

PERTURBATION THEORY OF EMBEDDED EIGENVALUES



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RESUMÉ – DANSK

I denne afhandling studeres problemer i forbindelse med perturbationsteori af indlejrede egenverdier. Den første del omhandler den translationsinvariante, massive Nelson-model og beviser, at indlejrede egenverdier af Hamilton-operatorens fibre må være mindst to gange differentiable som funktion af fiberparameteren. Beviset bygger på en udvikling af orden $2 + \alpha$ mht. til fiberparameteren. De væsentligste tekniske hindringer er, at fiber-operatorerne a priori ikke har tilstrækkelig regularitet mht. den konjugerede operator og at kommutatorer mellem den konjugerede operator og fiber-operatorerne ikke kan begrænses af fiber-operatoren. I anden del bevises en abstrakt analytisk perturbationsteori for indlejrede egenverdier. Et teknisk krav på Hamilton-operatorerne er dog, at alle itererede kommutatorer skal kunne begrænses af Hamilton-operatoren, hvilket udelukker Nelson-modellen. Strategien i artiklen er at bruge spektraldeformationsteknikker hvor den unitære gruppe er genereret af den konjugerede operator. Et Mourre-estimat gør det muligt at bevise, at det essentielle spektrum af de transformerede Hamilton-operatorer forsvinder, og således tillader brug af Kato-teori.

RESUMÉ – ENGLISH

In this thesis problems connected with perturbation theory of embedded eigenvalues are studied. The first part deals with the case of the translation invariant massive Nelson model and establishes that embedded eigenvalues of the fiber Hamiltonians have to depend at least twice differentiable on the fiber parameter. This is done by deriving an expansion to order $2 + \alpha$ w.r.t. the fiber parameter. The main technical obstacles are that a priori the fiber Hamiltonians do not have sufficient regularity w.r.t. the conjugate operator and that commutators of the conjugate operator with the fiber Hamiltonians cannot be bounded by the Hamiltonians again. In the second part an abstract analytic perturbation theory for embedded eigenvalues is established. However a technical requirement on the Hamiltonians is that all iterated commutators should be bounded by the Hamiltonian thus excluding the Nelson model. The strategy of the paper is to use spectral deformation techniques where the unitary group is generated by the conjugate operator. A Mourre estimate allows to prove that the essential spectrum of the transformed Hamiltonians recedes thus permitting the use of Kato theory.

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CHAPTER 1: Overview

1 General Motivation of the Thesis: Perturbation Theory of Embedded Eigenvalues

The question how the spectrum of a family of operators $\{H_\kappa\}_\kappa$ acting on a Hilbert space depends on the parameter κ is most likely an old one, since it contains the question how the eigenvalues of a family of matrices depend on a parameter κ . In the case of bounded operators on a Hilbert space \mathcal{H} the notion of a family $\{H_\kappa\}_\kappa$ depending analytically on κ is somewhat natural, since the set $\mathcal{B}(\mathcal{H})$ of all bounded operators on \mathcal{H} equipped with the operator norm is a Banach space. A function $f : U \rightarrow X$, where $U \subset \mathbb{C}$ is open, with values in some Banach space X is called *weakly analytic*, if the functions $z \mapsto x^*(f(z))$ are analytic for all $x^* \in X^*$. If X is a Hilbert space, this property is equivalent to the notion of *strong analyticity*, that is analyticity of the functions $z \mapsto f(z)\psi$ for all $\psi \in X$. The continuing interest in perturbation theory over the last 50 years or so is partly due to its immense importance in quantum mechanical problems. Schrödinger's equation for the Hydrogen atom admits an explicit solution and the spectrum of the corresponding operator can be found. At a first glance the theoretical predictions seem to be confirmed by experiments as they reproduce the famous Rydberg formula. However, phenomena like the Zeeman and the Stark effect cause the energy levels of the atom to split which in turn can be observed in more accurate experiments. These effects have, from a conceptual viewpoint, a natural interpretation as a perturbation to the known solution of the Hydrogen atom, since the corresponding additions to the interaction potential always come with a small pre-factor in one way or the other.

However, the situation of unbounded operators is more complicated than the case of bounded operators, so the problem of formulating the perturbation problem in a mathematically precise way is interesting in itself. One of the basic ideas in this business is that for discrete eigenvalues with finite multiplicities the perturbation problem is essentially reduced to its finite dimensional counterpart for matrices after restricting the operator to this subspace. These ideas have been systematically summed up by Kato in his now classic textbook *Perturbation Theory for Linear Operators*, [33]. Another excellent and more streamlined exhibition can be found in another classic textbook of mathematical physics, see [45].

Despite the success it should be mentioned that the application of the theory comes with some problems. First of all the calculation of the eigenvalues $\lambda(\kappa)$ as power series of κ , the so-called Rayleigh-Schrödinger series, is difficult, due to the complexity of the coefficients. In several physically interesting cases this perturbation series is not even convergent and one can only extract some meaning out of it in an asymptotic sense. This particular problem is usually referred to as asymptotic perturbation theory. Kato's as well as Reed and Simons' book give an introduction to this topic, it is however discussed in greater detail in an overview paper by Hunziker, see [28]. We will not follow this particular line of research and

leave the discussion of asymptotic perturbation theory at this point.

A more important issue is that for an eigenvalue λ to be treated in a perturbative way by Kato's theory it has to be isolated and thus excludes the situation, where an eigenvalue with finite multiplicity is an element of the essential spectrum. One of the main complications in such a situation is that the projection P onto the eigenspace cannot be constructed by means of the Riesz projection

$$P = \frac{1}{2\pi i} \int_{\gamma(\lambda)} (H - z)^{-1} dz,$$

since a path $\gamma(\lambda)$ around λ which does not intersect the spectrum of H does not exist. One way of dealing with such a situation goes back to two papers by Aguilar and Combes and Balslev and Combes, [3, 6]. There the technique of spectral deformation is used to investigate the structure of the essential spectrum of Schrödinger operators H . More precisely, for certain unitary groups $U(\theta)$ the operator $H_\theta = U(\theta)HU(\theta)^{-1}$ obtained by conjugating H with $U(\theta)$ admits an analytic continuation to a strip in the complex plane around the real axis. The virtue of this transformation is that for complex θ the essential spectrum of H swings out into the complex plane with an angle controlled by the imaginary part of θ and can in this way uncover embedded eigenvalues or threshold values. We will describe this approach in more detail later in the thesis and thus refrain from going into more details for the time being.

An example of an operator with an embedded eigenvalue is $-\Delta_{x_1} - \Delta_{x_2} - 2|x_1|^{-1} - 2c_2|x_2|^{-1}$, see [45]. It describes a Helium atom in the limit of infinite mass of the nucleus and neglect of repulsion amongst the electrons. In [49] Simon treats the addition of the electron-electron interaction within the context of perturbation theory under certain analyticity assumptions. Many of the ideas presented in the mentioned paper rely on the concept of spectral deformation.

Some years later Agmon, Herbst and Skibsted published a paper on the perturbations of embedded eigenvalues in the quantum mechanical N -body problem, [2]. The authors do not work with an analyticity assumption but focus on the so-called Fermi Golden Rule which they use to show that embedded eigenvalues disappear under small perturbations. For our purposes the paper is interesting, because of its use of Mourre-theory which now is one of the standard methods within spectral theory.

More recent results in perturbation theory of embedded eigenvalues can be found in a paper by Faupin, Møller and Skibsted, see [15]. In their paper Mourre theory and the limiting absorption principle are used to prove an expansion of the perturbed eigenvalue w.r.t. the perturbation parameter up to second order. Even though their results are formulated in an abstract way, they are designed to be applied in the context of massless Pauli Fierz Hamiltonians and the massless Nelson model in particular.

In this thesis we aim to obtain similar results on perturbation theory of embedded eigenvalues. One major part of the thesis is devoted to the massive and translation invariant Nelson model. This model describes a non-relativistic quantum field of massive bosons interacting with a single electron through linear coupling. This model has a fiber decomposition w.r.t. total momentum and we address the following question: If the fiber Hamiltonians $H(\xi)$ exhibit bands of embedded eigenvalues λ_ξ in the energy momentum spectrum below the two-boson threshold, how do these eigenvalues depend on ξ ? By using recent results in Mourre theory of this model and related topics, [39, 40], an expansion of $\lambda_{\xi+\zeta}$ for small ζ up to terms of order $|\zeta|^{2+\alpha}$, where $\alpha \in (0, 1)$, is proven. Such a result obviously implies that the map $\zeta \mapsto \lambda_{\xi+\zeta}$ is twice differentiable.

This result is of interest in its own right for the spectral theory of such models. The twice differentiability of ground states enabled Møller and Rasmussen in [39, 44] to obtain a Mourre estimate away from threshold energies in the essential spectrum below the two-boson threshold. Note that the result on the validity of the Mourre estimate in the PhD thesis of the first author was weaker than the result obtained in the follow up paper by Møller and Rasmussen. Recently Dybalski and Møller have proven asymptotic completeness in the massive Nelson model in the energy regime in which the Mourre estimate holds. This results motivates the desire to establish these type of estimates also in higher energy regions. In a sense one would like to proceed step by step and continue to push the analysis to higher regimes. Another central ingredient to push the results on asymptotic completeness in a similar fashion requires knowledge of the behavior of the threshold sets. In particular these should not fill up the whole essential spectrum by forming a dense set. Such a situation could be argued to be impossible, if it was known that embedded eigenvalues were not just twice differentiable but rather analytic functions w.r.t. the fiber parameter. Even though it was our initial ambition to obtain smoothness rather than twice differentiability by proving arbitrary expansions of $\lambda_{\xi_0+\zeta}$ below the two-boson threshold, we were not able to achieve it due to technical complications.

However, the desire to possibly return to this problem in future research lead to the following question: Which abstract assumptions are actually needed to obtain analyticity results for parameter dependence of embedded eigenvalues? A partial answer to this question is given in the second part of this thesis. The starting point is a self-adjoint operator H together with a unitary group $U(\theta)$ generated by a self-adjoint operator A . To start the analysis the assumption that all iterated commutators $i^k \text{ad}_A^k(H)$ remain H -bounded and satisfy $\|i^k \text{ad}_A^k(H)(H + i)^{-1}\| \leq C^k k!$ for some $C > 0$ is shown to be equivalent to the statement that the operator $H_\theta = U(\theta)HU(\theta)^{-1}$ extends analytically to a strip in the complex plane. By using a version of Mourre's estimate absence of essential spectrum of H_θ can then be shown in a certain region around the eigenvalue of interest. The real (and in this region discrete) eigenvalues of H_θ correspond to eigenvalues of H and thus

allow a study via the usual Kato theory if H is substituted by an analytic family $H(\xi)$ of type A at least in the case, where the perturbation parameter ξ is drawn from a one dimensional set. It should be noted however, that this method yields a perturbation theory for all eigenvalues of $H_\theta(\xi) = U(\theta)H(\xi)U(\theta)^{-1}$ and thus opens up the possibility to study resonances.

It should be noted however that this result cannot be applied to the massive Nelson model, since there the commutator $\text{ad}_A(H(\xi))$ is only bounded by $(H(\xi) + c)^{3/2}$ for appropriately chosen constant $c > 0$. Nevertheless, we aim to apply the obtained result to study resonances or eigenvalues in discrete Schrödinger operators.

2 Conjugate Operators, Mourre Theory, Limiting Absorption Principle

2.1 Commutator Calculus

In this section we mainly follow the presentation of [21]. Let \mathcal{H} be a Hilbert space, A be a self-adjoint operator with domain $D(A)$ and $S \in \mathcal{B}(\mathcal{H})$.

Definition 2.1. Define $W_t := e^{itA}$ and let $k \in \mathbb{N}$. We say that $S \in C^k(A)$, if the maps $t \mapsto W_{-t}SW_t\psi$ are elements of $C^k(\mathbb{R}, \mathcal{H})$ for all $\psi \in \mathcal{H}$.

Let $\psi, \psi' \in D(A)$ and define

$$\langle \psi, i[A, S]\psi' \rangle := \langle A\psi iS\psi' \rangle - \langle S^*\psi, iA\psi \rangle.$$

The factor i is only added so that the operator which will be implementing the form later on is self-adjoint for $S = S^*$. Suppose that there exists $C > 0$ such that

$$|\langle \psi, i[A, S]\psi' \rangle| \leq C\|\psi\|\|\psi'\|.$$

Then the sesqui-linear form $i[A, S]$ has a continuous extension $i[A, S]^\circ$ to the whole of \mathcal{H} and there is a bounded operator $i\text{ad}_A(S) \in \mathcal{B}(\mathcal{H})$ such that

$$\langle \psi, i[A, S]\psi' \rangle := \langle \psi, i\text{ad}_A(S)\psi' \rangle.$$

Suppose further that $i\text{ad}_A(S) \in C^1(A)$. Then the double commutator form $i[A, i\text{ad}_A(S)]$ extends from $D(A) \times D(A)$ to a bounded form on $\mathcal{H} \times \mathcal{H}$ and there is a bounded operator $i^2\text{ad}_A^2(S)$ implementing it. If this procedure can be iterated k times, we say that S admits k bounded commutators with A . It turns out that

Lemma 2.2. *Let $k \in \mathbb{N}$. $S \in C^k(A)$, if and only if S admits k bounded commutators with A .*

We will thus use these two equivalent characterizations of the set $C^k(A)$ interchangeably without further comment. Some of the important features of the set $C^1(A)$ are collected in the following Lemma.

Lemma 2.3.

1. $S \in C^1(A) \Leftrightarrow S^* \in C^1(A)$. In this case $\text{iad}_A(S^*) = (\text{iad}_A(S))^*$.
2. $S_1, S_2 \in C^1(A) \Rightarrow S_1 S_2 \in C^1(A)$.
3. $S \mapsto \text{iad}_A(S)$ is a linear map from $C^1(A)$ to $\mathcal{B}(\mathcal{H})$ which is closed in the weak operator topology.
4. $S \mapsto \text{iad}_A(S)$ is a derivation: $\text{iad}_A(S_1 S_2) = S_1 \text{iad}_A(S_2) + \text{iad}_A(S_1) S_2$.

The situation is more involved for unbounded, self-adjoint operators T . The basic idea is to define the commutators via the resolvent of T .

Definition 2.4. Let T be self-adjoint on $D(T)$. We say that T is of class $C^k(A)$, if there exists $z \in \rho(T)$ such that $(T - z)^{-1} \in C^k(A)$.

Again there is a connection between the $C^1(A)$ class and commutator forms.

Lemma 2.5. Let T be self-adjoint on $D(T) \subset \mathcal{H}$. Then T is of class $C^1(A)$, if and only if the following two conditions hold:

1. $\exists C > 0 \forall \psi, \psi' \in D(A) \cap D(T) : |\langle \psi, i[A, S]v \rangle| \leq C \|\psi\|_T \|\psi'\|_T$.
2. There exists $z \in \rho(T)$ such that $\{\psi \in D(A) \mid (T - z)^{-1} \in D(A)\}$ is a core for A .

For convenience we denote by $D(T)$ the Banach space obtained by equipping $D(T)$ with graph norm. We write $D(T)^*$ for its dual space. Moreover, the following formula holds:

$$\text{iad}_A((T - z)^{-1}) = -(T - z)^{-1} \text{iad}_A(T) (T - z)^{-1}, \quad (2.1)$$

where

$$\text{iad}_A(T) \in \mathcal{B}(D(T), D(T)^*)$$

and the first resolvent appearing on the right hand side is extended to a bounded operator from $D(T)^*$ to \mathcal{H} . Recall that $D(T) \subset \mathcal{H} \subset D(T)^*$ with continuous and dense inclusions. Operators for which $\text{ad}_A(T)$ takes values in \mathcal{H} rather than $D(T)^*$ are of a natural interest, since (2.1) holds in a more literal sense, that is none of the objects have to be extended to spaces larger than \mathcal{H} . In this case $\text{iad}_A(T) \in \mathcal{B}(D(T), \mathcal{H})$, that is $\text{iad}_A(T)(T - z)^{-1}$ extends to a bounded operator on \mathcal{H} by the closed graph theorem.

The commutator calculus presented here is frequently used in spectral theory. The case $\text{iad}_A(T)(T - z)^{-1}$ typically arises in the theory of Schrödinger operators and is therefore well-studied. However the case, where $\text{ad}_A(T)$ takes values in \mathcal{H} but can only be extended to a bounded operator from $D(T^n)$ to \mathcal{H} for some $n > 1$ are more difficult to deal with. In our framework they arise in the spectral analysis of massive, translation invariant Nelson models.

2.2 Mourre Estimate

The following estimate is at the heart of what is now known as Mourre theory. Mourre used the commutator calculus in his paper, [41], to establish this type of estimate in the quantum 3-body problem and thereby managed to prove absence of singular continuous spectrum.

A self-adjoint operator T which is of class $C^1(A)$ is said to satisfy a *Mourre estimate* around λ , if there exists a bounded interval J' , a constant $C_M > 0$ and a compact operator K such that for all closed intervals $J \subset J'$ the estimate

$$E_J(T)i[A, T]E_J(T) \geq C_M E_J(T) + K, \quad (2.2)$$

holds in the sense of quadratic forms. Here, $C_M > 0$ is the so-called Mourre-constant and $E_J(T)$ denotes the spectral projection of T on J . In this scenario A is said to be a *conjugate operator* to H at λ . If the choice $K = 0$ can be made for some J , that is

$$E_J(T)i[A, T]E_J(T) \geq C_M E_J(T), \quad (2.3)$$

the Mourre estimate is said to be *strict*.

There exist several versions of this estimate. We shall discuss some of them. The first and maybe easiest one replaces the projections $E_J(T)$ by a smoothed out version. More precisely, let $f \in C_0^\infty(\mathbb{R})$ be such that $f1_J = f$. Then $f(T)i[A, T]f(T) = f(T)E_J(T)i[A, T]E_J(T)f(T)$ implies

$$f(T)i[A, T]f(T) \geq C_M f(T)^2 + K',$$

where $K' = f(T)Kf(T)$ is still compact.

2.3 Local Commutators, Regularity of Eigenstates

A possible way to try and deal with the situation in which the commutators $i\text{ad}_A(T)$ are not bounded by $(S - z)^{-1}$ anymore is to localize the operator T by a local version $f(T)$, where $f \in C_0^\infty(\mathbb{R})$ and is linear in a neighborhood of the energy region of interest. The following proposition justifies this idea.

Proposition 2.6. *If T is a self adjoint operator of class $C^1(A)$, then $f(H) \in C^1(A)$ for all $f \in C_0^\infty(\mathbb{R})$.*

This leads to a natural generalization of the sets $C^k(A)$.

Definition 2.7. Let T be a self-adjoint operator on \mathcal{H} . For $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}$ open we write T is of class $C_{\text{loc}}^k(A; \Omega)$, if $f(T) \in C^k(A)$ for all $f \in C_0^\infty(\Omega)$. We define $C_{\text{loc}}^k(A) := C_{\text{loc}}^k(A; \mathbb{R})$. For an operator T which is of class $C_{\text{loc}}^k(A)$ we will also simply say that T is locally of class $C^k(A)$.

An interesting consequence of an operator T to be locally of class $C^{k+1}(A)$ is that eigenstates of T with eigenvalue λ are elements of $D^{(k)}$ provided that there is a Mourre-estimate (2.2) around λ . This result, among others, has been obtained by Møller and Westrich in 2010. Note that the original formulation requires a slightly modified version of the estimate (2.2), see Theorem 1.6 in [40]. However (2.2) implies the estimate used in their formulation. A proof of this fact can also be found in their paper.

Theorem 2.8 (Regularity of Eigenstates). *Suppose that T satisfies a Mourre estimate around λ . Let ψ be an eigenvector of T with eigenvalue λ . If $f(T) \in C^{k+1}(A)$ for all $f \in C_0^\infty(\mathbb{R})$ and some $k \in \mathbb{N}$, then $\psi \in D(A^k)$.*

2.4 Limiting Absorption Principles

Another important consequence of an operator to be of class $C^2(A)$ is the so-called limiting absorption principle. More precisely the operator $\langle A \rangle^{-s}(T - z)^{-1}\langle A \rangle^{-s}$ has a limit as $|\operatorname{Im}(z)| \rightarrow 0$ for $k \geq 2$ and $s > 1/2$. These types of limits have been studied extensively in the literature. We refer to Mourre's paper, [41] and several of its generalizations [32], [4]. A Generalization to the local $C^k(A)$ classes has been proven by Sahbani in [47]. In this paper refinements of these classes are used to obtain optimal results which makes it rather difficult to access. However, in [19] the authors discuss a version of Sahbani's result which is more fitting in our framework and we will follow their presentation.

Assumption 2.9.

1. T is a self-adjoint operator on \mathcal{H} which is of class $C_{\text{loc}}^2(A; \Omega)$
2. For all $\lambda \in \Omega$ there is an open interval J containing λ such that (2.3) holds.

Theorem 2.10 (Limiting Absorption Principle). *Suppose Assumption 2.9 holds and let $s > 1/2$, and $\psi, \psi' \in \mathcal{H}$. Then the limit*

$$\langle \psi, \langle A \rangle^{-s}(T - \lambda \pm i0)^{-1}\langle A \rangle^{-s}\psi' \rangle := \langle \psi, \lim_{\epsilon \rightarrow 0} \langle A \rangle^{-s}(T - \lambda \pm i\epsilon)^{-1}\langle A \rangle^{-s}\psi' \rangle$$

exists uniformly in λ on every compact subset of Ω .

This theorem allows us to define a bounded operator

$$\langle A \rangle^{-s}(T - \lambda \pm i0)^{-1}\langle A \rangle^{-s} := \lim_{\epsilon \rightarrow 0} \langle A \rangle^{-s}(T - \lambda \pm i\epsilon)^{-1}\langle A \rangle^{-s}$$

via the help of sesquilinear forms. Another result which is of particular interest in perturbation theory concerns Hölder-continuity w.r.t. λ of these limit operators.

Theorem 2.11 (Hölder Continuity). *Suppose Assumption 2.9 holds and let $s \in (1/2, 1)$. The map*

$$\lambda \mapsto \langle A \rangle^{-s} (T - \lambda \pm i0)^{-1} \langle A \rangle^{-s}$$

is Hölder continuous in Ω with Hölder-exponent $\alpha_s = s - 1/2$.

If we replace the singular operator T by a family of self-adjoint operators T_κ and the preceding theorems hold uniformly in κ as well, one can usually prove joint Hölder continuity in λ and κ . A nice application in the context of the quantum N -body problem can be found in [31].

3 The Feshbach-Schur Method

The method presented in this section deals with the question when an operator given by some 2×2 block decomposition is invertible, provided one of the diagonal blocks is invertible. The method presented here is also known as the Grushin problem. An easily accessible presentation can be found in [26]. A nice overview article by Zworski and Sjöstrand, [53], shows connections to other type of problems.

The method has also been successfully used in the context of spectral theory for non-relativistic QED by Bach, Fröhlich and Sigal, [5]. An application to spectral theory of Pauli-Fierz operators can be found in [11]. The last reference is of particular interest to us, since it stresses an abstract connection between Mourre theory and the Feshbach method. A version of this connection will be used to obtain a perturbation theory of embedded mass-shells below the 1-boson threshold in massive translation invariant Nelson models. The Feshbach method also plays a role in the second part of the thesis to conclude absence of essential spectrum in a region provided that a part of the block decomposition has empty spectrum in the same region.

We follow the presentation of [5]. Suppose we are given a self-adjoint operator T_0 with dense domain $D(T_0)$ and an orthogonal projection P with $\text{Ran}(P) \subset D(T_0)$ which commutes with T_0 . Put $\bar{P} = 1 - P$. Suppose that W is T_0 -bounded and that the operator $T := T_0 + W$ is self-adjoint with domain $D(T) = D(T_0)$.

Further assume that $\bar{P}T\bar{P} - z\bar{P}$ is invertible on $\bar{P}\mathcal{H}$ and that the operators

$$\begin{aligned} &(\bar{P}T\bar{P} - z\bar{P})^{-1}\bar{P}, \quad PWP(\bar{P}T\bar{P} - z\bar{P})^{-1}\bar{P}, \quad (\bar{P}T\bar{P} - z\bar{P})^{-1}\bar{P}WP \\ &PWP(\bar{P}T\bar{P} - z\bar{P})^{-1}\bar{P}WP, \quad PWP \end{aligned}$$

all extend to bounded operators on \mathcal{H} . Then we can define the operator

$$F_P(T - z) := (T_0 - z)P + PWP + PWP(\bar{P}T\bar{P} - z\bar{P})^{-1}\bar{P}WP. \quad (3.4)$$

By the assumptions made on T_0 and W $F_P(T - z)$ is a closed operator on $\bar{P}\mathcal{H}$ with domain $D(F_P(T - z)) = D(T_0P)$. Indeed, simply note that $F_P(T - z) - T_0P$

is bounded on $\overline{P\mathcal{H}}$. The assumptions given here are too strong, see the discussion in [5], in particular Lemma II.2.

The operator $F_P(T - z)$ is usually referred to as the *Feshbach map*. Its importance lies in the fact that it is invertible, if and only if $T - z$ is invertible. More importantly, since $F_P(T - z)$ is defined on the smaller space $P\mathcal{H}$ the eigenvalue problem is potentially easier to solve. Note however, that if $F_P(T - z)\psi = 0$, (3.4) shows that the equation determining z on $\overline{P\mathcal{H}}$ becomes nonlinear in z .

The next proposition appears as Theorem II.1 in [5] and sums up the properties of the Feshbach map.

Proposition 3.1 (Feshbach map).

1. $0 \in \rho(F_P(T - z)) \Leftrightarrow 0 \in \rho(T - z)$. In this case $F_P(T - z)^{-1} = P(T - z)^{-1}P$.
2. If $T\psi = z\psi$ for some $\psi \neq 0$, then $F_P(T - z)P\psi = 0$.
3. If $F_P(T - z)P\psi' = 0$ for some $\psi' \neq 0$, then $T\psi'' = z\psi''$, where $\psi'' = (P - (\overline{P}T\overline{P} - z\overline{P}))^{-1}\overline{P}W\psi'$.
4. $\dim \ker(T - z) = \dim \ker(F_P(T - z))$.

Define $\overline{R}_z := (\overline{P}T\overline{P} - z\overline{P})^{-1}$. If $[F_P(T - z)]^{-1}$ exists, the following block decomposition for $(T - z)^{-1}$ can be proven:

$$\begin{aligned} P(T - z)^{-1}P &= [F_P(T - z)]^{-1} \\ P(T - z)^{-1}\overline{P} &= -[F_P(T - z)]^{-1}PW\overline{P}\overline{R}_z \\ \overline{P}(T - z)^{-1}P &= -\overline{R}_z\overline{P}WP[F_P(T - z)]^{-1} \\ \overline{P}(T - z)^{-1}\overline{P} &= \overline{R}_z + \overline{R}_z\overline{P}WP[F_P(T - z)]^{-1}PW\overline{P}\overline{R}_z. \end{aligned}$$

4 Perturbation Theory

4.1 Kato's Analytic Perturbation Theory

In this chapter we sum up Kato's analytic perturbation theory developed in his classic textbook, [33]. Another nice review can be found in [45]. The question addressed there can easily be motivated in the case of 2×2 matrices. Consider the matrix

$$B(\xi) := \begin{pmatrix} 2 & \xi \\ \xi & 2 \end{pmatrix}$$

for $\xi \in \mathbb{C}$. $B(0)$ clearly is a self-adjoint matrix with only eigenvalue $\lambda_0 = 2$ and two dimensional eigenspace. A natural question one can ask is how the spectrum, that is the eigenvalues, of the family of matrices $B(\xi)$ will depend on ξ . In this situation it is of course trivial to calculate that $(\lambda(\xi) - 2)^2 = \xi^2$ and thus $\lambda(\xi) - 2$ is given by one of the branches of the complex square root of ξ^2 .

It turns out that this situation is generic for all families $\{B(\xi)\}_{\xi \in U}$ of bounded operators, $U \subset \mathbb{C}$ open, depending analytically on the parameter ξ . In this case analytic dependence means that the map $\xi \mapsto B(\xi)$ is an analytic function with values in the Banach space of all bounded operators on some finite dimensional space X .

The situation for general bounded or even unbounded operators on an infinite dimensional space is more subtle to describe. First of all the spectrum of an operator need not consist of isolated points anymore and the dimension of eigenspaces might be infinite. However, if the operator has eigenvalues of finite multiplicity and the spectrum can be separated by a closed curve in a part which contains only finitely many isolated eigenvalues of finite multiplicity in its interior and a remainder in the exterior part, the restriction of the operator to the spectral subspace generated by these isolated points will reduce the situation to that of bounded operators on finite dimensional spaces. The remainder of this section is devoted to reviewing how these ideas can be made precise. Since it is more condensed, we follow the presentation in [45].

Definition 4.1. Let $U \subset \mathbb{C}$ be open and connected. Suppose that for every $\beta \in U$ there exists a closed operator $T(\beta)$. The family $\{T(\beta)\}_{\beta \in U}$ is called *analytic of type A*, if the following two conditions are satisfied.

1. $D(\beta) = D$ for some dense set $D \subset \mathcal{H}$
2. For all $\psi \in D$ the map $\beta \mapsto T(\beta)\psi$ is analytic.

Since the operators we are considering are typically self-adjoint for real values of β , we assume

$$\forall \beta \in U \cap \mathbb{R} : T(\beta)^* = T(\beta).$$

In our trivial example we have seen that a doubly degenerate eigenvalue split into two nondegenerate eigenvalues which, as functions of β , were branches of an analytic function. As claimed the next theorem states that this situation is somehow generic, even for perturbations of unbounded operators.

Theorem 4.2 (Analytic Perturbation of Discrete Eigenvalues). *Let $\{T(\beta)\}_{\beta \in U}$ be analytic of type A, self adjoint for $\beta \in \mathbb{R}$. Let $\lambda_0 \in \mathbb{R}$ be a discrete eigenvalue of $T := T(0)$ of multiplicity m . Then there exists an open neighborhood \mathcal{O} of 0 and functions $\lambda_1(\beta), \dots, \lambda_m(\beta)$ which are analytic in \mathcal{O} and satisfy $\lambda_j(0) = \lambda_0$ such that the numbers $\lambda_1(\beta), \dots, \lambda_m(\beta)$ are eigenvalues of $T(\beta)$. Moreover, these functions need not be distinct and the eigenvalues $\lambda_1(\beta), \dots, \lambda_m(\beta)$ are all eigenvalues of $T(\beta)$ for $\beta \in \mathcal{O}$.*

4.2 Embedded Eigenvalues

For a closed operator T we call λ_0 an embedded eigenvalue of T , if λ_0 is an eigenvalue and $(\lambda_0 - \epsilon, \lambda_0 + \epsilon) \cap \sigma(T) \neq \{\lambda_0\}$ for all $\epsilon > 0$. Now suppose $\{T(\beta)\}_{\beta \in U}$

is analytic of type A. If λ_0 is an embedded eigenvalue of the operator $T = T(0)$, Kato's theory is not applicable, since it requires us to draw a little circle around λ_0 which does not intersect $\sigma(T)$. However, through a combination of the Feshbach method and the limiting absorption principle an expansion of the eigenvalue up to a certain order in the perturbation parameter can be achieved.

To illustrate this idea we will assume that the β -dependence is linear. More precisely, we modify the example given in Section 3 in the following way:

$$T(\beta) = T_0 + (1 + \beta)W.$$

Let λ_0 be an embedded eigenvalue of T and denote by P the projection onto the corresponding eigenspace. Put $\bar{P} = 1 - P$. Assume that there exists an interval $J \ni \lambda_0$ such that $\bar{P}T(\beta)\bar{P}$ does not have an eigenvalue in J for $|\beta|$ small. Moreover, suppose that there exists a self-adjoint operator A conjugate to $T(\beta)$ for small β . Suppose that $\bar{P}A\bar{P}$ is strictly conjugate to $\bar{P}T(\beta)\bar{P}$, that is

$$\theta(\bar{P}T(\beta)\bar{P})[\bar{P}A\bar{P}, \bar{P}T(\beta)\bar{P}]\theta(\bar{P}T(\beta)\bar{P}) \geq C\theta(\bar{P}T(\beta)\bar{P})^2.$$

Now suppose that the limiting absorption principles as well as the results on Hölder continuity extend to this scenario. We assume that there exists a constant $C > 0$ and $\alpha \in (0, 1)$ independent of β, λ such that the limit

$$\langle A \rangle^{-s} (\bar{P}T(\beta)\bar{P} - \lambda \pm i0)^{-1} \langle A \rangle^{-s} := \lim_{\epsilon \rightarrow 0} \langle A \rangle^{-s} (\bar{P}T(\beta)\bar{P} - (\lambda \pm i\epsilon)\bar{P})^{-1} \langle A \rangle^{-s}$$

exists for $\lambda \in J$,

$$\|\langle A \rangle^{-s} (\bar{P}T(\beta)\bar{P} - \lambda \pm i0)^{-1} \langle A \rangle^{-s}\| \leq C$$

and the map $(\lambda, \beta) \mapsto \langle A \rangle^{-s} (\bar{P}T(\beta)\bar{P} - \lambda \pm i0)^{-1} \langle A \rangle^{-s}$ is jointly Hölder continuous in λ, β with Hölder exponent α uniformly in a neighborhood of $(\lambda_0, 0)$.

We want to show that, if $\lambda(\beta)$ is an eigenvalue of $T(\beta)$ for small β we obtain an expansion of $\lambda(\beta)$ up to terms of order $|\beta|^2(|\beta|^\alpha + |\lambda(\beta) - \lambda_0|^\alpha)$. First of all we obtain for $\epsilon > 0$ that the Feshbach map $F_P(T(\beta) - \lambda(\beta) - i\epsilon)$ exists. If $U_1 := PW\bar{P}\langle A \rangle^s$ and $U_2 := \langle A \rangle^s \bar{P}WP$ extend to bounded operators for $s > 1/2$, we obtain

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} F_P(T(\beta) - \lambda(\beta) - i\epsilon) \\ &= (\lambda_0 - \lambda(\beta))P + PWP \\ &+ (1 + \beta)^2 U_1 \langle A \rangle^{-s} (\bar{P}T(\beta)\bar{P} - \lambda \pm i0)^{-1} \langle A \rangle^{-s} U_2. \end{aligned}$$

If on top of that the limit $F_P(T(\beta)) := \lim_{\epsilon \rightarrow 0} F_P(T(\beta) - \lambda(\beta) - i\epsilon)$ has a nontrivial kernel, the equation $F_P(T(\beta))\psi = 0$ for nonzero ψ and joint Hölder continuity of the boundary value of the resolvent then yield the desired expansion for $\lambda(\beta)$.

5 Spectral Deformation

We present a general theory of spectral deformation given by Hunziker in [29]. Let H be a self-adjoint operator on a Hilbert space \mathcal{H} and $U(\theta)$, $\theta \in \mathbb{R}$, be a unitary group such that the operator

$$H_\theta = U(\theta)HU(\theta)^{-1}$$

extends to an analytic family in an open strip of width $R > 0$ around the real axis. The extension of the unitary operators $U(\theta)$ itself can be defined via functional calculus. First note that by Stone's theorem there exists a self adjoint operator A which generates the unitary group. Hence

$$\langle \psi, U(\theta)\psi \rangle = \int_{\mathbb{R}} e^{-is\theta} d\mu_\psi^A(s).$$

This expression now extends to complex θ whenever $d\mu_\psi^A$ has compact support. Such vectors form a core of $U(\theta)$ for complex θ . Suppose H has an embedded eigenvalue λ of multiplicity $m < \infty$ and denote by P the corresponding eigenprojection. If $\lambda \notin \sigma(H_\theta)$ for some θ with $\text{Im}(\theta) \neq 0$, then λ is a discrete eigenvalue of H_θ by definition.

The eigenprojection $P(\theta)$ onto the corresponding eigenspace of H_θ extends to an analytic family of operators on the whole strip of width $R > 0$. Thus, the range of this family has constant dimension:

$$\dim[\text{Ran}(P(\theta))] = m.$$

The range of P is contained in $D(U(\theta)^{-1})$ and the relation

$$P(\theta) = U(\theta)PU(\theta)^{-1} \tag{5.5}$$

holds on $D(U(\theta)^{-1})$. These relations are of particular use, if H is replaced by an analytic family of operators $H(\xi)$. Then, if the operator $H_\theta(\xi) = U(\theta)H(\xi)U(\theta)^{-1}$ is an analytic family for every fixed θ . Then Kato's analytic perturbation theory can be applied to the isolated eigenvalue λ of $H_\theta(\xi)$. The equation (5.5) then establishes a link to the original eigenspace and real eigenvalues of $H_\theta(\xi)$ are also eigenvalues of $H(\xi)$.

6 Direct Integrals

In this section we present the basic theory of direct integrals. Our presentation follows [45, 46]. In particular, we assume that all fiber Hilbert spaces are given by a fixed Hilbert space \mathcal{H}' . First we discuss some notions of strong and weak measurability.

Definition 6.1 (Vector Valued Measurable Functions). Let $(E, \|\cdot\|)$ be a Banach space, (M, \mathfrak{A}, μ) a measure space and $f : M \rightarrow E$ a function.

1. Denote by $S(M)$ the set of functions $g : M \rightarrow E$ taking only finitely many values $\{e_1, \dots, e_n\} \subset E$ and $g^{-1}(\{e_j\}) \in \mathfrak{A}$. f is called *strongly measurable*, if

$$\exists \{g_n\}_{n \in \mathbb{N}} \subset S(M) : \|f(x) - g_n(x)\| \rightarrow 0 \quad \mu\text{-a.e.}$$

2. f is called *Borel measurable*, if $f^{-1}(\mathcal{O}) \in \mathfrak{A}$ for every open set $\mathcal{O} \subset E$.
3. f is called *weakly measurable*, if $x \mapsto \rho(f(x))$ is a measurable (complex valued) function for all $\rho \in E^*$.

We define the set

$$\mathcal{M}_w(M, E) := \{g : M \rightarrow E \mid g \text{ weakly measurable}\}.$$

It can be shown that in this situation strong measurability \Rightarrow Borel measurability \Rightarrow weak measurability. If E is a separable Hilbert space, the notions are equivalent.

Proposition 6.2. Let \mathcal{H}' be a separable Hilbert space, (M, \mathfrak{A}, μ) a measure space and $f : M \rightarrow \mathcal{H}'$ a function. Then the following statements are equivalent:

1. f is strongly measurable.
2. f is Borel measurable.
3. f is weakly measurable.

