and analytically tractable) parameterization of the joint distribution of forward variance swap rates and the underlying asset on a set of tenor dates. At this level the only assumption we are making is that this joint distribution is infinitely divisible [26] and our model is just a flexible parameterization of the Lévy triplet of the distribution. This makes it particularly easy to price any payoff which involves these variables only at tenor dates.

The assumption of a common jump factor which affects the variance swaps and the underlying in opposite directions is not only analytically convenient but quite realistic from an empirical perspective. As revealed in Table 1, while the unconditional correlation of daily returns of the VIX index and the S&P 500 from September 22nd, 2003 to February 27th, 2009 is -74% , the *conditional* correlation between the two sub-series for daily moves in the S&P 500 less than 0.5% drops to -0.45 while the conditional correlation of moves greater than 5% , which can be interpreted as jumps, is -93% , close to -100% . This observation is also in agreement with the findings reported by Tauchen and Todorov [27] using non-parametric methods. This feature, which has no equivalent in diffusion-based stochastic volatility models such as [7, 21], is a generic property of our framework.

Given the dynamics (13) of the underlying asset, the quadratic variation over the time interval $[T_0, T_m]$ is given by

$$[\log S]_{T_m} - [\log S]_{T_0} = \sum_{i=0}^{m-1} \sigma_i^2 (T_{i+1} - T_i) + \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}} u_i(x, V_{T_i}^i)^2 J(ds dx). \quad (14)$$

By taking expectation on (14) the variance swap rate in (2) equals

$$V_{T_0}^{T_m} = \sum_{i=0}^{m-1} \frac{T_{i+1} - T_i}{T_m - T_0} \left(\mathbb{E}[\sigma_i^2 | \mathcal{F}_{T_0}] + \mathbb{E} \left[\int_{\mathbb{R}} u_i(x, V_{T_i}^i)^2 \nu(dx) | \mathcal{F}_{T_0} \right] \right),$$

and the forward variance at time t for the interval $[T_i, T_{i+1}]$ equals

$$V_t^i = \mathbb{E}[\sigma_i^2 | \mathcal{F}_t] + \mathbb{E} \left[\int_{\mathbb{R}} u_i(x, V_{T_i}^i)^2 \nu(dx) | \mathcal{F}_t \right]. \quad (15)$$

Equation (15) has to hold at all times, but since V_t^i is a martingale we just need to ensure that at time T_i

$$V_{T_i}^i = \sigma_i^2 + \int_{\mathbb{R}} u_i(x, V_{T_i}^i)^2 \nu(dx). \quad (16)$$

Having fixed $\nu(dx)$ it is seen that any observed forward variance structure can be matched by an appropriate choice of the σ_i s which leaves the parameters in u_i free to calibrate to option prices.

3.3 Pricing of Vanilla Options

Let us now explain the procedure for pricing European calls and puts on S in this framework. The aim is to compute in an efficient manner the value of call options of various strikes and maturities at, some initial date $t = 0$:

$$C(0, S_0, K, T_m) = e^{-\int_0^{T_m} r_s ds} E[(S_{T_m} - K)^+ | \mathcal{F}_0]. \quad (17)$$

Denote by $\mathcal{F}_t^{(Z, J)}$ the filtration generated by the Wiener process Z and the Poisson random measure J . By first conditioning on the factors driving the variance swap curve and using the iterated expectation property

$$C(0, S_0, K, T_m) = e^{-\int_0^{T_m} r_s ds} E[E[(S_{T_m} - K)^+ | \mathcal{F}_{T_m}^{(Z, J)}] | \mathcal{F}_0] \quad (18)$$

we obtain a *mixing formula* for valuing call options:

Proposition 1 *The value $C(0, S_0, K, T_m)$ of a European call option with maturity T_m and strike K is given by*

$$C(0, S_0, K, T_m) = E[C^{BS}(U_{T_m}, K, T_m; \sigma_*)], \quad (19)$$

where $C^{BS}(S, K, T; \sigma)$ denotes the Black-Scholes formula for a call option with strike K and maturity T :

$$\sigma_*^2 = \frac{1}{T_m} \sum_{i=0}^{m-1} \sigma_i^2 (1 - \rho^2) (T_{i+1} - T_i), \quad (20)$$

and U_{T_m} is a $\mathcal{F}_{T_m}^{(Z, J)}$ -measurable random variable given by

$$U_{T_m} = S_0 \exp \left\{ \sum_{i=0}^{m-1} - \left(\frac{1}{2} \sigma_i^2 \rho^2 + \int_{\mathbb{R}} \left(e^{u_i(x, V_{T_i}^i)} - 1 \right) \nu(dx) \right) (T_{i+1} - T_i) \right. \\ \left. \rho (Z_{T_{i+1}} - Z_{T_i}) \sigma_i + \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}} u_i(x, V_{T_i}^i) J(dx ds) \right\} \quad (21)$$

and the expectation in (19) is taken with respect to the law of (Z, J) .

Proof. Conditional on $\mathcal{F}_{T_m}^{(Z, J)}$, the increments of Z and the paths of the variance swap rates, and in particular $V_{T_i}^i$, thus the corresponding σ_i are known for $i = 0, 1, \dots, m$ from equation (16). Moreover

$$W_{T_{i+1}} - W_{T_i} \stackrel{d}{=} \rho (Z_{T_{i+1}} - Z_{T_i}) + \sqrt{(1 - \rho^2)} \left(\hat{W}_{T_{i+1}} - \hat{W}_{T_i} \right),$$

where the Wiener processes Z and \hat{W} are independent. The process S_t can therefore be decomposed as

$$S_{T_m} = U_{T_m} \exp \left\{ \int_0^{T_m} (r_s - q_s) ds + \sum_{i=0}^{m-1} \left(-\frac{1}{2} \sigma_i^2 (1 - \rho^2) (T_{i+1} - T_i) \right. \right. \\ \left. \left. + \sigma_i \sqrt{1 - \rho^2} \left(\hat{W}_{T_{i+1}} - \hat{W}_{T_i} \right) \right) \right\},$$

where $\sigma_i^2 = V_{T_i}^i - \int_{\mathbb{R}} u_i(x, V_{T_i}^i)^2 \nu(dx)$ and

$$U_{T_m} = S_0 \exp \left\{ \sum_{i=0}^{m-1} - \left(\frac{1}{2} \sigma_i^2 \rho^2 + \int_{\mathbb{R}} \left(e^{u_i(x, V_{T_i}^i)} - 1 \right) \nu(dx) \right) (T_{i+1} - T_i) \right. \\ \left. \rho (Z_{T_{i+1}} - Z_{T_i}) \sigma_i + \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}} u_i(x, V_{T_i}^i) J(dx ds) \right\}$$