In this case we simply call f measurable, since no confusion can arise.

We can now define the Hilbert space of square integrable \mathcal{H}' -valued functions on M .

Definition 6.3. Let \mathcal{H}' be a separable Hilbert space and (M, \mathfrak{A}, μ) a measure space. Define the set $L^2(M, d\mu; \mathcal{H}')$ of all measurable functions $f : M \rightarrow \mathcal{H}'$ that satisfy

$$\int_M \|f(x)\|_{\mathcal{H}'}^2 d\mu(x) < \infty.$$

Equipped with the scalar product

$$\langle f, g \rangle_{L^2(M, d\mu; \mathcal{H}')} := \int_M \langle f(x), g(x) \rangle_{\mathcal{H}'} d\mu(x)$$

this set is a Hilbert space that we denote by \mathcal{H} .

Note that in the case $\mu = \delta_{m_1} + \dots + \delta_{m_k}$ for Dirac measures δ_{m_j} concentrated on points $m_j \in M$ we obtain that $L^2(M, d\mu; \mathcal{H}')$ is isomorphic to the k -fold direct sum of \mathcal{H}' . Therefore, $L^2(M, d\mu; \mathcal{H}')$ has the natural interpretation as a kind of continuous direct sum with identical summands. In order to stress this interpretation we put

$$\mathcal{H} := \int_M^\oplus \mathcal{H}' d\mu := L^2(M, d\mu; \mathcal{H}')$$

and call this new Hilbert space a *direct integral with constant fibers*. We will need to study operators on such fiber integrals. We start with the bounded case to fix some notation.

$$L^\infty(M, d\mu; \mathcal{B}(\mathcal{H}')) := \{A \in \mathcal{M}_w(M, \mathcal{B}(\mathcal{H}')) \mid \|A\|_\infty < \infty\},$$

where

$$\|A\|_\infty := \text{ess sup } \|A(x)\|_{\mathcal{B}(\mathcal{H}')}$$

The set of bounded operators A on \mathcal{H} which can in some way be decomposed w.r.t. the fiber decomposition is of special interest. We define

Definition 6.4 (Decomposable Operators). Let A be a bounded operator on \mathcal{H} . A is called *decomposable* w.r.t. the direct integral $\mathcal{H} = \int_M^\oplus \mathcal{H}' d\mu$, if there exists a function $A(\cdot) \in L^\infty(M, d\mu; \mathcal{B}(\mathcal{H}'))$ such that

$$\forall \psi \in \mathcal{H} : (A\psi)(x) = A(x)\psi(x) \quad \mu\text{-a.e.}$$

In this case we write

$$A = \int_M^\oplus A(x) d\mu(x)$$

and call $A(x)$ the *fibers* of A .

It turns out that every element $A(\cdot) \in L^\infty(M, d\mu; \mathcal{B}(\mathcal{H}'))$ uniquely defines a bounded operator on the direct integral \mathcal{H} . We define unbounded operators on direct integrals next.

Definition 6.5. A function $A(\cdot)$ from M to the set of self adjoint operators on \mathcal{H}' (not necessarily bounded) is called measurable, if and only if the function $(A(\cdot) + i)^{-1}$ is measurable. In this situation we define an operator A on $\mathcal{H} = \int_M^\oplus \mathcal{H}' d\mu$ by putting

$$D(A) = \left\{ \psi \in \mathcal{H} \mid \psi(x) \in D(A) \text{ } \mu\text{-a.e.}, \int_M \|A(x)\psi(x)\|_{\mathcal{H}'}^2 d\mu(x) < \infty \right\}$$

and

$$(A\psi)(x) = A(x)\psi(x).$$

We write

$$A = \int_M^{\oplus} A(m) d\mu(x).$$

and call $A(m)$ the *fibers* of A .

It turns out that such an operator A is self-adjoint, if all of its fibers $A(x)$ are self-adjoint.

7 Fock Space

7.1 Construction of Fock space

Fock space is one of the central mathematical objects in dealing with systems that might admit an infinite number of particles. The idea is roughly that a quantum system of N particles is described by the N -fold tensor product of the single particle Hilbert space. By taking a direct sum of all such N particle spaces, we obtain a set which describes a system with arbitrary numbers of particles. We start by giving a more precise abstract definition of this notion.

Let \mathfrak{h} be a Hilbert space. Define

$$\otimes^n \mathfrak{h} := \mathfrak{h} \otimes \mathfrak{h} \otimes \cdots \otimes \mathfrak{h}.$$

and the set

$$\mathcal{F}(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} \otimes^n \mathfrak{h},$$

where $\otimes^0 \mathfrak{h} := \mathbb{C}$. It is called *Fock space* over \mathfrak{h} . Thus an element ψ of $\mathcal{F}(\mathfrak{h})$ is a sequence $\psi = (\psi_0, \psi_1, \psi_2, \dots)$, where $\psi_n \in \otimes^n \mathfrak{h}$. Define the scalar product

$$\langle \psi, \psi' \rangle_{\mathcal{F}(\mathfrak{h})} := \psi_0^* \psi'_0 + \sum_{n=1}^{\infty} \langle \psi_n, \psi'_n \rangle_{\otimes^n \mathfrak{h}}.$$

Under the additional requirement

$$\|\psi\| = \langle \psi, \psi \rangle_{\mathcal{F}(\mathfrak{h})} < \infty$$

$\mathcal{F}(\mathfrak{h})$ equipped with $\langle \cdot, \cdot \rangle_{\mathcal{F}(\mathfrak{h})}$ becomes a Hilbert space. It is separable, if \mathfrak{h} is. The vector

$$\Omega = (1, 0, 0, \dots)$$

is called the *vacuum vector*. From a physical point of view it is natural to single out the subspaces which are invariant under permutations and the ones which produce a sign after such an operation. They are called bosonic and fermionic Fock space respectively. Since we will only need the bosonic Fock space, we focus on its construction.

On the subset of simple tensors of $\otimes^n \mathfrak{h}$ we define the symmetrization operator by

$$\mathcal{S}_n \psi_1 \otimes \cdots \otimes \psi_n := \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \psi_{\pi(1)} \otimes \cdots \otimes \psi_{\pi(n)},$$

where \mathfrak{S}_n denotes the group of permutations of n elements. It extends from to a self adjoint bounded operator on $\otimes^n \mathfrak{h}$ which we denote by \mathcal{S}_n as well. The pre-factor $n!^{-1}$ guarantees that \mathcal{S}_n is an orthogonal projection:

$$\mathcal{S}_n^* = \mathcal{S}_n = \mathcal{S}_n^2.$$

By putting

$$\mathcal{S} := \sum_{n=0}^{\infty} \mathcal{S}_n$$

we obtain a symmetrization operator on the whole Fock space. We then define the *symmetric* or *bosonic Fock space* over \mathfrak{h} by

$$\mathcal{F}_s(\mathfrak{h}) = \mathcal{S}\mathcal{F}(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} \mathcal{S}_n \otimes^n \mathfrak{h}.$$

7.2 Operators on Fock Space

In this section we define annihilation and creation operators, the Segal fields and the second quantization. Once more we follow the presentation of [45, 46]. Let T be a self adjoint operator on \mathfrak{h} with domain $D(T)$. Define the operator

$$T^{(n)} := T \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \mathbb{1} \otimes T \otimes \cdots \otimes \mathbb{1} + \cdots + \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes T$$

on $\otimes_{k=1}^n D(T)$. Moreover, we put $A^{(0)} = 0$. Define the set of *finite particle vectors* by

$$\mathcal{F}_0 := \{\psi \in \mathcal{F}(\mathfrak{h}) \mid \exists n \in \mathbb{N} \forall m \geq n : \psi_m = 0\}$$

and let

$$D_T := \{\psi \in \mathcal{F}_0 \mid \forall n \in \mathbb{N} : \psi_n \in \otimes_{k=1}^n D(T)\}.$$

Then D_T is dense in $\mathcal{F}(\mathfrak{h})$ and the symmetric operator

$$d\Gamma(T) = \sum_{k=0}^{\infty} T^{(k)}$$

is essentially self-adjoint on D_T . It is called the *second quantization* of T . Moreover, $d\Gamma(T)|_{\mathcal{F}_s(\mathfrak{h})}$ is essentially self adjoint on $\mathcal{F}_s(\mathfrak{h}) \cap D_T$. An important choice is $T = \mathbb{1}$, $D(T) = \mathfrak{h}$. Its second quantization

$$N := d\Gamma(\mathbb{1})$$

is called the *number operator*. Note that N acts on $\otimes^n \mathfrak{h}$ by multiplication with n which explains the name. For $g \in \mathfrak{h}$ define a map $b^-(g) : \otimes^n \mathfrak{h} \rightarrow \otimes^{n-1} \mathfrak{h}$ by

$$b^-(g)\psi_1 \otimes \cdots \otimes \psi_n = \langle g, \psi_1 \rangle \psi_2 \otimes \cdots \otimes \psi_n.$$

$b^-(g)$ can be extended to a bounded map $\otimes^n \mathfrak{h} \rightarrow \otimes^{n-1} \mathfrak{h}$ by linearity and the estimate $\|b^-(g)\psi'\| \leq \|g\|\|\psi'\|$. Furthermore, define $b^-(g)\psi_0 = 0$ for all $\psi_0 \in \otimes^0 \mathfrak{h} := \mathbb{C}$. Then $b^-(g)$ extends to a bounded map $\mathcal{F}_s(\mathfrak{h}) \rightarrow \mathcal{F}_s(\mathfrak{h})$. We then define the *annihilation operator* on $\mathcal{F}_{0,s} := \mathcal{F}_0 \cap \mathcal{F}_s(\mathfrak{h})$ by

$$a(g) = (N + 1)^{\frac{1}{2}} b^-(g).$$

Now define an operator $b^+(g) : \otimes^n \mathfrak{h} \rightarrow \otimes^{n+1} \mathfrak{h}$ by

$$b^+(g)\psi_1 \otimes \cdots \otimes \psi_n = g \otimes \psi_1 \otimes \cdots \otimes \psi_n.$$

The *creation operator* is then defined on $\mathcal{F}_{0,s}$ by

$$a^*(g) = \mathcal{S}b^+(g)(N + 1)^{\frac{1}{2}}.$$

Both, $a(g)$ and $a^*(g)$ are closable and their closures are denoted by the same symbols respectively. The operators satisfy the *canonic commutation relations*

$$[a(g), a(g')] = [a^*(g), a^*(g')] = 0, \quad [a(g), a^*(g')] = \langle g, g' \rangle \mathbb{1}.$$

It follows that $a(g)$ and $a^*(g)$ have the same domain and we can thus define the Segal field operator by

$$\Phi(g) := \frac{1}{\sqrt{2}}(a(g) + a^*(g)).$$

The second *quantization functor* Γ is defined by

$$\Gamma(\mathfrak{h}) = \mathcal{F}, \quad \Gamma(T) = \bigoplus_{n=1}^{\infty} \otimes^n T,$$

where $T \in \mathcal{B}(\mathfrak{h})$ and $\otimes^n T$ is defined on simple tensors by

$$[\otimes^n T](\psi_1 \otimes \cdots \otimes \psi_n) = T\psi_1 \otimes \cdots \otimes T\psi_n$$

and extends to a bounded operator on $\otimes^n \mathfrak{h}$. Note that

$$\Gamma(e^{iT}) = e^{i\mathrm{d}\Gamma(T)}.$$

CHAPTER 2: Second Order Perturbation Theory in Massive Nelson Models

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1 Introduction

The model discussed here describes the interaction of a particle with a scalar field of massive bosons. Due to E. Nelson's paper [42] it is often referred to as the massive Nelson model. Its structure has also been discussed by Cannon in [7] some years after. However, Nelson himself refers to several other authors in his paper. It should be noted that for a certain (constant) choice of the boson dispersion relation we obtain H. Fröhlich's model for the polaron, [16]. Fröhlich's paper predates Nelson's and the Fröhlich Hamiltonian can be derived independently from first principles.

The term polaron refers to an electron moving through a crystal which leads to local deformation of the crystal structure due to the opposite charge of atom cores and the electron. Hence the movement of the electron will cause certain vibrations throughout the crystal which in their quantized version are bosons also called phonons. The moving electron can then be pictured as being surrounded by a cloud of phonons which will affect its mobility, thus leading to an effective electron mass. In some sense it might be helpful to imagine the general situation as some kind of analogue, the electron somehow interacting with an abstract bosonic field while being surrounded by a boson cloud. A nice review of the polaron's scientific history is given in [37]. With this picture in mind we will from now on refer to the abstract particle in the Nelson model simply as an electron.

Regarding the treatment of the Nelson model we should also mention the work of J. Fröhlich in [17, 18] and Dereziński and Gérard in [10], since these authors obtained results independent of the coupling strength, that is no smallness of the coupling function is required. In the present paper we also follow this general direction in that the coupling function is first fixed to be an arbitrary compactly supported and smooth function for computational aspects. The obtained results are then extended to a larger class.

Since we have mentioned the polaron, we should comment on the difference to what is usually referred to as the Nelson model. Generally, it should be noted that the polaron case has to be treated differently from a technical point of view, since its constant dispersion relation causes difficulties in spectral theory. More precisely, a boson dispersion relation like $\sqrt{m^2 + |k|^2}$, $m > 0$, which grows as $|k| \rightarrow \infty$ makes the derivation of central objects in spectral theory such as a Mourre estimate possible. Therefore, results by Møller and Rasmussen on the spectral theory of the model first avoided the constant dispersion relation case, see [36, 44]. In their follow up paper [39] the authors greatly improve their previous results and were able to avoid problems coming from a bounded boson dispersion relation by forcing the electron dispersion relation to be unbounded with bounded second derivatives instead. This assumption is somewhat natural, since the free electron dispersion relation is quadratic.

In this thesis we will fix such a quadratic dispersion relation for the electron, first

and foremost due to computational simplicity. The methods presented here are valid for both, the unbounded and the bounded case of boson dispersion relation. A technical complication when dealing with the massive Nelson model is that the Hamiltonian H describing it is translation invariant, that is it commutes with the operator of total momentum. This implies that the Hamiltonian is in fact a fibered operator, where each fiber operator acts on Fock space, see Section 2.1 for details. This diagonalization goes back to Lee, Low and Pines in [34]. A rigorous formulation in the context of Fock space can be found in [39] and will be explained in a later part of the thesis. Even though the specific quadratic choice enables us to carry out very explicit calculations and is therefore somewhat simple, it still causes some technical problems in the context of Mourre theory. More precisely, the commutator forms with the conjugate operator and the fiber Hamiltonians cannot be implemented by an operator which is bounded by the fiber Hamiltonians $H(\xi)$ again. Therefore, methods developed in the context of Schrödinger operators are typically not applicable.

In the translation invariant case the structure of the spectrum of the fiber Hamiltonians $H(\xi)$, where $\xi \in \mathbb{R}^\nu$ denotes the total momentum, has been studied by many authors and we merely focus on the results used in this paper. A good overview is given in the introductions of [36, 39]. In the already mentioned papers [17, 18] by J. Fröhlich and and a paper by Spohn, [54], a proof of the HVZ Theorem appears. In [36] an HVZ theorem for the essential spectrum is proven by Møller in the case of unbounded boson dispersion relation. Moreover, as in Spohn's paper, [54], the existence of a non-degenerate ground state for all ξ in dimensions $\nu = 1, 2$ is shown. Dimensions $\nu = 3, 4$ can also be treated, but there the existence of a ground state for all ξ cannot be established. However, if it should exist it is non-degenerate as well.

There seem to be no results on the existence of eigenvalues between the ground state of $H(\xi)$ and the beginning of its essential spectrum. Since the fiber operators form an analytic family of type A such eigenvalues have to be analytic functions of ξ . These functions are usually referred to as isolated *mass shells*. It should be noted that their only possible accumulation point lies at the bottom of the essential spectrum. In order to construct a conjugate operator known from Mourre theory this fact is exploited in the sense that the isolated mass shells are required to be twice continuously differentiable, see [39].

In a sense the situation is similar when moving from the bottom of the discrete spectrum into the bottom of the essential spectrum, even though it is not known whether so-called embedded mass shells exist. An embedded mass shell is a function $\xi \mapsto \lambda_\xi$, where all (ξ, λ_ξ) are taken from a subset of the energy momentum spectrum in which all λ_ξ are embedded eigenvalues of the fiber Hamiltonians $H(\xi)$. If these objects should exist however, they have to be accounted for in analogy to the isolated mass shells when moving to higher energy regimes. More precisely, one would expect that high energy versions of the Mourre estimate will require

the same regularity in the embedded case as in the isolated one. This regularity problem is the main topic of discussion in this thesis. In particular, our main result, Theorem 4.6 gives an affirmative answer, in that it provides an expansion of $\lambda_{\xi+\zeta}$ in ζ up to terms of order $|\zeta|^{2+\alpha}$, where ζ is chosen small, all eigenvalues are assumed to be non-degenerate. We are confident that this last restriction can be removed and the result thus be extended to eigenvalues of constant (but arbitrary) finite multiplicity.

In this context we should mention that the examination of the nature of the spectrum is usually carried out to establish asymptotic completeness. Indeed, the previously mentioned discussion enabled Dybalski and Møller to established asymptotic completeness in the Nelson model below the 2-boson threshold under rather general assumptions, see [13]. Note that in this situation twice differentiability suffices to deal with the embedded mass shells except possibly accumulation points which have to be avoided. To show that these accumulation points are not filling up the whole energy regime as a dense set analyticity of the mass shells would be needed. It was the initial plan behind this paper to go into this direction and prove smoothness as an intermediate result by extending the mentioned expansion of $\lambda_{\xi_0+\zeta}$. Therefore, Theorem 5.34, a key technical result, is formulated for arbitrary k instead of the case $k = 3$ needed to obtain the expansion. Unfortunately, due to technical complications this project could not be accomplished in time and was thus discarded from the PhD thesis. However, the first crucial step to push the analysis to the orders $3 + \alpha$ appeared to be almost within our reach and we will pursue this direction of research in the future.

2 Main Results

2.1 The Model

Massive translation invariant Nelson models describe an electron linearly coupled to a bosonic field. The electron's Hilbert space is

$$\mathcal{K} := L^2(\mathbb{R}_x^\nu),$$

where the index x signifies that we are working in position space. The free Hamiltonian for the electron is given by $\Omega(p) := p^2$, where $p = -i\nabla$ on \mathcal{K} . Note that the dispersion relation Ω can be chosen more general but due to computational difficulties arising from an abstract choice of function we restrict ourselves to a quadratic term. The Hilbert space of a single boson is denoted by

$$\mathfrak{h} := L^2(\mathbb{R}_k^\nu),$$

where the index k indicates that the variable for functions in \mathfrak{h} is to be understood as a momentum. Define the bosonic Fock space over \mathfrak{h} by $\mathcal{F} = \mathcal{F}_s(\mathfrak{h})$. The full

Hilbert space of the composite system consisting of the electron plus the bosonic field is defined by

$$\mathcal{H} := \mathcal{K} \otimes \mathcal{F}.$$

The dispersion relation of the bosonic particle is denoted by ω . We employ the following general assumptions on ω under which a Mourre estimate was proven by Møller and Rasmussen in [39].

Assumption 2.1 (Boson Dispersion Relation). *Let $\omega \in C^\infty(\mathbb{R}^\nu)$ be real-analytic. Furthermore, suppose that ω satisfies the following extra conditions.*

1. $\exists m > 0 : \inf_{k \in \mathbb{R}^\nu} \omega(k) = m.$
2. $\forall k_1, k_2 \in \mathbb{R}^\nu : \omega(k_1 + k_2) < \omega(k_1) + \omega(k_2).$
3. ω is rotation invariant.
4. $\forall \alpha \in \mathbb{N}^\nu : \sup_{k \in \mathbb{R}^\nu} |\partial^\alpha \omega(k)| < \infty$
5. $\exists c > 0 \forall m \in \{1, 2\} : |k|^m |\nabla^m \omega(k)| \leq c \omega(k).$
6. $\lim_{|k| \rightarrow \infty} \omega(k) = \infty$ or $\sup_k \omega(k) < \infty.$

A possible choice for such a function is $\omega(k) := \sqrt{k^2 + m^2}$ or $\omega(k) = m$, where $m > 0$. Since a concrete choice of ω will not play any role in our arguments, we keep it in this generality. Furthermore, we have to choose a coupling function that implements the interaction between the electron and the bosonic field.

Assumption 2.2 (Coupling Function). *Depending on the situation we will typically assume one of the three conditions on the coupling function $g \in L^2(\mathbb{R}^\nu)$.*

1. $g \in C_0^\infty(\mathbb{R}^\nu).$
2. $g \in L_{\text{uv}}^2(\mathbb{R}^\nu)$, where $L_{\text{uv}}^2(\mathbb{R}^\nu)$ is defined in (2.1).
3. $\exists k \in \mathbb{N}_0 : g \in H_{\text{uv}}^k(\mathbb{R}^\nu)$, where $H_{\text{uv}}^k(\mathbb{R}^\nu)$ is defined in (2.2).

We define the space of square integrable functions decaying to arbitrary order by

$$L_{\text{uv}}^2(\mathbb{R}^\nu) := \bigcap_{n=0}^{\infty} L^2(\mathbb{R}^\nu, (1 + |k|)^{2n} dx). \quad (2.1)$$

The space of coupling functions with arbitrary decay at infinity and k local weak derivatives is defined by

$$H_{\text{uv}}^k(\mathbb{R}^\nu) := L_{\text{uv}}^2(\mathbb{R}^\nu) \cap H_{\text{loc}}^k(\mathbb{R}^\nu). \quad (2.2)$$

For the first half of the paper we will restrict ourselves to coupling functions $g \in C_0^\infty(\mathbb{R}^\nu)$, since it allows for rather explicit calculations of iterated commutators,

see Section 3. It can be shown that the sets in (2.2) and (2.1) can be equipped with a locally convex topology turning them into Fréchet spaces, see Section 5.1. It turns out that $g \in C_0^\infty(\mathbb{R}^\nu)$ is a dense subset in this topology and that commutator forms, where general coupling functions are used, can be approximated by their smooth and compactly supported counterparts.

We define the free and the interacting Hamiltonian on \mathcal{H} by

$$\begin{aligned} H_0 &:= \mathbb{1}_{\mathcal{K}} \otimes d\Gamma(\omega) + p^2 \otimes \mathbb{1}_{\mathcal{F}}, \\ H &:= H_0 + V_g, \end{aligned}$$

where the operator V_g is given by

$$V_g(x) := \frac{1}{2} \int_{\mathbb{R}^\nu} \left(e^{-ik \cdot x} g(k) \mathbb{1}_{\mathcal{K}} \otimes a^*(k) + e^{ik \cdot x} \overline{g(k)} \mathbb{1}_{\mathcal{K}} \otimes a(k) \right) d^\nu k.$$

The operator of total momentum

$$P := -i\nabla \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{K}} \otimes d\Gamma(k)$$

commutes with H and H_0 . Thus, both of these operators are translation invariant and can be represented as a direct integral. More precisely, there exists a unitary transformation $I_{\text{LLP}} : \mathcal{H} \rightarrow L^2(\mathbb{R}^\nu, \mathcal{F})$ such that

$$\begin{aligned} I_{\text{LLP}} H_0 I_{\text{LLP}}^* &= \int_{\mathbb{R}^\nu}^{\oplus} H_0(\xi) d^\nu \xi, \\ I_{\text{LLP}} H I_{\text{LLP}}^* &= \int_{\mathbb{R}^\nu}^{\oplus} H(\xi) d^\nu \xi, \end{aligned}$$

where $H_0(\xi)$ and $H(\xi)$ are operators acting on \mathcal{F} given explicitly by

$$\begin{aligned} H_0(\xi) &= d\Gamma(\omega) + (k - d\Gamma(k))^2, \\ H(\xi) &= H_0(\xi) + \Phi(g), \\ \Phi(g) &= \frac{1}{2}(a^*(g) + a(g)). \end{aligned}$$

The transformation I_{LLP} was first used by Lee, Low and Pines in [34]. It can be written as

$$I_{\text{LLP}} := (\mathbb{F} \otimes \mathbb{1}_{\mathcal{F}}) \circ \Gamma_{\mathcal{K} \otimes \mathcal{F}}(e^{-ik \cdot x}),$$

where $\Gamma_{\mathcal{K} \otimes \mathcal{F}}$ is an analogue of the second quantization functor Γ . It has been defined by Møller and Rasmussen in [39]. More precisely, for a bounded operator $D \in \mathcal{B}(\mathcal{K} \otimes \mathcal{F})$ with $\|D\| \leq 1$ it is given by

$$\Gamma_{\mathcal{K} \otimes \mathcal{F}}(D) := (1 \otimes \mathcal{S})G(D)(1 \otimes \mathcal{S}),$$

where \mathcal{S} is the symmetrization operator on Fock space and $G(D) = \bigoplus_{n=0}^{\infty} G^{(n)}(D)$ acting on $\mathcal{K} \otimes \bigoplus_{n=0}^{\infty} \otimes^n \mathfrak{h}$, $G^{(0)}(D) = D$ and $G^{(n)}(D) = D_1 \cdots D_n$. The D_j are bounded operators on $\mathcal{K} \otimes \otimes^n \mathfrak{h}$ defined by

$$D_j := [\mathcal{E}_j^{(n)}]^* D \otimes 1_{\otimes^{n-1} \mathfrak{h}} \mathcal{E}_j^{(n)}.$$

Finally, $\mathcal{E}_j^{(n)}$ is defined via its action on simple tensors by

$$\mathcal{E}_j^{(n)}(f \otimes \psi_1 \otimes \cdots \otimes \psi_j \otimes \cdots \otimes \psi_n) = f \otimes \psi_j \otimes \psi_1 \otimes \cdots \otimes \psi_{j-1} \otimes \psi_{j+1} \otimes \cdots \otimes \psi_n,$$

where $f \in \mathcal{K}$ and $\psi_1, \dots, \psi_n \in \mathfrak{h}$. There are several cores one can use for the fiber operators. For our purposes the set

$$\mathcal{C}_0^\infty := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} C_{0,\text{sym}}^\infty(\mathbb{R}^{n\nu}),$$

where $C_{0,\text{sym}}^\infty(\mathbb{R}^m)$ denotes the smooth functions of compact support which are symmetric under permutation of coordinates.

We quickly sum up some results on Mourre-theory needed to state the main theorem of the paper in the next section. Under (a generalizations of) our assumptions a Mourre estimate has been established by Møller and Rasmussen in [39].

Define the operator

$$a_\xi := \frac{1}{2} (v_\xi \cdot i\nabla_k i\nabla_k v_\xi)$$

which is essentially self-adjoint on $C_0^\infty(\mathbb{R}^\nu)$ for every smooth and compactly supported vector field $v_\xi \in C_0^\infty(\mathbb{R}^\nu, \mathbb{R}^\nu)$. A proof of this statement can be found in [44]. We now define the *conjugate operator* A_ξ at fiber ξ by

$$A_\xi := d\Gamma(a_\xi).$$

Proposition 2.3 (Regularity w.r.t. A). *Let $\xi \in \mathbb{R}^\nu$. There exists an open neighborhood $\mathcal{O}(\xi)$ of ξ such that for all $\xi' \in \mathcal{O}(\xi)$ the operators $H(\xi')$ are of class $C^2(A_\xi)$.*

It requires several definitions to properly state the set in which the Mourre estimate established by Møller and Rasmussen in [39] holds. They have thus been moved and into Section 2.3 in order not to obscure the presentation.

Theorem 2.4 (Mourre Estimate). *Let $(\xi, \lambda) \in \mathcal{E}^{(1)} \setminus \mathcal{T}^{(1)}$, where the sets $\mathcal{E}^{(1)}$ and $\mathcal{T}^{(1)}$ are defined in (2.4) and (2.5). Denote by $\mathcal{O}(\xi)$ the neighborhood from Proposition 2.3. Define $I_{\lambda,\kappa} := (\lambda - \kappa, \lambda + \kappa)$ for $\kappa > 0$. There exists $\kappa > 0, c > 0$, and a compact self-adjoint operator K such that $I_{\lambda,\kappa} \subset \mathcal{E}^{(1)}(\xi') \setminus \mathcal{T}^{(1)}(\xi')$ and*

$$\mathbf{1}_{I_{\lambda,\kappa}}(H(\xi')) i[H(\xi'), A] \mathbf{1}_{I_{\lambda,\kappa}}(H(\xi')) \geq c \mathbf{1}_{I_{\lambda,\kappa}}(H(\xi')) + K, \quad (2.3)$$

where $\xi' \in \mathcal{O}(\xi)$.

2.2 Main Results

Assumption 2.5.

1. Suppose there exists $U \subset \mathbb{R}^\nu$ such that the operators $H(\xi)$ have a family of embedded eigenvalues $\lambda_\xi \in \sigma(H(\xi)) \cap \mathcal{E}^{(1)}(\xi) \setminus \mathcal{T}^{(1)}(\xi)$ of multiplicity one.
2. Further assume that this family is isolated. More precisely, suppose that there exists $d > 0$ such that for all $\xi \in U$ we have that $\text{dist}(\lambda_\xi, \sigma_p(H(\xi)) \setminus \{\lambda_\xi\}) \geq d$, where $\sigma_p(H(\xi))$ denotes the set of eigenvalues of $H(\xi)$.

Notation 2.6 (Conjugate Operator for a Fixed Fiber).

1. Throughout the rest of the paper we fix ξ_0 in U . Moreover, we fix the neighborhood $\mathcal{O}_0 := \mathcal{O}(\xi_0)$ given by Proposition 2.3.
2. From now on we put $A := A_{\xi_0}$.

Notation 2.7 (Eigenstate and Eigenprojection).

1. Denote by η the normalized eigenstate of $H(\xi_0)$, that is $H(\xi_0)\eta = \lambda_{\xi_0}\eta$ and $\|\eta\| = 1$.
2. Put $P := |\eta\rangle\langle\eta|$ and $\bar{P} = 1 - P$.

We first state a result on finite local regularity w.r.t. the conjugate operator A .

Theorem 2.8 (Finite Local Regularity). *Suppose that the coupling function g satisfies $g \in H_{\text{uv}}^k(\mathbb{R}^\nu)$. Then*

$$f(H(\xi)) \in C^k(A)$$

for all $f \in C_0^\infty(\mathbb{R})$.

For the proof, see Theorem 4.6 which asserts the statement in the compactly supported and smooth case. The full statement reappears as Theorem 5.34 and is basically proven by generalizing the proof of Theorem 4.6 to the situation of general coupling functions in $H_{\text{uv}}^k(\mathbb{R}^\nu)$.

Obviously, the theorem implies the following Corollary which is interesting in its own right.

Corollary 2.9. *Let $f \in C_0^\infty(\mathbb{R})$ and $g \in C_0^\infty(\mathbb{R}^\nu)$, where g is the coupling function. Then*

$$\forall k \in \mathbb{N} : f(H(\xi)) \in C^k(A).$$

In Appendix E it is shown that the projected fiber Hamiltonians $\overline{P}H(\xi)\overline{P}$ have the same property. It should further be noted that the previous statement has a consequence on the structure of the eigenstate η due to regularity of eigenstates, see [40], provided that there is a Mourre estimate in a neighborhood of the corresponding eigenvalue. More precisely, the mentioned result then implies the validity of the implication

$$g \in H_{\text{uv}}^{k+1}(\mathbb{R}^\nu) \Rightarrow \eta \in D(A^k).$$

Let $\xi_0, \xi_0 + \zeta \in U$. With this implication we can prove the following Proposition for the projected Hamiltonian $\overline{P}H(\xi + \zeta)\overline{P}$.

Proposition 2.10. *There exists an open neighborhood \mathcal{V} of ξ_0 and $r > 0$ such that for $\xi \in \mathcal{V}$ and $|\zeta| < r$ $\overline{P}H(\xi + \zeta)\overline{P}$ is of class $C^2(\overline{P}A\overline{P})$ and satisfies a strict Mourre estimate (that is $K = 0$) of the type (2.3).*

For $\epsilon > 0$ we define

$$\overline{R}_{x+i\epsilon}(\xi) = (\overline{P}H(\xi)\overline{P} - (x + i\epsilon)\overline{P})^{-1}$$

and put $\overline{A} := \overline{P}A\overline{P}$. Note that \overline{A} is symmetric and closed with domain $D(A)$. This notation will not lead to confusion, since the context dictates the choice of ξ in the definition of \overline{A} .

Theorem 2.11 (Limiting Absorption Principle, Hölder Continuity). *Let $s > 1/2$. Then there exists an open interval J containing λ_{ξ_0} , an open neighborhood \mathcal{V} of ξ_0 satisfying $\mathcal{V} \subset \mathcal{O}_0$ and constants $C, r > 0$ independent of ζ, ϵ such that*

$$\forall \xi \in \mathcal{V} \forall |\zeta| < r : \sup_{x \in J, |\epsilon| < 1} \|\langle \overline{A} \rangle^{-s} \overline{R}_{x+i\epsilon}(\xi + \zeta) \langle \overline{A} \rangle^{-s}\| \leq C$$

and the limit

$$\langle \overline{A} \rangle^{-s} \overline{R}_{x+i0}(\xi + \zeta) \langle \overline{A} \rangle^{-s} := \lim_{\epsilon \rightarrow 0} \langle \overline{A} \rangle^{-s} \overline{R}_{x+i\epsilon}(\xi + \zeta) \langle \overline{A} \rangle^{-s}$$

exists as a bounded operator on $\overline{P}\mathcal{H}$ for all $x \in J$, $\xi \in \mathcal{V}$ and $|\zeta| < r$. If $s \in (1/2, 1)$ we have that there exists $\alpha \in (1/2, 1)$ and a constant $C' > 0$ such that

$$\begin{aligned} & \|\langle \overline{A} \rangle^{-s} \overline{R}_{x+i0}(\xi + \zeta) \langle \overline{A} \rangle^{-s} - \langle \overline{A} \rangle^{-s} \overline{R}_{x'+i0}(\xi + \zeta') \langle \overline{A} \rangle^{-s}\| \\ & \leq C' (|x - x'|^\alpha + |\zeta - \zeta'|^\alpha) \end{aligned}$$

for all $x, x' \in J$ and $|\zeta|, |\zeta'| < 1$. Moreover, C' is independent of $\xi \in \mathcal{V}$.

The proof of this statement can be found in Section 6.4. It is then used in an substantial way to derive the next statement which is the main theorem of this paper. The proof itself relies on abstract considerations which can be found throughout the literature. A first step consists of showing the statement for $\zeta = 0$ and ξ in a neighborhood \mathcal{V}' of ξ_0 . This neighborhood derives from a slight

modification of Gérard's proof of the limiting absorption principle in [22]. A quick discussion on how Gérard's result can be modified to fit into a parameter dependent context is given in Appendix D. By choosing an open subset $\mathcal{V} \subset \mathcal{V}'$ containing ξ_0 we can achieve that $\xi' + \zeta \in \mathcal{V}' \cap \mathcal{U}(\xi)$ for $|\zeta|$ sufficiently small.

Theorem 2.12 (Second Order Perturbation Theory). *Let $\xi_0 \in U$ and $s \in (1/2, 1)$. There exists an open neighborhood \mathcal{V} of ξ_0 satisfying $\mathcal{V} \subset \mathcal{O}_0$ and $r > 0$ such that for all $\xi \in \mathcal{V}$ and all $|\zeta| < r$*

$$\lambda_{\xi+\zeta} = \lambda_\xi + \sum_{\sigma=1}^{\nu} \zeta_\sigma \beta_\sigma(\xi) + \sum_{\sigma,\sigma'=1}^{\nu} \zeta_\sigma \zeta_{\sigma'} \beta_{\sigma,\sigma'}(\xi) + O(|\zeta|^{2+\alpha}).$$

Where we have defined

$$\beta_\sigma(\xi) := 2\xi_\sigma - 2\langle \eta, d\Gamma(k_\sigma)\eta \rangle, \quad U_\sigma\eta := \overline{P\langle \overline{A} \rangle^s} d\Gamma(k_\sigma)\eta$$

and

$$\begin{aligned} \beta_{\sigma,\sigma'}(\xi) := & \delta_{\sigma,\sigma'} + 4\xi_\sigma \xi_{\sigma'} \langle \overline{P\langle \overline{A} \rangle^s} \eta, \langle \overline{A} \rangle^{-s} \overline{R}_{\lambda_\xi+i0}(\xi) \langle \overline{A} \rangle^{-s} \overline{P\langle \overline{A} \rangle^s} \eta \rangle \\ & + 4\xi_{\sigma'} \langle \overline{P\langle \overline{A} \rangle^s} \eta, \langle \overline{A} \rangle^{-s} \overline{R}_{\lambda_\xi+i0}(\xi) \langle \overline{A} \rangle^{-s} U_\sigma \eta \rangle \\ & + 4\xi_{\sigma'} \langle U_\sigma \eta, \langle \overline{A} \rangle^{-s} \overline{R}_{\lambda_\xi+i0}(\xi) \langle \overline{A} \rangle^{-s} \overline{P\langle \overline{A} \rangle^s} \eta \rangle \\ & - 4\langle U_\sigma \eta, \langle \overline{A} \rangle^{-s} \overline{R}_{\lambda_\xi+i0}(\xi) \langle \overline{A} \rangle^{-s} U_{\sigma'} \eta \rangle. \end{aligned}$$

Moreover, the error term can be estimated independently of ξ : There exists $C > 0$ independent of $\xi \in \mathcal{V}$ such that

$$|O(|\zeta|^{2+\alpha})| \leq C|\zeta|^{2+\alpha}.$$

The proof combines most of the results presented here and is given at the end of Section 6.5. It rests on a combination of the Feshbach-Schur method with the results on Hölder continuity of the resolvent boundary values. The above expansion finally implies the desired differentiability.