is $\mathcal{F}_{T_m}^{(Z, J)}$ -measurable. In distribution we have

$$S_{T_m} \stackrel{d}{=} U_{T_m} e^{\int_0^{T_m} (r_s - q_s) ds + \mu_* T_m + \sigma_* \hat{W}_{T_m}},$$

where σ_* is given in (20) and

$$\mu_* = \frac{1}{T_m} \sum_{i=0}^{m-1} -\frac{1}{2} \sigma_i^2 (1 - \rho^2) (T_{i+1} - T_i), \quad (22)$$

where we notice that $\mu_* = -\frac{1}{2}(\sigma_*)^2$. Hence, given $\mathcal{F}_{T_m}^{(Z,J)}$, the inner conditional expectation in (18) reduces to the evaluation of the Black-Scholes formula

$$E[(S_{T_m} - K)_+ | \mathcal{F}_{T_m}^{(Z,J)}] = C^{BS}(U_{T_m}, K, T_m; \sigma_*),$$

where σ_* depends on all the u_i s through (16) and (20). ■

This result has interesting consequences for pricing and calibration of vanilla contracts. Note that the outer expectation can be computed by Monte Carlo simulation of the Z and J : with N simulated sample paths for Z and J we obtain the following approximation

$$\hat{C}_N = \frac{1}{N} \sum_{k=1}^N C^{BS}(U_{T_m}^{(k)}, K, T_m; \sigma_*(k)) \xrightarrow{N \rightarrow \infty} C(0, S_0, K, T_m). \quad (23)$$

Since the averaging is done over the variance swap factors Z and J , this is a deterministic function of the parameters in the u_i s. This will prove useful when calibrating the model using option data, since we *do not* have to run the N Monte Carlo simulations for each calibration trial.

Equation (23) thus allows to calibrate the model to the *entire* implied volatility surface in an efficient manner, in contrast to the Bergomi model where it is only possible to calibrate to at-the-money slopes of the implied volatilities (ATM skews).

3.4 Examples

Different classes of models in the above framework can be obtained by various parameterizations of the Lévy measure $\nu(dx)$ describing the jumps and for the functions u_i appearing in (13). In this paper we implement two model parameterizations; normal jumps and double exponential jumps in the variance swaps and the index.

3.4.1 Normally Distributed Jumps

In the first example, we specify the Lévy measure as

$$\nu(dx) = \lambda \frac{1}{\delta \sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\delta^2}} dx, \quad (24)$$

where m and δ^2 is the mean and variance in a normal distribution and λ the intensity of the jumps. With $\nu(dx)$ specified the characteristic function

in (11) can be calculated (see (41) in Appendix A) and prices of options on the variance swaps can be computed by Fourier transform methods.

We now consider the following specification for u_i , which relates the jumps in the variance swap to the jumps in the underlying asset

$$u_i(x, V_{T_i}^i) = \left(\frac{V_{T_i}^i}{V_0^i} \right)^{\frac{1}{2}} b_i x, \quad (25)$$

but any functional form can be used as long as (16) leads to positive values for σ_i .

This gives us the σ_i s at time T_i

$$\begin{aligned} \sigma_i^2 &= V_{T_i}^i - \frac{V_{T_i}^i}{V_0^i} \int_{\mathbb{R}} (b_i x)^2 \nu(dx) \\ &= V_{T_i}^i - \lambda \frac{V_{T_i}^i}{V_0^i} (b_i^2 m^2 + b_i^2 \delta^2). \end{aligned}$$

In order to achieve non-negative values for σ_i^2 we require

$$\lambda (b_i^2 m^2 + b_i^2 \delta^2) \leq V_0^i. \quad (26)$$

3.4.2 Exponentially Distributed Jumps

To introduce an asymmetry in the tail of the jump size distribution we can use a double exponential distribution [23]:

$$\nu(dx) = \lambda \left(p \alpha_+ e^{-\alpha_+ x} \mathbf{1}_{x \geq 0} + (1-p) \alpha_- e^{-\alpha_- |x|} \mathbf{1}_{x < 0} \right) dx, \quad (27)$$

where p denote the probability of a positive jump, $1/\alpha_+$ and $1/\alpha_-$ the expected positive and negative jump sizes, respectively. In this specification the characteristic function for the forward variance swap is given by (43).

As above, we specify the u_i function as in (25), which gives rise to the following constraint on the σ_i s

$$\sigma_i^2 = V_{T_i}^i - \lambda \frac{V_{T_i}^i}{V_0^i} \left(\frac{2pb_i^2}{\alpha_+^2} + \frac{2(1-p)b_i^2}{\alpha_-^2} \right).$$

To ensure positive σ_i s we constrain the calibration by

$$\lambda \left(\frac{2pb_i^2}{\alpha_+^2} + \frac{2(1-p)b_i^2}{\alpha_-^2} \right) \leq V_0^i. \quad (28)$$

The parameters b_i in the above specifications can take any value as long as (26) respectively (28) are satisfied. We will see later, how the calibration of b_i entails that jumps in the underlying have opposite direction to the

jumps in the VIX futures. Together with negative correlation between Z and W , this feature enables the model to generate positive skews for the implied volatility of VIX options and negative skews for the implied volatility of calls and puts on the underlying index, as observed in empirical data. This property is achieved without the need to add extra volatility factors, as proposed recently in e.g. [10, 20, 21] where multi-factor mean-reverting dynamics on volatility and volatility of volatility are imposed to accommodate this behavior, resulting in a loss of tractability.

3.5 Impact of Jumps on the Valuation of Variance Swaps

As noted above, recent price history across most asset classes has pointed to the importance of discontinuities in the evolution of prices; this is supplemented by an increasing body of statistical evidence for jumps in price dynamics [1, 2, 3, 17]. Before moving on we would like to digress on the effect of jumps in the underlying index dynamics when pricing variance swaps.

As noted by Neuberger [25], if the underlying asset follows continuous dynamics, its quadratic variation and hence also the payoff from a variance swap can be replicated by continuous rebalancing of a position in the underlying asset and a static position in a log contract, which in turn can be replicated by a static portfolio of calls and puts [12, 19]:

$$\frac{1}{T-t} ([\log S]_T - [\log S]_t) = \frac{2}{T-t} \left(\int_t^T \frac{dS_t}{S_t} - \log \frac{S_T}{S_t} \right)$$

and then under discrete sampling

$$\frac{M}{k} \sum_{i=1}^k \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 = 2 \frac{M}{k} \sum_{i=1}^k \left(\frac{\Delta S_i}{S_{i-1}} - \log \frac{S_i}{S_{i-1}} \right).$$

The left hand side is the realized variance paid and the right hand side is the payoff from the hedging strategy. Figure 4 depicts the day to day profit-loss from holding a long position in the replicating portfolio and a short position in the three month variance swap. Variance Point (VP) is defined as realized variance multiplied by 10,000. As can be seen, the replicating portfolio performs relatively well most of the days, but days in which the index moved considerably show an error of up to -1.5 VP. Table 2 shows the error from holding the portfolio relative to the realized variance over the three month period. The maximum error from following this strategy arises in the June 2008 - September 2008 variance swap and results in an error of -0.243%. These four real examples indicate that the error from assuming continuous dynamics have been historically small.