Proposition 2.13 (Twice Differentiable Parameter Dependence). *Let $\xi_0 \in U$, $s \in (1/2, 1)$ and \mathcal{V} be the neighborhood of ξ_0 in Theorem 2.12. Define a map $\Lambda : \mathcal{V} \rightarrow U$ by $\Lambda(\xi) = \lambda_\xi$. Then $\Lambda \in C^2(\mathcal{V}, U)$.*

PROOF: Denote by e_σ the usual basis vector for \mathbb{R}^ν and by $\delta_{x,y}$ the Kronecker delta. First note that due to

$$\begin{aligned} \lambda_{\xi+he_\sigma} &= \lambda_\xi + h \sum_{\sigma_1=1}^{\nu} \delta_{\sigma,\sigma_1} \beta_{\sigma_1}(\xi) + h^2 \sum_{\sigma_1,\sigma_2=1}^{\nu} \delta_{\sigma,\sigma_1} \delta_{\sigma,\sigma_2} \beta_{\sigma_1,\sigma_2}(\xi) \\ &+ O(|h|^{2+\alpha}) \\ &= \lambda_\xi + h\beta_\sigma(\xi) + h^2\beta_{\sigma,\sigma}(\xi) + O(|h|^{2+\alpha}), \end{aligned}$$

Λ is differentiable at every $\xi \in \mathbb{R}^\nu$ and $\partial_\sigma \Lambda(\xi) = \beta_\sigma(\xi)$. Since $\xi \mapsto \beta_\sigma(\xi)$ is continuous, so is $\partial_\sigma \Lambda$ and thus $\Lambda \in C^1(\mathcal{V}, U)$. In order to prove twice differentiability, we have to control

$$\frac{1}{h} (\beta_\sigma(\xi + he_{\sigma'}) - \beta_\sigma(\xi)).$$

Note that

$$\begin{aligned} \frac{\beta_\sigma(\xi)}{h} &= \frac{\lambda_{\xi+he_\sigma} - \lambda_\xi}{h^2} - \beta_{\sigma,\sigma}(\xi) + O(|h|^\alpha) \\ \frac{\beta_\sigma(\xi + he_{\sigma'})}{h} &= \frac{\lambda_{\xi+he_\sigma+he_{\sigma'}} - \lambda_{\xi+he_{\sigma'}}}{h^2} - \beta_{\sigma,\sigma}(\xi - he_{\sigma'}) + O(|h|^\alpha) \end{aligned}$$

and an easy computation which uses the expansion in Theorem 2.12 shows that

$$\begin{aligned} &\frac{1}{h} (\beta_\sigma(\xi + he_{\sigma'}) - \beta_\sigma(\xi)) \\ &= \beta_{\sigma,\sigma'}(\xi) + \beta_{\sigma',\sigma}(\xi) + \beta_{\sigma,\sigma}(\xi + he_{\sigma'}) - \beta_{\sigma,\sigma}(\xi) + O(|h|^\alpha). \end{aligned}$$

Due to Hölder-continuity of $\beta_{\sigma,\sigma}(\xi)$, we obtain

$$\partial_{\sigma'} \partial_\sigma \Lambda(\xi) = \beta_{\sigma,\sigma'}(\xi) + \beta_{\sigma',\sigma}(\xi)$$

and thus the continuity of $\partial_{\sigma'} \partial_\sigma \Lambda(\xi)$ on \mathcal{V} . \square

2.3 Remarks on the Spectral Theory of the Model

In order to state results on Mourre theory in this model, we have to introduce some notation. We follow the presentation in [39]. We define the energy momentum spectrum by

$$\Sigma = \{(\xi, E) \mid E \in \sigma(H(\xi))\}$$

and the bottom of the spectrum of the fiber Hamiltonian $H(\xi)$ by

$$\Sigma_0(\xi) := \inf \sigma(H(\xi)).$$

The energy necessary to support the interacting system at total momentum $\xi - \sum_{j=1}^n k_j$ and n non-interacting bosons with momenta k_1, \dots, k_n is given by

$$\Sigma_0^{(n)}(\xi, \underline{k}) = \Sigma_0(\xi - k_1 + \dots + k_n) + \sum_{j=1}^n \omega(k_j),$$

where $\underline{k} = (k_1, \dots, k_n) \in \mathbb{R}^{n\nu}$. The minimal energy needed for such a situation to occur is

$$\Sigma_0^{(n)}(\xi) = \inf_{\underline{k} \in \mathbb{R}^{n\nu}} \Sigma_0^{(n)}(\xi, \underline{k}).$$

These numbers are called n -boson threshold. Note however that the term carries a double meaning in the sense that it can denote the minimal energy needed to

support the interacting system and n free bosons but it can also mean a threshold in the following sense that it denotes an energy at which the system can form an interacting bound state and n bosons with zero breakup velocity. With the above notation we put

$$\mathcal{E}^{(1)}(\xi) = (\Sigma_0^{(1)}(\xi), \Sigma_0^{(2)}(\xi)) \quad \text{and} \quad \mathcal{E}^{(1)} = \{(\xi, \lambda) \in \mathbb{R}^\nu \times \mathbb{R} \mid \lambda \in \mathcal{E}^{(1)}(\xi)\}. \quad (2.4)$$

The HVZ theorem can be written as

$$\sigma_{\text{ess}}(\xi) = [\Sigma_0^{(1)}(\xi), \infty).$$

Hence the 1-boson threshold also serves as the start of the essential spectrum. It is therefore somehow natural to write

$$\Sigma_{\text{ess}}(\xi) = \Sigma_0^{(1)}(\xi).$$

The discrete or isolated part of the energy momentum spectrum is defined as

$$\Sigma_{\text{iso}} := \{(\xi, E) \in \Sigma \mid E < \Sigma_{\text{ess}}(\xi)\}.$$

Since the possibility of energy levels to cross cannot be ruled out at this generality, we define the set of level crossings by

$$\mathcal{X} := \{(\xi, E) \in \Sigma_{\text{iso}} \mid \forall n \in \mathbb{N} : \Sigma_{\text{iso}} \cap B((\xi, E); n^{-1}) \text{ is not a graph}\},$$

where $B((\xi, E); r)$ denotes the ball of radius $r > 0$ centered at $(\xi, E) \in \mathbb{R}^\nu \times \mathbb{R}$. The connected components of \mathcal{X} are connected to each other by real analytic manifolds S . The collection of such manifolds is denoted by \mathcal{S} . More precisely, the set \mathcal{S} consists of tuples (\mathcal{A}, S) , where \mathcal{A} is a ν dimensional annulus or an open ball centered at 0, and $S : \mathcal{A} \rightarrow \mathbb{R}$ is a function satisfying $\Sigma_0(\xi) \leq S(\xi) < \Sigma_{\text{ess}}(\xi)$. For $(\mathcal{A}, S) \in \mathcal{S}$ we define

$$S^{(1)}(\xi, k) = S(\xi - k) + \omega(k).$$

The threshold set at momentum $\xi \in \mathbb{R}^\nu$ is

$$\mathcal{T}^{(1)}(\xi) = \mathcal{T}_{\mathcal{S}}^{(1)}(\xi) \cup \mathcal{T}_{\parallel}^{(1)}(\xi) \cup \mathcal{T}_{\nparallel}^{(1)}(\xi).$$

The different components are defined as follows.

$$\begin{aligned} \mathcal{T}_{\mathcal{S}}^{(1)}(\xi) &:= \{E \in \mathbb{R} \mid \exists (\mathcal{A}, S) \in \mathcal{S}, k \in \mathcal{A} + \xi : E = S^{(1)}(\xi, k), \\ &\quad \nabla_k S^{(1)}(\xi, k) = 0\} \end{aligned}$$

For $\xi \in \mathbb{R}^\nu$ define $u(\xi)$ to be a unit vector such that $\xi = su(\xi)$ for some $s \in \mathbb{R}$. Finally, we put

$$\begin{aligned} \mathcal{T}_{\parallel}^{(1)}(\xi) &:= \{E \in \mathbb{R} \mid \exists r \in \mathbb{R}(\xi - ru(\xi), E - \omega(ru(\xi))) \in \mathcal{X}\}, \\ \mathcal{T}_{\nparallel}^{(1)}(\xi) &:= \{E \in \mathbb{R} \mid \exists k \in \mathbb{R}^\nu : (\xi - k, E - \omega(k)) \in \mathcal{X}, \nabla_k \omega(k) = 0\} \end{aligned}$$

and

$$\mathcal{T}^{(1)} = \mathcal{T}_S^{(1)} \cup \mathcal{T}_{\parallel}^{(1)} \cup \mathcal{T}_{\nparallel}^{(1)}, \quad (2.5)$$

where $\mathcal{T}_S^{(1)} = \{(\xi, E) \mid E \in \mathcal{T}_S^{(1)}(\xi)\}$ etc. At last we have accumulated all the definitions necessary to formulate the Mourre estimate Theorem 2.4 established by Møller and Rasmussen, see [39], Theorem 3.18.

3 Operators Acting on a Core

3.1 Bounds on a Core

In this section we have to deal with products of field operators and second quantized multiplication operators frequently. It is thus convenient to introduce some notation for it.

Notation 3.1.

$$\begin{aligned} \prod_{i=1}^n \Phi(g_i) &:= \Phi(g_1)\Phi(g_2) \cdots \Phi(g_{n-1})\Phi(g_n), \\ \prod_{i=1}^n d\Gamma(u_i) &:= d\Gamma(u_1) d\Gamma(u_2) \cdots d\Gamma(u_{n-1}) d\Gamma(u_n), \end{aligned}$$

where the $u_i \in C^\infty(\mathbb{R}^\nu, \mathbb{C})$ are arbitrary functions and $g_i \in C_0^\infty(\mathbb{R})$. Note that these operators make sense at least on \mathcal{C}_0^∞ . Moreover, the operators in the second product all commute so that there is no ambiguity. Whenever there are products of field operators the symbol $\prod_{i=1}^n \Phi(g_i)$ means that they should be arranged such that the factors in the product respect the ordering of the indexes when read from left to right.

Definition 3.2. The set of smooth functions bounded by the photon dispersion relation is defined by

$$C_\omega^\infty := \{f \in C^\infty(\mathbb{R}^\nu) \mid \exists C > 0 : f \leq C\omega\}$$

and the set of all operators acting on \mathcal{C}_0^∞ by

$$\text{Op}(\mathcal{C}_0^\infty) := \{S \text{ linear operator} \mid D(S) = \mathcal{C}_0^\infty, S\mathcal{C}_0^\infty \subset \mathcal{C}_0^\infty\}.$$

1. Define the following subsets of $\text{Op}(\mathcal{C}_0^\infty)$:

$$\begin{aligned}\mathfrak{E}_{0,0,0} &:= \mathbb{C}\{1\} \\ \mathfrak{E}_{0,0,\alpha_3} &:= \text{Span}\left\{\prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_\sigma} \mid q_\sigma \in \mathbb{N}_0, \sum_{\sigma=1}^{\nu} q_{n_3,\sigma} = n_3 \leq \alpha_3\right\} \\ &\quad + \mathfrak{E}_{0,0,0}, \\ \mathfrak{E}_{0,\alpha_2,0} &:= \text{Span}\left\{\prod_{m=1}^{j_2} d\Gamma(u_m) \mid u_i \in C_\omega^\infty, 1 \leq i \leq j_2 \leq \alpha_2\right\} \\ &\quad + \mathfrak{E}_{0,0,0}, \\ \mathfrak{E}_{\alpha_1,0,0} &:= \text{Span}\left\{\prod_{i=1}^n \Phi(g_i) \mid g_1, \dots, g_{j_1} \in C_0^\infty(\mathbb{R}^\nu), j_1 \leq \alpha_1\right\} \\ &\quad + \mathfrak{E}_{0,0,0}.\end{aligned}$$

2. Note that operators in these sets can be multiplied without any issues concerning domains, since they only act on \mathcal{C}_0^∞ . We can thus define

$$\begin{aligned}\mathfrak{E}_{\alpha_1,\alpha_2,0} &:= \mathfrak{E}_{\alpha_1,0,0}\mathfrak{E}_{0,\alpha_2,0}, & \mathfrak{E}_{\alpha_1,0,\alpha_3} &:= \mathfrak{E}_{\alpha_1,0,0}\mathfrak{E}_{0,0,\alpha_3}, \\ \mathfrak{E}_{0,\alpha_2,\alpha_3} &:= \mathfrak{E}_{0,\alpha_2,0}\mathfrak{E}_{0,0,\alpha_3}, & \mathfrak{E}_{\alpha_1,\alpha_2,\alpha_3} &:= \mathfrak{E}_{\alpha_1,0,0}\mathfrak{E}_{0,\alpha_2,0}\mathfrak{E}_{0,0,\alpha_3}.\end{aligned}$$

3. We write

$$T \sim \mathfrak{E}_{\alpha_1,\alpha_2,\alpha_3},$$

if $\mathcal{C}_0^\infty \subset D(T)$ and

$$\exists \tilde{T} \in \mathfrak{E}_{\alpha_1,\alpha_2,\alpha_3} : T|_{\mathcal{C}_0^\infty} = \tilde{T}.$$

Remark 3.3.

1. The restriction $|u_i| \leq C_i \cdot \omega$ is made so that $d\Gamma(u_i)$ is $H_0(\xi)$ -bounded.
2. We require $g_i \in C_0^\infty(\mathbb{R}^\nu)$ so that $v_j k_\sigma^\ell g_i \in L^2(\mathbb{R}^\nu)$ for all i, j, ℓ, σ that can appear.
3. All functions are chosen to be smooth, so that $T \in \mathfrak{E}_{\alpha_1,\alpha_2,\alpha_3}$ implies $T\mathcal{C}_0^\infty \subset \mathcal{C}_0^\infty$.
4. It follows immediately from the definition that $\alpha_i \leq \beta_i$ implies

$$\mathfrak{E}_{\alpha_1,\alpha_2,\alpha_3} \subseteq \mathfrak{E}_{\beta_1,\beta_2,\beta_3}$$

for $\alpha_i, \beta_i \in \mathbb{N}_0$.

The spaces $\mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ allow us to control form identities on \mathcal{C}_0^∞ such as commutators by calculating them on the core and then order them such that all fields are on the left and all second quantized multiplication operators are on the right. This in turn will lead to (form) bounds in terms of graph norms of powers of the free Hamiltonian.

Definition 3.4 (Commutators on \mathcal{C}_0^∞). Let $T, S \in \text{Op}(\mathcal{C}_0^\infty)$. Then we define an operator $\text{ad}_S(T) \in \text{Op}(\mathcal{C}_0^\infty)$ by

$$\forall \psi \in \mathcal{C}_0^\infty : \text{ad}_S(T)\psi := ST\psi - TS\psi.$$

We draw first conclusions from our definitions. Let $T \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$, $S \in \text{Op}(\mathcal{C}_0^\infty)$ and $\psi, \psi' \in \mathcal{C}_0^\infty$. According to the definitions there exists $\tilde{T} \in \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ such that $T|_{\mathcal{C}_0^\infty} = \tilde{T}$. If $\mathcal{C}_0^\infty \subset D(T^*)$, we may compute

$$\begin{aligned} \langle \psi, [S, T]\psi' \rangle &= \langle S^*\psi, \tilde{T}\psi' \rangle - \langle T^*\psi, S\psi' \rangle = \langle \psi, S\tilde{T}\psi' \rangle - \langle \psi, \tilde{T}S\psi' \rangle \\ &= \langle \psi, \text{ad}_S(\tilde{T})\psi' \rangle \end{aligned}$$

Hence we can calculate the commutator form on $[S, T]$ on $\mathcal{C}_0^\infty \times \mathcal{C}_0^\infty$ and argue that it is given by the operator $\text{ad}_S(\tilde{T})$. However, it will in general be difficult to extract more information on the analytical properties of this operator. In particular, we would like to extend it to larger domains.

To this end the subspaces $\mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3} \subset \text{Op}(\mathcal{C}_0^\infty)$ are a quite useful technical tool, since their elements may be easily bounded by powers of the number operator and hence by powers of the free Hamiltonian $H_0(\xi)$. It is in general not clear, if $\text{ad}_S(\tilde{T}) \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ again. However, if we are in a position to explicitly compute $\text{ad}_S(\tilde{T})$ on \mathcal{C}_0^∞ , we can then try to re-write it in such a fashion that it will be an element of $\mathfrak{C}_{\alpha'_1, \alpha'_2, \alpha'_3}$ for some new constants α'_i . In this situation we may view the operation of taking a commutator with S as a map between sets $\mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ and $\mathfrak{C}_{\alpha'_1, \alpha'_2, \alpha'_3}$. In the following we slightly abuse notation and define a map

$$\begin{aligned} \text{ad}_S(\cdot) &: \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3} \rightarrow \text{Op}(\mathcal{C}_0^\infty) \\ T &\mapsto \text{ad}_S(T). \end{aligned}$$

The next Proposition deals with the question whether $\text{ad}_S(T) \sim \mathfrak{C}_{\alpha'_1, \alpha'_2, \alpha'_3}$ for certain choices of S . It will play a crucial role in our analysis.

If we further assume that T is a $H_0(\xi)^{m/2}$ -bounded operator for some $m \in \mathbb{N}$, an extension of the commutator form would require us to first extend to form to the larger domain. Since the form satisfies the generic bound

$$\begin{aligned} |\langle \psi, [S, T]\psi' \rangle| &\leq C \|\psi\|_{S, m} \|\psi'\|_{S, m}, \\ \|\psi\|_{S, m} &:= \|\psi\| + \|H_0(\xi)^{\frac{m}{2}}\psi\| + \|S\psi\|, \end{aligned}$$

this could be done if we knew that \mathcal{C}_0^∞ was a $\|\cdot\|_{S,m}$ -dense subset of $D(H^{m/2}) \cap D(S)$. We are interested in commutators with the conjugate operator A and $H(\xi)$. Since the domain of the conjugate operator A contains \mathcal{C}_0^∞ and $AC_0^\infty \subset \mathcal{C}_0^\infty$, we can replace S by A in the preceding argument.

Hence the key question we have to address in order to extend commutator forms like $[A, T]$ is whether or not \mathcal{C}_0^∞ is dense in $D(H^{m/2}) \cap D(S)$ w.r.t. the intersection norms $\|\cdot\|_m := \|\cdot\|_{A,m}$ for all $m \in \mathbb{N}$.

Proposition 3.5. *Let $g \in C_0^\infty(\mathbb{R}^\nu)$ and $\alpha_i \in \mathbb{N}_0$, where $i = 1, 2, 3$, but at least one $\alpha_i \neq 0$. Then*

$$\begin{aligned} \text{ad}_{d\Gamma(k_\sigma)}(\mathfrak{E}_{\alpha_1, \alpha_2, \alpha_3}) &\subseteq \mathfrak{E}_{\alpha_1, \alpha_2, \alpha_3}, \\ \text{ad}_{H_0(\xi)}(\mathfrak{E}_{\alpha_1, \alpha_2, \alpha_3}) &\subseteq \begin{cases} \mathfrak{E}_{\alpha_1, \alpha_2, \alpha_3+1} & , \alpha_1 \neq 0 \\ \mathfrak{E}_{0,0,0} & , \alpha_1 = 0 \end{cases}, \\ \text{ad}_A(\mathfrak{E}_{\alpha_1, \alpha_2, \alpha_3}) &\subseteq \begin{cases} \mathfrak{E}_{\alpha_1, \alpha_2, \alpha_3} + \mathfrak{E}_{\alpha_1, \alpha_2+1, \alpha_3-1} & , \alpha_3 \geq 1, \alpha_2 \geq 1 \text{ or } \alpha_1 \geq 1 \\ \mathfrak{E}_{0,1, \alpha_3-1} & , \alpha_3 \geq 1, \alpha_1 = \alpha_2 = 0 \\ \mathfrak{E}_{\alpha_1, \alpha_2, 0} & , \alpha_3 = 0 \end{cases} \\ \text{ad}_{\Phi(g)}(\mathfrak{E}_{\alpha_1, \alpha_2, \alpha_3}) &\subseteq \begin{cases} \mathfrak{E}_{\alpha_1-1, \alpha_2, \alpha_3} + \mathfrak{E}_{\alpha_1+1, \alpha_2-1, \alpha_3} & , \alpha_1, \alpha_2, \geq 1, \alpha_3 \in \mathbb{N}_0 \\ \mathfrak{E}_{\alpha_1-1, 0, 0} & , \alpha_1 \geq 1, \alpha_2 = \alpha_3 = 0 \\ \mathfrak{E}_{\alpha_1-1, 0, \alpha_3} + \mathfrak{E}_{\alpha_1+1, 0, \alpha_3-1} & , \alpha_1, \alpha_3 \geq 1, \alpha_2 = 0 \\ \mathfrak{E}_{1, \alpha_2-1, \alpha_3} & , \alpha_1 = 0, \alpha_2 \geq 1, \alpha_3 \in \mathbb{N}_0 \\ \mathfrak{E}_{1, 0, \alpha_3-1} & , \alpha_1 = \alpha_2 = 0, \alpha_3 \geq 1 \end{cases} \end{aligned}$$

In particular,

$$\text{ad}_{H(\xi)}(\mathfrak{E}_{\alpha_1, \alpha_2, \alpha_3}) \subseteq \text{ad}_{H_0(\xi)}(\mathfrak{E}_{\alpha_1, \alpha_2, \alpha_3}) + \text{ad}_{\Phi(g)}(\mathfrak{E}_{\alpha_1, \alpha_2, \alpha_3}).$$

PROOF: For the different operators we want to take commutators with, we only treat the cases, where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}_0$ are as general as possible, since the proofs of the special cases are more simple. Since the elements of the sets $\mathfrak{E}_{\alpha_1, \alpha_2, \alpha_3}$ are finite linear combinations, it suffices to carry out the calculations for summands of the form

$$\prod_{i=1}^{n_1} \Phi(g_i) \prod_{m=1}^{n_2} d\Gamma(v_m) \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3, \sigma}},$$

where $n_i \leq \alpha_i$ and $q_{n_3, 1} + \dots + q_{n_3, \nu} = n_3$.

Now note that $d\Gamma(k_\sigma)$ commutes with all other second quantized multiplication

operators. Hence

$$\begin{aligned} & \text{ad}_{\text{d}\Gamma(k_\sigma)} \left(\prod_{i=1}^{n_1} \Phi(g_i) \prod_{m=1}^{n_2} \text{d}\Gamma(v_m) \prod_{\sigma=1}^{\nu} \text{d}\Gamma(k_\sigma)^{q_{n_3, \sigma}} \right) \\ &= \text{ad}_{\text{d}\Gamma(k_\sigma)} \left(\prod_{i=1}^{n_1} \Phi(g_i) \right) \prod_{m=1}^{n_2} \text{d}\Gamma(v_m) \prod_{\sigma=1}^{\nu} \text{d}\Gamma(k_\sigma)^{q_{n_3, \sigma}}. \end{aligned}$$

Lemma A.5 implies

$$\begin{aligned} & \text{ad}_{\text{d}\Gamma(k_\sigma)} \left(\prod_{i=1}^{n_1} \Phi(g_i) \right) \\ &= \sum_{\ell=1}^{n_1} \Phi(g_1) \cdots \Phi(g_{n, \ell-1}) [\text{d}\Gamma(k_\sigma), \Phi(g_{n, \ell})] \Phi(g_{n, \ell+1}) \cdots \Phi(g_{n_1}) \\ &= -i \sum_{\ell=1}^{n_1} \Phi(g_1) \cdots \Phi(g_{n, \ell-1}) \Phi(ik_\sigma g_{n, \ell}) \Phi(g_{n, \ell+1}) \cdots \Phi(g_{n_1}) \end{aligned}$$

and the first statement follows.

In the next step we analyze how taking a commutator with $H_0(\xi)$ effects elements of $\mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$. Since $H_0(\xi)$ commutes with all second quantized multiplication operators, the only interesting case is $\alpha_1 \neq 0$. In this case we obtain

$$\begin{aligned} & \text{ad}_{H_0(\xi)} \left(\prod_{\ell=1}^{n_1} \Phi(g_\ell) \prod_{m=1}^{n_2} \text{d}\Gamma(v_m) \prod_{\sigma=1}^{\nu} \text{d}\Gamma(k_\sigma)^{q_{n_3, \sigma}} \right) \\ &= \left[H_0(\xi), \prod_{\ell=1}^{n_1} \Phi(g_\ell) \right] \prod_{m=1}^{n_2} \text{d}\Gamma(v_m) \prod_{\sigma=1}^{\nu} \text{d}\Gamma(k_\sigma)^{q_{n_3, \sigma}}. \end{aligned}$$

In order to keep the equations shorter we introduce

$$\mathcal{C}_{n_1} := [H_0(\xi), \Phi(g_1) \cdots \Phi(g_{n_1})].$$

Clearly,

$$\mathcal{C}_{n_1} = \sum_{\ell=1}^{n_1} \Phi(g_1) \cdots \Phi(g_{n, \ell-1}) [H_0(\xi), \Phi(g_{n, \ell})] \Phi(g_{n, \ell+1}) \cdots \Phi(g_{n_1}).$$

Thus, we have to calculate the single commutator

$$\begin{aligned}
& [\Phi(g), H_0(\xi)] \\
&= [\Phi(g), d\Gamma(\omega)] + [\Phi(g), (\xi - d\Gamma(k))^2] \\
&= -i\Phi(i\omega g) - 2[\Phi(g), \xi \cdot d\Gamma(k)] + [\Phi(g), d\Gamma(k)^2] \\
&= -i\Phi(i\omega g) + 2i\Phi(i\xi \cdot kg) - id\Gamma(k) \cdot \Phi(kg) - i\Phi(kg) \cdot d\Gamma(k) . \\
&= -i\Phi(i\omega g) + 2i\Phi(i\xi \cdot kg) - i \sum_{\sigma=1}^{\nu} (d\Gamma(k_{\sigma})\Phi(ik_{\sigma}g) + \Phi(ik_{\sigma}g)d\Gamma(k_{\sigma})) \\
&= -i\Phi(i\omega g) + 2i\Phi(i\xi \cdot kg) - i \sum_{\sigma=1}^{\nu} (i\Phi(ik_{\sigma}^2g) + 2\Phi(ik_{\sigma}g)d\Gamma(k_{\sigma})) \\
&= -i\Phi(i\omega g) + 2i\Phi(i\xi \cdot kg) + \Phi(i|k|^2g) - 2i \sum_{\sigma=1}^{\nu} \Phi(ik_{\sigma}g)d\Gamma(k_{\sigma}).
\end{aligned}$$

This last equation implies

$$\begin{aligned}
\mathcal{C}_{n_1} &= -i \sum_{\ell=1}^n \Phi(g_1) \cdots \Phi(g_{\ell-1}) \Phi(i\omega g_{\ell}) \Phi(g_{\ell+1}) \cdots \Phi(g_{n_1}) \\
&\quad + 2i \sum_{\ell=1}^n \Phi(g_1) \cdots \Phi(g_{\ell-1}) \Phi(i\xi \cdot kg_{\ell}) \Phi(g_{\ell+1}) \cdots \Phi(g_{n_1}) \\
&\quad + \sum_{\ell=1}^n \Phi(g_1) \cdots \Phi(g_{\ell-1}) \Phi(i|k|^2g_{\ell}) \Phi(g_{\ell+1}) \cdots \Phi(g_{n_1}) \\
&\quad - 2i \sum_{\ell=1}^n \sum_{\sigma=1}^{\nu} \Phi(g_1) \cdots \Phi(g_{\ell-1}) \Phi(ik_{\sigma}g_{\ell}) d\Gamma(k_{\sigma}) \Phi(g_{\ell+1}) \cdots \Phi(g_{n_1}).
\end{aligned}$$

All contributions except the last one are already in the correct form. By commuting $d\Gamma(k_{\sigma})$ through to the right we obtain

$$\begin{aligned}
& \Phi(g_{\ell-1}) \Phi(ik_{\sigma}g_{\ell}) d\Gamma(k_{\sigma}) \Phi(g_{\ell+1}) \cdots \Phi(g_{n_1}) \\
&= \Phi(g_{\ell-1}) \Phi(ik_{\sigma}g_{\ell}) \Phi(g_{\ell+1}) \cdots \Phi(g_{n_1}) d\Gamma(k_{\sigma}) \\
&\quad + \Phi(g_{\ell-1}) \Phi(ik_{\sigma}g_{\ell}) [d\Gamma(k_{\sigma}), \Phi(g_{\ell+1}) \cdots \Phi(g_{n_1})].
\end{aligned}$$

Another application of Lemma A.5 gives

$$\begin{aligned}
& \Phi(g_{\ell-1}) \Phi(ik_{\sigma}g_{\ell}) d\Gamma(k_{\sigma}) \Phi(g_{\ell+1}) \cdots \Phi(g_{n_1}) \\
&= \Phi(g_{\ell-1}) \Phi(ik_{\sigma}g_{\ell}) \Phi(g_{\ell+1}) \cdots \Phi(g_{n_1}) d\Gamma(k_{\sigma}) \\
&\quad + \Phi(g_{\ell-1}) \Phi(ik_{\sigma}g_{\ell}) \sum_{k=1}^n C_k \Phi(g_{\ell+1}) \cdots \Phi(g_{\ell+k-1}) \Phi(ik_{\sigma}g_{\ell+k}) \times \\
&\quad \quad \quad \times \Phi(g_{\ell+k+1}) \cdots \Phi(g_n).
\end{aligned}$$

Therefore, the $H_0(\xi)^{\frac{1}{2}}$ -boundedness of $d\Gamma(k_\sigma)$ implies

$$\text{ad}_{H_0(\xi)} \left(\prod_{\ell=1}^{n_1} \Phi(g_\ell) \prod_{m=1}^{n_2} d\Gamma(v_m) \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3, \sigma}} \right) \in \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3+1}.$$

We examine the commutators with the field operator $\Phi(g)$ next. Let $\alpha_1, \alpha_2 \geq 1$ and $\alpha_3 \in \mathbb{N}_0$. We compute

$$\begin{aligned} & \text{ad}_{\Phi(g)} \left(\prod_{\ell=1}^{n_1} \Phi(g_\ell) \prod_{m=1}^{n_2} d\Gamma(v_m) \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3, \sigma}} \right) \\ &= \left[\Phi(g), \prod_{\ell=1}^{n_1} \Phi(g_\ell) \right] \prod_{m=1}^{n_2} d\Gamma(v_m) \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3, \sigma}} \\ & \quad + \prod_{\ell=1}^{n_1} \Phi(g_\ell) \prod_{m=1}^{n_2} d\Gamma(v_m) \left[\Phi(g), \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3, \sigma}} \right] \\ & \quad + \prod_{\ell=1}^{n_1} \Phi(g_\ell) \left[\Phi(g), \prod_{m=1}^{n_2} d\Gamma(v_m) \right] \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3, \sigma}}. \end{aligned}$$

A two-fold application of Lemma A.4 gives

$$\begin{aligned}
& \text{ad}_{\Phi(g)} \left(\prod_{\ell=1}^{n_1} \Phi(g_\ell) \prod_{m=1}^{n_2} d\Gamma(v_m) \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3,\sigma}} \right) \\
&= \left[\Phi(g), \prod_{\ell=1}^{n_1} \Phi(g_\ell) \right] \prod_{m=1}^{n_2} d\Gamma(v_m) \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3,\sigma}} \\
&+ \prod_{\ell=1}^{n_1} \Phi(g_\ell) \left[\Phi(g), \prod_{m=1}^{n_2} d\Gamma(v_m) \right] \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3,\sigma}} . \\
&+ \prod_{\ell=1}^{n_1} \Phi(g_\ell) \prod_{m=1}^{n_2} d\Gamma(v_m) \left[\Phi(g), \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3,\sigma}} \right] \\
&= \sum_{j=1}^{n_1} 2i\text{Im}\langle g, g_j \rangle \prod_{\ell=1}^{j-1} \Phi(g_\ell) \prod_{\ell=j+1}^{n_1} \Phi(g_\ell) \prod_{m=1}^{n_2} d\Gamma(v_m) \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3,\sigma}} \\
&+ \sum_{i=1}^{n_2} \sum_{D \in P_i(N_{n_2})} C_{N_{n_2} \setminus D} \prod_{\ell=1}^{n_1} \Phi(g_\ell) \Phi(g_D) \prod_{j \in N_{n_2} \setminus D} d\Gamma(v_j) \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3,\sigma}} \\
&- \sum_{\sigma=1}^{\nu} \sum_{i=1}^{q_{n_3,\sigma}} \sum_{D \in P_i(N_{q_{n_3,\sigma}})} \sum_{d=1}^{n_2} \sum_{D' \in P_i(N_{n_2})} C_{N_{q_{n_3,\sigma}} \setminus D} C_{N_{n_2} \setminus D'} \prod_{\ell=1}^{n_1} \Phi(g_\ell) \times \\
&\quad \times \Phi((g_D)_{D'}) \prod_{m \in N_{n_2} \setminus D'} d\Gamma(v_m) \prod_{\ell \in N_{q_{n_3,\sigma}} \setminus D'} d\Gamma(k_\sigma)^\ell \prod_{\substack{\sigma'=1 \\ \sigma' \neq \sigma}}^{\nu} d\Gamma(k_{\sigma'})^{q_{n_3,\sigma'}} .
\end{aligned}$$

This equation clearly implies that

$$\begin{aligned}
& \text{ad}_{\Phi(g)} \left(\Phi(g_1) \cdots \Phi(g_{n_1}) \prod_{m=1}^{n_2} d\Gamma(v_m) \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3,\sigma}} \right) \\
&\in \mathfrak{C}_{\alpha_1-1, \alpha_2, \alpha_3} + \mathfrak{C}_{\alpha_1+1, \alpha_2-1, \alpha_3} .
\end{aligned}$$

Since $\text{ad}_{H(\xi)}(\cdot) = \text{ad}_{H_0(\xi)}(\cdot) + \text{ad}_{\Phi(g)}(\cdot)$, the mapping properties of $\text{ad}_{H(\xi)}(\cdot)$ now follow from what has been proven already.

In order to complete the proof it remains to analyze $\text{ad}_A(\cdot)$. Let $\alpha_i \geq 1$, $i = 1, 2, 3$

and calculate

$$\begin{aligned}
& \text{ad}_A \left(\prod_{\ell=1}^{n_1} \Phi(g_\ell) \prod_{j=1}^{n_2} d\Gamma(v_j) \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3,\sigma}} \right) \\
&= \left[A, \prod_{\ell=1}^{n_1} \Phi(g_\ell) \right] \prod_{j=1}^{n_2} d\Gamma(v_j) \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3,\sigma}} \\
&\quad + \prod_{\ell=1}^{n_1} \Phi(g_\ell) \left[A, \prod_{j=1}^{n_2} d\Gamma(v_j) \right] \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3,\sigma}} \\
&\quad + \prod_{\ell=1}^{n_1} \Phi(g_\ell) \prod_{j=1}^{n_2} d\Gamma(v_j) \left[A, \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3,\sigma}} \right].
\end{aligned}$$

By Lemma A.6 we obtain

$$\left[A, \prod_{\ell=1}^{n_1} \Phi(g_\ell) \right] \in \mathfrak{C}_{\alpha_1,0,0}$$

and similarly, since second quantization of $\vec{v} \cdot \nabla v_j$ leads to an $H_0(\xi)$ -bounded operator, we have that

$$\prod_{\ell=1}^{n_1} \Phi(g_\ell) \left[A, \prod_{j=1}^{n_2} d\Gamma(v_j) \right] \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3,\sigma}} \in \mathfrak{C}_{\alpha_1,\alpha_2,\alpha_3}.$$

The last term remaining has to be treated with more care. Recall that the u_m were only introduced to make products of operators of the type $d\Gamma(k_\sigma)^N$ appear more symmetric. Therefore, commuting A with some $d\Gamma(u_j)$ amounts to commuting A with some $d\Gamma(k_\sigma)$ which gives $[A, d\Gamma(k_\sigma)] = d\Gamma(i(\vec{v})_\sigma)$ by Remark A.1. luckily, second quantization of multiplication by a component of \vec{v} is $H_0(\xi)$ bounded so that

$$\prod_{\ell=1}^{n_1} \Phi(g_\ell) \prod_{j=1}^{n_2} d\Gamma(v_j) \left[A, \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3,\sigma}} \right] \in \mathfrak{C}_{\alpha_1,\alpha_2+1,\alpha_3-1}$$

which proves that $\text{ad}_A(\mathfrak{C}_{\alpha_1,\alpha_2,\alpha_3}) \subseteq \mathfrak{C}_{\alpha_1,\alpha_2,\alpha_3} + \mathfrak{C}_{\alpha_1,\alpha_2+1,\alpha_3-1}$. \square

For technical reasons we will be forced to control the adjoint T^* of some $T \in \mathfrak{C}_{\alpha_1,\alpha_2,\alpha_3}$. It turns out that $T^* \in \mathfrak{C}_{\alpha_1,\alpha_2,\alpha_3}$ again.

Lemma 3.6.

$$T \sim \mathfrak{C}_{\alpha_1,\alpha_2,\alpha_3} \implies T^* \sim \mathfrak{C}_{\alpha_1,\alpha_2,\alpha_3}.$$

PROOF: Let

$$T = \prod_{i=1}^{n_1} \Phi(g_i) \prod_{j=1}^{n_2} d\Gamma(v_j) \prod_{\sigma=1}^{\nu} d\Gamma(k_{\sigma})^{q_{n_3, \sigma}}$$

and $\psi, \psi' \in \mathcal{C}_0^{\infty}$, where $n_i \leq \alpha_i$. Then, by symmetry,

$$\langle \psi, T\psi' \rangle = \left\langle \prod_{j=1}^{n_2} d\Gamma(v_j) \prod_{\sigma=1}^{\nu} d\Gamma(k_{\sigma})^{q_{n_3, \sigma}} \Phi(g_{n_1}) \cdots \Phi(g_1) \psi, \psi' \right\rangle.$$

Obviously, in the case $n_1 = 0$ the statement follows immediately, since all second quantized multiplication operators are commuting on \mathcal{C}_0^{∞} . We may thus assume that $n_1 \geq 1$. In any case the previous equation shows that $\mathcal{C}_0^{\infty} \subset D(T^*)$. Define

$$S_{n_1} := \prod_{j=1}^{n_2} d\Gamma(v_j) \prod_{\sigma=1}^{\nu} d\Gamma(k_{\sigma})^{q_{n_3, \sigma}} \Phi(g_{n_1}) \cdots \Phi(g_1).$$

Clearly, $T^*|_{\mathcal{C}_0^{\infty}} = S_{n_1}$. A repeated application of Lemma A.4 shows that $S_{n_1} \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$. In particular, there exists $S'_{n_1} \in \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ such that $S_{n_1}|_{\mathcal{C}_0^{\infty}} = S'_{n_1}$. Hence $T^*|_{\mathcal{C}_0^{\infty}} = S'_{n_1}$. \square

Proposition 3.7. *Let $g \in C_0^{\infty}(\mathbb{R}^{\nu})$ and $m \in \mathbb{N}$. Then*

$$H(\xi)^m \sim \sum_{j=0}^m \sum_{k=0}^{m-j} \mathfrak{C}_{j, k, 2(m-j-k)}.$$

PROOF: We begin the inductive proof by noting that for $m = 1$

$$\begin{aligned} H(\xi) &= \Phi(g) + d\Gamma(\omega) + \xi^2 - 2 \sum_{\sigma=1}^{\nu} \xi_{\sigma} d\Gamma(k_{\sigma}) + \sum_{\sigma=1}^{\nu} d\Gamma(k_{\sigma})^2 \\ &\in \mathfrak{C}_{1,0,0} + \mathfrak{C}_{0,1,0} + \mathfrak{C}_{0,0,2} = \sum_{j=0}^1 \sum_{k=0}^{1-j} \mathfrak{C}_{j, k, 2(1-j-k)}. \end{aligned}$$

Assume that the assertion is correct for some $m \in \mathbb{N}$. We clearly have that

$$H(\xi)^{m+1} = \Phi(g)H(\xi)^m + H_0(\xi)H(\xi)^m.$$

Since left multiplication by $\Phi(g)$ takes $\mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ into $\mathfrak{C}_{\alpha_1+1, \alpha_2, \alpha_3}$ we may invoke the induction hypothesis to conclude that

$$\Phi(g)H(\xi)^m \sim \sum_{j=0}^m \sum_{k=0}^{m-j} \mathfrak{C}_{j+1, k, 2(m-j-k)} \subset \sum_{j=0}^{m+1} \sum_{k=0}^{m+1-j} \mathfrak{C}_{j, k, 2(m+1-j-k)},$$

where we have used that for two subsets $V, V' \subset \mathcal{H}$ we always have $V \subset V + V'$ provided that $0 \in V'$.