Sometimes this is taken as an argument that jumps are not significant when variance swaps are priced. This can very well be a false conclusion,

which can be realized from the following. When assuming continuous dynamics, the error that arises when valuing variance swap rates is given [14] by

$$-\frac{2}{T-t}\mathbb{E}\left[\int_t^T\int_{\mathbb{R}}\left(e^y-1-y-\frac{y^2}{2}\right)\nu(dy)ds\mid\mathcal{F}_t\right],$$

where y is the jump size in the index. That this expectation is small under the physical measure does not entail that the mean of the error under the *pricing* measure should be zero. This common fallacy is akin to saying that since crashes occur infrequently deep out-of-the-money puts should be priced at zero: recent experience shows that such mistaken assertions can be costly. In fact adding jumps with negative mean value to the log-price dynamics should increase the value of the variance swap rate, since being long a variance swap can be seen as an insurance against downward movements in the underlying asset and therefore have a risk premium attached to it [15].

4 The VIX Index

As mentioned, the market for options on the VIX index is well developed, and unless data on options on variance swaps is available, information stemming from this market should be used when the dynamics of the variance swaps are specified. For completeness we describe the VIX index in the first subsection, but the reader is referred to [16] and [14] for a detailed discussion. In the second subsection we derive our choice for the VIX futures dynamics given the dynamics of the forward variance swaps .

4.1 Description

The Chicago Board Options Exchange (CBOE) introduced the volatility index, VIX, in 1993, and it has since become the industry benchmark for market volatility. The VIX index provides investors with a quote on the expected market volatility over the next 30 calendar days. In September 2003 the VIX was revised such that the index is now calculated on the basis of put and call options on S&P 500 instead of the S&P 100 index. Furthermore, the index was changed to calculate the expected volatility on the basis of options in a wide range of strike prices, where the original VIX index was based on at-the-money strikes only.

The general formula used in the VIX calculation at time t is

$$VS_t^T = \frac{2}{T-t}\sum_i\frac{\Delta K_i}{K_i^2}e^{r_t^T(T-t)}Q(K_i,T;t) - \frac{1}{T-t}\left(\frac{F_t}{K_0} - 1\right)^2, \quad (29)$$

where T is time of expiration, K_i the strike price of the i th out-of-the money

option, ΔK_i the resolution of the strike grid:

$$\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2} .$$

For the lowest strike, ΔK_i is the difference between the lowest and second lowest strikes. This also applies to the highest strike. r_t^T is the risk-free interest rate to expiration, $Q(K_i, T; t)$ the midpoint between the bid and ask prices on the option with strike K_i and maturity T , either a call if $K_i > F_t$ or a put if $K_i < F_t$, where F_t is the forward S&P 500 index price. K_0 is the first strike below the forward index level F_t . Normally, no options will expire in exactly 30 days. Therefore, CBOE interpolates between $VS_t^{T_1}$ and $VS_t^{T_2}$

$$VIX_t = 100 \sqrt{\frac{365}{30} \left((T_1 - t) VS_t^{T_1} \frac{N_{T_2} - 30}{N_{T_2} - N_{T_1}} + (T_2 - t) VS_t^{T_2} \frac{30 - N_{T_1}}{N_{T_2} - N_{T_1}} \right)}, \quad (30)$$

where N_{T_1} and N_{T_2} denote the actual number of days to expiration for the two options maturities, $N_{T_1} < 30 < N_{T_2}$.

The VIX is often presented as an indicator of the expected volatility over the next 30 days. It is not immediately clear from either (29) or (30) that this is in fact the case. For notational simplicity assume that the jumps exhibit finite variation. Then it can be shown that the realized variance of returns can be decomposed into three components

$$\begin{aligned} [\log S]_T - [\log S]_t &= \frac{2}{T-t} \left(\int_0^{F_t} \frac{1}{K^2} (K - S_T)^+ dK \right. \\ &\quad \left. + \int_{F_t}^{\infty} \frac{1}{K^2} (S_T - K)^+ dK + \int_t^T \left(\frac{1}{F_{s-}} - \frac{1}{F_t} \right) dF_s \right. \\ &\quad \left. - \int_t^T \int_{\mathbb{R}} \left(e^y - 1 - y - \frac{y^2}{2} \right) J(dy ds) \right), \quad (31) \end{aligned}$$

where J is a Poisson random measure with Lévy measure $\nu(dx)$, driving the jumps in the underlying asset. Notice how the realized variance can be replicated by trading in options and futures up to a discontinuous jump component.

Taking expectations with respect to the pricing measure in (31) we obtain

$$\begin{aligned} V_t^T &= \frac{2}{T-t} e^{\int_t^T r_s ds} \int_0^{\infty} \frac{Q(K, T; t)}{K^2} dK \\ &\quad - \frac{2}{T-t} \mathbb{E} \left[\int_t^T \int_{\mathbb{R}} \left(e^y - 1 - y - \frac{y^2}{2} \right) \nu(dy) ds \mid \mathcal{F}_t \right]. \quad (32) \end{aligned}$$

Equation (29) is a discretization of the first term in (32). The extra term $\frac{1}{T-t} \left(\frac{F_t}{K_0} - 1 \right)^2$ in (29) is a contribution due to the discretization around F_t .

The square of the VIX index is thus a model free estimate of the expected volatility over the next 30 days given continuous dynamics of the underlying but under general price processes the error term is given by the last term in (32):

$$VIX_t^2 = V_t^{t+30\text{days}} + \frac{2}{T-t} \mathbb{E} \left[\int_t^T \int_{\mathbb{R}} \left(e^y - 1 - y - \frac{y^2}{2} \right) \nu(dy) ds \mid \mathcal{F}_t \right]. \quad (33)$$

4.2 VIX Futures

Let VIX_t^i denote the VIX futures price for the interval $[T_i, T_{i+1}]$ seen at time t . For $t = T_i$ we have

$$(VIX_{T_i}^i)^2 = V_{T_i}^i + 2 \int_{\mathbb{R}} \left(e^{u_i(x, V_{T_i}^i)} - 1 - u_i(x, V_{T_i}^i) - \frac{u_i(x, V_{T_i}^i)^2}{2} \right) \nu(dx).$$

We also have the martingale property for $t < T_i$

$$VIX_t^i = \mathbb{E} [VIX_{T_i}^i \mid \mathcal{F}_t]$$

By Jensen's inequality for convex functions

$$\begin{aligned} (VIX_t^i)^2 &\leq \mathbb{E} \left[(VIX_{T_i}^i)^2 \mid \mathcal{F}_t \right] \\ &= V_t^i + 2 \mathbb{E} \left[\int_{\mathbb{R}} \left(e^{u_i(x, V_{T_i}^i)} - 1 - u_i(x, V_{T_i}^i) - \frac{u_i(x, V_{T_i}^i)^2}{2} \right) \nu(dx) \mid \mathcal{F}_t \right], \end{aligned} \quad (34)$$

since V_t^i is a martingale. Equation (34) shows that there is a ‘‘convexity correction’’ connecting VIX futures to forward variance swap rates.

To obtain a tractable framework, we first characterize the dynamics of $\sqrt{V_t^i}$ then propose an approximation which has a closed-form characteristic function.

Proposition 2 *The process $\sqrt{V_t^i}$ has the multiplicative decomposition*

$$\sqrt{V_t^i} = \sqrt{V_0^i} M_t^i A_t^i$$

where A_t^i is a finite variation process and M_t^i is an exponential martingale given by

$$M_t^i = \exp \left\{ \int_0^t \eta_s^i ds + \frac{1}{2} \int_0^t \omega e^{-k_1(T_i-s)} dZ_s + \int_0^t \int_{\mathbb{R}} \frac{1}{2} e^{-k_2(T_i-s)} x J(dx ds) \right\}, \quad (35)$$

where

$$\eta_t^i = -\frac{1}{8} \omega^2 e^{-2k_1(T_i-t)} - \int_{\mathbb{R}} \left(\exp \left\{ \frac{1}{2} e^{-k_2(T_i-t)} x \right\} - 1 \right) \nu(dx). \quad (36)$$

The proof of this result is given in Appendix B.

Using this result we approximate VIX_t^i with

$$VIX_t^i \simeq VIX_0^i M_t^i = VIX_0^i \exp(Y_t^i) . \quad (37)$$

where $Y_t^i = \ln M_t^i$. This approximation leaves the initial level of the VIX^i and the volatility of VIX^i untouched and thus is relevant for pricing VIX derivatives. As in the case of the forward variance swaps, once the Lévy measure $\nu(dx)$ is specified, the characteristic function for the exponent in (37) can be easily computed, so options on the VIX index can be priced by Fourier transform methods.²

5 Implementation

In this section the model specifications described in Section 3.4 are implemented on data on the VIX and the S&P 500 implied volatility smiles.

5.1 Data

We assess the performance of the model using prices from August 20th, 2008 on a range of VIX put and call options for five maturities, VIX futures for the same maturities, the dividend yield on S&P 500, call and put options on S&P 500 for six maturities and for the various maturities we also have the corresponding prices of the futures on S&P 500, from which we also derive a discount curve. We also observe 3 month forward variance swap rates for various maturities. The variance swap rates have been converted to forward 1 month variance swap rates by simple linear extrapolation and these are depicted in Figure 3. Options for which the bid price is zero were removed.

5.2 Calibration

The calibration of the model consists of three steps:

1. Determine the parameters controlling the variance swap dynamics by calibration to VIX options using the characteristic function in (42) or (44) and Fourier transform methods.
2. Use the parameters from the first step to simulate N paths of the variance swaps and store the increments of Z , the jump times and jump sizes along with the $V_{T_i}^i$ s.
3. Calibrate to options on the index recursively by use of (23).

²The characteristic functions for the specifications in Section 3.4 are given in Appendix A.

In the first step we compute model prices by the fast Fourier transform technique described in [13] using the revised method in [18]. The calibration is performed by minimizing the sum of squared errors weighted by the inverse bid-ask spread across all maturities and strikes on out-of-the-money options on the VIX futures

$$SE = \sum_{\text{options}} \frac{1}{Q_{Ask} - Q_{Bid}} (Q_{Market,Mid} - Q_{Model})^2, \quad (38)$$

using a gradient-based minimization algorithm.

The resulting parameters for the two specifications are shown in Table 3 and it also includes the resulting average relative percentage bid-ask error from the calibration,

$$\text{Error} = \frac{1}{\#\{\text{options}\}} \sum_{\text{options}} \frac{\max\{(Q_{Model} - Q_{Ask})^+, (Q_{Bid} - Q_{Model})^+\}}{Q_{Market,Mid}}. \quad (39)$$

We see how both model specifications are able to achieve very low calibration error. Note the high value of k_1 compared to k_2 , which implies (in the risk-neutral dynamics) that the diffusion component Z is mainly causing fluctuations at the short end of the variance swap curve, while the jumps impact the entire curve.

Figures 5 and 6 depict the (Black) implied volatility for the bid, ask, mid and model prices of VIX options as a function of moneyness. We observe that model prices fit very well within the bid-ask spread for almost all the observations in both specifications. This performance should be compared with flat model implied volatilities in the Bergomi model [5] and downward sloping volatilities in the Heston model [22]. The Bergomi model [7] is able to generate positive skews on volatility of volatility via the use of a Markov functional mapping for the forward variances, but it should be noted that the model can only price vanilla options on the underlying index by Monte Carlo simulation.

Step 2 is done by discretization of (9). In this example we have used $2 \cdot 10^5$ simulated sample paths. Simulation of the jumps is straightforward, since they arrive at constant intensity and the jump sizes are computed by draws from a normal/exponential distribution times the scaling $e^{-k_2(T_i - \tau)}$, where τ is the jump time. For details on how to simulate the increments in the variance swaps due to the diffusion the reader is referred to [5].

In step 3 we have fixed the correlation parameter between the continuous components to $\rho = -0.45$.³ Then, the last step is implemented step-wise by matching the model prices (23) to market prices on the shortest maturity

³Instead one could also specify a correlation ρ_i on each interval and include that parameter in the calibration for each interval. We tried this, but the performances of the models did not improve with this added flexibility.

out-of-the-money options. Again, this is achieved by minimizing the SE (38). This yields b_0 . Then b_1 are found in the same way by calibrating to the next maturing options by the use of b_0 for the first interval in (21). The remaining b_i s are estimated in a similar manner.

The estimated parameters for the two different specifications of the jump size distributions are shown in Table 4 along with the pricing error in (39) for each maturity. The mean and standard deviations of the jumps before scaling with $(V_{T_i}^i/V_0^i)^{\frac{1}{2}}$ are also included. The values of b_i for $i = 0, \dots, 3$ relate to the monthly distributions of the index up to the maturity of the options expiring on December 19th, 2008. For the last two option maturities the interval between the expirations are three months and hence b_i for $i = 4, 5$ relate to the three month distributions of the index. Double exponential jumps perform slightly better on the first interval, but there is not a strong difference between the two specifications, showing that the model performance is not very sensitive to the choice of the jump size distribution.