Similarly, it should be clear that $H_0(\xi) \sim \mathfrak{C}_{0,1,0} + \mathfrak{C}_{0,0,2}$ and that right multiplication by elements of this space sends $\mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ into $\mathfrak{C}_{\alpha_1, \alpha_2+1, \alpha_3} + \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3+2}$. If we compute

$$H_0(\xi)H(\xi)^m = H(\xi)^m H_0(\xi) + \text{ad}_{H_0(\xi)}(H(\xi)^m)$$

and use Proposition 3.5 to control the commutator contribution, we can apply the last remark and see that

$$\begin{aligned} & H_0(\xi)H(\xi)^m \\ & \sim \sum_{j=0}^m \sum_{k=0}^{m-j} (\mathfrak{C}_{j,k+1,2(m-j-k)} + \mathfrak{C}_{j,k,2(m+1-j-k)} + \mathfrak{C}_{j,k,2(m-j-k)+1}) \\ & \subset \sum_{j=0}^m \sum_{k=0}^{m-j} (\mathfrak{C}_{j,k+1,2(m-j-k)} + \mathfrak{C}_{j,k,2(m+1-j-k)}) \\ & \subset \sum_{j=0}^{m+1} \sum_{k=0}^{m+1-j} \mathfrak{C}_{j,k,2(m+1-j-k)}, \end{aligned}$$

where we have made use of $\mathfrak{C}_{j,k,2(m-j-k)+1} \subseteq \mathfrak{C}_{j,k,2(m-j-k)+2}$. It is now easy to show that

$$\begin{aligned} & \sum_{j=1}^{m+1} \sum_{k=0}^{m+1-j} \mathfrak{C}_{j,k,2(m+1-j-k)} + \sum_{j=0}^m \sum_{k=0}^{m-j} (\mathfrak{C}_{j,k+1,2(m-j-k)} + \mathfrak{C}_{j,k,2(m+1-j-k)}) \\ & \subset \sum_{j=0}^{m+1} \sum_{k=0}^{m+1-j} \mathfrak{C}_{j,k,2(m+1-j-k)}. \end{aligned}$$

Since $H(\xi)^{m+1} = \Phi(g)H(\xi)^m + H_0(\xi)H(\xi)^m$ is an element of the left hand side, the argument is complete. \square

Lemma 3.8. *Let $m \in \mathbb{N}$. For all $n \in \mathbb{N}$ we have that*

$$[A^n, H(\xi)^m] = \text{ad}_{A^n}(H(\xi)^m)$$

as a form identity on \mathcal{C}_0^∞ , where

$$\text{ad}_{A^n}(H(\xi)^m)\psi := \sum_{j=0}^{n-1} \binom{n}{j} \text{ad}_A^{n-j}(H(\xi)^m)A^j\psi \quad (3.6)$$

for all $\psi \in \mathcal{C}_0^\infty$.

PROOF: The proof can be carried out by induction. The statement is clear for $n = 1$. Let $\psi, \psi' \in \mathcal{C}_0^\infty$. We assume that the statement is true for some $n \in \mathbb{N}$ and compute

$$\begin{aligned}
& \langle \psi, [A^{n+1}, H(\xi)^m] \psi' \rangle \\
&= \langle A\psi, [A^n, H(\xi)^m] \psi' \rangle + \langle \psi, [A, H(\xi)^m] A^n \psi' \rangle \\
&= \sum_{j=0}^{n-1} \binom{n}{j} [\langle \psi, \text{ad}_A^{n-j}(H(\xi)^m) A^{j+1} \psi' \rangle + \langle \psi, \text{ad}_A^{n+1-j}(H(\xi)^m) A^j \psi' \rangle] \\
&\quad + \langle \psi, \text{ad}_A(H(\xi)^m) A^n \psi' \rangle \\
&= \sum_{j=0}^n \binom{n+1}{j} \langle \psi, \text{ad}_A^{n+1-j}(H(\xi)^m) A^j \psi' \rangle,
\end{aligned}$$

where we have used that $AC_0^\infty \subset \mathcal{C}_0^\infty$ and $\text{ad}_A^i(H(\xi)^m)\mathcal{C}_0^\infty \subset \mathcal{C}_0^\infty$ for compactly supported coupling functions g . This completes the proof. \square

Definition 3.9. Define the set of smooth functions bounded by polynomials $p(|k|) = (1 + |k|)^\ell$ for some $\ell \in \mathbb{N}$.

$$\mathcal{C}_p^\infty := \left\{ f \in C^\infty(\mathbb{R}^\nu) \mid \forall n \in \mathbb{N} \exists (\ell, C) \in \mathbb{N}_0 \times \mathbb{R}^+ : \frac{|f^{(n)}(k)|}{(1 + |k|)^\ell} \leq C \right\}. \quad (3.7)$$

Note that $\mathcal{C}_\omega^\infty \subset \mathcal{C}_p^\infty$. For a function f we call M_f the multiplication operator by f acting on $L^2(\mathbb{R}^\nu)$ with the usual maximal domain. We define the set

$$\mathcal{M}(\mathcal{C}_p^\infty) := \bigcup_f \{M_f \mid f \in \mathcal{C}_p^\infty\}. \quad (3.8)$$

Furthermore, on $H_{\text{loc}}^k(\mathbb{R}^\nu)$ we may define the operator

$$D_\eta^\alpha := M_\eta \partial^\alpha, \quad (3.9)$$

where α is a multi-index with $|\alpha| \leq k$, $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^\nu)$. Note that in this notation $D_\eta^0 = M_\eta$. Moreover, we define

$$\mathcal{D}^k := \bigcup_{\substack{\alpha \in \mathbb{N}^\nu \\ |\alpha| \leq k}} \bigcup_{n=1}^{\infty} \{D_\eta^\alpha \mid \alpha \in \mathbb{N}^\nu, |\alpha| \leq k, \text{supp}(\eta) \subset B_n(0)\}. \quad (3.10)$$

Finally, the union of these two sets is denoted by

$$\mathcal{O}_{\text{uv}}^k := \mathcal{M}(\mathcal{C}_p^\infty) \cup \mathcal{D}^k. \quad (3.11)$$

These sets can be used to specify how commuting $H_0(\xi), \Phi(g), H(\xi)$ or A with $T \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ affects the compactly supported functions appearing when T acts on \mathcal{C}_0^∞ . In order to keep track of the dependence on these functions we introduce some notation.

Definition 3.10. Let $T \in \text{Op}(\mathcal{C}_0^\infty)$. In order to emphasize the dependence on the coupling functions $g_1, \dots, g_{n_1} \in \mathcal{C}_0^\infty(\mathbb{R})$ we write

$$T(g_1, \dots, g_{n_1}) \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3},$$

if $n_1 \leq \alpha_1$ and there exist $u_1, \dots, u_{n_2} \in \mathcal{C}_\omega^\infty$ and $q_{n_3, \sigma} \in \mathbb{N}_0$, where $\sigma = 1, \dots, \nu$ and $q_{n_3, 1} + \dots + q_{n_3, \nu} = n_3$ and $C_T \in \mathbb{C}$, such that

$$\forall \psi \in \mathcal{C}_0^\infty : T\psi = C_T \prod_{i=1}^{n_1} \Phi(g_i) \prod_{j=1}^{n_2} d\Gamma(u_j) \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3, \sigma}} \psi.$$

This definition allows us to reformulate Proposition 3.5 in the special case where $\text{ad}_{H_0(\xi)}(\cdot)$ etc. act on $T(g_1, \dots, g_{n_1}) \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$. In particular, it enables us to keep track of how the coupling functions change which will be used to extend certain arguments to more general functions later on.

Before stating the Lemma we introduce some more notation. Let $T(g_1, \dots, g_{n_1}) \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ be such that

$$T(g_1, \dots, g_{n_1})\psi = C_T \prod_{i=1}^{n_1} \Phi(g_i) \prod_{j=1}^{n_2} d\Gamma(u_j) \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3, \sigma}} \psi$$

for all $\psi \in \mathcal{C}_0^\infty$. Put

$$S(n_2, n_3) := \prod_{j=1}^{n_2} d\Gamma(u_j) \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3, \sigma}} \in \text{Op}(\mathcal{C}_0^\infty).$$

We define

$$T(g_1, \dots, \hat{g}_m, \dots, g_{n_1}) \sim \mathfrak{C}_{\alpha_1-1, \alpha_2, \alpha_3}$$

by setting

$$\begin{aligned} & T(g_1, \dots, \hat{g}_m, \dots, g_{n_1})\psi \\ & := C_T \Phi(g_1) \cdots \Phi(g_{m-1}) \Phi(g_{m+1}) \cdots \Phi(g_{n_1}) S(n_2, n_3) \psi. \end{aligned}$$

Lemma 3.11. Let $T(g_1, \dots, g_{n_1}) \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ and $\psi \in \mathcal{C}_0^\infty$.

1. There exist $N \in \mathbb{N}$ and operators $M_{f_{1, \ell}}, \dots, M_{f_{n_1, \ell}} \in \mathcal{M}(\mathcal{C}_p^\infty)$ and

$$T_\ell(M_{f_{1, \ell}} g_1, \dots, M_{f_{n_1, \ell}} g_{n_1}) \in \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3+1},$$

where $1 \leq \ell \leq N$, such that

$$\text{ad}_{H_0(\xi)}(T(g_1, \dots, g_{n_1}))\psi = \sum_{\ell=1}^N T_\ell(M_{f_{1, \ell}} g_1, \dots, M_{f_{n_1, \ell}} g_{n_1})\psi.$$

2. Let $g \in C_0^\infty$. Then there exist $N \in \mathbb{N}$, $M_{f_{n_1+1,\ell}} \in \mathcal{M}(C_p^\infty)$ and operators

$$T_\ell(g_1, \dots, g_n, M_{f_{n_1+1,\ell}}g) \sim \mathfrak{C}_{\alpha_1-1, \alpha_2, \alpha_3+1},$$

where $1 \leq \ell \leq N$, such that

$$\begin{aligned} \text{ad}_{\Phi(g)}(T(g_1, \dots, g_{n_1}))\psi &= \sum_{i=1}^n T(g_1, \dots, \hat{g}_i, \dots, g_n)\psi \\ &\quad + \sum_{\ell=1}^N T_\ell(g_1, \dots, g_n, M_{f_{n_1+1,\ell}}g)\psi. \end{aligned}$$

3. For every $\sigma \in \{1, \dots, \nu\}$ define $\delta_\sigma \in \mathbb{N}^\nu$ by $(\delta_\sigma)_j = \delta_{\sigma,j}$. There exist $N \in \mathbb{N}$ and operators $T_1(g_1, \dots, g_n), \dots, T_N(g_1, \dots, g_n) \sim \mathfrak{C}_{\alpha_1, \alpha_2+1, \alpha_3-1}$ such that

$$\begin{aligned} &\text{ad}_A(T(g_1, \dots, g_{n_1}))\psi \\ &= \sum_{i=1}^{n_1} \sum_{\sigma=1}^{\nu} T(g_1, \dots, (D_{v_\sigma}^0 + D_{v_\sigma}^{\delta_\sigma})g_i, \dots, g_{n_1})\psi \\ &\quad + \sum_{i=1}^N T_i(g_1, \dots, g_{n_1})\psi. \end{aligned}$$

PROOF: Due to $C_\omega^\infty \subset C_p^\infty$ and $k_\sigma \in C_p^\infty$ for all $\sigma = 1, \dots, \nu$, the first two statements are immediate consequences of the proof of Proposition 3.5. Likewise, since the vector field v has compact support and $a = \frac{1}{2}(v \cdot i\nabla + i\nabla \cdot v) = v \cdot i\nabla + \frac{i}{2}M_{\text{div}(v)}$, $D_{v_\sigma}^0, D_{v_\sigma}^1 \in \mathcal{D}^1$ and the third statement immediately follows from the proof of Proposition 3.5 as well. \square

Lemma 3.12. Let $m \in \mathbb{N}$ and $c > 0$ and $g \in C_0^\infty(\mathbb{R}^\nu)$. There exist $N_m \in \mathbb{N}$, $M_{f_{1,\ell}}, \dots, M_{f_{j_\ell,\ell}} \in \mathcal{M}(C_p^\infty)$, where $1 \leq \ell \leq N_m$ and $0 \leq j_\ell \leq m$, such that for all $\psi \in C_0^\infty$

$$H(\xi)^m \psi = H_0(\xi)^m \psi + \sum_{\ell=1}^{N_m} \sum_{j_\ell=1}^m T_\ell(M_{f_{1,\ell}}g, \dots, M_{f_{j_\ell,\ell}}g)\psi,$$

where for all $1 \leq \ell \leq N_m$

$$T_\ell(M_{f_{1,\ell}}g, \dots, M_{f_{j_\ell,\ell}}g) \sim \mathfrak{C}_{j_\ell, k_\ell, 2(m-j_\ell-k_\ell)}$$

for some $0 \leq k_\ell \leq m - j_\ell$.

PROOF: From Proposition 3.7 we already know that there exist $N_m \in \mathbb{N}$, $g_{1,\ell}, \dots, g_{j_\ell,\ell} \in C_p^\infty$, where $1 \leq \ell \leq N_m$ and $0 \leq j_\ell \leq m$, such that for all $\psi \in C_0^\infty$

$$H(\xi)^m \psi = H_0(\xi)^m \psi + \sum_{\ell=1}^{N_m} \sum_{j_\ell=0}^m T(g_{1,\ell}, \dots, g_{j_\ell,\ell})\psi,$$

where for all $1 \leq \ell \leq N_m$

$$T(g_{1,\ell}, \dots, g_{\ell,j_\ell}) \sim \mathfrak{C}_{j_\ell, k_\ell, 2(m-j_\ell-k_\ell)}.$$

for some $1 \leq k_\ell \leq m-j_\ell$. We thus have to show that $g_{1,\ell} = M_{1,\ell}g, \dots, g_{j_\ell,\ell} = M_{f_{j_\ell,\ell}}g$ for appropriately chosen operators $M_{1,\ell}, \dots, M_{j_\ell,\ell} \in \mathcal{M}(C_p^\infty)$.

The statement is clear for $m = 1$. The general case follows by induction from Lemma 3.11 and a similar argumentation as in Proposition 3.7. More precisely, Lemma 3.11 implies that in every step the coupling functions will at most accumulate one factor M_f for some $f \in C_\omega^\infty$. Since $M_f M_{f'} = M_{ff'}$ and C_ω^∞ is an algebra, the induction step can be carried out. \square

Corollary 3.13. *Let $T(g_1, \dots, g_{n_1}) \in \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ and $g \in C_0^\infty(\mathbb{R})$. Then*

$$\begin{aligned} \text{ad}_{H_0(\xi)+\Phi(g)}(T(h_1, \dots, h_{n_1})) &= \sum_{i=1}^{n_1} T(h_1, \dots, \hat{h}_i, \dots, h_{n_1}) \\ &\quad + \sum_{\ell=1}^N T_\ell(h_1, \dots, h_{n_1}, M_{f_{n_1+1,\ell}}h) \\ &\quad + \sum_{\ell=1}^N T(M_{f_{1,\ell}}h_1, \dots, M_{f_{n_1,\ell}}h_{n_1}). \end{aligned}$$

PROOF: Due to

$$\begin{aligned} &\text{ad}_{H_0(\xi)+\Phi(g)}(T(g_1, \dots, g_{n_1})) \\ &= \text{ad}_{H_0(\xi)}(T(g_1, \dots, g_{n_1})) + \text{ad}_{\Phi(g)}(T(g_1, \dots, g_{n_1})), \end{aligned}$$

we simply combine Lemma 3.11.1 and Lemma 3.11.2 to complete the proof. \square

3.2 Extension of the Bounds to Larger Domains

Lemma 3.14. *Let $n > 0$. For $x \in D(A) \cap D(H_0^n)$ define the intersection norm $\|x\|_{I,n} := \|x\| + \|Ax\| + \|H_0^n x\|$. Then:*

1. $D_n := (H_0 - \lambda)^{-n}D(A) \subseteq D(A) \cap D(H_0^n)$ is independent of the choice of $\lambda \leq -1$.
2. If $n \geq 1$, C_0^∞ is dense in D_n w.r.t. the intersection norm $\|\cdot\|_{I,n}$.

PROOF: Let $\lambda_1, \lambda_2 < 0$ and put $D_n(\lambda_i) := (H_0 - \lambda_i)^{-n}D(A)$ for $i = 1, 2$. Note that by the first resolvent identity

$$(H_0 - \lambda_1)^{-1} = (H_0 - \lambda_2)^{-1} (1 + (\lambda_1 - \lambda_2)(H_0 - \lambda_1)^{-1}).$$

For $n \in \mathbb{N}$ commutativity of the resolvents thus yields

$$(H_0 - \lambda_1)^{-n} = (H_0 - \lambda_2)^{-n} (1 + (\lambda_1 - \lambda_2)(H_0 - \lambda_1)^{-1})^n.$$

Since $(H_0 - \lambda_1)^{-1}$ preserves the domain of A so does $(H_0 - \lambda_1)^{-\ell}$ for all $\ell \in \mathbb{N}$. Combined with the previous equation this implies

$$(H_0 - \lambda_1)^{-n} D(A) \subseteq (H_0 - \lambda_2)^{-n} D(A).$$

Reversing roles of λ_1 and λ_2 gives the other inclusion and thus $D_n(\lambda_1) = D_n(\lambda_2)$. This proves the first assertion, if $n \in \mathbb{N}$.

Since the integer case is clear, it suffices to show that

$$(1 + (\lambda_1 - \lambda_2)(H_0 - \lambda_1)^{-1})^\alpha D(A) \subset D(A)$$

for $\alpha \in (0, 1)$. If $(1 + (\lambda_1 - \lambda_2)(H_0 - \lambda_1)^{-1})^\alpha \in C^1(A)$, the desired result follows, since all operators in $C^1(A)$ preserve $D(A)$. In order to see this first note that the choice of λ_i implies $\inf_{\lambda \in \sigma(H(\xi))} |\lambda - \lambda_i| \geq 1$ and hence $(\lambda_1 - \lambda_2)(H_0 - \lambda_1)^{-1}$ has norm less than $\frac{1}{2}$ for $|\lambda_1 - \lambda_2| < 1/2$. Furthermore, we abbreviate

$$R := (\lambda_1 - \lambda_2)(H_0 - \lambda_1)^{-1}$$

and compute

$$\begin{aligned} |\langle \psi, [A, (1 + R)^\alpha] \psi' \rangle| &\leq C_\alpha \int_0^\infty \frac{1}{t^\alpha} |\langle (1 + t + R)^{-1} \psi, [A, R] (1 + t + R)^{-1} \psi' \rangle| dt \\ &\leq \int_0^\infty \frac{C_\alpha \|\psi\| \|\psi'\| \|\text{ad}_A(R)\|}{t^\alpha (\frac{1}{2} + t)^2} dt \leq C'_\alpha \|\psi\| \|\psi'\| \|\text{ad}_A(R)\|, \end{aligned}$$

where we have used that $(H_0 - \lambda_1)^{-1}$ is of class $C^1(A)$ which immediately gives that R is of class $C^1(A)$ and the form $[A, R]$ is given by a bounded operator $\text{ad}_A(R)$.

We now prove the second statement. To this end let $\phi \in D_n$. We need to construct a sequence $\phi_k \in \mathcal{C}_0^\infty$ such that $\phi_k \rightarrow \phi$ w.r.t. $\|\cdot\|_I$. Due to $\phi \in D_n$, we may write $\phi = (H_0 + 1)^{-n} \psi$ for some $\psi \in D(A)$. Since \mathcal{C}_0^∞ is a core for A , there exists a sequence $(\psi_k)_k \subseteq D(A)$ such that $\|\psi - \psi_k\|_A \rightarrow 0$, where $\|\cdot\|_A$ denotes the graph norm of A .

We define $\phi_k := (H_0 + 1)^{-n} \psi_k \in D_n \subseteq D(A) \cap D(H^n)$. By the continuity of $(H_0 + 1)^{-n}$ it is then clear that $\|\phi_k - \phi\| \rightarrow 0$. As before we first cover the case where $n \in \mathbb{N}$. This assumption allows us to calculate

$$\|H_0^n(\phi_k - \phi)\| = \|H_0^n(H_0 + 1)^{-n}(\psi_k - \psi)\| \leq \|(\psi_k - \psi)\| \rightarrow 0.$$

Thus, the validity of the second assertion for $n \in \mathbb{N}$ follows, if we can establish that $\|A(\phi_k - \phi)\| \rightarrow 0$. Since ψ and all ψ_k are in $D(A)$ and $(H_0 + 1)^{-n}$ is of class $C^1(A)$, we may calculate

$$\begin{aligned} \|A(\phi_k - \phi)\| &= \|A(H_0 + 1)^{-n}(\psi_k - \psi)\| \\ &\leq \|(H_0 + 1)^{-n}A(\psi_k - \psi)\| + \|\text{ad}_A((H_0 + 1)^{-n})(\psi_k - \psi)\| \\ &\leq \max\{\|(H_0 + 1)^{-n}\|, \|\text{ad}_A((H_0 + 1)^{-n})\|\} \cdot \|\psi_k - \psi\|_A . \end{aligned}$$

Therefore, $\|A(\phi_k - \phi)\| \rightarrow 0$ which completes the proof of the case, where $n \in \mathbb{N}$.

Next assume that $n = m + 1/2$ for some $m \in \mathbb{N}$. The H_0^n -term in the intersection norm can be dealt with easily. Indeed,

$$\|H_0^n(\phi_k - \phi)\| = \|H_0^m(H_0 + 1)^{-m}H_0^{\frac{1}{2}}(H_0 + 1)^{-\frac{1}{2}}(\psi_k - \psi)\| \leq \|\psi_k - \psi\|.$$

In order to deal with the half power we compute

$$\begin{aligned} |\langle \eta, A(\phi_k - \phi) \rangle| &= |\langle \eta, A(H_0 + 1)^{-k-\frac{1}{2}}(\psi_k - \psi) \rangle| \\ &\leq |\langle \eta, (H_0 + 1)^{-\frac{1}{2}}A(H_0 + 1)^{-k}(\psi_k - \psi) \rangle| \\ &\quad + |\langle \eta, [A, (H_0 + 1)^{-\frac{1}{2}}](H_0 + 1)^{-k}(\psi_k - \psi) \rangle| \\ &\leq |\langle \eta, (H_0 + 1)^{-\frac{1}{2}}(H_0 + 1)^{-k}A(\psi_k - \psi) \rangle| \\ &\quad + |\langle \eta, (H_0 + 1)^{-\frac{1}{2}}\text{ad}_A((H_0 + 1)^{-k})(\psi_k - \psi) \rangle| \\ &\quad + |\langle \eta, [A, (H_0 + 1)^{-\frac{1}{2}}](H_0 + 1)^{-k}(\psi_k - \psi) \rangle| \end{aligned}$$

for $\eta \in \mathcal{C}_0^\infty$. The first two terms only involve bounded operators and thus give uniform expressions in η . If we can show that the form expression $[A, (H_0 + 1)^{-\frac{1}{2}}](H_0 + 1)^{-k}$ is given by a bounded operator, $\|A(\phi_k - \phi)\| \rightarrow 0$ and we are done.

Let $\eta, \eta' \in \mathcal{C}_0^\infty$. Since $H_0 + 1$ is strictly positive, we can make use of the integral representation formula in [14], Lemma 3.1, for $(H_0 + 1)^{-1/2}$. Moreover, $[A, H_0]$ is given by the operator $\text{ad}_A(H_0)$ on \mathcal{C}_0^∞ . Due to $(H_0 + 1)^{-1}\mathcal{C}_0^\infty \subset \mathcal{C}_0^\infty$, we may calculate

$$\begin{aligned} &|\langle \eta', [A, (H_0 + 1)^{-\frac{1}{2}}](H_0 + 1)^{-1}\eta \rangle| \\ &= C \int_0^\infty t^{-\frac{1}{2}} |\langle \eta', (H_0 + 1 + t)^{-1}[A, H_0](H_0 + 1 + t)^{-1}(H_0 + 1)^{-1}\eta \rangle| dt \\ &= C \int_0^\infty t^{-\frac{1}{2}} |\langle (H_0 + 1 + t)^{-1}\eta', \text{ad}_A(H_0)(H_0 + 1 + t)^{-1}(H_0 + 1)^{-1}\eta \rangle| dt . \end{aligned}$$

By direct computation on \mathcal{C}_0^∞ we can establish that

$$\text{ad}_A(H_0) = d\Gamma(iv \cdot \nabla \omega) + i \sum_{\sigma=1}^{\nu} \xi_\sigma d\Gamma(v_\sigma) + 2i \sum_{\sigma=1}^{\nu} d\Gamma(v_\sigma) d\Gamma(k_\sigma).$$

Thus, $\text{ad}_A(H_0)(H_0 + 1)^{-3/2}$ extends from \mathcal{C}_0^∞ to a bounded operator. This lets us compute

$$\begin{aligned} & C \int_0^\infty t^{-\frac{1}{2}} |\langle (H_0 + 1 + t)^{-1} \eta', \text{ad}_A(H_0)(H_0 + 1 + t)^{-1} (H_0 + 1)^{-1} \eta \rangle| dt \\ & \leq C' \int_0^\infty t^{-\frac{1}{2}} \| (H_0 + 1 + t)^{-1} \eta' \| \| (H_0 + 1)^{\frac{1}{2}} (H_0 + 1 + t)^{-1} \eta \| dt \\ & \leq C' \int_0^\infty \frac{t^{-\frac{1}{2}}}{(1+t)^{\frac{3}{2}}} dt \cdot \|\eta'\| \|\eta\| = C'' \|\eta'\| \|\eta\|. \end{aligned}$$

Hence we have shown that the form $[A, (H_0 + 1)^{-\frac{1}{2}}](H_0 + 1)^{-1}$ is bounded and thus given by a bounded operator $\text{ad}_A((H_0 + 1)^{\frac{1}{2}})(H_0 + 1)^{-1}$. As previously mentioned this completes the proof. \square

As a next step we prove that D_n can be used to approximate elements in the intersection domain $D(H_0^n) \cap D(A)$ w.r.t. the intersection topology.

Lemma 3.15. *Let $n \geq 1$. D_n is dense in the intersection domain $D(A) \cap D(H_0^n)$ equipped with the intersection topology given by $\|\cdot\|_{I,n}$.*

PROOF: Let $n \geq 1$ and write $n = m + \alpha$ for $m \in \mathbb{N}$ and $\alpha \in [0, 1)$. Let $\psi \in D(A) \cap D(H_0^n)$ and define

$$\psi_k := k^n (H_0 + k)^{-n} \psi.$$

for large enough k . Clearly, $\psi_k \in D_n$ for all $k \in \mathbb{N}$. Furthermore, recall that the operator $H_k = k^n (H_0 + k)^{-n}$ converges strongly to the identity and therefore $H_0^n (H_k - 1)\psi$ and $(H_k - 1)\psi$ converge to 0 for all $\psi \in D(H_0^n)$. Next, we show that $\|A(\psi_k - \psi)\| \rightarrow 0$. In order to see this let $\phi \in D(A) \cap D(H_0^n)$ and compute

$$\begin{aligned} |\langle \phi, A\psi_k - A\psi \rangle| &= |\langle \phi, Ak^n(H_0 + k)^{-n}\psi - A\psi \rangle| \\ &\leq |\langle \phi, k^n(H_0 + k)^{-n}A\psi - A\psi \rangle| + |\langle \phi, k^n[A, (H_0 + k)^{-n}]\psi \rangle| \\ &\leq \|\phi\| \|(H_k - 1)A\psi\| + |\langle \phi, k^n[A, (H_0 + k)^{-n}]\psi \rangle|. \end{aligned} \quad (3.12)$$

The first term converges to 0, because $H_k - 1$ converges strongly to 0. In order to deal with the last term we estimate

$$k^n |\langle \phi, [A, (H_0 + k)^{-m-\alpha}] \psi \rangle| \leq k^n |\langle \phi, [A, (H_0 + k)^{-\alpha}] (H_0 + k)^{-m} \psi \rangle| + k^n |\langle \phi, (H_0 + k)^{-\alpha} [A, (H_0 + k)^{-m}] \psi \rangle| \quad (3.13)$$

and deal with both terms separately. First note that the commutator form $[A, H_0^j]$ is implemented by a $H_0^{j+\frac{1}{2}}$ -bounded operator on \mathcal{C}_0^∞ for all $j \in \mathbb{N}$. This can be seen by directly computing H_0^j on \mathcal{C}_0^∞ via a binomial expansion. The additional half power in the bound comes from a $d\Gamma(k_\sigma)$ which is transformed into a $d\Gamma(v_\sigma)$ after commuting with A . Since \mathcal{C}_0^∞ is dense in D_n for all $n \geq 1$ by Lemma 3.14, the form and hence the operator extends to all D_n , $n \geq 1$. Due to this fact and $\psi \in D(H_0^n)$ the second term in (3.13) can be estimated as follows:

$$\begin{aligned} & k^n |\langle \phi, (H_0 + k)^{-\alpha} [A, (H_0 + k)^{-m}] \psi \rangle| \\ &= k^n |\langle \phi, (H_0 + k)^{-m-\alpha} [A, (H_0 + k)^m] (H_0 + k)^{-m} \psi \rangle| \\ &\leq \sum_{\ell=0}^{m-1} \binom{m}{\ell} k^{n+\ell} |\langle \phi, (H_0 + k)^{-m-\alpha} [A, H_0^{m-\ell}] (H_0 + k)^{-m} \psi \rangle| \\ &\leq C \sum_{\ell=0}^{m-1} \binom{m}{\ell} k^{n+\ell} \| (H_0 + k)^{-n} \phi \| \| (H_0 + 1)^{m-\ell+\frac{1}{2}} (H_0 + k)^{-m} \psi \| \\ &\leq C \| \phi \| \| (H_0 + 1) \psi \| \sum_{\ell=0}^{m-1} \binom{m}{\ell} k^\ell \| (H_0 + k)^{-\ell-\frac{1}{2}} \| \\ &\leq k^{-\frac{1}{2}} C \| \phi \| \| (H_0 + 1) \psi \| 2^{m-1}. \end{aligned}$$

Thus,

$$\lim_{k \rightarrow \infty} k^n \| \text{ad}_A((H_0 + k)^{-m})(H_0 + k)^{-\alpha} \psi \| = 0.$$

The first term in (3.13) vanishes for $\alpha = 0$. For $\alpha \in (0, 1)$ it can be dealt with in a similar fashion as in the proof of Lemma 3.14, where we proceed as in [14], Lemma 3.2, and use an integral representation for $(H_0 + k)^\alpha$ instead of the respective formula for $(H_0 + 1)^{-\alpha}$. Once again we use that the form bounds on

$[A, H_0]$ extend to $(H_0 + 1)^{-3/2}D(A)$ and compute

$$\begin{aligned}
& k^{m+\alpha} |\langle \phi, [A, (H_0 + k)^{-\frac{1}{2}}](H_0 + k)^{-m}\psi \rangle| \\
&= C_\alpha k^{m+\alpha} \int_0^\infty t^{-\alpha} \langle \phi, (H_0 + k + t)^{-1} [A, H_0] (H_0 + k + t)^{-1} (H_0 + k)^{-m}\psi \rangle dt \\
&\leq C'_\alpha k^{m+\alpha} \int_0^\infty t^{-\alpha} \| (H_0 + k + t)^{-1} \phi \| \| (H_0 + k + t)^{-1} (H_0 + k)^{-m} (H_0 + 1)^{\frac{3}{2}} \psi \| dt \\
&\leq C'_\alpha k^\alpha \| \phi \| \| (H_0 + 1)^{\frac{3}{2}} \psi \| \int_0^\infty t^{-\alpha} (k + t)^{-2} dt \\
&\leq k^{-1} C''_\alpha \| \phi \| \| (H_0 + 1)^{\frac{3}{2}} \psi \|.
\end{aligned}$$

We conclude that $|\langle \phi, A\psi_k - A\psi \rangle| \rightarrow 0$. This completes the proof. \square

We would now like to replace $D(H_0(\xi)^n)$ in the preceding discussion by $D((H(\xi) + c)^n)$. Unfortunately, the proofs depend on the relation $(H_0(\xi) + 1)^{-1} \mathcal{C}_0^\infty \subset \mathcal{C}_0^\infty$ which is not satisfied for $(H(\xi) + c)^{-1}$ anymore. To solve this issue we prove two technical Lemmas which let us compare both domains equipped with the respective graph norms.

Lemma 3.16. *Suppose that the coupling function $g \in C_0^\infty(\mathbb{R}^\nu)$ and let $c > 0$ large enough so $-c \in \rho(H)$. For any $m \in \mathbb{N}_0$ we have that*

$$B_m^g := (H_0 + 1)^{\frac{m}{2}} (H + c)^{-\frac{m}{2}} \in \mathcal{B}(\mathcal{H}).$$

PROOF: Since the coupling function g is fixed, we simply put $B_m^g = B_m$. We proceed inductively. The case $m = 0$ is clear. Assume that there exists $m \in \mathbb{N}$ such that $B_{m'} \in \mathcal{B}(\mathcal{H})$ for all $m' \in \mathbb{N} \cap [0, m]$. We compute

$$\begin{aligned}
B_{m+1} &= (H_0 + 1)^{\frac{m-1}{2}} (H_0 + 1) (H + c)^{-1} (H + c)^{-\frac{m-1}{2}} \\
&= (H_0 + 1)^{\frac{m-1}{2}} (H + c)^{-\frac{m-1}{2}} + (1 - c) (H_0 + 1)^{\frac{m-1}{2}} (H + c)^{-\frac{m+1}{2}} \\
&\quad + (H_0 + 1)^{\frac{m-1}{2}} \Phi(g) (H + c)^{-\frac{m+1}{2}} \\
&= B_{m-1} + (1 - c) B_{m-1} (H + c)^{-1} + (H_0 + 1)^{\frac{m-1}{2}} \Phi(g) (H + c)^{-\frac{m+1}{2}}.
\end{aligned}$$

By the induction hypothesis the first two contributions are bounded operators. In order to deal with the last term we compute formally

$$\begin{aligned}
(H_0 + 1)^{\frac{m-1}{2}} \Phi(g) (H + c)^{-\frac{m+1}{2}} &= \Phi(g) (H_0 + 1)^{\frac{m-1}{2}} (H + c)^{-\frac{m+1}{2}} \\
&\quad + [\Phi(g), (H_0 + 1)^{\frac{m-1}{2}}] (H + c)^{-\frac{m+1}{2}}. \tag{3.14}
\end{aligned}$$

We would like to make sense out of this computation in the sense of quadratic forms on \mathcal{C}_0^∞ and then argue that these forms are in fact bounded.

The $H_0^{\frac{1}{2}}$ -boundedness of $\Phi(g)$ implies that $\Phi(g)(H_0+1)^{(m-1)/2}$ is a $H_0^{m/2}$ -bounded operator. By induction hypothesis $D(H_0^{m/2}) \subset D((H+c)^{m/2})$ and hence

$$\Phi(g)(H_0+1)^{\frac{m-1}{2}}(H+c)^{-\frac{m+1}{2}}$$

is a bounded operator.

Thus, it remains to show that the last term on the right hand side of (3.14) can be defined on \mathcal{C}_0^∞ and extends to a bounded operator. In order to do this first assume that $m-1=2k$ for some integer $k \in \mathbb{N}$. It is easy to see that H_0 can act on the sets $\mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ by multiplication and that

$$(H_0+1)\mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3} \subset \mathfrak{C}_{\alpha_1, \alpha_2+1, \alpha_3} + \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3+2}.$$

This fact combined with $(H_0+1) \in \mathfrak{C}_{0,1,0} + \mathfrak{C}_{0,0,2}$ then implies

$$(H_0+1)^k \in \sum_{\substack{i,j \in \mathbb{N}_0 \\ i+j=k}} \mathfrak{C}_{0,i,2j}. \quad (3.15)$$

Indeed, the formula is true for $k=1$. Assume it holds for some $k \in \mathbb{N}$ and compute

$$(H_0+1)^{k+1} \in \sum_{\substack{i,j \in \mathbb{N}_0 \\ i+j=k}} (\mathfrak{C}_{0,i+1,2j} + \mathfrak{C}_{0,i,2j+2}) = \sum_{\substack{i,j \in \mathbb{N}_0 \\ i+j=k+1}} \mathfrak{C}_{0,i,2j}.$$

Using Proposition 3.5 we see that $\text{ad}_{\Phi(g)}((H_0+1)^k)$ is again an H_0^k -bounded operator which implies

$$\|\text{ad}_{\Phi(g)}((H_0+1)^{\frac{m-1}{2}})(H+c)^{-\frac{m+1}{2}}\eta\| \leq C\|B_{m-1}\| \|(H+c)^{-1}\|\|\eta\|.$$

for all $\eta \in \mathcal{C}_0^\infty$. Clearly, this upper bound implies that the operator extends to a bounded operator on the whole Hilbert space. This completes the proof for the case $m-1=2k, k \in \mathbb{N}$.