Figures 7 and 8 show the result of the calibration to the S&P 500 options. The fit of the two specifications are practically indistinguishable and overall the calibrations perform well for both with somewhat less success on the shortest maturity.

We conclude this section with an examination of the size of the jump error term when valuing the forward variance swap rates V^i under the assumption of continuous dynamics given the models implemented in this paper are “true”. Recall that the error is given by

$$\varepsilon_i = -2\mathbb{E} \left[\int_{\mathbb{R}} \left(e^{u_i(x, V_{T_i}^i)} - 1 - u_i(x, V_{T_i}^i) - \frac{u_i(x, V_{T_i}^i)^2}{2} \right) \nu(dx) \mid \mathcal{F}_0 \right]. \quad (40)$$

In Table 5 the absolute errors implied by the calibration of the two model specifications are computed for each interval. Moreover, the errors relative to the initial forward variance swap value are reported. For the first interval it is 2.2%/2.3% and then it roughly increases as i increases. The reported errors for the last two intervals are taken as the average of the three monthly errors relative to the average of the three monthly forward variances covering the same intervals.

6 Conclusion

We have presented a model for the joint dynamics of a set of forward variance swap rates along with the underlying index. Using Lévy processes as building blocks, this model leads to a tractable pricing framework for variance swaps, VIX futures and vanilla call/put options, which makes calibration of the model to such instruments feasible. This tractability feature distinguishes our model from previous attempts [5, 4, 21] which only allow

for full Monte Carlo pricing of vanilla options.

Our model reproduces salient empirical features of variance swap dynamics, in particular the strong negative correlation of large index moves with large moves in the VIX and the positive skew observed in implied volatilities of VIX options, by introducing a common jump component in the variance swaps and the underlying asset. Using two different specifications for the jump size distribution (Lévy measure) we have illustrated the feasibility of the numerical implementation, as well as the capacity of the model to match a complete set of market prices of vanilla options and options on the VIX. Our model can be used to price and hedge various payoffs sensitive to forward volatility, such as cliquet or forward start options, as well as volatility derivatives, in a manner consistent with market prices of simpler instruments such as calls, puts or variance swaps which are typically used for hedging them.

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A Characteristic Functions

A.1 Normally Distributed Jumps

Variance Swaps

We have from [26] that the characteristic function of $X_{T_i}^i$ is given by

$$\begin{aligned} \mathbb{E} \left[e^{iuX_{T_i}^i} \right] &= \exp \left\{ -\frac{1}{2}u^2 \int_0^{T_i} \omega^2 e^{-2k_1(T_i-s)} ds + iu \int_0^{T_i} \mu_s^i ds \right. \\ &\quad \left. + \int_0^{T_i} \int_{\mathbb{R}} \left(\exp \left\{ iue^{-k_2(T_i-s)}x \right\} - 1 \right) \nu(dx) dt \right\}. \end{aligned}$$

Notice now that

$$\int_0^{T_i} e^{-2k_1(T_i-s)} ds = \frac{1 - e^{-2k_1T_i}}{2k_1}$$

and

$$\mathbb{E} [e^Y] = e^{m + \frac{\delta^2}{2}}$$

for $Y \sim N(m, \delta^2)$. Inserting this into the expression for the characteristic function we arrive at

$$\begin{aligned} \mathbb{E} \left[e^{iuX_{T_i}^i} \right] &= \exp \left\{ -\frac{1}{2}\omega^2 iu \frac{1 - e^{-2k_1T_i}}{2k_1} - \frac{1}{2}\omega^2 u^2 \frac{1 - e^{-2k_1T_i}}{2k_1} \right. \\ &\quad \left. - iu\lambda \int_0^{T_i} \left(\exp \left\{ e^{-k_2(T_i-s)}m + \frac{1}{2}e^{-2k_2(T_i-s)}\delta^2 \right\} - 1 \right) ds \right. \\ &\quad \left. + \lambda \int_0^{T_i} \left(\exp \left\{ iue^{-k_2(T_i-s)}m - \frac{1}{2}u^2 e^{-2k_2(T_i-s)}\delta^2 \right\} - 1 \right) ds \right\}. \end{aligned} \tag{41}$$

VIX Futures

Recall from (37) that $VIX_t^i \simeq VIX_0^i M_t^i$ where $M_t^i = \exp(Y_t^i)$ is a positive exponential martingale. The characteristic function of

$$Y_{T_i}^i = \int_0^{T_i} \eta_s^i ds + \frac{1}{2} \int_0^{T_i} \omega e^{-k_1(T_i-s)} dZ_s + \int_0^{T_i} \int_{\mathbb{R}} \frac{1}{2} e^{-k_2(T_i-s)} x J(dx ds)$$

can be found in the same way as for X^i . It is given by

$$\begin{aligned} \mathbb{E} \left[e^{iuY_{T_i}^i} \right] &= \exp \left\{ -\frac{1}{8}\omega^2 iu \frac{1 - e^{-2k_1T_i}}{2k_1} - \frac{1}{8}\omega^2 u^2 \frac{1 - e^{-2k_1T_i}}{2k_1} \right. \\ &\quad \left. - iu\lambda \int_0^{T_i} \left(\exp \left\{ \frac{1}{2}e^{-k_2(T_i-s)}m + \frac{1}{8}e^{-2k_2(T_i-s)}\delta^2 \right\} - 1 \right) ds \right. \\ &\quad \left. + \lambda \int_0^{T_i} \left(\exp \left\{ \frac{1}{2}iue^{-k_2(T_i-s)}m - \frac{1}{8}u^2 e^{-2k_2(T_i-s)}\delta^2 \right\} - 1 \right) ds \right\}. \end{aligned} \tag{42}$$

A.2 Exponentially Distributed Jumps

Variance Swaps

The characteristic function in (11) takes the form

$$\begin{aligned} \mathbb{E} \left[e^{iuX_{T_i}^i} \right] &= \exp \left\{ -\frac{1}{2}\omega^2 iu \frac{1 - e^{-2k_1 T_i}}{2k_1} - \frac{1}{2}\omega^2 u^2 \frac{1 - e^{-2k_1 T_i}}{2k_1} \right. \\ &\quad - iu\lambda \int_0^{T_i} \left(\frac{p\alpha_+}{\alpha_+ - e^{-k_2(T_i-s)}} + \frac{(1-p)\alpha_-}{\alpha_- + e^{-k_2(T_i-s)}} - 1 \right) ds \\ &\quad \left. + \lambda \int_0^{T_i} \left(\frac{p\alpha_+}{\alpha_+ - iue^{-k_2(T_i-s)}} + \frac{(1-p)\alpha_-}{\alpha_- + iue^{-k_2(T_i-s)}} - 1 \right) ds \right\}. \end{aligned} \quad (43)$$

VIX Futures

Recall from (37) that $VIX_t^i \simeq VIX_0^i M_t^i$ where $M_t^i = \exp(Y_t^i)$ is a positive exponential martingale. In this specification the characteristic function of $Y_{T_i}^i$ is given by

$$\begin{aligned} \mathbb{E} \left[e^{iuY_{T_i}^i} \right] &= \exp \left\{ -\frac{1}{8}\omega^2 iu \frac{1 - e^{-2k_1 T_i}}{2k_1} - \frac{1}{8}\omega^2 u^2 \frac{1 - e^{-2k_1 T_i}}{2k_1} \right. \\ &\quad - iu\lambda \int_0^{T_i} \left(\frac{p\alpha_+}{\alpha_+ - \frac{1}{2}e^{-k_2(T_i-s)}} + \frac{(1-p)\alpha_-}{\alpha_- + \frac{1}{2}e^{-k_2(T_i-s)}} - 1 \right) ds \\ &\quad \left. + \lambda \int_0^{T_i} \left(\frac{p\alpha_+}{\alpha_+ - \frac{1}{2}iue^{-k_2(T_i-s)}} + \frac{(1-p)\alpha_-}{\alpha_- + \frac{1}{2}iue^{-k_2(T_i-s)}} - 1 \right) ds \right\}. \end{aligned} \quad (44)$$

B Proof of proposition 2

We can express V_t^i in (9) as

$$\begin{aligned} V_t^i &= V_0^i + \int_0^t V_{s-}^i \omega e^{-k_1(T_i-s)} dZ_s \\ &\quad + \int_0^t \int_{\mathbb{R}} V_{s-}^i \left(\exp \left\{ e^{-k_2(T_i-s)} x \right\} - 1 \right) J(ds dx) \\ &\quad - \int_0^t \int_{\mathbb{R}} V_{s-}^i \left(\exp \left\{ e^{-k_2(T_i-s)} x \right\} - 1 \right) \nu(dx) ds. \end{aligned}$$

Applying the Itô formula to $\sqrt{V_t^i}$ we obtain

$$\begin{aligned} \sqrt{V_t^i} &= \sqrt{V_0^i} + \frac{1}{2} \int_0^t (V_{s-}^i)^2 \omega^2 e^{-2k_1(T_i-s)} \left(-\frac{1}{4} (V_{s-}^i)^{-\frac{3}{2}} \right) ds \\ &\quad - \int_0^t \int_{\mathbb{R}} V_{s-}^i \left(\exp \left\{ e^{-k_2(T_i-s)} x \right\} - 1 \right) \frac{1}{2} (V_{s-}^i)^{-\frac{1}{2}} \nu(dx) ds \\ &\quad + \int_0^t V_{s-}^i \omega e^{-k_1(T_i-s)} \frac{1}{2} (V_{s-}^i)^{-\frac{1}{2}} dZ_s \\ &\quad + \int_0^t \int_{\mathbb{R}} (V_{s-}^i)^{\frac{1}{2}} \left(\exp \left\{ \frac{1}{2} e^{-k_2(T_i-s)} x \right\} - 1 \right) J(ds dx) \\ &= \sqrt{V_0^i} + \frac{1}{2} \int_0^t \sqrt{V_{s-}^i} \omega e^{-k_1(T_i-s)} dZ_s \\ &\quad - \int_0^t \int_{\mathbb{R}} \sqrt{V_{s-}^i} \left(\frac{1}{2} \left(\exp \left\{ e^{-k_2(T_i-s)} x \right\} - 1 \right) - \frac{1}{8} \omega^2 e^{-2k_1(T_i-s)} \right) \nu(dx) ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \sqrt{V_{s-}^i} \left(\exp \left\{ \frac{1}{2} e^{-k_2(T_i-s)} x \right\} - 1 \right) J(ds dx). \end{aligned}$$

This can be written as

$$\begin{aligned} \sqrt{V_t^i} &= \sqrt{V_0^i} \exp \left\{ \int_0^t \hat{\mu}_s^i ds + \frac{1}{2} \int_0^t \omega e^{-k_1(T_i-s)} dZ_s \right. \\ &\quad \left. + \int_0^t \int_{\mathbb{R}} \frac{1}{2} e^{-k_2(T_i-s)} x J(dx ds) \right\}, \end{aligned} \quad (45)$$

where

$$\hat{\mu}_t^i = -\frac{1}{4} \omega^2 e^{-2k_1(T_i-t)} - \frac{1}{2} \int_{\mathbb{R}} \left(\exp \left\{ e^{-k_2(T_i-t)} x \right\} - 1 \right) \nu(dx).$$

The stochastic integrals in (45) being processes with independent increments, a straightforward application of the exponential formula for Poisson random measures [18, Prop. 3.6.] yields the multiplicative decomposition in Proposition 2 where M is given by (35)-(36).

C Tables

Table 2: Error from following the replicating strategy relative to the realized variance over the three month windows.

	Sep-Dec 2007	Dec 2007-Mar 2008	Mar-Jun 2008	Jun-Sep 2008
Error	-0.137%	-0.025%	0.060%	-0.243%

Table 3: Calibrated parameters for the two models from the VIX volatility smiles on August 20th, 2008. The top panel corresponds to the normally distributed jumps and the bottom to the double exponentially distributed jumps.

Normal jumps							
λ	ω	k_1	k_2	m	δ	Error (%)	
3.5201	2.0389	21.9623	2.0743	0.5394	0.2468	0.64	
Double exponential jumps							
λ	ω	k_1	k_2	p	α_+	α_-	Error (%)
13.5938	1.9765	22.3033	2.2020	0.8663	4.2457	19.9055	0.85

Table 4: Model parameters calibrated from the S&P 500 volatility smiles on August 20th, 2008 with the correlation between the two Brownian components set to -0.45.

i	0	1	2	3	4	5
<u>Gaussian jumps</u>						
b_i	-0.151	-0.159	-0.152	-0.173	-0.187	-0.193
$b_i m$	-0.081	-0.086	-0.082	-0.093	-0.101	-0.104
$ b_i \delta $	0.037	0.039	0.038	0.043	0.046	0.048
Error (%)	8.8	0.6	1.1	1.8	1.9	2.7
<u>Double exponential jumps</u>						
b_i	-0.153	-0.159	-0.149	-0.169	-0.184	-0.192
$\left(\frac{b_i p}{\alpha_+} - \frac{b_i(1-p)}{\alpha_-}\right)$	-0.030	-0.031	-0.029	-0.033	-0.036	-0.038
$\left(\frac{b_i^2 p}{\alpha_+^2} + \frac{b_i^2(1-p)}{\alpha_-^2}\right)^{\frac{1}{2}}$	0.034	0.035	0.033	0.037	0.040	0.042
Error (%)	6.9	0.8	1.4	2.0	2.1	2.7