For $m-1=2k+1, k \in \mathbb{N}$ we cannot simply apply the mapping properties of $\text{ad}_{\Phi(g)}(\cdot)$ in Proposition 3.5, due to the presence of a factor $(H_0+1)^{1/2}$. In any case it is still true that for $\eta, \eta' \in \mathcal{C}_0^\infty$

$$\begin{aligned} |\langle \eta, [\Phi(g), (H_0+1)^{k+\frac{1}{2}}]\eta' \rangle| &\leq |\langle \eta, [\Phi(g), (H_0+1)^{\frac{1}{2}}](H_0+1)^k \eta' \rangle| \\ &\quad + |\langle \eta, (H_0+1)^{\frac{1}{2}}[\Phi(g), (H_0+1)^k]\eta' \rangle|. \end{aligned} \quad (3.16)$$

In order to deal with the first term we use the representation formula of [14] once more. Let $\eta'' \in \mathcal{C}_0^\infty$ and calculate

$$\begin{aligned}
& |\langle \eta, [\Phi(g), (H_0 + 1)^{\frac{1}{2}}] \eta'' \rangle| \\
& \leq C \int_0^\infty t^{\frac{1}{2}} |\langle \eta, (H_0 + 1 + t)^{-1} [\Phi(g), H_0] (H_0 + 1 + t)^{-1} \eta'' \rangle| dt \\
& \leq C \int_0^\infty t^{\frac{1}{2}} \| (H_0 + 1 + t)^{-1} \eta \| \| (H_0 + 1 + t)^{-1} H_0 \eta'' \| dt \\
& \leq C \| \eta \| \| H_0 \eta'' \| \int_0^\infty \frac{t^{\frac{1}{2}}}{(1+t)^2} dt \leq C' \| \eta \| \| (H_0 + 1) \eta'' \|.
\end{aligned}$$

Due to $H_0 \mathcal{C}_0^\infty \subset \mathcal{C}_0^\infty$, we may put $\eta'' = (H_0 + 1)^k \eta'$ and obtain the estimate

$$|\langle \eta, [\Phi(g), (H_0 + 1)^{\frac{1}{2}}] (H_0 + 1)^k \eta' \rangle| \leq 2C \| \eta \| \| (H_0 + 1)^{k+1} \eta' \|.$$

Thus, the sesquilinear form $[\Phi(g), (H_0 + 1)^{\frac{1}{2}}] (H_0 + 1)^k$ extends to $\mathcal{H} \times D(H_0^k)$ by continuity and density. Again by the induction hypothesis $D((H + 1)^{m/2}) \subset D((H_0 + c)^{m/2})$, where $k + 1 = m/2$. This gives meaning to (3.14) and implies

$$\begin{aligned}
& |\langle \eta, [\Phi(g), (H_0 + 1)^{\frac{1}{2}}] (H_0 + 1)^k (H + c)^{-k - \frac{3}{2}} \eta' \rangle| \\
& \leq 2C \| \eta \| \| (H_0 + 1)^{k+1} (H + c)^{-k-1} (H + c)^{-\frac{1}{2}} \eta' \| \\
& \leq 2C \| B_m (H + c)^{-\frac{1}{2}} \| \| \eta \| \| \eta' \|, \tag{3.17}
\end{aligned}$$

where we have used that $k + 1 = m/2$ and that $B_m (H + c)^{-\frac{1}{2}}$ is a bounded operator by induction hypothesis. This shows that the first term in (3.16) gives a bounded contribution.

Next we deal with the second term in (3.16). Recall that $\text{ad}_{\Phi(g)}(\mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}) \subset \mathfrak{C}_{\alpha_1-1, \alpha_2, \alpha_3} + \mathfrak{C}_{\alpha_1+1, \alpha_2, \alpha_3}$. This clearly implies

$$\text{ad}_{\Phi(g)}((H_0 + 1)^k) \in \sum_{\substack{i, j \in \mathbb{N}_0 \\ i+j=k}} \mathfrak{C}_{1, i, 2j}.$$

as operators on \mathcal{C}_0^∞ . Hence there are functions u_1, \dots, u_{k-1} and integers $q_{1, \sigma}, \dots, q_{2k, \sigma}$ for $\sigma \in \{1, \dots, \nu\}$ such that $d\Gamma(u_j)$ is H_0 -bounded and $\text{ad}_{\Phi(g)}((H_0 + 1)^k)$ is given on \mathcal{C}_0^∞ as linear combinations of elements of the type

$$\Phi(g_1) \prod_{i=1}^n d\Gamma(u_i) \prod_{\sigma=1}^\nu d\Gamma(k_\sigma)^{m, \sigma},$$

where $n \leq k$, $1 \leq m \leq 2k$, $q_{m,1} + \dots + q_{m,\sigma} \leq 2k$ and $n + (q_{m,1} + \dots + q_{m,\sigma})/2 = k$. Therefore, $(H_0 + 1)^{\frac{1}{2}} \text{ad}_{\Phi(g)}((H_0 + 1)^k)$ is given on \mathcal{C}_0^∞ by linear combinations of the type

$$\begin{aligned} & [\Phi(g_1), (H_0 + 1)^{\frac{1}{2}}] \prod_{i=1}^n d\Gamma(u_i) \prod_{\sigma=1}^\nu d\Gamma(k_\sigma)^{m,\sigma}, \\ & \Phi(g_1)(H_0 + 1)^{\frac{1}{2}} \prod_{i=1}^n d\Gamma(u_i) \prod_{\sigma=1}^\nu d\Gamma(k_\sigma)^{m,\sigma}. \end{aligned}$$

By the above argumentation $[\Phi(g_1), (H_0 + 1)^{\frac{1}{2}}]$ and $\Phi(g_1)(H_0 + 1)^{\frac{1}{2}}$ are both H_0 -bounded. The restrictions on n and the $q_{m,\sigma}$ now imply that $[\Phi(g_1), (H_0 + 1)^{\frac{1}{2}}] \prod_{i=1}^n d\Gamma(u_i) \prod_{\sigma=1}^\nu d\Gamma(k_\sigma)^{m,\sigma}$ and $\Phi(g_1)(H_0 + 1)^{\frac{1}{2}} \prod_{i=1}^n d\Gamma(u_i) \prod_{\sigma=1}^\nu d\Gamma(k_\sigma)^{m,\sigma}$ are H_0^k -bounded. As we have argued above this implies that the quadratic forms extend to $\mathcal{H} \times D(H_0^k)$ which gives meaning to the following computation:

$$\begin{aligned} & |\langle \eta, (H_0 + 1)^{\frac{1}{2}} [\Phi(g), (H_0 + 1)^k] (H + c)^{-k - \frac{3}{2}} \eta' \rangle| \\ & \leq |\langle \eta, (H_0 + 1)^k (H + c)^{-k - \frac{3}{2}} \eta' \rangle| \leq C' \|B_m (H + c)^{-\frac{1}{2}}\| \|\eta\| \|\eta'\|. \end{aligned} \quad (3.18)$$

Combining (3.16)-(3.18) finally give a rigorous meaning to the second term in (3.14). Moreover, (3.18) extends to the whole Hilbert space by density and continuity and hence the sesquilinear form extends to a bounded operator denoted by $(H_0 + 1)^{\frac{1}{2}} \text{ad}_{\Phi(g)}((H_0 + 1)^k) (H + c)^{-k - \frac{3}{2}}$. Therefore, (3.16) extends to the whole Hilbert space. This in turn establishes that all terms in (3.14) are bounded in the case $m - 1 = 2k + 1$. This finishes the inductive proof. \square

Lemma 3.17. *Suppose that the coupling function $g \in \mathcal{C}_0^\infty(\mathbb{R}^\nu)$ and let $c > 0$ large enough so $-c \in \rho(H)$. For any $m \in \mathbb{N}_0$ we have that*

$$F_m^g := (H + c)^{\frac{m}{2}} (H_0 + 1)^{-\frac{m}{2}} \in \mathcal{B}(\mathcal{H}).$$

PROOF: The lemma is a rather easy consequence of Lemma 3.16 and its proof. There, one of the main steps is to establish that $\Phi(g)(H_0 + 1)^{\frac{m-1}{2}}$ and $[\Phi(g), (H_0 + 1)^{\frac{m-1}{2}}]$ both extend to $H_0^{\frac{m}{2}}$ -bounded operators.

We prove the lemma by induction. First of all observe that the case $m = 0$ is clear. Now assume that there exists $m \in \mathbb{N}$ such that $F_{m'}^g$ are bounded operators for all $m' \in \mathbb{N}_0 \cap [0, m]$. As in Lemma 3.16 we compute

$$\begin{aligned} F_{m+1}^g &= (H + c)^{\frac{m+1}{2}} (H_0 + 1)^{-\frac{m+1}{2}} \\ &= F_{m-1}^g + (c - 1) F_{m-1}^g (H_0 + 1)^{-1} + (H + c)^{\frac{m-1}{2}} \Phi(g)(H_0 + 1)^{\frac{m+1}{2}} \\ &= F_{m-1}^g + (c - 1) F_{m-1}^g (H_0 + 1)^{-1} + B_{m-1}^g (H_0 + 1)^{\frac{m-1}{2}} \Phi(g)(H_0 + 1)^{-\frac{m+1}{2}}. \end{aligned}$$

By the $H_0^{\frac{m}{2}}$ -boundedness of $\Phi(g)(H_0 + 1)^{\frac{m-1}{2}}$, Lemma 3.16 and the induction hypothesis all operators are bounded and thus F_{m+1}^g is bounded. \square

We sum up some immediate but important consequences of this section in a couple of remarks.

Remark 3.18.

1. For any $m \in \mathbb{N}_0$ we have that $D(H_0^{\frac{m}{2}}) = D((H + c)^{\frac{m}{2}})$.
2. The norms $\|x\|_{H_0, m} = \|(H_0 + 1)^{\frac{m}{2}}x\|$ and similarly $\|x\|_{H, m} = \|(H + c)^{\frac{m}{2}}x\|$ for large enough $c > 0$ are equivalent on $D(H^{\frac{m}{2}})$.

Remark 3.19. Let $q(\cdot, \cdot)$ be a sesquilinear form on \mathcal{C}_0^∞ . Suppose further that there exists $T \in \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ such that $n_T := n_1/2 + n_2 + n_3/2 \geq 3/2$ and

$$q(\phi, \psi) = \langle \phi, T\psi \rangle.$$

Then q extends uniquely to a sesquilinear form \tilde{q} which is continuous on $D((H + c)^{n_T})$ w.r.t. the graph norm $\|\cdot\|_{n_T} := \|\cdot\|_{(H+c)^{n_T}}$. Moreover \tilde{q} is given by the extension \tilde{T} of the operator T to an operator in $\mathcal{B}((H + c)^{n_T}, \mathcal{H})$, that is

$$\tilde{q}(\phi, \psi) = \langle \phi, \tilde{T}\psi \rangle$$

3.3 Iterated Commutators on a Core

The statements in the previous section can be used to associate \mathcal{H} -valued operators to iterated commutators calculated on the core \mathcal{C}_0^∞ .

Notation 3.20. We fix $c > 0$ such that $-c \in \rho(H)$.

Definition 3.21 (Iterated Commutators on \mathcal{C}_0^∞ I). We introduce the abbreviation $\mathcal{T} := \{T_1, T_2\}$, where $T_1 := H$ and $T_2 := A$. For $n \in \mathbb{N}$, define

$$\mathfrak{J}_n := \{\underline{w} = (\underline{w}(1), \dots, \underline{w}(n)) \in \{1, 2\}^n \mid \underline{w}(1) = 2\}.$$

For $\underline{w} \in \mathfrak{J}_n$ we define its ℓ -th truncation $\underline{w}^{(\ell)}$ by

$$\underline{w}^{(\ell)} := (\underline{w}(1), \dots, \underline{w}(\ell)) \in \{1, 2\}^\ell.$$

The ℓ -th truncation $\text{ad}_{\mathcal{T}}^{\underline{w}^{(\ell)}}((H + c)^m)$ of the mixed commutator corresponding to \underline{w} on \mathcal{C}_0^∞ is iteratively defined by

$$\text{ad}_{\mathcal{T}}^{\underline{w}^{(1)}}((H + c)^m) := A(H + c)^m - (H + c)^m A$$

for $\ell = 1$ and

$$\text{ad}_{\mathcal{T}}^{\underline{w}^{(\ell)}}((H + c)^m) := T_{\underline{w}^{(\ell)}} \text{ad}_{\mathcal{T}}^{\underline{w}^{(\ell-1)}}((H + c)^m) - \text{ad}_{\mathcal{T}}^{\underline{w}^{(\ell-1)}}((H + c)^m) T_{\underline{w}^{(\ell)}}$$

for $\ell \geq 2$. All equations hold in the sense of operators on \mathcal{C}_0^∞ . The mixed commutator corresponding to $\underline{w} \in \mathfrak{J}_n$ is then defined as

$$\text{ad}_{\mathcal{T}}^{\underline{w}}((H+c)^m) := \text{ad}_{\mathcal{T}}^{\underline{w}^{(n)}}((H+c)^m).$$

This operator is sometimes simply referred to as a mixed or iterated commutator. It is convenient to define $\mathfrak{J}_0 := 0$ and $\text{ad}_{\mathcal{T}}^0((H+c)^m) := (H+c)^m$ for all $\underline{w} \in \mathfrak{J}_0$.

Remark 3.22. By the mapping properties of the maps $\text{ad}_H(\cdot)$ and $\text{ad}_A(\cdot)$ on $\text{Op}(\mathcal{C}_0^\infty)$ it is clear by induction that there exists a constant $C_{\underline{w}} > 0$ such that for all $\psi \in \mathcal{C}_0^\infty$

$$\|\text{ad}_{\mathcal{T}}^{\underline{w}}((H+c)^m)\psi\| \leq C_{\underline{w}}\|(H+c)^{m+\frac{n}{2}}\psi\| \leq C'_{\underline{w}}\|(H+c)^M\psi\|,$$

where $M \in \mathbb{N}$ with $M \geq m + \frac{n}{2}$. In particular, $\text{ad}_{\mathcal{T}}^{\underline{w}}((H+c)^m)$ extends to a bounded operator from both $D((H+c)^{m+\frac{n}{2}})$ and $D((H+c)^M)$ with values in the Hilbert space \mathcal{H} . We do not introduce a new symbol for these extensions and will also simply call it a mixed or iterated commutator.

Notation 3.23. Throughout the rest of this thesis the symbol \mathcal{T} always denotes the set $\mathcal{T} := \{T_1, T_2\}$, where $T_1 := H$ and $T_2 := A$.

Definition 3.24. Let $n \in \mathbb{N}_0$, $\underline{w} \in \mathfrak{J}_n$. For $n \geq 1$ the overall amount of taken A commutators is defined by

$$\mathbf{n}_{\underline{w}}^A := |\{1 \leq i \leq n \mid \underline{w}(i) = 2\}|.$$

For $n = 0$ we simply put $\mathbf{n}_{\underline{w}}^A = 0$.

Lemma 3.25. Let $m, n \in \mathbb{N}_0$, $g \in \mathcal{C}_0^\infty(\mathbb{R})$ and $\underline{w} \in \mathfrak{J}_n$. There exists $N_{\underline{w}}$ and operators $O_{1,1}^{(\ell)}, \dots, O_{1,n+1}^{(\ell)}, \dots, O_{j_\ell,1}^{(\ell)}, \dots, O_{j_\ell,n+1}^{(\ell)} \in \mathcal{M}(\mathcal{C}_p^\infty) \cup \mathcal{D}^1$, where $1 \leq \ell \leq N_{\underline{w}}$ and $0 \leq j_\ell \leq n+1$ such that for all $\psi \in \mathcal{C}_0^\infty$

$$\text{ad}_{\mathcal{T}}^{\underline{w}}((H+c)^m)\psi = \sum_{\ell=1}^{N_{\underline{w}}} T(\tilde{O}_1^{(\ell)}g, \dots, \tilde{O}_{j_\ell}^{(\ell)}g)\psi,$$

where

$$T(\tilde{O}_1^{(\ell)}g, \dots, \tilde{O}_{j_\ell}^{(\ell)}g) \in \mathfrak{C}_{\alpha_1^\ell, \alpha_2^\ell, \alpha_3^\ell}$$

with $\alpha_1^\ell/2 + \alpha_2^\ell + \alpha_3^\ell/2 \leq m + n/2$ and

$$\tilde{O}_1^{(\ell)} := \prod_{i=1}^{n+1} O_{1,i}^{(\ell)}, \dots, \tilde{O}_{j_\ell}^{(\ell)} := \prod_{i=1}^{n+1} O_{j_\ell,i}^{(\ell)}.$$

Moreover, for all ℓ at most $\mathbf{n}_A^{\underline{w}}$ of the operators $O_{i,1}^{(\ell)}, \dots, O_{i,n+1}^{(\ell)}$ are in \mathcal{D}^1 .

PROOF: The statement can be proven by induction in n , the order of the iterated commutator. For $n = 0$ the statement reduces to Corollary 3.12 and has thus already been proven.

Let $\psi \in \mathcal{C}_0^\infty$ and assume the claim is correct for some $n \in \mathbb{N}_0$. Suppose $\underline{w} \in \mathfrak{J}_{n+1}$ and hence $\underline{w}^{(n)} \in \mathfrak{J}_n$. By definition of the iterated commutator and the induction hypothesis this implies

$$\begin{aligned} \text{ad}_{\mathcal{T}}^{\underline{w}}((H+c)^m)\psi &= \text{ad}_{T_{\underline{w}(n+1)}} \left(\text{ad}_{\mathcal{T}}^{\underline{w}^{(n)}}((H+c)^m) \right) \psi \\ &= \sum_{\ell=1}^{N_{n,\underline{w}}} \text{ad}_{T_{\underline{w}(n+1)}}(T(\tilde{O}_1^{(\ell)}g, \dots, \tilde{O}_{j_\ell}^{(\ell)}g))\psi. \end{aligned}$$

Assume $\underline{w}(n+1) = 1$. By Corollary 3.11 the operators in the last step of the equation can at most gain one additional field with coupling function Og for some $O \in \mathcal{M}(C_p^\infty)$ or existing fields may gain at most one additional factor $O \in \mathcal{M}(C_p^\infty)$ on top of the existing operators $\tilde{O}_i^{(\ell)}$. This shows the statement for $\underline{w}(n+1) = 1$.

Likewise for $\underline{w}(n+1) = 2$ the worst contribution is an additional operator $O \in \mathcal{D}^1$. Moreover in this case $\mathbf{n}_{\underline{w}}^A = n+1$ and at most $n+1$ operators in a given factor $O\tilde{O}_i^{(\ell)}$ are differentiations. This completes the proof. \square

Note that the last Lemma immediately shows the $H_0^{n/2+m}$ -boundedness of $\text{ad}_{\mathcal{T}}^{\underline{w}}((H+c)^m)$.

4 Regularity With Respect to the Conjugate Operator

4.1 Large Powers of the Resolvent

Notation 4.1.

1. $H := H(\xi)$ and $H_c := H + c$ for $c > 0$ large enough so that $-c \in \rho(H)$
2. For all $z \in \rho(H)$ put

$$R_z := (H(\xi) - z)^{-1} \equiv (H - z)^{-1}.$$

3. Furthermore, recall that we put $\mathcal{T} := \{T_1, T_2\}$, where $T_1 := H$ and $T_2 := A$.

In Remark 3.22 we have already seen that

$$\text{ad}_{\mathcal{T}}^{\underline{w}}((H+c)^m) \in \mathfrak{B}(D((H+c)^M), \mathcal{H}) \quad (4.19)$$

for any integer $M \geq m + n/2$. It turns out that for large enough values of M the product of this operators with R_z^M is actually in $C^1(\mathbb{A})$.

Lemma 4.2. *Let $n, m \in \mathbb{N}$, $\underline{w} \in \{1, 2\}^n$ and $Q_{\underline{w}} \geq m + \text{Int}(n/2) + 1$, where $\text{Int}(r)$ denotes the smallest integer that is bigger than $r \in \mathbb{R}$. Then*

$$\text{ad}_{\mathcal{T}}^{\underline{w}}((H + c)^m)R_{-c}^{Q_{\underline{w}}} \in C^1(A).$$

In particular,

$$\begin{aligned} & \text{ad}_A \left(\text{ad}_{\mathcal{T}}^{\underline{w}}((H + c)^m)R_{-c}^{Q_{\underline{w}}} \right) \\ &= \text{ad}_A \left(\text{ad}_{\mathcal{T}}^{\underline{w}}(H_c^m) \right) R_{-c}^{Q_{\underline{w}}} \\ & \quad + \sum_{k=0}^1 (-1)^{k+1} \text{ad}_{\mathcal{T}}^{\underline{w}}(H_c^m) R_{-c}^{Q_{\underline{w}}-k} \text{ad}_H^{1-k}(\text{ad}_A(H_c^{Q_{\underline{w}}})) R_{-c}^{M_{\underline{w}}+1} \end{aligned}$$

as a bounded operator on \mathcal{H} .

PROOF: In order to keep the formulas shorter we abbreviate

$$S^{\underline{w}} := \text{ad}_{\mathcal{T}}^{\underline{w}}(H_c^m).$$

Let $\eta, \eta' \in \mathcal{C}_0^\infty$ and calculate

$$\begin{aligned} \langle \eta, [A, S^{\underline{w}}R_{-c}^{Q_{\underline{w}}}] \eta' \rangle &= \langle A\eta, S^{\underline{w}}R_{-c}^{Q_{\underline{w}}}\eta' \rangle - \langle (S^{\underline{w}}R_{-c}^{Q_{\underline{w}}})^* \eta, A\eta' \rangle \\ &= \langle A\eta, S^{\underline{w}}R_{-c}^{Q_{\underline{w}}}\eta' \rangle - \langle (S^{\underline{w}})^* \eta, R_{-c}^{Q_{\underline{w}}} A\eta' \rangle, \end{aligned}$$

where in the second step we have used that $T^*(S^{\underline{w}})^* \subset (TS^{\underline{w}})^*$ whenever $TS^{\underline{w}}$ and $T^*(S^{\underline{w}})^*$ are defined. Since we are assuming that the coupling function is smooth and compactly supported, $S^{\underline{w}}$ and its adjoint $(S^{\underline{w}})^*$ map \mathcal{C}_0^∞ into itself by Lemma 3.6. Due to $R_{-c}\mathcal{C}_0^\infty \subset D(A)$, we may thus compute

$$\begin{aligned} \langle \eta, [A, S^{\underline{w}}R_{-c}^{Q_{\underline{w}}}] \eta' \rangle &= \langle A\eta, S^{\underline{w}}R_{-c}^{Q_{\underline{w}}}\eta' \rangle - \langle (S^{\underline{w}})^* \eta, R_{-c}^{Q_{\underline{w}}} A\eta' \rangle \\ & \quad - \langle (S^{\underline{w}})^* \eta, AR_{-c}^{Q_{\underline{w}}}\eta' \rangle + \langle (S^{\underline{w}})^* \eta, AR_{-c}^{Q_{\underline{w}}}\eta' \rangle \\ &= \langle \eta, \text{ad}_A(S^{\underline{w}})R_{-c}^{Q_{\underline{w}}}\eta' \rangle \\ & \quad + \langle (S^{\underline{w}})^* \eta, R_{-c}^{Q_{\underline{w}}} \text{ad}_A((H + c)^{Q_{\underline{w}}})R_{-c}^{Q_{\underline{w}}}\eta' \rangle. \end{aligned}$$

Note that $\text{ad}_A(S^{\underline{w}})R_{-c}^{Q_{\underline{w}}}$ is a bounded operator on \mathcal{H} , because $\text{ad}_A(S^{\underline{w}})$ extends to a bounded operator from $D(H_c^{M_{\underline{w}}+1/2})$, where $M_{\underline{w}} := m + \text{Int}(n/2)$, into \mathcal{H} by Proposition 3.5. We are thus left with the second term in the previous equation.

$\text{ad}_A(H_c^{Q_{\underline{w}}})$ is a bounded operator from $D(H_c^{M_{\underline{w}}+3/2})$ into \mathcal{H} so that it is not clear a priori that $R_{-c}^{Q_{\underline{w}}} \text{ad}_A((H + c)^{Q_{\underline{w}}})R_{-c}^{Q_{\underline{w}}}$ extends from \mathcal{C}_0^∞ to a bounded operator. The main idea of the proof is to use the rearrangement formula in Lemma A.7 to move

a resolvent from the left to the right and thereby obtain a bounded expression. With the help of this lemma we calculate

$$\begin{aligned} & \langle (S^w)^* \eta, R_{-c}^{Q_w} \text{ad}_A((H+c)^{Q_w}) R_{-c}^{Q_w} \eta' \rangle \\ &= \langle \eta, S^w R_{-c}^{Q_w-1} \text{ad}_A((H+c)^{Q_w}) R_{-c}^{Q_w+1} \eta' \rangle \\ & \quad - \langle \eta, S^w R_{-c}^{Q_w} \text{ad}_H(\text{ad}_A((H+c)^{Q_w})) R_{-c}^{Q_w+1} \eta' \rangle. \end{aligned}$$

Note that $\text{ad}_H(\text{ad}_A(H_c^{Q_w}))$ is bounded from $D(H^{Q_w+})$ into \mathcal{H} . Thus, there is a $C_w > 0$ which depends on w through the various operator norms involved, such that for all $\eta, \eta' \in \mathcal{C}_0^\infty$

$$|\langle \eta, [A, S^w R_{-c}^{Q_w}] \eta' \rangle| \leq C_w \|\eta\| \|\eta'\|.$$

By density of \mathcal{C}_0^∞ in \mathcal{H} this estimate extends to \mathcal{H} and hence $S^w R_{-c}^{Q_w}$ is of class $C^1(A)$. The second part of the statement follows, since the form is given by the correct operators on \mathcal{C}_0^∞ and because these are bounded they extend to \mathcal{H} and uniquely represent the extension of the form. \square

Definition 4.3.

1. Let $i, k \in \mathbb{N}$ and $b \in \mathbb{N}_0$. Define

$$\prod_{p=1}^i \text{ad}_{\mathcal{T}}^{w_p}(H_c^{M_p}) := \text{ad}_{\mathcal{T}}^{w_i}(H_c^{M_i}) \text{ad}_{\mathcal{T}}^{w_{i-1}}(H_c^{M_{i-1}}) \cdots \text{ad}_{\mathcal{T}}^{w_1}(H_c^{M_1})$$

and

$$\begin{aligned} \mathfrak{U}_{i,k,b} := \text{Span} \{ & R_{-c}^B \prod_{p=1}^j \text{ad}_{\mathcal{T}}^{w_p}(H_c^{M_p}) R_{-c}^{Q_p} \mid B \geq b, 1 \leq j \leq i, \sum_{p=1}^i \mathfrak{n}_{w_p}^A = k \\ & Q_p \geq M_p + \text{Int}(|w_p|/2) + 1, \}, \end{aligned}$$

$$\mathfrak{U}_{0,0,b} := \text{Span} \{ R_{-c}^B \mid B \geq b \},$$

$$\mathfrak{U}_{0,k,b} := 0,$$

where 0 means the 0-operator in the last equation.

2. Let $i \in \mathbb{N}$, $w_1 \in \mathfrak{J}_{n(1)}, \dots, w_i \in \mathfrak{J}_{n(i)}$, $M_1, \dots, M_i \in \mathbb{N}$ and $Q_1, \dots, Q_i \in \mathbb{N}$ with $Q_j \geq M_j + \text{Int}(|w_j|/2)$. We introduce the notation

$$\prod_{p=1}^i \text{ad}_{\mathcal{T}}^{w_p}(H_c^{M_p}) R_{-c}^{Q_p} := \text{ad}_{\mathcal{T}}^{w_i}(H_c^{M_i}) R_{-c}^{Q_i} \cdots \text{ad}_{\mathcal{T}}^{w_1}(H_c^{M_1}) R_{-c}^{Q_1}. \quad (4.20)$$

Since the operators $\text{ad}_{\mathcal{T}}^{w_p}(H_c^{M_p})$ need not commute, the symbol $\prod_{p=1}^i \text{ad}_{\mathcal{T}}^{w_p}(H_c^{M_p})$ will always mean that the product is to be written in decreasing order from left to right, that is the highest index appears on the very left and the lowest index on the very right.

Note that this notation is somewhat contrary to $\prod_i \Phi(g_i)$ due to the order in which the indexes appear. But since these symbols will not be used simultaneously, no confusion can arise. Moreover, note that if we define

$$I_j := \prod_{p=1}^j \text{ad}_{\mathcal{T}}^{w_p}(H_c^{M_p}) R_{-c}^{Q_p}$$

for $j \geq 1$ and assume $0 = k = \sum_{p=1}^i \mathbf{n}_{w_p}^A$, we would necessarily have $T = 0$. Indeed, $k = 0$ would imply $\mathbf{n}_{w_p}^A = 0$ which in turn implies that $\text{ad}_{\mathcal{T}}^{w_p}(H_c^{M_p}) = 0$ for all p . This motivates the definition $\mathfrak{U}_{0,k,b} = 0$ for $k \neq 0$.

Clearly, $\mathfrak{U}_{i,k,b} \subset \mathcal{B}(\mathcal{H})$. Since the operators which are of class $C^1(A)$ form an algebra, every element of $\mathfrak{U}_{i,k,b}$ is of class $C^1(A)$ by Lemma 4.2. This implies that taking a commutator with A defines a map $\text{ad}_A(\cdot) : \mathfrak{U}_{i,k,b} \rightarrow \mathcal{B}(\mathcal{H})$.

Lemma 4.4. *Let $i, k \in \mathbb{N}_0$, $b \in \mathbb{N}$ and suppose $b \geq 3 \cdot 2^i$. Then*

$$\text{ad}_A(\mathfrak{U}_{i,k,b}) \subset \mathfrak{U}_{i+1,k+1,b-3 \cdot 2^i}.$$

PROOF: The statement is proven by induction in i . In the case $i = 0$ and $k \neq 0$ there is nothing to show. Let $i = k = 0$, $B \geq b \in \mathbb{N} \cap [3, \infty)$ and $U_{0,0,B} := R_{-c}^B \in \mathfrak{U}_{0,0,b}$. Due to $B \geq 3$, we may compute on \mathcal{C}_0^∞

$$\begin{aligned} \text{ad}_A(U_{0,0,B}) &= -R_{-c}^B \text{ad}_A((H+c)^B) R_{-c}^B \\ &= -\sum_{\ell=0}^3 \binom{3}{\ell} (-1)^\ell R_{-c}^{B-\ell} \text{ad}_H^{3-\ell}(\text{ad}_A((H+c)^B)) R_{-c}^{B+3}. \end{aligned}$$

The equation extends to the whole Hilbert space by continuity and density of \mathcal{C}_0^∞ . Moreover, $\text{ad}_A(U_{0,0,B}) \in \mathfrak{U}_{1,1,b-3}$ which proves the statement in the case $i = 0$.

Now suppose there exists $i \in \mathbb{N}$ such that the statement is correct for all $i' \leq i - 1$. Let $b \geq 3 \cdot 2^i$ and

$$U_{i,k,b} := R_{-c}^B \prod_{p=1}^j \text{ad}_{\mathcal{T}}^{w_p}(H_c^{M_p}) \in \mathfrak{U}_{i,k,b}.$$

In order to reduce notation we define

$$S^{w_p} := \text{ad}_{\mathcal{T}}^{w_p}(H_c^{M_p}), \quad p = 1, \dots, i.$$

The main idea of the proof is to use Lemma 4.2 and the algebra structure of the $C^1(A)$ class to obtain an expression for $\text{ad}_A(U_{i,k,b})$ which is then shown to be an element of $\mathfrak{U}_{i+1,k+1,b-3\cdot 2^i}$. Since $\text{ad}_A(\cdot)$ defines a derivation on $C^1(A)$, $\text{ad}_A(U_{i,k,b})$ is a sum of several terms. By the induction hypothesis all contributions of the form

$$R_{-c}^B S^{w_i} R_{-c}^{Q_i} \cdots \text{ad}_A \left(S^{w_\ell} R_{-c}^{Q_\ell} \right) \cdots S^{w_2} R_{-c}^{Q_2} S^{w_1} R_{-c}^{Q_1},$$

where $2 \leq \ell \leq i$, are elements of $\mathfrak{U}_{i+1,k+1,b-3\cdot 2^i}$. Hence, it remains to examine the term

$$R_{-c}^B S^{w_i} R_{-c}^{Q_i} \cdots S^{w_2} R_{-c}^{Q_2} \text{ad}_A \left(S^{w_1} R_{-c}^{Q_1} \right),$$

where the restriction $B \geq 3 \cdot 2^i$ plays a crucial role. With our abbreviations Lemma 4.2 takes the form

$$\begin{aligned} \text{ad}_A \left(S^{w_1} R_{-c}^{Q_1} \right) &= \text{ad}_A \left(S^{w_1} \right) R_{-c}^{Q_1} \\ &+ \sum_{k=0}^1 (-1)^{k+1} S^{w_1} R_{-c}^{Q_1-k} \text{ad}_H^{1-k} \left(\text{ad}_A \left(H_c^{Q_1} \right) \right) R_{-c}^{Q_1+1}. \end{aligned} \quad (4.21)$$

Thus, there are two main contributions to consider: $\text{ad}_A(S^{w_j})R_{-c}^{Q_j}$ and the sum.

We deal with the sum first. The fewest amount of resolvents occurs for $k = 1$ which we take as the starting point of our analysis. Then we argue that the $k = 0$ contribution can be dealt with in a similar fashion but with less severe restrictions on B . Define the set

$$W_i := \left\{ (k_i, \dots, k_0) \in \mathbb{N}^{i+1} \mid k_j \in [0, 3 \cdot 2^j], j = 1, \dots, i, k_0 \in [0, 2] \right\}.$$

We claim that for any $i \in \mathbb{N}$ and all

$$\begin{aligned} &R_{-c}^B S^{w_i} R_{-c}^{Q_i} \cdots S^{w_2} R_{-c}^{Q_2} S^{w_1} R_{-c}^{Q_1-1} \text{ad}_A \left(H_c^{Q_1} \right) R_{-c}^{Q_1+1} \\ &= \sum_{(k_i, \dots, k_0) \in W_i} C_{k_0, \dots, k_i} R_{-c}^{B-k_i} S(k_i, \dots, k_0) \text{ad}_H^{2-k_0} (S_0) R_{-c}^{Q_1+3}, \end{aligned} \quad (4.22)$$

where

$$\begin{aligned} S(k_i, \dots, k_0) &:= \prod_{j=1}^i \text{ad}_H^{3 \cdot 2^j - k_j} (S^{w_j}) R_{-c}^{Q_j + 3 \cdot 2^j - k_{j-1} - \delta_{1,j}}, \\ S_0 &:= \text{ad}_A(H_c^{Q_1}), \quad B \geq 3 \cdot 2^i, \\ C_{k_0, \dots, k_i} &= \binom{3 \cdot 2^i}{k_i} \cdots \binom{2}{k_0} (-1)^{k_0 + \dots + k_i}. \end{aligned}$$

Clearly, if (4.22) is valid, then

$$R_{-c}^B S^{w_i} R_{-c}^{Q_i} \cdots S^{w_2} R_{-c}^{Q_2} S^{w_1} R_{-c}^{Q_1-1} \text{ad}_A \left(H_c^{Q_1} \right) R_{-c}^{Q_1+1} \in \mathfrak{U}_{i+1,k+1,b-3\cdot 2^i}$$

and it thus suffices to prove (4.22) in order to deal with this term.

To establish the validity of (4.22) we use an inductive proof. Let $i = 1$ and $B_1 \geq 6$. A two-fold application of the rearrangement formula in Lemma A.7 gives

$$\begin{aligned}
& R_{-c}^B S^{w_1} R_{-c}^{Q_1-1} \text{ad}_A (H_c^{Q_1}) R_{-c}^{Q_1+1} \\
&= \sum_{k_1=0}^6 \binom{6}{k_1} (-1)^{k_1} R_{-c}^{B_1-k_1} \text{ad}_H^{6-k_1} (S^{w_1}) R_{-c}^{Q_1+6-1} \text{ad}_A (H_c^{Q_1}) R_{-c}^{Q_1+1} \\
&= \sum_{k_1=0}^6 \sum_{k_0=0}^2 \binom{6}{k_1} \binom{2}{k_0} (-1)^{k_1+k_2} R_{-c}^{B_1-k_1} \text{ad}_H^{6-k_1} (S^{w_1}) R_{-c}^{Q_1+6-1-k_0} \\
&\quad \times \text{ad}_H^{2-k_0} (\text{ad}_A (H_c^{Q_1})) R_{-c}^{Q_1+3}.
\end{aligned}$$

Hence, the claim is true for $i = 1$. Now assume it is true for some $i \in \mathbb{N}$ and choose $B \geq 3 \cdot 2^{i+1}$. Consider a similar operator as before consisting out of $i + 1$ products of the form $S^{w_j} R_{-c}^{Q_j}$ instead of just i . Then another application of the rearrangement formula implies

$$\begin{aligned}
& R_{-c}^B S^{w_{i+1}} R_{-c}^{Q_{i+1}} S^{w_i} R_{-c}^{Q_i} \cdots S^{w_1} \text{ad}_A (H_c^{Q_1}) R_{-c}^{Q_1+1} \\
&= \sum_{k_{i+1}=0}^{3 \cdot 2^{i+1}} \binom{3 \cdot 2^{i+1}}{k_{i+1}} (-1)^{k_{i+1}} R_{-c}^{B-k_{i+1}} \text{ad}_H^{3 \cdot 2^{i+1}-k_{i+1}} (S^{w_{i+1}}) R_{-c}^{Q_{i+1}+3 \cdot 2^{i+1}} \\
&\quad \times S^{w_i} R_{-c}^{Q_i} \cdots S^{w_1} R_{-c}^{Q_1-1} \text{ad}_A (H_c^{Q_1}) R_{-c}^{Q_1+1}.
\end{aligned}$$

Note that the H -commutator with $S^{w_{i+1}}$ is bounded by $Q_{i+1} + 3 \cdot 2^i$ resolvents. Since there are $Q_{i+1} + 3 \cdot 2^i + 3 \cdot 2^i$ resolvent to its right, there are sufficiently many resolvents left for us to use the induction hypothesis with $B = 3 \cdot 2^i$ and the remaining i factors in each product. This proves (4.22).