Table 5: The errors arising from approximating the index dynamics with a continuous process given the model specifications described in this paper are “true”.

i	0	1	2	3	4	5
<u>Gaussian jumps</u>						
$\varepsilon_i \times 10^2$	0.100	0.119	0.133	0.157	0.225	0.252
$\frac{\varepsilon_i}{V_0^2}$ (%)	2.2	2.4	2.5	2.8	3.8	4.1
<u>Double exponential jumps</u>						
$\varepsilon_i \times 10^2$	0.105	0.120	0.131	0.157	0.218	0.252
$\frac{\varepsilon_i}{V_0^2}$ (%)	2.3	2.4	2.4	2.8	3.6	4.1

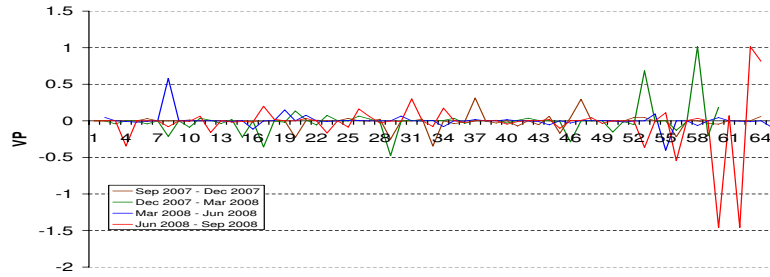


Figure 4: The profit-loss from day to day in variance points (VP) from holding a long position of the hedge portfolio and a short variance swap for four different three month variance swaps. The x -axis is trading days during the three month period.

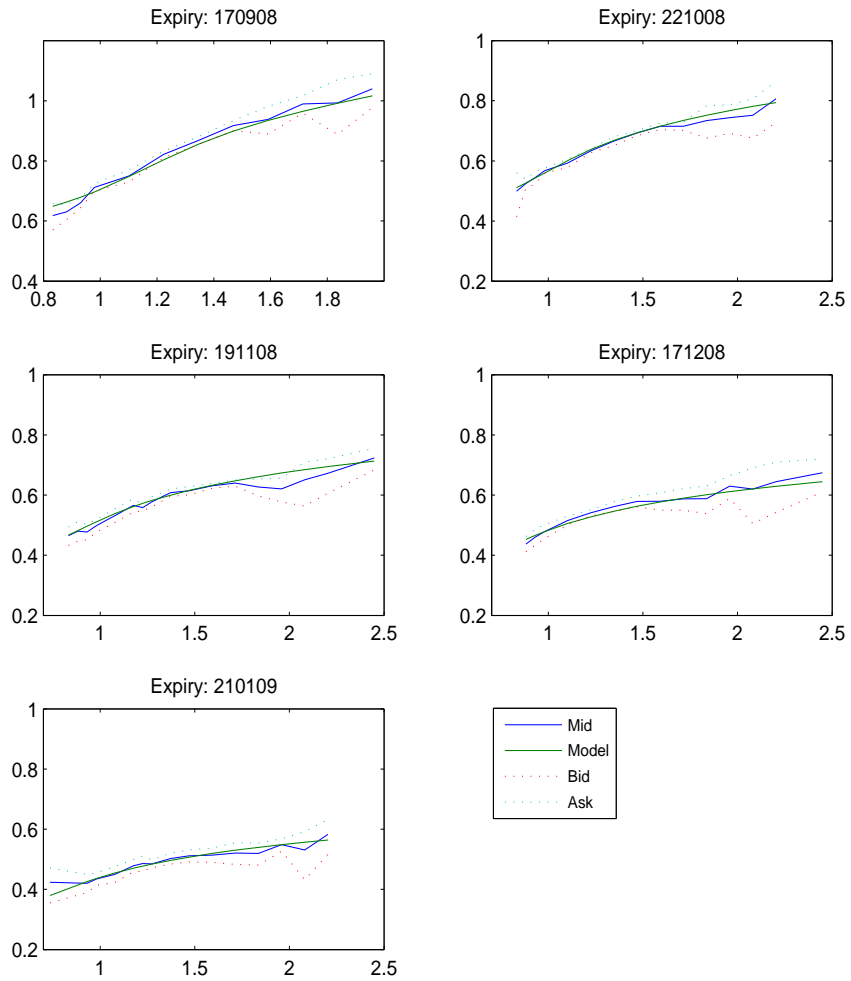


Figure 5: VIX implied volatility smiles on August 20th 2008 for the model with normally distributed jump sizes plotted against moneyness $m = K/VIX_t$ on the x axis.

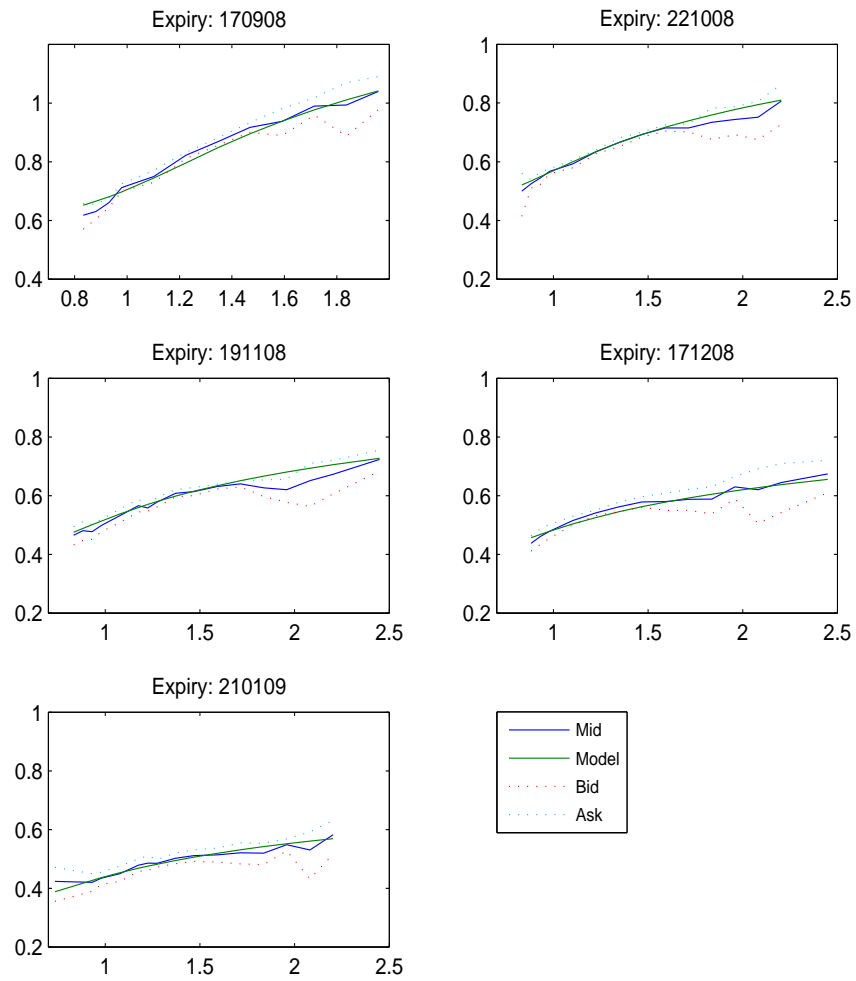


Figure 6: VIX implied volatility smiles on August 20th 2008 for the model with double exponentially distributed jump sizes plotted against moneyness $m = K/VIX_t$ on the x axis.