The contribution arising from the term $k = 0$ in (4.21), can be treated in complete analogy to the case, where $k = 1$. Define the set

$$W'_i := \{(k_0, \dots, k_i) \in \mathbb{N}^{i+1} \mid k_j \in [0, 2^{j+1}], j = 0, \dots, i\}$$

We can establish inductively that for any $i \in \mathbb{N}$

$$\begin{aligned}
& R_{-c}^B S^{w_i} R_{-c}^{Q_i} \cdots S^{w_2} R_{-c}^{Q_2} S^{w_1} R_{-c}^{Q_1} \text{ad}_H (\text{ad}_A (H_c^{Q_1})) R_{-c}^{Q_1+1} \\
&= \sum_{(k_0, \dots, k_i) \in W'_i} C'_{k_0, \dots, k_i} R_{-c}^{B-k_i} \mathcal{S}'(k_0, \dots, k_i) \text{ad}_H^{3-k_0} (S_0) R_{-c}^{Q_1+3}, \quad (4.23)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{S}'(k_0, \dots, k_i) &:= \prod_{j=1}^i \text{ad}_H^{2^{j+1}-k_j} (S^{w_j}) R_{-c}^{Q_i+2^{j+1}-k_{j-1}}, \\
C'_{k_0, \dots, k_i} &= \binom{2^{i+1}}{k_i} \cdots \binom{2}{k_0} (-1)^{k_0+\dots+k_i}, \quad B \geq 2^{i+1}.
\end{aligned}$$

Once more, it should be clear that

$$R_{-c}^B S^{w_i} R_{-c}^{Q_i} \cdots S^{w_2} R_{-c}^{Q_2} S^{w_1} R_{-c}^{Q_1} \operatorname{ad}_H (\operatorname{ad}_A (H_c^{Q_1})) R_{-c}^{Q_1+1} \in \mathfrak{U}_{i+1, k+1, b-3 \cdot 2^i},$$

if (4.23) holds. As we have mentioned above the proof of (4.23) is very similar to the proof of (4.22) and will thus be omitted. However, to illustrate why the summation indexes are drawn from W'_i rather than W_i we provide the induction start. Let $B \geq 4$. A two-fold application of Lemma A.7 gives

$$\begin{aligned} & R_{-c}^B S^{w_1} R_{-c}^{Q_1} \operatorname{ad}_H (\operatorname{ad}_A ((H+c)^{Q_1})) R_{-c}^{Q_1+1} \\ &= \sum_{k_1=0}^4 \sum_{k_0=0}^2 \binom{4}{k_1} \binom{2}{k_0} (-1)^{k_1+k_0} R_{-c}^{B-k_1} \operatorname{ad}_H^{4-k_1} (S^{w_1}) R_{-c}^{Q_1+4-k_0} \operatorname{ad}_H^{3-k_0} (S_0) R_{-c}^{Q_1+3}. \end{aligned}$$

Last but not least yet another slightly modified version of the preceding arguments shows that for $B \geq 2^i$

$$\begin{aligned} & R_{-c}^B S^{w_i} R_{-c}^{Q_i} \cdots S^{w_2} R_{-c}^{Q_2} \operatorname{ad}_A (S^{w_1}) R_{-c}^{Q_1} \\ &= \sum_{(k_1, \dots, k_i) \in W''_i} C''_{k_1, \dots, k_i} R_{-c}^{B-k_i} S''(k_1, \dots, k_i) \operatorname{ad}_H^{3-k_0} (S_0) R_{-c}^{Q_1+2}, \end{aligned} \quad (4.24)$$

where

$$\begin{aligned} S''(k_1, \dots, k_i) &= \prod_{j=1}^i \operatorname{ad}_H^{2^j - k_j} (S^{w_j}) R_{-c}^{Q_i + 2^j - k_{j-1} + \delta_{1,j}}, \\ C_{k_1, \dots, k_i} &= \binom{2^i}{k_i} \cdots \binom{2}{k_1} (-1)^{k_1 + \dots + k_i}, \\ W''_i &:= \{(k_1, \dots, k_i) \in \mathbb{N}^i \mid k_j \in [0, 2^j]\}. \end{aligned}$$

As before the difference in the summation set is due to a different induction start.

Hence all contributions arising from (4.21) have been shown to be elements of $\mathfrak{U}_{i+1, k+1, b-3 \cdot 2^i}$. Since any $U'_{i, k, b} \in \mathfrak{U}_{i+1, k+1, b-3 \cdot 2^i}$ is a finite linear combination of elements of the form $U_{i, k, b}$, the proof is complete. \square

The previous Lemma enables us to prove the main statement of this section: For any $k \in \mathbb{N}$ there exists a sufficiently large $m_k \in \mathbb{N}$ such that $R^{m_k} \in C^1(A)$.

Proposition 4.5 (Large Resolvent Powers are in $C^k(A)$ I). *Let $k \in \mathbb{N}_0$ and suppose that $m \in \mathbb{N}$ satisfies $m \geq m_k := \sum_{j=0}^k 3 \cdot 2^j$. Then $(H(\xi) + c)^{-m}$ is of class $C^{k+1}(A)$. In particular,*

$$\operatorname{ad}_A^k ((H(\xi) + c)^{-m}) \in \mathfrak{U}_{k, k, m-m_k}. \quad (4.25)$$

PROOF: Since $(H(\xi) + c)^m \in \mathfrak{U}_{0,0,m}$, the statement is an immediate consequence of Lemma 4.4. Indeed, due to $m \geq m_k := 3 + 3 \cdot 2 + \dots + 3 \cdot 2^k$, a k -fold application of the inclusion $\text{ad}_A(\mathfrak{U}_{i,k,b}) \subset \mathfrak{U}_{i+1,k+1,b-3 \cdot 2^i}$ completes the proof. \square

It should be noted that this result is most likely not optimal. For example it is known from Morten Grud Rasmussen's PhD thesis, see [44], Proposition 2.5, that $(H(\xi) - z)^{-1} \in C^2(A)$, whereas Proposition 4.5 states that $(H(\xi) - z)^{-m} \in C^2(A)$ for $m \geq m_1 = 3 \cdot 2 + 3 = 9$. However the mentioned result is of a different nature than ours, since it is not clear that $(H(\xi) - z)^{-1} \in C^2(A)$ implies $(H(\xi) - z)^{-m} \in C^2(A)$ for all $m \in \mathbb{N}$.

4.2 Local Regularity

One of our main results is stated right in the beginning. It guarantees that for any smooth and compactly supported function f the bounded operator $f(H(\xi))$ is of class $C^k(A)$ for any $k \in \mathbb{N}$.

Theorem 4.6 ($f(H(\xi)) \in C^k(A)$ I). *Let the coupling function $g \in C_0^\infty(\mathbb{R}^\nu)$ and suppose that $f \in C_0^\infty(\mathbb{R})$. Then $f(H(\xi)) \in C^k(A)$ for all $k \in \mathbb{N}$.*

The remaining part of the section is devoted to the proof of this statement. The most important tool used in the proof is that $(H + c)^{-m}$ is of class $C^k(A)$ for sufficiently large $m \in \mathbb{N}$ and any $k \in \mathbb{N}$, see Proposition 4.5. In Lemma 4.9 we prove that for any $k \in \mathbb{N}$ $(H + c)^{-m}(H - z)^{-1} \in C^k(A)$ for sufficiently large m . The proof of Lemma 4.9 is similar to the strategy used in Lemma 4.4 and we refer to the proof of the latter whenever analogies in the argumentation arise. Furthermore, the proof of Theorem 4.6 and Proposition 4.5 follow more or less the same strategy and we will refer to the proof of the Proposition as often as possible to avoid repetitions.

Notation 4.7.

1. Throughout this section we simply write H and H_0 instead of $H(\xi)$ and $H_0(\xi)$ respectively.
2. For any $z \in \rho(H)$ we define $R_z(H) := (H - z)^{-1}$. If no confusions can arise, we drop the H dependence and write R_z instead of $R_z(H)$.
3. Let $i \in \mathbb{N}$, $\theta \in \{0, 1\}$, $\underline{w}_1 \in \mathfrak{J}_{n(1)}, \dots, \underline{w}_i \in \mathfrak{J}_{n(i)}$, $M_1, \dots, M_i \in \mathbb{N}$ and $Q_1, \dots, Q_i \in \mathbb{N}$ with $Q_j \geq M_j + \text{Int}(|\underline{w}_j|/2)$. For any $z \in \rho(H)$ we introduce the notation

$$\prod_{p=1}^i R_z^{\theta_p} \text{ad}_{\mathcal{T}}^{\underline{w}_p}(H_c^{M_p}) R_{-c}^{Q_p} := R_z^\theta \text{ad}_{\mathcal{T}}^{\underline{w}_i}(H_c^{M_i}) R_{-c}^{Q_i} \dots R_z^\theta \text{ad}_{\mathcal{T}}^{\underline{w}_1}(H_c^{M_1}) R_{-c}^{Q_1}. \quad (4.26)$$

The symbol $\prod_{p=1}^i \text{ad}_{\mathcal{T}^p}^{w_p}(H_c^{M_p})$ will always mean that the product is to be written in decreasing order from left to right, that is the highest index appears on the very left and the lowest index on the very right.

Definition 4.8. Let $i, k, b \in \mathbb{N}$. Define

$$\begin{aligned} \mathfrak{V}_{i,k,b}^z &:= \text{Span}\{R_{-c}^B \left(\prod_{p=1}^i R_z^\theta \text{ad}_{\mathcal{T}^p}^{w_p}(H_c^{M_p}) R_{-c}^{Q_p} \right) R_z \mid B \geq b, 1 \leq j \leq i, \sum_{p=1}^i \mathbf{n}_{w_p}^A = k \\ &\quad \theta \in \{0, 1\}, Q_p \geq M_p + \text{Int}(|w_p|/2) + 1, \}, \\ \mathfrak{V}_{0,0,b}^z &:= \text{Span}\{R_{-c}^B R_z \mid B \geq b\}, \\ \mathfrak{V}_{0,k,b}^z &:= 0. \end{aligned}$$

As in the case of $\mathfrak{U}_{i,k,b}$ we clearly have that $\mathfrak{V}_{i,k,b}^z \subset \mathcal{B}(\mathcal{H})$ and that every element of $\mathfrak{V}_{i,k,b}^z$ is of class $C^1(A)$. Again, this implies that taking a commutator with A defines a linear map $\text{ad}_A : \mathfrak{V}_{i,k,b}^z \rightarrow \mathcal{B}(\mathcal{H})$.

Lemma 4.9. Let $i, k \in \mathbb{N}_0$, $b \in \mathbb{N}$ and suppose $b \geq 5 \cdot 2^i$. Then

$$\text{ad}_A(\mathfrak{V}_{i,k,b}) \subset \mathfrak{V}_{i+1,k+1,b-5 \cdot 2^i}.$$

PROOF: We introduce the abbreviation

$$S_{w_i} := \text{ad}_{\mathcal{T}^i}^{w_i}((H+c)^{M_i}).$$

We prove the statement by induction. For $i = k = 0$ and $B \geq b \geq 5$ we put $V_{0,0,B} := R_{-c}^B R_z \in \mathfrak{V}_{0,0,b}$ and compute

$$\text{ad}_A(V_{0,0,b}) = \text{ad}_A(R_{-c}^B) R_z - R_{-c}^B R_z \text{ad}_A(H) R_z$$

on \mathcal{C}_0^∞ . The first term can be dealt with as in Lemma 4.4 which shows that $\text{ad}_A(R_{-c}^B) R_z \in \mathfrak{V}_{1,1,b-3}$. As for the second term we compute

$$R_{-c}^B \text{ad}_A(H) = (-1)^B \sum_{\ell=0}^5 \binom{5}{\ell} (-1)^\ell R_{-c}^{B-\ell} \text{ad}_H^{5-\ell}(\text{ad}_A(H)) R_{-c}^5 \quad (4.27)$$

and thus $R_{-c}^B R_z \text{ad}_A(H) R_z \in \text{ad}_A(R_{-c}^B) R_z \in \mathfrak{V}_{1,1,b-5}$. This completes the induction start.

To perform the induction step suppose that the statement is correct for $i' \leq i-1$ and let $i, k \in \mathbb{N}$ and $B \geq b \geq 5 \cdot 2^i$. Put

$$V_{i,k,b} := R_{-c}^B R_z^\theta S_{w_i} R_{-c}^{Q_i} R_z^\theta \cdots R_z^\theta S_{w_1} R_{-c}^{Q_1} R_z \in \mathfrak{V}_{i,k,b}^z.$$

After taking a commutator with A we have to redistribute resolvents in order to obtain expressions which are linear combinations of operators of the form $V_{i+1,k+1,b}$. As in Lemma 4.4 the induction hypothesis implies that we have two main contributions to consider:

$$R_{-c}^B R_z^\theta S_{w_i} R_{-c}^{Q_i} \cdots R_z^\theta \text{ad}_A \left(S_{w_1} R_{-c}^{Q_1} \right) R_z$$

and

$$R_{-c}^B R_z^\theta S_{w_i} R_{-c}^{Q_i} R_z^\theta \cdots R_z^\theta S_{w_1} R_{-c}^{Q_1} \text{ad}_A(R_z).$$

As in the case $i = 0$ the first of the two relevant cases can be dealt with as in Lemma 4.4, since $B \geq 5 \cdot 2^i > 3 \cdot 2^i$. In order to deal with the second term we compute

$$\begin{aligned} & R_{-c}^B R_z^\theta S_{w_i} R_{-c}^{Q_i} R_z^\theta \cdots R_z^\theta S_{w_1} R_{-c}^{Q_1} \text{ad}_A(R_z) \\ &= -R_{-c}^B R_z^\theta S_{w_i} R_{-c}^{Q_i} R_z^\theta \cdots R_z^\theta S_{w_1} R_{-c}^{Q_1} R_z \text{ad}_A(H) R_z. \end{aligned} \quad (4.28)$$

Observe that it requires 5 additional resolvents to turn $R_z \text{ad}_A(H) R_z$ into an operator in $C^1(A)$, if we do not want to incorporate R_z into the calculation.

Due to structural similarities we would like to deal with (4.28) as in Lemma 4.4. (4.27) suggests that this would require us to adapt (4.22)-(4.24) to the current situation. This however is straight-forward due to the choice $B \geq 5 \cdot 2^i$. We omit the details to avoid unnecessary repetition. \square

Lemma 4.10. *Let $k \in \mathbb{N}_0$, $m \in \mathbb{N}$ and suppose that $m \geq m'_k := \sum_{j=0}^k 5 \cdot 2^j$. Then $(H(\xi) + c)^{-m} (H(\xi) + z)^{-1}$ is of class $C^{k+1}(A)$ for any $z \in \rho(H(\xi))$. Moreover,*

$$\text{ad}_A^k((H(\xi) + c)^{-m} (H(\xi) + z)^{-1}) \in \mathfrak{Y}_{k,k,m-m'_k}^z. \quad (4.29)$$

PROOF: As in the proof of Proposition 4.5 the statement follows from a k -fold application of Lemma 4.9, since $(H(\xi) + c)^{-m} (H(\xi) + z)^{-1} \in \mathfrak{Y}_{0,0,m}$ and $m \geq m'_k$. \square

PROOF OF THEOREM 4.6: Suppose $f \in C_0^\infty(\mathbb{R})$ and $k \in \mathbb{N}$. By Lemma 4.10 there exists $m'_k \in \mathbb{N}$ such that for all $m \geq m'_k$ $(H - c)^{-m} (H - z)^{-1} \in C^k(A)$ and $\text{ad}_A^k((H - c)^{-m} (H - z)^{-1}) \in \mathfrak{Y}_{k,k,m-m_k}^z$.

We start by deriving an estimate of the norm of a typical element of $\mathfrak{Y}_{k,k,m-m_k}^z$. Let

$$R_{-c}^B \prod_{p=1}^k (R_z^\theta \text{ad}_{\mathcal{T}}^{w_p} (H_c^{M_p}) R_{-c}^{Q_p}) R_z \in \mathfrak{Y}_{k,k,m-m_k}^z$$

and estimate

$$\begin{aligned}
& \|R_{-c}^B \prod_{p=1}^k (R_z^\theta \operatorname{ad}_{\mathcal{T}^p}^{w_p}(H_c^{M_p}) R_{-c}^{Q_p}) R_z\| \\
& \leq \|R_{-c}^B\| \|R_z\|^{k\theta+1} \prod_{p=1}^k \| \operatorname{ad}_{\mathcal{T}^p}^{w_p}(H_c^{M_p}) R_{-c}^{Q_p} \| \\
& \leq |\operatorname{Im}(z)|^{-k\cdot\theta-1} \|R_{-c}^B\| \prod_{p=1}^k \| \operatorname{ad}_{\mathcal{T}^p}^{w_p}(H_c^{M_p}) R_{-c}^{Q_p} \|.
\end{aligned}$$

By Lemma 4.10 there exist $j_k \in \mathbb{N}$, integers $1 \leq p_1, \dots, p_k \leq k$, integers $B_1, \dots, B_{j_k}, \geq m - m_k$, constants C_1, \dots, C_{j_k} and operators

$$\begin{aligned}
V_{p_1, k, B_1}^z & := R_{-c}^{B_1} \prod_{q=1}^{p_1} (R_z^{\theta_1} \operatorname{ad}_{\mathcal{T}^q}^{w_q}(H_c^{M_q}) R_{-c}^{Q_q}) R_z \in \mathfrak{V}_{k, k, m-m_k}^z \\
& \vdots \\
V_{p_{j_k}, k, B_{j_k}}^z & := R_{-c}^{B_{j_k}} \prod_{q=1}^{p_{j_k}} (R_z^{\theta_{j_k}} \operatorname{ad}_{\mathcal{T}^q}^{w_q}(H_c^{M_q}) R_{-c}^{Q_q}) R_z \in \mathfrak{V}_{k, k, m-m_k}^z
\end{aligned}$$

such that

$$\operatorname{ad}_A^k((H+c)^m) = \sum_{i=1}^{j_k} C_i V_{p(i), k, m-m_k}^z.$$

From the previous estimate we deduce that

$$\begin{aligned}
\left\| \sum_{i=1}^{j_k} C_i V_{p(i), k, m-m_k}^z \right\| & \leq \sum_{i=1}^{j_k} C_i |\operatorname{Im}(z)|^{-p_i \cdot \theta_i - 1} \|R_{-c}^B\| \prod_{q=1}^{p_i} \| \operatorname{ad}_{\mathcal{T}^q}^{w_q}(H_c^{M_q}) R_{-c}^{Q_q} \| \\
& = \sum_{i=1}^{j_k} C'_i \cdot |\operatorname{Im}(z)|^{-p_i \cdot \theta_i - 1}
\end{aligned}$$

for some constants $C'_i > 0$.

Fix $m \geq m'_k$. The main strategy of the proof is to use the Helffer-Sjöstrand formula to see that

$$f(H) = f_m(H)(H+c)^{-m} = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}_m}{\partial \bar{z}}(z) (H-z)^{-1} (H+c)^{-m} dz,$$

where $f_m \in C_0^\infty(\mathbb{R})$ is given by $f_m(x) := f(x)(x+c)^m$. The preceding estimates can now be used to bound the integral. To this end we choose an almost analytic

extension \tilde{f}_m of f_m with compact support such that

$$\forall N \in \mathbb{N} \exists D_N > 0 : |\tilde{f}_m(z)| \leq D_N |z|^N,$$

see [9]. We can thus estimate

$$\begin{aligned} & \left\| \int_{\mathbb{C}} \frac{\partial \tilde{f}_m}{\partial \bar{z}}(z) \operatorname{ad}_A^k((H - z)^{-1}(H + c)^{-m}) dz \right\| \\ & \leq \sum_{i=1}^{j_k} C'_i D_{p_i \cdot \theta_i + 1} \int_{\operatorname{supp}(\frac{\partial \tilde{f}_m}{\partial \bar{z}})} \frac{|\operatorname{Im}(z)|^{p_i \cdot \theta_i + 1}}{|\operatorname{Im}(z)|^{p_i \cdot \theta_i + 1}} dz < \infty. \end{aligned}$$

With this statement at our disposal we may resume the proof of the $C^k(A)$ property. Clearly, $f(H) \in C^1(A)$, see for instance [21]. Now suppose there is a $k \in \mathbb{N}$ such that $f(H)$ is in $C^k(A)$ for any smooth and compactly supported real-valued function f . Fix $m \geq m_{k+1} > m_k$ with m_k defined as above and choose any $f \in C_0^\infty(\mathbb{R})$. With the same definition as above this implies

$$\begin{aligned} \operatorname{ad}_A^k(f(H)) &= \operatorname{ad}_A^k(f_m(H)(H + c)^{-m}) \\ &= -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}_m}{\partial \bar{z}}(z) \operatorname{ad}_A^k((H - z)^{-1}(H + c)^{-m}) dz, \end{aligned}$$

where we have used the strong closedness of $C^1(A)$ in $\mathcal{B}(\mathcal{H})$ and that $\operatorname{ad}_A^k((H - z)^{-1}(H + c)^{-m})$ yields an integrable expression. But by our choice of m the integrand admits another commutator with A which will also give an integrable expression, since the same type of estimates hold. Using the strong closedness of the $C^1(A)$ class once more, we can conclude that $f(H)$ is in fact in $C^{k+1}(A)$. \square

Lemma 4.11. *Let $\lambda_\xi \in \mathcal{E}^{(1)}(\xi) \setminus \mathcal{T}^{(1)}(\xi)$ be an eigenvalue of $H(\xi)$ and $\eta \in D(H(\xi))$ be a corresponding eigenvector, that is $H(\xi)\eta = \lambda_\xi \eta$. Then*

1. $\eta \in D(A^n)$ for all $n \in \mathbb{N}$.
2. Let $n \in \mathbb{N}$. Then $A^n \eta \in D(H(\xi)^m)$ for all $m \in \mathbb{N}$.

PROOF: By Theorem 4.6 we have that $f(H(\xi)) \in C^k(A)$ for all $k \in \mathbb{N}$ and for all $f \in C_0^\infty(\mathbb{R})$. By the regularity of eigenstates established in [40] $\eta \in D(A^k)$ for all $k \in \mathbb{N}$. This proves the first part of the statement. The second part can be proven by induction in $n \in \mathbb{N}$. The case $n = 1$ is true, since

$$H(\xi)^m A \eta = \lambda_\xi^m A \eta + \operatorname{ad}_A(H(\xi)^m) \eta,$$

where we have used that the form identity $[A, H(\xi)^m] = \operatorname{ad}_A(H(\xi)^m)$ extends from C_0^∞ to $D(A) \cap D(H(\xi)^{m+1})$. Now suppose that there exists $n \in \mathbb{N}$ such

that $A^{n'}\eta \in D(H(\xi)^m)$ for all $m \in \mathbb{N}$ and all $n' \leq n$. Since $\text{ad}_A^\ell(H(\xi)^m)$ is $(H(\xi) + c)^{m+\ell/2}$ -bounded, we have that

$$A^j\eta \in D(\text{ad}_A^{n-j}(H(\xi)^m))$$

for all $j \in \mathbb{N} \cap [1, n]$. Thus, the induction hypothesis implies that (3.6) in Lemma 3.8 extends from $\psi \in \mathcal{C}_0^\infty$ to η . This implies that

$$H(\xi)^m A^{n+1}\eta = \lambda_\xi^m A^{n+1}\eta + \sum_{j=0}^n \binom{n+1}{j} \text{ad}_A^{n+1-j}(H(\xi)^m) A^j\eta$$

and hence $A^{n+1}\eta \in D(H(\xi)^m)$. \square

5 Extension to More General Coupling Functions

Lemma 5.1. *Let \mathfrak{H} be a Hilbert space, $\psi \in \Gamma_{\text{fin}}(\mathfrak{H})$ and $f_1, \dots, f_k \in \mathfrak{H}$. Denote by $N_\psi := \max\{n \in \mathbb{N} \mid \psi^{(n)} \neq 0\} \in \mathbb{N}$. Then there exists $C_n > 0$ independent of f_1, \dots, f_k such that*

$$\left\| \prod_{i=1}^{\ell} \Phi(f_i)\psi \right\| \leq C_n (N_\psi + 1)^{\frac{\ell}{2}} \cdot \prod_{i=1}^{\ell} \|f_i\|. \quad (5.30)$$

PROOF: We use the estimate

$$\|(N+1)^p \prod_{i=1}^{\ell} \Phi(f_i) (N+1)^{-p-\ell/2}\| \leq C_{n,p} \prod_{i=1}^{\ell} \|f_i\|$$

proven in CITE DER/GER for the special case $p = 0$. \square

Lemma 5.2. *Let $\psi \in \Gamma_{\text{fin}}(\mathfrak{H})$ and $f_1, \dots, f_k \in \mathfrak{H}$. Choose sequences $(f_{j,n})_n$ such that $\|f_j - f_{j,n}\| \rightarrow 0$. Then*

$$\lim_{n \rightarrow \infty} \|\Phi(f_1) \cdots \Phi(f_\ell)\psi - \Phi(f_{1,n}) \cdots \Phi(f_{\ell,n})\psi\| = 0. \quad (5.31)$$

PROOF: The main idea is to rewrite the terms in (5.31) in a telescopic sum and use (5.30) in order to show that the limit is 0. For $\ell \in \mathbb{N}$ we abbreviate $C_\ell = (N_\psi + 1)^{\frac{\ell}{2}}$ and compute

$$\begin{aligned} & \|\Phi(f_1) \cdots \Phi(f_\ell)\psi - \Phi(f_{1,n}) \cdots \Phi(f_{\ell,n})\psi\| \\ &= \left\| \sum_{k=1}^{\ell} \Phi(f_{1,n}) \cdots \Phi(f_{k-1,n}) \Phi(f_k - f_{k,n}) \Phi(f_k) \cdots \Phi(f_\ell)\psi \right\| \\ &\leq \sum_{k=1}^{\ell} \|\Phi(f_{1,n}) \cdots \Phi(f_{k-1,n}) \Phi(f_k - f_{k,n}) \Phi(f_k) \cdots \Phi(f_\ell)\psi\| \\ &\leq \|\psi\| C_\ell \sum_{k=1}^{\ell} \|f_{1,n}\| \cdots \|f_{k-1,n}\| \|f_{k,n} - f_k\| \|f_{k+1}\| \cdots \|f_\ell\|. \end{aligned}$$

There is a constant $D > 0$ such that $\|f_{j,n}\| \leq D$ for all $j = 1, \dots, \ell$, since $f_{j,n}$ are convergent sequences and hence uniformly bounded in n . Therefore, $\|\Phi(f_1) \cdots \Phi(f_\ell)\psi - \Phi(f_{1,n}) \cdots \Phi(f_{\ell,n})\psi\|$ converges to 0 as $n \rightarrow \infty$. \square

5.1 A Fréchet Space of Coupling Functions

Throughout this chapter we will make use of the notation introduced in Definition 3.9. Recall the definitions

$$L_{\text{uv}}^2(\mathbb{R}^\nu) := \bigcap_{n=0}^{\infty} L^2(\mathbb{R}^\nu, (1 + |k|)^{2n} dx).$$

and

$$H_{\text{uv}}^k(\mathbb{R}^\nu) := L_{\text{uv}}^2(\mathbb{R}^\nu) \cap H_{\text{loc}}^k(\mathbb{R}^\nu).$$

It is sometimes convenient to define $H_{\text{uv}}^0(\mathbb{R}^\nu) := L_{\text{uv}}^2(\mathbb{R}^\nu)$. On $H_{\text{uv}}^k(\mathbb{R}^\nu)$ we define the semi-norms

$$\|f\|_n := \|(1 + |k|)^n f\|, \quad \|f\|_{n,\alpha} := \|1_{B_n(0)} \partial^\alpha f\|,$$

where $\alpha \in \mathbb{N}_0^k$ with $|\alpha| \leq k$ and $B_n(0) \subset \mathbb{R}^\nu$ is the open ball around 0 with radius n .

Remark 5.3. It should be clear that $M_f H_{\text{uv}}^k(\mathbb{R}^\nu) \subset H_{\text{uv}}^k(\mathbb{R}^\nu)$ for all $f \in C_p^\infty$ and all $k \in \mathbb{N}$. Moreover, $D_\eta^\alpha H_{\text{uv}}^k(\mathbb{R}^\nu) \subset H_{\text{uv}}^{k-|\alpha|}$ for all multi-indices α with $|\alpha| \leq k$.

Proposition 5.4. *The vector space $H_{\text{uv}}^k(\mathbb{R}^\nu)$ equipped with the topology generated by the countable family of seminorms $\bigcup_{n=0}^{\infty} \bigcup_{|\alpha| \leq k} \{\|\cdot\|_n, \|\cdot\|_{n+1,\alpha}\}$ is a Fréchet space.*

PROOF: It is clear that these norms give rise to a locally convex topological vector space. Since the family of norms is countable and separating, the topology is metrizable with the standard choice of translation invariant metric d . Next we show that the metric space equipped with this topology is actually complete.

Choose a sequence g_j in $H_{\text{uv}}^k(\mathbb{R}^\nu)$ which is Cauchy w.r.t. the metric d . Since this means that the sequence is Cauchy w.r.t. to any individual semi-norm, we have that g_j is Cauchy in $L^2(\mathbb{R}^\nu)$ and hence there exists $g \in L^2(\mathbb{R}^\nu)$ such that $g_j \rightarrow g$ in $L^2(\mathbb{R}^\nu)$.

Furthermore, note that for any $f \in H_{\text{uv}}^k(\mathbb{R}^\nu)$ we can reconstruct the norm of the k -th Sobolev space on $B_n(0)$ by

$$\|f\|_{H^k(B_n(0))} = \sum_{|\alpha| \leq k} \|f\|_{n,\alpha}.$$

Hence the sequence g_j is also Cauchy w.r.t. all $H^k(B_n(0))$ and completeness of these spaces implies that there exists a $H^k(B_n(0))$ limit $h^{(n)}$. Further note that

$$\|1_{B_n(0)}g_j - h^{(n)}\|_{L^2(\mathbb{R}^\nu)} \leq \|g_j - h^{(n)}\|_{H^k(B_n(0))}.$$

Uniqueness of limits implies that $h^{(n)} = 1_{B_n(0)}g$. Hence the limit function g is an element of $H_{\text{loc}}^k(\mathbb{R}^\nu)$ for all n .

It remains to show that $(1 + |k|)^n g$ is square integrable for all n . As above we argue that since $(1 + |k|)^n g_j$ is a Cauchy sequence in L^2 , it has an L^2 -limit $f^{(n)}$. Note that $f^{(0)} = g$ by definition. We claim that $f^{(n)} = (1 + |k|)^n g$ for all n . Calculate

$$\begin{aligned} \|(1 + |k|)^{-n} f^{(n)} - g_j\|_{L^2(\mathbb{R}^\nu)}^2 &= \int_{\mathbb{R}^\nu} \frac{|f^{(n)}(k) - g_j(1 + |k|)^n|^2}{(1 + |k|^2)^{2n}} d^\nu k \\ &\leq \|f^{(n)} - g_j(1 + |k|)^n\|_{L^2(\mathbb{R}^\nu)}^2 \longrightarrow 0. \end{aligned}$$

This implies $g_j \rightarrow (1 + |k|)^{-n} f^{(n)}$ in L^2 and since limits are unique we obtain

$$\forall n \in \mathbb{N} : f^{(n)} = (1 + |k|)^n g.$$

This completes the proof, since $f^{(n)} \in L^2$. \square

Furthermore, the compactly supported, smooth functions are dense in H_{uv}^k .

Proposition 5.5. $C_0^\infty(\mathbb{R})$ is dense in $H_{\text{uv}}^k(\mathbb{R}^\nu)$.

PROOF: We first show that the compactly supported functions in $H_{\text{uv}}^k(\mathbb{R}^\nu)$ are a dense subset of $H_{\text{uv}}^k(\mathbb{R}^\nu)$ and then establish that $C_0^\infty(\mathbb{R})$ is dense in the compactly supported functions in $H_{\text{uv}}^k(\mathbb{R}^\nu)$.

Let $f \in H_{\text{uv}}^k(\mathbb{R}^\nu)$ and choose a function $\eta \in C_0^\infty(\mathbb{R})$ which satisfies

1. $\eta(x) = 1$ for $|x| \leq 1$.
2. $\eta(x) = 0$ for $|x| \geq 2$.

Note that there is $M > 0$ such that $|\partial^\alpha \eta(x)| \leq M$ for all multiindices with $|\alpha| \leq k$. Define $\eta_j(x) := \eta(j^{-1}x)$ and note that $|\partial^\alpha \eta_j(x)| \leq Mj^{-|\alpha|}$ for all $|\alpha| \leq k$. Clearly, $\eta_j f \in H_{\text{uv}}^k(\mathbb{R}^\nu)$ for all j and has compact support. Let $n \in \mathbb{N}$ and calculate

$$\|f - \eta_j f\|_n^2 = \int_{\mathbb{R}^\nu} |1 - \eta_j(x)|^2 |f(x)|^2 (1 + |x|)^{2n} d^\nu x.$$

Since $1 - \eta_j$ converges pointwise to 0 and $|f|^2(1 + |x|)^{2n} \in L^1(\mathbb{R}^\nu)$ by assumption, we may apply Lebesgue's Theorem and conclude that $\|f - \eta_j f\|_n \rightarrow 0$ as $j \rightarrow \infty$ for all n . Similarly, there exists a constant $C > 0$ such that

$$\|f - \eta_j f\|_{n,\alpha} \leq \|1_{B^{-n}(0)}(1 - \eta_j)\partial^\alpha f\|_2 + \sum_{\substack{\beta < \alpha \\ \beta \in \mathbb{N}_0^\nu}} \binom{\alpha}{\beta} \|\partial^\alpha f \partial^{\alpha-\beta} \eta_j\|,$$

where we have used that $f \in H_{\text{loc}}^k(\mathbb{R}^\nu)$. Obviously, $\partial^{\alpha-\beta}\eta_j$ converges pointwise to 0 and $\|f - f\eta_j\|_{n,\alpha} \rightarrow 0$ for $j \rightarrow \infty$ follows from Lebesgue's Theorem. This establishes that the compactly supported functions in $H_{\text{uv}}^k(\mathbb{R}^\nu)$ are a dense subset of $H_{\text{uv}}^k(\mathbb{R}^\nu)$.

In order to show that $C_0^\infty(\mathbb{R}^\nu)$ is a dense subset of the compactly supported functions in $H_{\text{uv}}^k(\mathbb{R}^\nu)$ we use mollifiers. Let $f \in H_{\text{uv}}^k(\mathbb{R}^\nu)$, $\text{supp}(f) \subset \mathbb{R}^\nu$ compact and define a function $\theta(x) := \exp((1 - |x|^2)^{-1})$ for $|x| < 1$ and 0 for $|x| \geq 1$. Moreover, define $\theta_j(x) := j^\nu \theta(jx)$ and $f_j := \theta_j * f$, where $\theta_j * (\cdot)$ denotes convolution with θ_j . Since f and all θ_j have compact support, so do all f_j and therefore $f_j \in C_0^\infty(\mathbb{R}^\nu)$ for all j . We have that $f_j \rightarrow f$ in $H^k(\mathbb{R}^\nu)$, see e.g. [1]. Thus,

$$\|f_j - f\|_{n,\alpha}^2 = \int_{B_n(0)} \|\partial^\alpha(f - f_j)\|^2 dx \leq \|f - f_j\|_{H^k(\mathbb{R}^\nu)}^2 \rightarrow 0$$

for all n and all α with $|\alpha| \leq k$. Since $f_j \rightarrow f$ pointwise almost everywhere, there exists a compact set $K \subset \mathbb{R}^\nu$ such that $\text{supp}(f), \text{supp}(f_j) \subset K$. Therefore,

$$\begin{aligned} \|f_j - f\|_n^2 &= \int_K |f(x) - f_j(x)|^2 (1 + |x|)^{2n} dx \\ &\leq \sup_{x \in K} [(1 + |x|)^{2n}] \|f - f_j\|_{L^2(\mathbb{R}^\nu)}^2 \rightarrow 0 \end{aligned}$$

for all n . This completes the proof by an $\epsilon/3$ argument. \square

5.2 Commutators

Notation 5.6. In order to distinguish between fiber Hamiltonians with compactly supported and smooth coupling functions and functions in $H_{\text{uv}}^k(\mathbb{R}^\nu)$ we define

$$H_f(\xi) := H_0(\xi) + \Phi(f)$$

for any $f \in L^2(\mathbb{R}^\nu)$. We abbreviate further and simply put $H_f \equiv H_f(\xi)$. Moreover the symbols g, g_j, \dots are reserved to denote compactly supported and smooth functions and h, h_j, \dots to denote functions in $H_{\text{uv}}^k(\mathbb{R}^\nu)$.

Let $\eta \in C_0^\infty(\mathbb{R}^\nu)$, $n, k \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^\nu$, $|\alpha| \leq k$ and $f \in H_{\text{uv}}^k(\mathbb{R}^\nu)$. There exists $m \in \mathbb{N}$ such that $\text{supp}(\eta) \subset B_m(0)$. This allows us to compute

$$\|D_\eta^\alpha f\|_n = \|(1 + |k|)^n \eta 1_{B_m(0)} D\alpha f\| \leq \|(1 + |k|)^n \eta\|_\infty \|f\|_{m,\alpha}.$$

For all $\beta \in \mathbb{N}_0^\nu$, $|\beta| \leq k - |\alpha|$ we perform a similar computation:

$$\begin{aligned} \|D_\eta^\alpha f\|_{n,\beta} &= \|1_{B_n(0)} D^\beta (\eta D^\alpha f)\| \leq \sum_{\substack{\gamma \in \mathbb{N}_0^\nu \\ \gamma \leq \beta}} \binom{\beta}{\gamma} \|1_{B_n(0)} D^{\beta-\gamma} \eta D^{\gamma+\alpha} f\| \\ &\leq \sum_{\substack{\gamma \in \mathbb{N}_0^\nu \\ \gamma \leq \beta}} \binom{\beta}{\gamma} \|D^{\beta-\gamma} \eta\|_\infty \|f\|_{n,\gamma+\alpha}. \end{aligned}$$

Hence, $D_\eta^\alpha : \mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu) \rightarrow \mathbb{H}_{\text{uv}}^{k-|\alpha|}(\mathbb{R}^\nu)$ is continuous as a linear map between locally convex spaces.