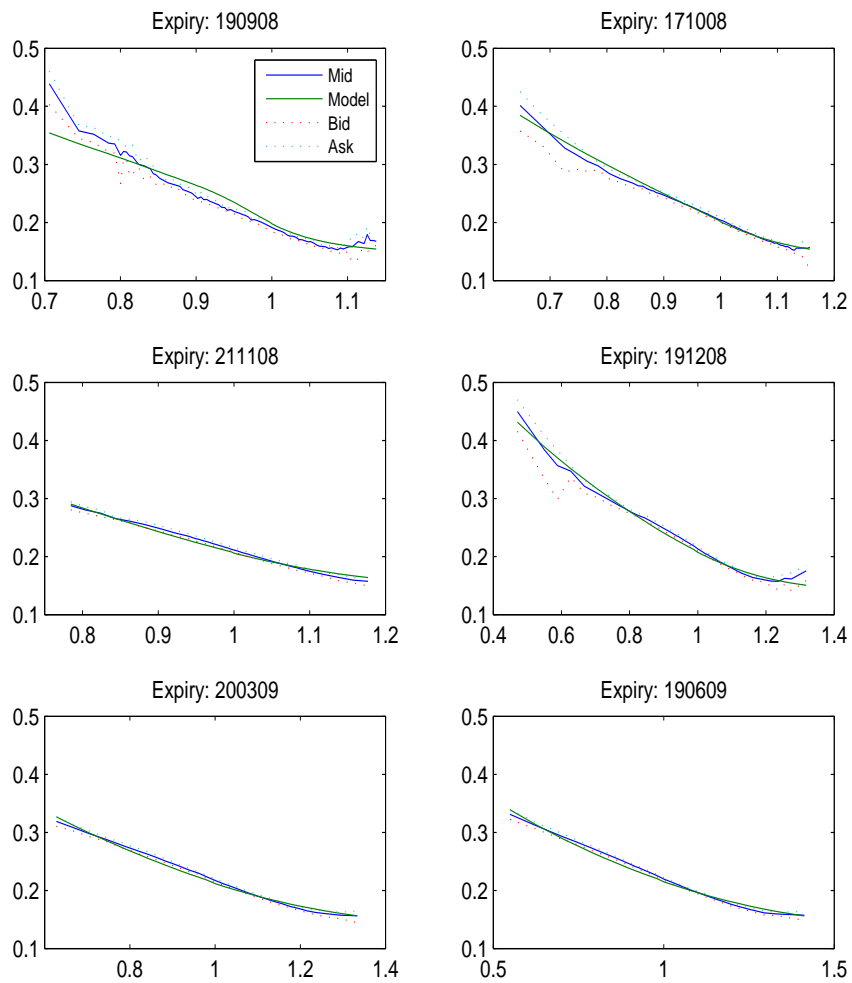


Figure 7: S&P 500 implied volatility smiles on August 20th 2008 for the model with normally distributed jump sizes plotted against moneyness $m = K/S_t$ on the x axis.

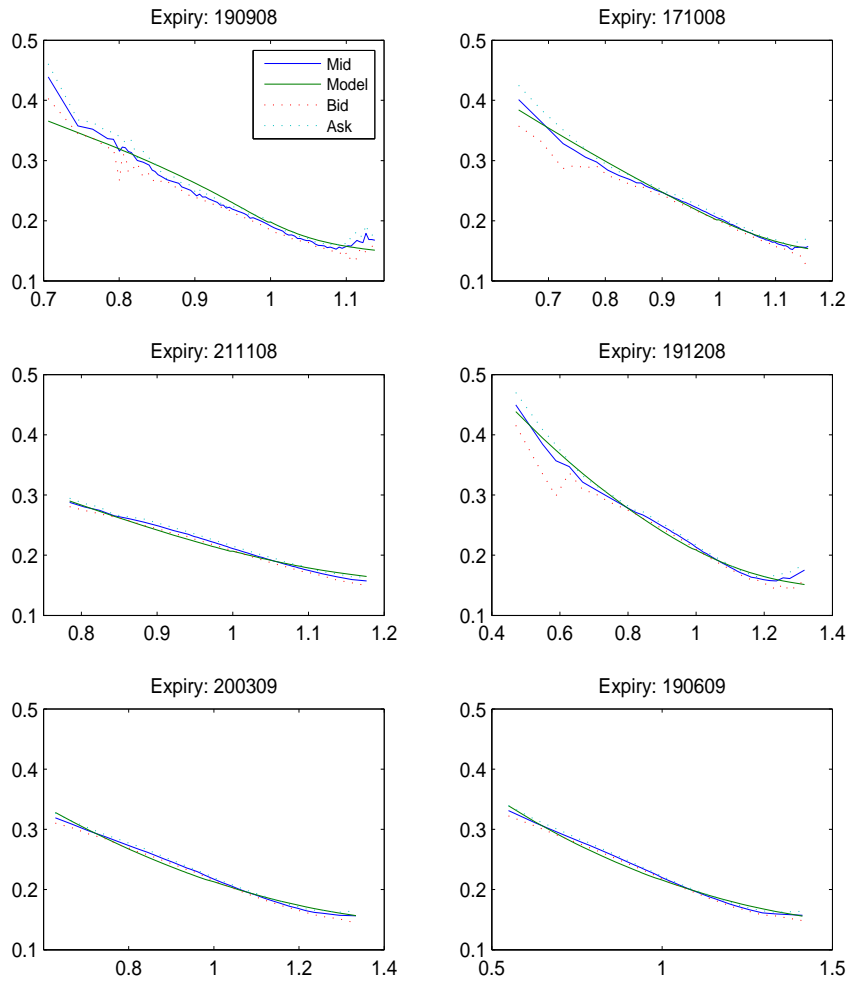


Figure 8: S&P 500 implied volatility smiles on August 20th 2008 for the model with double exponentially distributed jump sizes plotted against monyness $m = K/S_t$ on the x axis.

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