Likewise, let $g \in C_p^\infty$, $f \in \mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu)$ and $|\beta| \leq k$. Due to $g \in C_p^\infty$, we have that for every $\gamma \in \mathbb{N}_0^\nu$ there exists $C_\gamma > 0$ and $\ell_\gamma \in \mathbb{N}_0$ such that $|D^\gamma f(k)| \leq C_\gamma (1 + |k|)^{\ell_\gamma}$. We calculate

$$\|gf\|_n = \|(1 + |k|)^n gf\| \leq C_0 \|f\|_{n+\ell_0}$$

and

$$\begin{aligned} \|gf\|_{n,\beta} &= \|1_{B_n(0)} D^\beta (gf)\| \leq \sum_{\substack{\gamma \in \mathbb{N}_0^\nu \\ \gamma \leq \beta}} \binom{\beta}{\gamma} \|1_{B_n(0)} D^\gamma g D^{\beta-\gamma} f\| \\ &\leq \sum_{\substack{\gamma \in \mathbb{N}_0^\nu \\ \gamma \leq \beta}} \binom{\beta}{\gamma} C_\gamma \|1_{B_n(0)} (1 + |k|)^{\ell_\gamma}\|_\infty \|f\|_{n,\beta-\gamma}. \end{aligned}$$

This establishes that $M_g : \mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu) \rightarrow \mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu)$ is a continuous linear map between locally convex spaces.

Lemma 5.7 (Approximation Lemma). *Let $n \in \mathbb{N}$ and $O_1, \dots, O_n \in \mathcal{O}_{\text{uv}}^k$ (see Definition 3.9).*

1. For every $\psi \in \mathcal{C}_0^\infty$ the map

$$K_\psi : \mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu) \times \dots \times \mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu) \longrightarrow \mathcal{H}$$

given by

$$(h_1, \dots, h_n) \mapsto \Phi(h_1) \cdots \Phi(h_n) \psi$$

is continuous w.r.t. the product topology on $\mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu) \times \dots \times \mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu)$.

2. Define

$$[K_\psi \circ (O_1, \dots, O_n)](h_1, \dots, h_n) := K_\psi(O_1 h_1, \dots, O_n h_n).$$

The map

$$(h_1, \dots, h_n) \mapsto [K_\psi \circ (O_1, \dots, O_n)](h_1, \dots, h_n)$$

is also continuous w.r.t. the product topology on $\mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu) \times \dots \times \mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu)$.

3. Let $h_1, \dots, h_{n_1} \in H_{uv}^k(\mathbb{R}^\nu)$, $u_1, \dots, u_{n_2} \in C_0^\infty(\mathbb{R}^\nu)$ (see Definition 3.2) define

$$T(h_1, \dots, h_{n_1}) := \prod_{i=1}^{n_1} \Phi(h_i) \prod_{j=1}^{n_2} d\Gamma(u_j) \prod_{\sigma=1}^{\nu} d\Gamma(k_\sigma)^{q_{n_3, \sigma}}, \quad (5.32)$$

where $\sum_{\sigma} q_{n_3, \sigma} = n_3$, as an operator on \mathcal{C}_0^∞ . Then the map

$$K'_\psi : H_{uv}^k \times \dots \times H_{uv}^k \longrightarrow \mathcal{H}$$

defined by

$$K'_\psi(h_1, \dots, h_{n_1}) = T(h_1, \dots, h_{n_1})\psi$$

is continuous for all $\psi \in \mathcal{C}_0^\infty$.

PROOF: Note that the second and third part immediately follow from the first and so it suffices to prove the first part. Since the maps $O_i \in \mathcal{O}_{uv}^k$ are continuous, the second statement follows directly from the first. By equation (5.30) and for $\psi \in \mathcal{C}_0^\infty$ we have

$$\|K_\psi(h_1, \dots, h_{n_1})\| = \left\| \prod_{i=1}^n \Phi(h_i) \psi \right\| \leq C_\psi \prod_{i=1}^n \|h_i\|_0. \quad (5.33)$$

Since Fréchet spaces are metrizable, sequential continuity is equivalent to continuity. This completes the proof of the first statement. \square

As in the case of compactly supported coupling functions we would like to define spaces of operators like $\mathfrak{E}_{\alpha_1, \alpha_2, \alpha_3}$ but with coupling functions taken from $H_{uv}^k(\mathbb{R}^\nu)$ instead of $C_0^\infty(\mathbb{R}^\nu)$. Since $\Phi(h)$ does not necessarily map \mathcal{C}_0^∞ into itself, this requires some adjustment of notation. Define

$$\text{Op}_{uv}(\mathcal{C}_0^\infty) := \{S \text{ linear operator} \mid D(S) = \mathcal{C}_0^\infty\}.$$

This set is still a vector space but unlike $\text{Op}(\mathcal{C}_0^\infty)$ it has no algebraic structure anymore. In any case we clearly have $\text{Op}(\mathcal{C}_0^\infty) \subset \text{Op}_{uv}(\mathcal{C}_0^\infty)$.

Definition 5.8.

1. Define the following subset of $\text{Op}_{uv}(\mathcal{C}_0^\infty)$:

$$\mathfrak{E}_{\alpha_1, 0, 0}^{uv, k} := \text{Span} \left\{ \prod_{i=1}^n \Phi(h_i) \mid h_1, \dots, h_{j_1} \in H_{uv}^k(\mathbb{R}^\nu), j_1 \leq \alpha_1 \right\} + \mathfrak{E}_{0, 0, 0}.$$

2. Due to $T\mathcal{C}_0^\infty \subset \mathcal{C}_0^\infty$ for all $T \in \mathfrak{E}_{0, \alpha_2, \alpha_3}$, the compositions $\prod_{i=1}^n \Phi(h_i)T$ can be defined on $D(T) = \mathcal{C}_0^\infty$ for such T and all $n \in \mathbb{N}$. This allows us to define

$$\mathfrak{E}_{\alpha_1, \alpha_2, \alpha_3}^{uv, k} := \mathfrak{E}_{\alpha_1, 0, 0} \mathfrak{E}_{0, \alpha_2, \alpha_3} \subset \text{Op}_{uv}(\mathcal{C}_0^\infty).$$

Moreover, we put

$$\mathfrak{E}_{\alpha_1, \alpha_2, \alpha_3}^{uv} = \mathfrak{E}_{\alpha_1, \alpha_2, \alpha_3}^{uv, 0}.$$

3. Let T be an operator on \mathcal{H} such that $\mathcal{C}_0^\infty \subset D(T)$. We write

$$T \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}^{\text{uv}, k},$$

if $T \in \text{Op}_{\text{uv}}(\mathcal{C}_0^\infty)$ and

$$\exists \tilde{T} \in \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}^{\text{uv}, k} : T|_{\mathcal{C}_0^\infty} = \tilde{T}.$$

Due to the similarities in the definitions of $\mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ and $\mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}^{\text{uv}}$ we might expect that similar statements as in the compactly supported case hold in the generalized scenario as well. This is indeed the case and all proofs follow the same strategy: We consider a sequence g_j in $\mathcal{C}_0^\infty(\mathbb{R}^\nu)$ which converges against h in \mathbb{H}_{uv}^k and use the Approximation Lemma 5.7 in combination with the (already proven) statement in the compactly supported case to finish the proof. Since the idea behind every proof is basically the same we only demonstrate the scheme in two selected cases to avoid repetitions.

Definition 5.9. Let $k \in \mathbb{N}_0$, $h \in \mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu)$, $S \in \{H_0, \Phi(h), H_h\}$ and $T(h_1, \dots, h_{n_1}) \in \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}^{\text{uv}, k}$. If the form $[S, T(h_1, \dots, h_{n_1})]$ is implemented by an operator on \mathcal{C}_0^∞ , it is called $\text{ad}_S(T(h_1, \dots, h_{n_1}))$. For $k \geq 1$ S may also be equal to A .

Lemma 5.10. Let $h \in \mathbb{H}_{\text{uv}}(\mathbb{R}^\nu)$, $S \in \{H_0, \Phi(h), H_h\}$ and choose $T(h_1, \dots, h_{n_1}) \in \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}^{\text{uv}}$. Then the form $[S, T(h_1, \dots, h_{n_1})]$ is implemented by an operator on \mathcal{C}_0^∞ . Moreover, the following three statements hold.

1. Let $\alpha_1 \neq 0$. There exist $N \in \mathbb{N}$, operators $M_{f_{1,\ell}}, \dots, M_{f_{n_1,\ell}} \in \mathcal{M}(\mathcal{C}_p^\infty)$ and

$$T_\ell(M_{f_{1,\ell}} h_1, \dots, M_{f_{n_1,\ell}} h_{n_1}) \in \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3 + 1},$$

where $1 \leq \ell \leq N$, such that

$$\text{ad}_{H_0(\xi)}(T(h_1, \dots, h_{n_1})) = \sum_{\ell=1}^N T_\ell(M_{f_{1,\ell}} h_1, \dots, M_{f_{n_1,\ell}} h_{n_1}).$$

2. Let $\alpha_1, \alpha_2 \geq 1$. Define

$$T(h_1, \dots, \hat{h}_i, \dots, h_{n_1}) := T(h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_{n_1}) \in \mathfrak{C}_{\alpha_1 - 1, \alpha_2, \alpha_3},$$

where the functions $v_1, \dots, v_{n_2} \in \mathcal{C}_\omega^\infty$ and the constants $q_{n_3,1}, \dots, q_{n_3,\nu} \in \mathbb{N}_0$ in the definition of $T(h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_{n_1})$ coincide with the ones in the definition of $T(h_1, \dots, h_{n_1})$.

There exist $N \in \mathbb{N}$, $M_{f_{n_1+1,\ell}} \in \mathcal{M}(\mathcal{C}_p^\infty)$ and operators

$$T_\ell(h_1, \dots, h_n, M_{f_{n_1+1,\ell}} h) \in \mathfrak{C}_{\alpha_1 + 1, \alpha_2 - 1, \alpha_3},$$

where $1 \leq \ell \leq N$, such that

$$\begin{aligned} \text{ad}_{\Phi(h)}(T(h_1, \dots, h_{n_1})) &= \sum_{i=1}^n T(h_1, \dots, \hat{h}_i, \dots, h_n) \\ &\quad + \sum_{\ell=1}^N T_\ell(h_1, \dots, h_n, M_{f_{n_1+1, \ell}} h). \end{aligned}$$

3. Let $\alpha_3 \geq 0$ and $\alpha_1 \geq 0$ or $\alpha_2 \geq 0$. For every $\sigma \in \{1, \dots, \nu\}$ define $\delta_\sigma \in \mathbb{N}^\nu$ by $(\delta_\sigma)_j = \delta_{\sigma, j}$. If $h_1, \dots, h_{n_1} \in \mathbb{H}_{\text{uv}}^1$, there exist $N \in \mathbb{N}$ and operators $T_1(h_1, \dots, h_n), \dots, T_N(h_1, \dots, h_n) \in \mathfrak{C}_{\alpha_1, \alpha_2+1, \alpha_3-1}^{\text{uv}}$ such that

$$\begin{aligned} &\text{ad}_A(T(h_1, \dots, h_{n_1})) \\ &= \sum_{i=1}^{n_1} \sum_{\sigma=1}^{\nu} T(h_1, \dots, (D_{v_\sigma}^0 + D_{v_\sigma}^{\delta_\sigma})h_i, \dots, h_{n_1}) \\ &\quad + \sum_{i=1}^N T_i(h_1, \dots, h_{n_1}). \end{aligned}$$

PROOF: We start with the first statement. Let $(g_j^\ell)_j \subset C_0^\infty(\mathbb{R})$ be a sequence converging to h_ℓ , $\ell \in \{1, \dots, n_1\}$, in the topology of $\mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu)$ and let $M_{f_1, \ell}, \dots, M_{f_{n_1}, \ell}$ by the operators of Corollary 3.11. Then Lemma 5.7.3 and Corollary 3.11 imply that

$$\begin{aligned} &|\langle \psi, [H_0, T(h_1, \dots, h_{n_1})]\psi' \rangle - \langle \psi, \sum_{\ell=1}^N T(M_{f_1, \ell} h_1, \dots, M_{f_{n_1}, \ell} h_{n_1})\psi' \rangle| \\ &\leq |\langle \psi, [H_0, T(h_1, \dots, h_{n_1})]\psi' \rangle - \langle \psi, [H_0, T(g_j^1, \dots, g_j^{n_1})]\psi' \rangle| \\ &\quad + |\langle \psi, [H_0, T(g_j^1, \dots, g_j^{n_1})]\psi' \rangle - \langle \psi, \sum_{\ell=1}^N T(M_{f_1, \ell} h_1, \dots, M_{f_{n_1}, \ell} h_{n_1})\psi' \rangle| \\ &\leq |\langle \psi, [H_0, T(h_1, \dots, h_{n_1})]\psi' \rangle - \langle \psi, [H_0, T(g_j^1, \dots, g_j^{n_1})]\psi' \rangle| \\ &\quad + \sum_{\ell=1}^N |\langle \psi, (T(M_{f_1, \ell} h_1, \dots, M_{f_{n_1}, \ell} h_{n_1}) - T(M_{f_1, \ell} g_j^1, \dots, M_{f_{n_1}, \ell} g_j^{n_1}))\psi' \rangle|. \end{aligned}$$

This proves the first statement. The proof of the third statement is similar and is thus omitted.

The second assertion differs from the other two in a structural fashion, because both, the operator $T(h_1, \dots, h_{n_1})$ and $\Phi(h)$, are defined via elements of $\mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu)$. However this only requires us to go through the above approximation procedure

twice. Since this kind of proof re-appears below, we provide the details. Let $(g_j) \subset C_0^\infty(\mathbb{R})$ converge against h in the topology of $H_{\text{uv}}^k(\mathbb{R}^\nu)$. Abbreviate

$$S(f_1, \dots, f_{n_1}, f) := \sum_{i=1}^n T(f_1, \dots, \hat{f}_i, \dots, f_{n_1}) \\ + \sum_{\ell=1}^N T_\ell(f_1, \dots, \hat{f}_i, \dots, f_{n_1}, M_{f_{n_1+1, \ell}} f).$$

For any $j \in \mathbb{N}$ we compute

$$|\langle \psi, [\Phi(h), T(h_1, \dots, h_{n_1})] \psi' \rangle - \langle \psi, S(h_1, \dots, h_{n_1}, h) \psi' \rangle| \\ \leq |\langle \psi, [\Phi(h), T(h_1, \dots, h_{n_1})] \psi' \rangle - \langle \psi, [\Phi(g_j), T(h_1, \dots, h_{n_1})] \psi' \rangle| \\ + |\langle \psi, [\Phi(g_j), T(h_1, \dots, h_{n_1})] \psi' \rangle - \langle \psi, [\Phi(g_j), T(g_j^1, \dots, g_j^{n_1})] \psi' \rangle| \\ + |\langle \psi, S(h_1, \dots, h_{n_1}, h) \psi' \rangle - \langle \psi, S(g_j^1, \dots, g_j^{n_1}, g_j) \psi' \rangle|$$

in complete analogy to the beginning of the proof. Since we clearly have that $S(g_j^1, \dots, g_j^{n_1}, g_j) \psi \rightarrow S(h_1, \dots, h_{n_1}, h) \psi$ in \mathcal{H} by Lemma 5.7.3, the second statement now follows by an $\epsilon/3$ -argument. \square

Corollary 5.11. *Let $T(h_1, \dots, h_{n_1}) \in \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}^{uv, 0}$ and $h \in H_{\text{uv}}(\mathbb{R}^\nu)$. Adopt the notation of Lemma 5.10.1 and Lemma 5.10.2. Then*

$$\text{ad}_{H_h}(T(h_1, \dots, h_{n_1})) = \sum_{i=1}^{n_1} T(h_1, \dots, \hat{h}_i, \dots, h_{n_1}) \\ + \sum_{\ell=1}^N T_\ell(h_1, \dots, h_{n_1}, M_{f_{n_1+1, \ell}} h) \\ + \sum_{\ell=1}^N T(M_{f_{1, \ell}} h_1, \dots, M_{f_{n_1, \ell}} h_{n_1}).$$

PROOF: Simply note that for $\psi, \psi' \in C_0^\infty$

$$\langle \psi, [H_h, T(h_1, \dots, h_{n_1})] \psi' \rangle = \langle \psi, [H_0, T(h_1, \dots, h_{n_1})] \psi' \rangle \\ + \langle \psi, [\Phi(h), T(h_1, \dots, h_{n_1})] \psi' \rangle.$$

The statement now follows from Lemma 5.10.1 and Lemma 5.10.2. \square

Corollary 5.12. *Let $k \in \mathbb{N}_0$, $h \in H_{\text{uv}}^k(\mathbb{R}^\nu)$ and $m \in \mathbb{N}$. There exist $N_m \in \mathbb{N}$, $M_{f_{1, \ell}}, \dots, M_{f_{j_\ell, \ell}} \in \mathcal{M}(C_p^\infty)$, where $1 \leq \ell \leq N_m$ and $0 \leq j_\ell \leq m$, such that for all $\psi \in C_0^\infty$*

$$H_h(\xi)^m \psi = H_0(\xi)^m \psi + \sum_{\ell=1}^{N_m} \sum_{j_\ell=1}^m T_\ell(M_{f_{1, \ell}} h, \dots, M_{f_{j_\ell, \ell}} h) \psi,$$

where for all $1 \leq \ell \leq N_m$

$$T_\ell(M_{f_{1,\ell}}h, \dots, M_{f_{j_\ell,\ell}}h) \in \mathfrak{C}_{j_\ell, k_\ell, 2(m-j_\ell-k_\ell)}^{\text{uv}, k}$$

for some $0 \leq k_\ell \leq m - j_\ell$.

PROOF: We prove the statement by induction in m . The case $m = 1$ is clear. Let $\psi \in \mathcal{C}_0^\infty$ and calculate

$$\begin{aligned} H_h(\xi)^{m+1}\psi &= H_h(\xi)H_0(\xi)^m + \sum_{\ell=1}^{N_m} \sum_{j_\ell=1}^m H_h(\xi)T_\ell(M_{f_{1,\ell}}h, \dots, M_{f_{j_\ell,\ell}}h)\psi \\ &= H_0(\xi)^{m+1}\psi + \Phi(h)H_0(\xi)^m\psi \\ &\quad + \sum_{\ell=1}^{N_m} \sum_{j_\ell=1}^m \Phi(h)T_\ell(M_{f_{1,\ell}}h, \dots, M_{f_{j_\ell,\ell}}h)\psi \\ &\quad + \sum_{\ell=1}^{N_m} \sum_{j_\ell=1}^m H_0(\xi)T_\ell(M_{f_{1,\ell}}h, \dots, M_{f_{j_\ell,\ell}}h)\psi \end{aligned}$$

Due to (3.15), the first two terms are already of the correct form. The same holds for every term in the first of the two sums, since

$$\Phi(h)T_\ell(M_{f_{1,\ell}}h, \dots, M_{f_{j_\ell,\ell}}h) \in \mathfrak{C}_{j_\ell+1, k_\ell, 2(m+1-(j_\ell+1)-k_\ell)}^{\text{uv}, k}.$$

As for the last sum, we compute

$$\begin{aligned} H_0(\xi)T_\ell(M_{f_{1,\ell}}h, \dots, M_{f_{j_\ell,\ell}}h)\psi &= T_\ell(M_{f_{1,\ell}}h, \dots, M_{f_{j_\ell,\ell}}h)H_0(\xi)\psi \\ &\quad + \text{ad}_{H_0(\xi)}(T_\ell(M_{f_{1,\ell}}h, \dots, M_{f_{j_\ell,\ell}}h))\psi. \end{aligned}$$

Note that

$$\begin{aligned} T_\ell(M_{f_{1,\ell}}h, \dots, M_{f_{j_\ell,\ell}}h)H_0(\xi) &\in \mathfrak{C}_{j_\ell, k_\ell+1, 2(m+1-j_\ell-(k_\ell+1))}^{\text{uv}, k} \\ &\quad + \mathfrak{C}_{j_\ell, k_\ell, 2(m+1-j_\ell-k_\ell)}^{\text{uv}, k} \end{aligned}$$

and

$$\text{ad}_{H_0(\xi)}(T_\ell(M_{f_{1,\ell}}h, \dots, M_{f_{j_\ell,\ell}}h))\psi = \sum_{\ell'=1}^{N'} T_{\ell, \ell'}(M_{f'_{1,\ell}}M_{f_{1,\ell}}h, \dots, M_{f'_{j_\ell,\ell}}M_{f_{j_\ell,\ell}}h)$$

for functions $f'_{1,\ell}, \dots, f'_{j_\ell,\ell} \in \mathbb{N}$ and

$$T_{\ell, \ell'}(M_{f'_{1,\ell}}M_{f_{1,\ell}}h, \dots, M_{f'_{j_\ell,\ell}}M_{f_{j_\ell,\ell}}h) \in \mathfrak{C}_{j_\ell, k_\ell, 2(m-j_\ell-k_\ell)+1}^{\text{uv}, k}.$$

Since $\mathfrak{C}_{j_\ell, k_\ell, 2(m-j_\ell-k_\ell)+1}^{\text{uv}, k} \subset \mathfrak{C}_{j_\ell, k_\ell, 2(m+1-j_\ell-k_\ell)}^{\text{uv}, k}$, this completes the proof. \square

Corollary 5.13 (Convergence in Strong Resolvent Sense). *Let $k \in \mathbb{N}_0$ and $h \in H_{uv}^k(\mathbb{R}^\nu)$. Let $(g_n)_n \subset C_0^\infty(\mathbb{R}^\nu)$ such that $g_n \rightarrow h$ in the topology of $H_{uv}^k(\mathbb{R}^\nu)$. Then $H_{g_n} \rightarrow H_h$ in the strong resolvent sense.*

PROOF: By Corollary 5.12 and Lemma 5.7 $H_{g_n}\psi \rightarrow H_h\psi$ for all $\psi \in C_0^\infty$. Since $\psi \in C_0^\infty$ is a common core for all H_{g_n} and H_h , this implies convergence in the strong resolvent sense, see e.g. [46]. \square

Observe that the previous lemma combined with Corollary 5.12 immediately implies that the form $[A, H_h]$ is implemented by an operator on C_0^∞ . In fact, we combine our results to learn more about its structure.

Corollary 5.14. *Let $h \in H_{uv}^1(\mathbb{R}^\nu)$ and adopt the notation of Corollary 5.12. There exist $N \in \mathbb{N}$, operators $T_{\ell,i}(M_{f_{1,\ell}}h, \dots, M_{f_{j_\ell,\ell}}h) \in \mathfrak{C}_{j_\ell, k_\ell+1, 2(m-k_\ell-j_\ell)-1}^{uv,k}$, $i \in \{1, \dots, N\}$, such that*

$$\begin{aligned} & \text{ad}_A((H_h(\xi) + c)^m) \\ &= \sum_{\ell=1}^{N_m} \sum_{j_\ell=0}^m \sum_{i=1}^{j_\ell} \sum_{\sigma=1}^{\nu} T_{\ell,i}(M_{f_{1,\ell}}h, \dots, (D_{v_\sigma}^0 + D_{v_\sigma}^{\delta_\sigma})M_{f_{i,\ell}}h, \dots, M_{f_{j_\ell,\ell}}h) \\ & \quad + \sum_{\ell=1}^{N_m} \sum_{j_\ell=0}^m \sum_{i=1}^N T_{\ell,i}(M_{f_{1,\ell}}h, \dots, M_{f_{j_\ell,\ell}}h) \end{aligned}$$

PROOF: Combine Lemma 5.10 with Corollary 5.12. \square

Lemma 5.15. *Let $T(h_1, \dots, h_{n_1}) \in \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}^{uv}$ and define $n := \alpha_1/2 + \alpha_2 + \alpha_3/2$. Then the operator $T(h_1, \dots, h_{n_1})$ is $(H_0 + 1)^n$ -bounded and thus extends to an operator T' on $D((H_0 + 1)^n)$ which we denote by $T(h_1, \dots, h_{n_1})$ again.*

PROOF: Choose sequences $(g_j^i)_j \subset C_0^\infty(\mathbb{R})$ so that $g_j^i \rightarrow h_i$ w.r.t. the topology of $H_{uv}(\mathbb{R}^\nu)$ for all $i \in \{1, \dots, n_1\}$. Note that since $g_j^i \rightarrow h_i$ in L^2 as well, there is a constant $C > 0$ so that $\|g_j^i\| \leq C\|h_i\|$. Let $\psi \in C_0^\infty$ and $\epsilon > 0$. There exists $j \in \mathbb{N}$ such that

$$\|T(h_1, \dots, h_{n_1})\psi - T(g_j^1, \dots, g_j^{n_1})\psi\| < \epsilon.$$

With this choice of j we calculate

$$\begin{aligned} \|T(h_1, \dots, h_{n_1})\psi\| &\leq \|T(h_1, \dots, h_{n_1})\psi - T(g_j^1, \dots, g_j^{n_1})\psi\| + \|T(g_j^1, \dots, g_j^{n_1})\psi\| \\ &< \epsilon + C^{n_1} \prod_{i=1}^{n_1} \|h_i\| \|(H_0 + 1)^n \psi\|, \end{aligned}$$

where we have used (5.33) in the last step. Since $\epsilon > 0$ was chosen arbitrarily, this concludes the proof. \square

Clearly, the previous statement enables us to extend all commutators which we have shown to be implemented by operators in $\mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}^{\text{uv}, k}$ for some $\alpha_i, k \in \mathbb{N}_0$ in this section to $D((H_0 + 1)^m)$ for appropriately chosen $m \in \mathbb{N}_0/2$.

In order to further extend these operators to $D((H_h + c)^n)$ for $h \in H_{\text{uv}}(\mathbb{R}^\nu)$ we have to extend Lemma 3.16 and Lemma 3.17 to this particular situation.

Lemma 5.16. *Let $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $h \in H_{\text{uv}}^k$.*

1. $B_m^h := (H_0 + 1)^{\frac{m}{2}}(H_h + c)^{-\frac{m}{2}} \in \mathcal{B}(\mathcal{H})$.
2. $F_m^h := (H_h + c)^{\frac{m}{2}}(H_0 + 1)^{-\frac{m}{2}} \in \mathcal{B}(\mathcal{H})$.

PROOF: Let $(g_n)_n \subset C_c^\infty(\mathbb{R}^\nu)$ be a sequence converging to h in H_{uv}^k . Since $(H_0 + 1)^{-\frac{m}{2}}$ preserves \mathcal{C}_0^∞ and $H_{g_n}\psi \rightarrow H_h\psi$ for all $\psi \in \mathcal{C}_0^\infty$, we immediately have that

$$\lim_{n \rightarrow \infty} (H_{g_n} + c)^k (H_0 + 1)^{-k} \psi = F_{2k} \psi.$$

By Lemma 3.17 all $(H_{g_n} + c)^k (H_0 + 1)^{-k}$ are bounded operators. Corollary 5.12, Lemma 5.15 and L^2 -convergence of g_n imply that there exists a constant $C > 0$ independent of n such that for all $\psi \in \mathcal{H}$

$$\|(H_{g_n} + c)^k (H_0 + 1)^{-k} \psi\| \leq C \|\psi\|.$$

Boundedness of F_{2k} now follows from the uniform boundedness principle. That all other F_k are bounded follows from interpolation.

In order to prove the first statement note that we have already proven all ingredients necessary to go through all steps in the proof of Lemma 3.16. By copying this proof word by word we can thus establish the validity of the first assertion. To avoid repetitions we will not present the details here. \square

As in the case of compactly supported coupling functions this result implies the following corollary.

Corollary 5.17. *Let $h \in H_{\text{uv}}(\mathbb{R}^\nu)$, $m \in \mathbb{N}_0$ and $c > 0$ large enough.*

1. *We have that $D(H_0^{\frac{m}{2}}) = D((H_h + c)^{\frac{m}{2}})$.*
2. *The norms $\|x\|_{H_0, m} = \|(H_0 + 1)^{\frac{m}{2}} x\|$ and $\|x\|_{H, m} = \|(H_h + c)^{\frac{m}{2}} x\|$ are equivalent on $D((H_h + c)^{\frac{m}{2}}) = D(H_0^{\frac{m}{2}})$.*

As a consequence the form extensions obtained earlier in this sections actually extend to the domain of $(H_h + c)^{\frac{m}{2}}$ for some integer power m . Since this fact is crucial for the remaining part of this section, we state it as a separate remark.

We are now in a position to define iterated commutators for coupling functions in $h \in H_{\text{uv}}^k$ in complete analogy to Definition 3.21.

Definition 5.18 (Iterated Commutators on \mathcal{C}_0^∞). Let $h \in H_{uv}$. We introduce the abbreviation $\mathcal{Q} := \{Q_1, Q_2\}$, where $Q_1 := H_h$ and $Q_2 := A$. Moreover, let $c > 0$ be large enough such that $-c \in \rho(H)$. For $n \in \mathbb{N}$, define

$$\mathfrak{J}_n := \{\underline{w} = (\underline{w}(1), \dots, \underline{w}(n)) \in \{1, 2\}^n \mid \underline{w}(1) = 2\}.$$

For $\underline{w} \in \mathfrak{J}_n$ we define its ℓ -th truncation $\underline{w}^{(\ell)}$ by

$$\underline{w}^{(\ell)} := (\underline{w}(1), \dots, \underline{w}(\ell)) \in \{1, 2\}^\ell$$

and the amount of taken A -commutators by

$$\mathbf{n}_{\underline{w}}^A := |\{j \in \{1, \dots, n\} \mid \underline{w}(j) = 2\}|.$$

If $h \in H_{uv}^k$ for $k = \mathbf{n}_{\underline{w}}^A$, the ℓ -th truncation $\text{ad}_{\mathcal{Q}}^{\underline{w}^{(\ell)}}((H_h + c)^m)$ of the mixed commutator corresponding to \underline{w} on \mathcal{C}_0^∞ is iteratively defined by

$$\text{ad}_{\mathcal{Q}}^{\underline{w}^{(1)}}((H_h + c)^m) := \text{ad}_A((H_h + c)^m)$$

for $\ell = 1$, where the operator on the right hand side implements the commutator form $[A, (H_h + c)^m]$ on \mathcal{C}_0^∞ . For $\ell \geq 2$ we define

$$\text{ad}_{\mathcal{Q}}^{\underline{w}^{(\ell)}}((H_h + c)^m) := \text{ad}_{T_{\underline{w}^{(\ell)}}} \left(\text{ad}_{\mathcal{Q}}^{\underline{w}^{(\ell-1)}}((H_h + c)^m) \right)$$

to be the operator which implements the form

$$[T_{\underline{w}^{(\ell)}}, \text{ad}_{\mathcal{Q}}^{\underline{w}^{(\ell-1)}}((H_h + c)^m)]$$

on \mathcal{C}_0^∞ . The mixed commutator corresponding to $\underline{w} \in \mathfrak{J}_n$ is then defined as

$$\text{ad}_{\mathcal{Q}}^{\underline{w}}((H_h + c)^m) := \text{ad}_{\mathcal{Q}}^{\underline{w}^{(n)}}((H_h + c)^m).$$

This operator is sometimes simply referred to as a mixed or iterated commutator. It is convenient to define $\mathfrak{J}_0 := \{0\}$ and $\text{ad}_{\mathcal{Q}}^{\underline{w}}((H_h + c)^m) := (H_h + c)^m$ for $\underline{w} \in \mathfrak{J}_0$.

In the next remark we stress that the previous construction obtains similar properties to the case of compactly supported and smooth coupling functions. We will make extensive use of it in this paper.

Remark 5.19.

1. We show inductively that the forms $[T_{\underline{w}^{(\ell)}}, \text{ad}_{\mathcal{Q}}^{\underline{w}^{(\ell-1)}}((H_h + c)^m)]$ are all given by $H_0^{m+\frac{\ell}{2}}$ -bounded operator by using Corollary 5.12 and then applying Lemma 5.10 repeatedly.

2. The requirement that $h \in \mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu)$ for $k = \mathbf{n}_{\underline{w}}^A$ guarantees that we may in fact apply Lemma 5.10 for k times, due to Remark 5.3.
3. We thus have that $\text{ad}_{\mathcal{Q}}^{\underline{w}}((H_h + c)^m)$ extends from \mathcal{C}_0^∞ to a bounded operator on $D(H_0^{m+\frac{n}{2}})$ with values in \mathcal{H} . By Corollary 5.17 this extension even defines a bounded \mathcal{H} -valued operator on $D((H_h + c)^{m+\frac{n}{2}})$. If we denote this extension with the same symbol again, we have argued that

$$\text{ad}_{\mathcal{Q}}^{\underline{w}}((H_h + c)^m) \in \mathcal{B}(D(H_h^M), \mathcal{H})$$

for all $M \in \mathbb{N}$ with $M \geq m + \frac{n}{2}$.

The first remark seems to imply that we should be able to prove an analogue of Lemma 3.25 for coupling functions in $\mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu)$ for large enough k . This is indeed the case and can be achieved by basically copying the inductive argument in 3.25 word by word, where the corresponding statements for the $\mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu)$ case have to be substituted for the respective Corollaries in the compactly supported case. We will omit the proof and simply state the result.

Proposition 5.20. *Let $m, n \in \mathbb{N}_0$, $\underline{w} \in \mathfrak{J}_n$ and $h \in \mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu)$ for $k = \mathbf{n}_{\underline{w}}^A$. Then there exists $N_{\underline{w}}$ and operators $O_{1,1}^{(\ell)}, \dots, O_{1,n+1}^{(\ell)}, \dots, O_{j_\ell,1}^{(\ell)}, \dots, O_{j_\ell,n+1}^{(\ell)} \in \mathcal{M}(\mathcal{C}_p^\infty) \cup \mathcal{D}^1$, where $1 \leq \ell \leq N_{\underline{w}}$ and $0 \leq j_\ell \leq n+1$ such that for all $\psi \in \mathcal{C}_0^\infty$*

$$\text{ad}_{\mathcal{T}}^{\underline{w}}((H_h + c)^m)\psi = \sum_{\ell=1}^{N_{\underline{w}}} T(\tilde{O}_1^{(\ell)}h, \dots, \tilde{O}_{j_\ell}^{(\ell)}h)\psi,$$

where

$$T(\tilde{O}_1^{(\ell)}h, \dots, \tilde{O}_{j_\ell}^{(\ell)}h) \in \mathfrak{C}_{\alpha_1^\ell, \alpha_2^\ell, \alpha_3^\ell}^{uw}$$

with $\alpha_1^\ell/2 + \alpha_2^\ell + \alpha_3^\ell/2 \leq m + n/2$ and

$$\tilde{O}_1^{(\ell)} := \prod_{i=1}^{n+1} O_{1,i}^{(\ell)}, \dots, \tilde{O}_{j_\ell}^{(\ell)} := \prod_{i=1}^{n+1} O_{j_\ell,i}^{(\ell)}.$$

Moreover, for all ℓ at most $\mathbf{n}_{\underline{w}}^A$ of the operators $O_{i,1}^{(\ell)}, \dots, O_{i,n+1}^{(\ell)}$ are in \mathcal{D}^1 .

Corollary 5.21. *Let $Q \in \mathbb{N}$ with $Q \geq m + \text{Int}(\frac{n}{2})$, $m, n \in \mathbb{N}_0$ and $\underline{w} \in \mathfrak{J}_n$. Moreover suppose that $h \in \mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu)$ for $k = \mathbf{n}_{\underline{w}}^A$. Define the maps*

$$S_{m,Q}^{\underline{w}}(\cdot)\psi : \mathbb{H}_{\text{uv}}^k \longrightarrow \mathcal{H},$$

for all $\psi \in \mathcal{H}$ by

$$S_{m,Q}^{\underline{w}}(h)\psi := \text{ad}_{\mathcal{T}}^{\underline{w}}((H_h + c)^m)(H_h - z)^{-Q}\psi,$$

where $z \in \rho(H_h)$. Then all maps $h \mapsto S_{m,Q}^{\underline{w}}(\cdot)\psi(h)$ are continuous w.r.t. the topology of $\mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu)$.

PROOF: By Corollary 5.17 $\text{ad}_{\mathcal{T}}^w((H_h + c)^m)$ is $(H_h + c)^{m+n/2}$ -bounded. We use Proposition 5.20 to compute

$$\|S_{m,Q}^w(h)\psi\| < C \left(\sum_{\ell=1}^{N_w} C_\ell \prod_{i=1}^{j_\ell} \|D_i^\ell h\| \right) \|H_0^{m+\frac{n}{2}}(H_h - z_j)^{-Q}\|\|\psi\|.$$

which implies the continuity of $S_{m,Q}^w(\cdot)\psi$. \square

Remark 5.22. Moreover, for $z_1, \dots, z_Q \in \rho(H_h)$ we could show in the exact same way as in the corollary that the map

$$h \mapsto \tilde{S}_{m,Q}^w(h)\psi := \text{ad}_{\mathcal{T}}^w((H_h + c)^m) \prod_{j=1}^Q (H_h - z_j)^{-1}\psi \quad (5.34)$$

is continuous w.r.t. the topology of $H_{uv}^k(\mathbb{R}^\nu)$ for $k \geq \mathfrak{n}_w^A$.

Remark 5.23. Suppose that we are given $w_i \in \mathfrak{I}_{n_i}$, $h_i \in H_{uv}^{k_i}$, where $k_i = \mathfrak{n}_{w_i}^A$, $n_i, m_i \in \mathbb{N}$, $Q_i \geq m + \text{Int}(n_i/2)$ for $i = 1, 2$. If we put $\psi = S_{m_2, Q_2}^{w_2}(h_2)\psi'$, we obtain

$$\begin{aligned} & \|S_{m_1, Q_1}^{w_1}(h)\psi\| \\ & \leq 2 \left(\sum_{\ell=1}^{N_{w_1}} C_\ell \prod_{i=1}^{j_\ell} \|D_i^\ell h_1\| \right) \|H_0^{m_1+\frac{n_1}{2}} \prod_{j=1}^{Q_1} (H_{h_1} - z_j)^{-1}\| \|S_{m_2, Q_2}^{w_2}(h_2)\psi'\| \\ & \leq 4 \prod_{a=1}^2 \left[\left(\sum_{\ell=1}^{N_{w_a}} C_\ell \prod_{i=1}^{j_\ell} \|D_i^\ell h_a\| \right) \|H_0^{m_2+\frac{n_2}{2}} \prod_{j=1}^{Q_2} (H_{h_2} - z_j)^{-1}\| \right] \|\psi'\| \end{aligned}$$

which is the continuity of the map which maps the pair (h_1, h_2) onto the vector

$$\text{ad}_{\mathcal{T}}^{w_1}((H_{h_1} + c)^{m_1}) \prod_{j=1}^{Q_1} (H_{h_1} - z_j)^{-1} \text{ad}_{\mathcal{T}}^{w_2}((H_{h_2} + c)^{m_2}) \prod_{j=1}^{Q_2} (H_{h_2} - z_j)^{-1}\psi.$$

We denote this map by the suggestive expression

$$S_{m_1, Q_1}^{w_1}(h_1)S_{m_2, Q_2}^{w_2}(h_2)\psi.$$

Obviously, this procedure can be iterated to any finite number of factors $\text{ad}_{\mathcal{T}}^w((H_h + c)^{m_2})(H_{h_2} + c)^{-Q_2}$. For our purposes the case, where all functions h_2 are the same will be most interesting.

5.3 Large Powers of the Resolvent

Since we deal with general coupling functions $h \in H_{uv}(\mathbb{R}^\nu)$ throughout this section, we have to adjust our notation to this case.

Notation 5.24. For the rest of this section we define

$$R_{-c} := (H_h + c)^{-1}$$

for any $h \in H_{uv}$, where the constant $c > 0$ is picked large enough.

Lemma 5.25. *Let $n, m \in \mathbb{N}$, $\underline{w} \in \mathfrak{J}_n$ and $Q_{\underline{w}} \in \mathbb{N}$ with $Q_{\underline{w}} \geq m + \text{Int}(n/2) + 1$, where $\text{Int}(r)$ denotes the smallest integer larger than $r \in \mathbb{R}$. Suppose that $h \in H_{uv}^k(\mathbb{R}^\nu)$, where $k = \mathbf{n}_{\underline{w}}^A$. Then $S_{m, Q_{\underline{w}}}^{\underline{w}}(h) \in C^1(A)$, that is*

$$\text{ad}_A \left(S_{m, Q_{\underline{w}}}^{\underline{w}}(h) \right) \in \mathfrak{B}(\mathcal{H}).$$

In particular,

$$\text{ad}_A \left(S_{m, Q_{\underline{w}}}^{\underline{w}}(h) \right) = S_{m, Q_{\underline{w}}}^{(2, \underline{w})}(h) + \sum_{k=0}^1 (-1)^{k+1} S_{m, Q_{\underline{w}}-k}^{\underline{w}}(h) S_{Q_{\underline{w}}, Q_{\underline{w}}+1}^{(1-k, 2)}(h) \quad (5.35)$$

as a bounded operator on \mathcal{H} , where $(2, \underline{w}) = (2, \underline{w}(1), \dots, \underline{w}(n)) \in \mathfrak{J}_{n+1}$.

PROOF: First of all note that by the choice of $Q_{\underline{w}}$ the operator $S_{m, Q_{\underline{w}}}^{\underline{w}}(h)$ is bounded. Hence we need to show that the commutator form $[A, S_{m, Q_{\underline{w}}+1}^{\underline{w}}(h)]$ is bounded. For this it suffices to show that (5.35) holds in the sense of forms on $D(A)$.

By Remark 5.23 we have that $S_{m, Q_{\underline{w}}+1}^{\underline{w}}(g_j) \rightarrow S_{m, Q_{\underline{w}}+1}^{\underline{w}}(h)$ and hence

$$\begin{aligned} \langle \psi, [A, S_{m, Q_{\underline{w}}+1}^{\underline{w}}(g_j)] \psi' \rangle &= \langle A\psi, S_{m, Q_{\underline{w}}+1}^{\underline{w}}(g_j) \psi' \rangle - \langle \psi, S_{m, Q_{\underline{w}}+1}^{\underline{w}}(g_j) A\psi' \rangle \\ &\longrightarrow \langle \psi, [A, S_{m, Q_{\underline{w}}+1}^{\underline{w}}(h)] \psi' \rangle \end{aligned}$$

Recall that we have already proven (5.35) for compactly supported and smooth coupling functions. By Corollary 5.21

$$\text{ad}_A \left(\text{ad}_{\mathcal{T}}^{\underline{w}}((H_{g_n} + c)^m) \right) R_{g_n}^{Q_{\underline{w}}+1} \psi = S_{m, Q_{\underline{w}}+1}^{(2, \underline{w})}(g_n) \psi \longrightarrow S_{m, Q_{\underline{w}}+1}^{(2, \underline{w})}(h) \psi$$

for any $\psi \in \mathcal{H}$. Likewise, for $k \in \{0, 1\}$ we may apply Remark 5.23 to obtain that

$$S_{m, Q_{\underline{w}}-k}^{\underline{w}}(g_n) S_{Q_{\underline{w}}, Q_{\underline{w}}+1}^{(1-k, 2)}(g_n) \longrightarrow S_{m, Q_{\underline{w}}-k}^{\underline{w}}(h) S_{Q_{\underline{w}}, Q_{\underline{w}}+1}^{(1-k, 2)}(h).$$

In total we have shown that both, the expressions for the commutator forms and the respective (trial) operators implementing the forms converge against each other. This shows (5.35) and thus completes the proof. \square

Similar to the case of compactly supported coupling functions, see Definition 4.3, we now define a span of products of operators of the type appearing in the last lemma.

Notation 5.26.

1. Let $j, n(1), \dots, n(j) \in \mathbb{N}$ and $\underline{w}_1 \in \mathfrak{J}_{n(1)}, \dots, \underline{w}_j \in \mathfrak{J}_{n(j)}$. For $1 \leq p \leq j$ we suppose that $M_p, Q_p \in \mathbb{N}$, $h \in H_{\text{uv}}^k(\mathbb{R}^\nu)$ for $k = \max_{p \in \{1, \dots, j\}} \mathfrak{n}_{\underline{w}_p}^A$ and $Q_p \geq M_p + \text{Int}(|\underline{w}_p|/2)$. We then define

$$\prod_{p=1}^j S_{M_p, Q_p}^{\underline{w}_p}(h) := S_{M_j, Q_j}^{\underline{w}_j}(h) S_{M_{j-1}, Q_{j-1}}^{\underline{w}_{j-1}}(h) \cdots S_{M_1, Q_1}^{\underline{w}_1}(h).$$

See Remark 5.23 for the definition of the right hand side.

2. Note that since $C_0^\infty(\mathbb{R}^\nu) \subset H_{\text{uv}}^k(\mathbb{R}^\nu)$ for all $k \in \mathbb{N}$, the previous definition is also meaningful for functions $g \in C_0^\infty(\mathbb{R}^\nu)$.
3. The context usually dictates the choice of $j, n(i), \underline{w}_i, \dots$ and we will thus not list them whenever no confusion can arise.

Definition 5.27. Let $i, k, b \in \mathbb{N}$ and $\ell \geq k$. Define

$$\mathfrak{U}_{i,k,b}^{\text{uv},\ell} := \text{Span}\left\{ R_h^B \prod_{p=1}^j S_{M_p, Q_p}^{\underline{w}_p}(h) \mid h \in H_{\text{uv}}^\ell(\mathbb{R}^\nu), 1 \leq j \leq i, \sum_{p=1}^i \mathfrak{n}_{\underline{w}_p}^A = k, B \geq b, \right. \\ \left. Q_p \geq M_p + \text{Int}(|\underline{w}_p|/2) + 1 \right\},$$

$$\mathfrak{U}_{0,0,b}^{\text{uv},\ell} := \text{Span}\{R_{-c}^B \mid B \geq b\},$$

$$\mathfrak{U}_{0,k,b}^{\text{uv},\ell} := 0.$$

Moreover, we define

$$\text{ad}_A(\mathfrak{U}_{i,k,b}^{\text{uv},\ell}) := \text{Span}\left\{ \text{ad}_A(R_h^B \prod_{p=1}^j S_{M_p, Q_p}^{\underline{w}_p}(h)) \mid R_h^B \prod_{p=1}^j S_{M_p, Q_p}^{\underline{w}_p}(h) \in \mathfrak{U}_{i,k,b}^{\text{uv},\ell} \right\}.$$

For any $\ell \in \mathbb{N}$ Definition 4.3 and Definition 5.27 are connected by the inclusion

$$\mathfrak{U}_{i,k,b} \subset \mathfrak{U}_{i,k,b}^{\text{uv},\ell}$$

due to $C_0^\infty(\mathbb{R}^\nu) \subset H_{\text{uv}}^k(\mathbb{R}^\nu)$ for any k . As a next step we want to prove that Lemma 4.4 and Proposition 4.5 extend to more general coupling functions. To this end we will use the same strategy as in the previous Lemma, exploit the continuity of the maps $S_{M_p, Q_p}^{\underline{w}_p}(\cdot)$ and the existing results for the compactly supported case.

Lemma 5.28. *Let $i, k, b \in \mathbb{N}$, suppose that $b \geq 3 \cdot 2^i$ and $\ell \geq k + 1$. Then*

$$\text{ad}_A(\mathfrak{U}_{i,k,b}^{\text{uv},\ell}) \subset \mathfrak{U}_{i+1, k+1, b-3 \cdot 2^i}^{\text{uv},\ell}.$$

PROOF: Since $\mathfrak{U}_{i,k,b}^{\text{uv},\ell}$ consists of linear combinations of elements of the form $\prod_{p=1}^j S_{M_p, Q_p}^{\underline{w}_p}(h)$ and the operation of taking a commutator is linear, it suffices to check the statement for elements of the type $\prod_{p=1}^j S_{M_p, Q_p}^{\underline{w}_p}(h)$.

The proof is carried out by induction. The induction start at $k = 0$ is a straight forward generalization of the corresponding part in Lemma 4.4 to coupling functions in $L_{\text{uv}}^2(\mathbb{R}^\nu)$.

Let $\ell \geq k + 1$, $h \in H_{\text{uv}}^\ell(\mathbb{R}^\nu)$, $\psi \in \mathcal{H}$ and $\prod_{p=1}^j S_{M_p, Q_p}^{\underline{w}_p}(h) \in \mathfrak{U}_{i,k,b}^{\text{uv},\ell}$. Choose a sequence $(g_n)_n \subset C_0^\infty(\mathbb{R}^\nu)$ such that $g_n \rightarrow h$ w.r.t. the topology of $H_{\text{uv}}^\ell(\mathbb{R}^\nu)$. Since the result in Lemma 4.4 does not depend on the exact choice of compactly supported and smooth coupling function, we conclude that there exist C_1, \dots, C_ℓ , j_1, \dots, j_N , $n_\ell(1), \dots, n_\ell(j_\ell) \in \mathbb{N}$, where $1 \leq j_1, \dots, j_N \leq i + 1$, and $\underline{w}_{\ell,1} \in \mathfrak{I}_{n_\ell(1)}, \dots, \underline{w}_{\ell,j_\ell} \in \mathfrak{I}_{n_\ell(j_\ell)}$ and $\mathbf{n}_{\underline{w}_{\ell,1}}^A + \dots + \mathbf{n}_{\underline{w}_{\ell,j_\ell}}^A = k + 1$ for all ℓ , and $M_{\ell,p_\ell}, Q_{\ell,p_\ell} \in \mathbb{N}$, where $\ell \in \{1, \dots, N\}$, $p_\ell \in \{1, \dots, j_\ell\}$ and $Q_{\ell,p_\ell} \geq M_{\ell,p_\ell} + \text{Int}(|\underline{w}_{\ell,p_\ell}|/2) + 1$, such that for every $n \in \mathbb{N}$:

$$\text{ad}_A \left(\prod_{p=1}^j S_{M_p, Q_p}^{\underline{w}_p}(g_n) \right) \psi = \sum_{\ell=1}^N C_\ell \prod_{p_\ell=1}^{j_\ell} S_{M_{\ell,p_\ell}, Q_{\ell,p_\ell}}^{\underline{w}_{\ell,p_\ell}}(g_n) \psi \quad (5.36)$$

We study the right hand side of (5.36) first. It is clear from Remark 5.23 that

$$\sum_{\ell=1}^N \prod_{p_\ell=1}^{j_\ell} S_{M_{\ell,p_\ell}, Q_{\ell,p_\ell}}^{\underline{w}_{\ell,p_\ell}}(g_n) \psi \rightarrow \sum_{\ell=1}^N \prod_{p_\ell=1}^{j_\ell} S_{M_{\ell,p_\ell}, Q_{\ell,p_\ell}}^{\underline{w}_{\ell,p_\ell}}(h) \psi .$$

In order to show convergence of the left hand side of (5.36) we compute

$$\begin{aligned} & |\langle \psi, [A, \prod_{p=1}^j S_{M_p, Q_p}^{\underline{w}_p}(g_n)] \psi' \rangle - \langle \psi, [A, \prod_{p=1}^j S_{M_p, Q_p}^{\underline{w}_p}(h)] \psi' \rangle | \\ & \leq \|A\psi\| \left\| \prod_{p=1}^j S_{M_p, Q_p}^{\underline{w}_p}(g_n) \psi' - \prod_{p=1}^j S_{M_p, Q_p}^{\underline{w}_p}(h) \psi' \right\| \\ & \quad + \|\psi\| \left\| \prod_{p=1}^j S_{M_p, Q_p}^{\underline{w}_p}(g_n) A \psi' - \prod_{p=1}^j S_{M_p, Q_p}^{\underline{w}_p}(h) A \psi' \right\| \end{aligned}$$

for $\psi, \psi' \in D(A)$. Since the upper bound converges to 0 as $n \rightarrow \infty$ by Remark 5.23, we have shown that

$$\langle \psi, [A, \prod_{p=1}^j S_{M_p, Q_p}^{\underline{w}_p}(h)] \psi' \rangle = \langle \psi, \sum_{\ell=1}^N C_\ell \prod_{p_\ell=1}^{j_\ell} S_{M_{\ell,p_\ell}, Q_{\ell,p_\ell}}^{\underline{w}_{\ell,p_\ell}}(h) \psi' \rangle$$

for all $\psi, \psi' \in D(A)$. The right hand side is now given by a bounded operator and thus the form extends by continuity and density to a continuous form on the whole of \mathcal{H} uniquely implemented by the operator $\text{ad}_A(\prod_{p=1}^j S_{M_p, Q_p}^{\underline{w}_p}(h))$.

However, uniqueness implies that

$$\mathrm{ad}_A\left(\prod_{p=1}^j S_{M_p, Q_p}^{w_p}(h)\right) = \sum_{\ell=1}^N \prod_{p_\ell=1}^{j_\ell} S_{M_{\ell, p_\ell}, Q_{\ell, p_\ell}}^{w_{\ell, p_\ell}}(h),$$

because it does so on the dense set $D(A)$. To complete the argument we simply note that since

$$\sum_{\ell=1}^N \prod_{p_\ell=1}^{j_\ell} S_{M_{\ell, p_\ell}, Q_{\ell, p_\ell}}^{w_{\ell, p_\ell}}(g_n) \in \mathfrak{U}_{i+1, k+1, b-3 \cdot 2^i}$$

for all $n \in \mathbb{N}$, it follows that

$$\sum_{\ell=1}^N \prod_{p_\ell=1}^{j_\ell} S_{M_{\ell, p_\ell}, Q_{\ell, p_\ell}}^{w_{\ell, p_\ell}}(h) \in \mathfrak{U}_{i+1, k+1, b-3 \cdot 2^i}^{\mathrm{uv}, \ell}.$$

This concludes the proof. \square

With the preceding arguments we have generalized all ingredients which needed in the proof of Proposition 4.5 to general coupling functions in $H_{\mathrm{uv}}^\ell(\mathbb{R}^\nu)$ for some ℓ . It is therefore not surprising that we are able to prove

Proposition 5.29 (The $C^k(A)$ Property for Large Resolvent Powers II). *Let $k \in \mathbb{N}_0$, $\ell \geq k$ and suppose that $m \in \mathbb{N}$ satisfies $m \geq m_k := \sum_{j=0}^k 3 \cdot 2^j$. Then for every $h \in H_{\mathrm{uv}}^{k+1}(\mathbb{R}^\nu)$ the operator $(H_h(\xi) + c)^{-m}$ is in $C^{k+1}(A)$. Moreover,*

$$\mathrm{ad}_A^k((H_h(\xi) + c)^{-m}) \in \mathfrak{U}_{k, k, m-m_k}^{\mathrm{uv}, \ell}.$$

PROOF: It is trivial to check the statement for $k = 0$. The induction step uses Lemma 5.28 and is a word by word copy of Proposition 4.5. \square

5.4 Local Regularity

As in the proof of Theorem 4.6 we will need to adapt notation in order to extend the proof from the previous chapter to the case, where there may be resolvents at two different points $-c, z \in \rho(H)$.

Notation 5.30.

1. For $h \in H_{\mathrm{uv}}(\mathbb{R}^\nu)$ and $z \in \rho(H_h)$ we define

$$R_z := (H_h - z)^{-1}.$$

2. Let $\theta \in \{0, 1\}$, $i \in \mathbb{N}$, $n(1), \dots, n(i) \in \mathbb{N}$, $\underline{w}_1 \in \mathfrak{I}_{n(1)}, \dots, \underline{w}_i \in \mathfrak{I}_{n(i)}$, $m_1, \dots, m_i \in \mathbb{N}$ and $Q_1, \dots, Q_i \in \mathbb{N}$ with $Q_j \geq m_j + \text{Int}(|\underline{w}_j|/2) + 1$. Suppose $h \in \mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu)$, where $k \geq \max\{\mathbf{n}_{\underline{w}_1}, \dots, \mathbf{n}_{\underline{w}_i}\}$. For any $z \in \rho(H)$ we define

$$\prod_{p=1}^i R_z^\theta S_{m_p, Q_p}^{\underline{w}_p}(h) := R_z^\theta S_{m_i, Q_i}^{\underline{w}_i}(h) \cdots R_z^\theta S_{m_1, Q_1}^{\underline{w}_1}(h).$$

Definition 5.31. Let $i, k, b \in \mathbb{N}$ and $\ell \geq k$. Define

$$\mathfrak{V}_{i,k,b}^{\text{uv},\ell} := \{R^B(-c) [\prod_{p=1}^i R_h(z)^\theta S_{m_p, Q_p}^{\underline{w}_p}(h)] R(z) \mid h \in \mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu), 1 \leq j \leq i, \\ \sum_{p=1}^i \mathbf{n}_{\underline{w}_p}^A = k, B \geq b, \theta \in \{0, 1\}, Q_p \geq M_p + \text{Int}(|\underline{w}_p|/2) + 1\}.$$

Moreover, for any ℓ we define

$$\mathfrak{V}_{0,0,b}^{\text{uv},\ell} := \text{Span}\{R_{-c}^B R_z \mid B \geq b\}, \quad \mathfrak{V}_{0,k,b}^{\text{uv},\ell} := 0.$$

Lemma 5.32. Let $i, k, b \in \mathbb{N}$ and suppose that $b \geq 5 \cdot 2^i$ and $\ell \geq k + 1$. Then

$$\text{ad}_A(\mathfrak{V}_{i,k,b}^{\text{uv},\ell}) \subset \mathfrak{V}_{i+1,k+1,b-5 \cdot 2^i}^{\text{uv},\ell}.$$

PROOF: The proof uses the same approximation strategy as Lemma 5.28. We fix $V_{i,k,b} \in \mathfrak{V}_{i,k,b}^{\text{uv},\ell}$ for a coupling function $h \in \mathbb{H}_{\text{uv}}^k(\mathbb{R}^\nu) \setminus \{C_0^\infty(\mathbb{R}^\nu)\}$. We may approximate the form expressions $[A, V_{i,k,b}]$ on C_0^∞ by limits of compactly supported coupling functions. Since the statement is correct for compactly supported and smooth coupling functions, these approximations are implemented by bounded operators which are already of the correct form. This then implies that the form expressions in the general case are bounded and hence implemented by bounded operators. Finally, we use that by uniqueness these operators have to coincide with the limits of the operators in the compactly supported case and hence are of the correct form. Since this is more or less a word-by-word copy of the proof of Lemme 5.28, we omit the details. \square

Lemma 5.33. Let $k \in \mathbb{N}_0, h \in \mathbb{H}_{\text{uv}}^{k+1}(\mathbb{R}^\nu)$, $m \in \mathbb{N}$ and suppose that $m \geq m_k := \sum_{j=0}^k 5 \cdot 2^j$. Then $(H_h + c)^{-m} (H_h - z)^{-1}$ is of class $C^{k+1}(A)$ for any $z \in \rho(H_h)$. Moreover, for $\ell \geq k + 1$ we have that

$$\text{ad}_A^k((H_h + c)^{-m} (H_h - z)^{-1}) \in \mathfrak{V}_{i,k,b-m_k}^{\text{uv},\ell}.$$

PROOF: As in the case of Proposition 5.29 it is trivial to verify the case $k = 0$ for $m \geq 5$ and $h \in \mathbb{H}_{\text{uv}}(\mathbb{R}^\nu)$ by Lemma A.7 in the same fashion as in Proposition 4.5. Since all ingredients used in the proof of Proposition 4.5 have been generalized

to the case of coupling functions in $H_{uv}^k(\mathbb{R}^\nu)$, the rest of the proof can be carried out in complete analogy to the proof of this proposition and can thus be omitted. \square

We have thus proven all necessary technical ingredients needed to extend Theorem 4.6 to coupling functions $h \in H_{uv}^k(\mathbb{R}^\nu)$.

Theorem 5.34 ($C^k(A)$ -Property for $f(H_h(\xi))$ with $h \in H_{uv}^k(\mathbb{R}^\nu)$). *Let $h \in H_{uv}^k(\mathbb{R}^\nu)$ and suppose that $f \in C_0^\infty(\mathbb{R}^\nu)$. Then $f(H_h(\xi)) \in C^k(A)$.*

With the help of the usual approximation procedure and Theorem 5.34 we can mimic the proof of Lemma 4.11 and obtain a generalization of this statement. Note that unlike in the case, where $g \in C_0^\infty(\mathbb{R}^\nu)$, $h \in H_{uv}^k(\mathbb{R}^\nu)$ implies less regularity of the eigenstate η , since we are not necessarily allowed to compute arbitrary amounts of derivatives of h .

Lemma 5.35. *Let $h \in H_{uv}^{k+1}(\mathbb{R}^\nu)$. $\eta \in D(H_h(\xi))$ be an eigenvector, that is $H_h(\xi)\eta = \lambda_\xi\eta$. Then*

1. $\eta \in D(A^k)$.
2. Let $k' \leq k$. Then $A^{k'}\eta \in D(H(\xi)^m)$ for all $m \in \mathbb{N}$.

Proposition 5.36 (Regularity of Eigenstates). *Let $h \in H_{uv}^{k+1}(\mathbb{R}^\nu)$ and suppose that there exist $\xi \in \mathbb{R}^\nu$, $\eta \in D(H(\xi))$ and $\lambda_\xi \in \sigma(H(\xi)) \cap \mathcal{E}^{(1)}(\xi) \setminus \mathcal{T}^{(1)}(\xi)$ with $H(\xi)\eta = \lambda_\xi\eta$. Then $\eta \in D(A^k)$.*

PROOF: By Theorem 5.34 $h \in H_{uv}^{k+1}(\mathbb{R}^\nu)$ implies that $H(\xi)$ is locally of class $C^{k+1}(A)$. This implies the statement by Theorem 1.6 in [40]. \square

6 The Feshbach-Schur Method

6.1 Small Total Momenta as a Perturbation

Throughout the rest of Section 6 we assume that the coupling function satisfies at least

$$h \in L_{uv}^2(\mathbb{R}^\nu)$$

unless stated otherwise. Moreover, we fix the following notation.

Notation 6.1.

1. Recall that $\xi_0 \in U$, $\lambda_{\xi_0} \in \sigma_p(H(\xi_0)) \cap \mathcal{E}^{(1)}(\xi_0) \setminus \mathcal{T}^{(1)}(\xi_0)$ with normalized eigenvector η , that is $H(\xi_0)\eta = \lambda_{\xi_0}\eta$ and $\|\eta\| = 1$.
2. $P := |\eta\rangle\langle\eta|$, $\bar{P} := 1 - P$.

3. Pick $\xi_0 \in U$ and fix $\kappa > 0$ such that (2.3) holds for all $\xi \in \mathcal{O}_0$.
4. For $\xi \in \mathbb{R}$ define $\overline{H}(\xi + \zeta) := \overline{P}H(\xi + \zeta)\overline{P}$.
5. $\overline{A} := \overline{P}A\overline{P}$.
6. For any $\xi \in \mathbb{R}^\nu$ and $z \in \mathbb{C}$ with $\text{Im}(z) > 0$ we define

$$\overline{R}_z(\xi) := (\overline{P}H(\xi)\overline{P} - z\overline{P})^{-1} \upharpoonright_{\text{Ran}(\overline{P})}.$$

Let $\xi, \zeta \in \mathbb{R}^\nu$. Then

$$H(\xi + \zeta) = H_0(\xi) + \Phi(h) + V_\zeta, \quad V_\zeta := \zeta^2 + 2\xi \cdot \zeta - 2\zeta \cdot d\Gamma(k)$$

6.2 Regularity of the projected Hamiltonian

Let T be a closed operator on \mathcal{H} with dense domain $D(T)$ and suppose that $\eta \in D(T) \cap D(T^*)$. Then, on $D(T)$ we may define

$$\overline{PT}\overline{P} = T - |\eta\rangle\langle T^*\eta| - |T\eta\rangle\langle\eta| + \langle\eta, T\eta\rangle P.$$

Clearly, $\overline{PT}\overline{P}$ with domain $D(T)$ is again a closed operator. By Theorem 5.34 and regularity of eigenstates, Proposition 5.36, $h \in \mathbb{H}_{\text{uv}}^2(\mathbb{R}^\nu)$ implies $\eta \in D(A)$ and hence $\overline{P}D(A) \subset D(A)$. Therefore, \overline{A} is a closed operator on $D(A)$. Likewise $\overline{H}(\xi + \zeta)$ is a closed operator on $D(H(\xi_0))$, since V_ζ is $H(\xi_0)$ -bounded and symmetric.

Lemma 6.2. *Let $k \in \mathbb{N}$, $T, T' \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}^{\text{uv}, k}$ and suppose T, T' are $H(\xi)^m$ -bounded for some $m \in \mathbb{N}$. Then $\eta \in D(T), D(T')$ and the following two statements hold.*

1. Let $h \in \mathbb{H}_{\text{uv}}^{k+1}(\mathbb{R}^\nu)$. Then $|T\eta\rangle\langle T'\eta| \in \mathcal{C}^k(\overline{A})$.
2. Let $\ell, \ell' \in \mathbb{N}$, put $L := \max\{\ell, \ell'\}$ and suppose that $h \in \mathbb{H}_{\text{uv}}^{k+L+1}(\mathbb{R}^\nu)$. Then $|TA^\ell\eta\rangle\langle T'A^{\ell'}\eta| \in \mathcal{C}^k(\overline{A})$.

PROOF: We prove the first statement by using induction in $k \in \mathbb{N}$. For $k = 1$ we have $\eta \in D(A)$ by the regularity of eigenstates, see Theorem 1.6 in [40]. Hence, we are in a situation in which (6.57) holds. This shows that $|T\eta\rangle\langle T'\eta| \in \mathcal{C}^1(\overline{A})$.

Now assume that the first assertion holds for some $k \in \mathbb{N}$ and suppose that $h \in \mathbb{H}_{\text{uv}}^{k+2}(\mathbb{R}^\nu)$ and $T, T' \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}^{\text{uv}, k+1}$. Clearly, the induction hypothesis implies that $|T\eta\rangle\langle T'\eta| \in \mathcal{C}^k(\overline{A})$. By Lemma 5.35 and the assumption on T, T' we can define the sets

$$\begin{aligned} V_n &:= \bigcup_{0 \leq i_1 + i_2 \leq n} \{\text{ad}_A^{i_1}(T)A^{i_2}\eta, TA^{i_1}\eta\}, \\ V'_n &:= \bigcup_{0 \leq i_1 + i_2 \leq n} \{\text{ad}_A^{i_1}(T')A^{i_2}\eta, T'A^{i_1}\eta\}, \\ F_n &:= \{|\zeta\rangle\langle\zeta'| \mid \zeta \in V_n, \zeta' \in V'_n\} \end{aligned}$$

for every $n \in \mathbb{N}$ with $n \leq k + 1$. As an intermediate step we show that in this situation there exist $m(\ell) \in \mathbb{N}$, finite rank operators $|\zeta_1\rangle\langle\zeta'_1|, \dots, |\zeta_{m(\ell)}\rangle\langle\zeta'_{m(\ell)}| \in F_\ell$ and constants $C_1, \dots, C_{m(\ell)} \in \mathbb{C}$ for every $\ell \leq k$ such that

$$\text{ad}_{\bar{A}}^\ell(|T\eta\rangle\langle T'\eta|) = \sum_{i=1}^{m(\ell)} C_i |\zeta_i\rangle\langle\zeta'_i| \quad (6.37)$$

as bounded operators. We prove this statement by induction as well. For $\ell = 1$ an easy computation shows that $\text{ad}_{\bar{A}}^1(|T\eta\rangle\langle T'\eta|) = [\bar{A}, |T\eta\rangle\langle T'\eta|]$ is of the correct form. Assume that $|T\eta\rangle\langle T'\eta| \in C^{\ell+1}(\bar{A})$ and (6.37) holds for $|T\eta\rangle\langle T'\eta| \in C^\ell(\bar{A})$. Let $\psi, \psi' \in D(A)$ and compute

$$\langle\psi, [\bar{A}, \text{ad}_{\bar{A}}^\ell(|T\eta\rangle\langle T'\eta|)]\psi'\rangle = \sum_{i=1}^{m(\ell)} C_i \langle\psi, [\bar{A}, |\zeta_i\rangle\langle\zeta'_i|]\psi'\rangle$$

There are four possible configurations for each ζ_i, ζ'_i which need to be checked. Let $1 \leq i \leq m(\ell)$ and assume that $\zeta_i = \text{ad}_A^{i_1}(T)A^{i_2}\eta$ and $\zeta'_i = T'A^{i_3}\eta$, where $1 \leq i_1, i_2, i_3 \leq \ell$. Then as a form on $D(A)$

$$\begin{aligned} \langle\psi, [A, |\zeta_i\rangle\langle\zeta'_i|]\psi'\rangle &= \langle\psi, |A \text{ad}_A^{i_1}(T)A^{i_2}\eta\rangle\langle\zeta'_i|\psi'\rangle - \langle\psi, |\zeta_i\rangle\langle AT'A^{i_3}\eta|\psi'\rangle \\ &= \langle\psi, |\text{ad}_A^{i_1}(T)A^{i_2+1}\eta\rangle\langle\zeta'_i|\psi'\rangle + \langle\psi, |\text{ad}_A^{i_1+1}(T)A^{i_2}\eta\rangle\langle\zeta'_i|\psi'\rangle \\ &\quad - \langle\psi, |\zeta_i\rangle\langle T'A^{i_3+1}\eta|\psi'\rangle - \langle\psi, |\zeta_i\rangle\langle \text{ad}_A(T)A^{i_3}\eta|\psi'\rangle. \end{aligned}$$

Hence $\text{ad}_A(|\zeta_i\rangle\langle\zeta'_i|)$ is implemented by a linear combination of finite rank operators $|\iota_i\rangle\langle\iota'_i| \in F_{\ell+1}$. The remaining cases can be checked in the exact same fashion and hence

$$\text{ad}_{\bar{A}}^{\ell+1}(|T\eta\rangle\langle T'\eta|) = \sum_{i=1}^{m(\ell+1)} C'_i |\iota_i\rangle\langle\iota'_i|,$$

where $m(\ell + 1) \in \mathbb{N}$, $|\iota_i\rangle\langle\iota'_i| \in F_{\ell+1}$ and $C'_1, \dots, C'_{m(\ell+1)} \in \mathbb{C}$. This proves (6.37) holds for any $\ell \in \mathbb{N}$ provided that $|T\eta\rangle\langle T'\eta| \in C^\ell(\bar{A})$.

Now we return to the initial induction hypothesis that $|T\eta\rangle\langle T'\eta| \in C^k(\bar{A})$ for some $k \in \mathbb{N}$. Taking another commutator and using (6.37) we compute

$$\langle\psi, [\bar{A}, \text{ad}_{\bar{A}}^k(|T\eta\rangle\langle T'\eta|)]\psi'\rangle = \langle\psi, \sum_{i=1}^{m(\ell)} C_i [\bar{A}, |\zeta_i\rangle\langle\zeta'_i|]\psi'\rangle.$$

Since we have already shown that $|\zeta_i\rangle\langle\zeta'_i| \in C^1(\bar{A})$ earlier in the proof, this shows that $[\bar{A}, \text{ad}_{\bar{A}}^k(|T\eta\rangle\langle T'\eta|)]$ is given by a bounded operator which proves the first statement.

The second statement directly follows from a slight generalization of (6.37). Indeed, if we replace $|T\eta\rangle\langle T'\eta|$ by $|TA^\ell\eta\rangle\langle T'A^\ell\eta|$, we can repeat the proof of (6.37) in this generalized setting and thus draw the same conclusions. \square

Remark 6.3. Note that a slight adaptation of the proof also yields that $|T\eta\rangle\langle T'\eta| \in C^k(A)$.

Lemma 6.4. Let $\ell \geq k+1$ and suppose the coupling function satisfies $h \in H_{uv}^\ell(\mathbb{R}^\nu)$. Then there exist $T_j \in C^{k-j}(A)$, where $j \in N_k := \{1, \dots, k\}$, such that

$$\text{ad}_A^j(d\Gamma(k_\sigma)) = \bar{P} \text{ad}_A^j(d\Gamma(k_\sigma))\bar{P} + T_{k-j}$$

extends from C_0^∞ to $D(H_h(\xi))$ for all $j \in N_k$. Here, we have put $C^0(A) := \mathfrak{B}(\mathcal{H})$.

PROOF: In order to carry out the proof by induction in k we prove the more precise statement that for all $k \in \mathbb{N}$ and $j \in N_k$ there exist $n_j \in \mathbb{N}$ and $H_h(\xi)$ -bounded $T_{j,i} \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ such that

$$T_j \in \text{Span}\{|T_{j,i}A^\ell\eta\rangle\langle T_{j,i'}A^{\ell'}\eta| \mid i, i' \in N_{n_j}, \ell, \ell' \in \mathbb{N}_0, 0 \leq \ell + \ell' \leq j\}. \quad (6.38)$$

Let $\psi, \psi' \in C_0^\infty$, $k = 1$ and $\ell \geq 2$. By regularity of eigenstates, see Theorem 1.6 in [40], we have that $\eta \in D(A)$. This allows us to compute

$$\begin{aligned} \langle \psi, [\bar{A}, d\Gamma(k_\sigma)]\psi' \rangle &= \langle \psi, [P, d\Gamma(k_\sigma)]A\bar{P}\psi' \rangle + \langle \psi, \bar{P}[A, d\Gamma(k_\sigma)]\bar{P}\psi' \rangle \\ &\quad + \langle \psi, \bar{P}A[P, d\Gamma(k_\sigma)]\psi' \rangle \\ &= \langle \psi, |\eta\rangle\langle d\Gamma(k_\sigma)A\eta| \bar{P}\psi' \rangle + \langle \psi, |\eta\rangle\langle \text{ad}_A(d\Gamma(k_\sigma))\eta| \bar{P}\psi' \rangle \\ &\quad - \langle \psi, |d\Gamma(k_\sigma)\eta\rangle\langle A\eta| \bar{P}\psi' \rangle + \langle \psi, \bar{P} \text{ad}_A(d\Gamma(k_\sigma))\bar{P}\psi' \rangle \\ &\quad + \langle \psi, \bar{P}|A\eta\rangle\langle d\Gamma(k_\sigma)\eta|\psi' \rangle - \langle \psi, \bar{P}|d\Gamma(k_\sigma)A\eta\rangle\langle \eta|\psi' \rangle \\ &\quad - \langle \psi, \bar{P}|\text{ad}_A(d\Gamma(k_\sigma))\eta\rangle\langle \eta|\psi' \rangle. \end{aligned}$$

By Lemma 6.2 all contributions except $\bar{P} \text{ad}_A(d\Gamma(k_\sigma))\bar{P}$ are given by elements of $\mathfrak{B}(\mathcal{H}) = C^0(A)$ which are elements of the set appearing in (6.38), where $\ell + \ell' \in \{0, 1\}$.

Suppose the statement is true for all $k' \leq k$ and (6.38) holds for all T_j , $j \in N_k$. Choose $\ell \geq k+2$. Let $j \in N_k$ and calculate

$$\langle \psi, [\bar{A}, \text{ad}_A^k(d\Gamma(k_\sigma))]\psi' \rangle = \langle \psi, [\bar{A}, \text{ad}_A^k(d\Gamma(k_\sigma))]\psi' \rangle + \langle \psi, [\bar{A}, T_k]\psi' \rangle.$$

Again by [40] this implies $\eta \in D(A^{k+1})$. Lemma 6.2 thus implies that $T_j \in C^{k+1-j}$ and therefore $[\bar{A}, T_k]$ extends to a bounded operator for all $j \leq k$. Moreover, this operator is an element of the set in (6.38), where $\ell + \ell' \leq k+1$. In order to deal with the first term we compute

$$\begin{aligned} \langle \psi, [\bar{A}, \text{ad}_A^k(d\Gamma(k_\sigma))]\psi' \rangle &= \langle \psi, [P, \text{ad}_A^k(d\Gamma(k_\sigma))]A\bar{P}\psi' \rangle \\ &\quad + \langle \psi, \bar{P} \text{ad}_A^{k+1}(d\Gamma(k_\sigma))\bar{P}\psi' \rangle \\ &\quad + \langle \psi, \bar{P}A[P, \text{ad}_A^k(d\Gamma(k_\sigma))]\psi' \rangle \end{aligned}$$

The second contribution is already of the correct form. The first and second can be dealt with similarly to the induction start. We omit the details to avoid repetition. \square

Definition 6.5. Recall the notation $I_{\lambda, \kappa} = (\lambda - \kappa, \lambda + \kappa)$. Let U be as in Assumption 2.5, pick $\xi \in U$ and fix $\kappa > 0$ such that (2.3) holds. In particular, $I_{\lambda_\xi, \kappa} \subset \mathcal{E}^{(1)}(\xi) \setminus \mathcal{T}^{(1)}(\xi)$. We define the set of functions

$$\mathcal{L}_{\mathcal{E}^{(1)} \setminus \mathcal{T}^{(1)}}(\lambda_\xi) := \left\{ \theta \in C_0^\infty(\mathbb{R}) \mid \exists \kappa' \in (0, \kappa) : \text{supp}(\theta) \subset I_{\lambda_\xi, \kappa'} \text{ and } \theta|_{I_{\lambda_\xi, \kappa'}} \equiv 1 \right\}.$$

Proposition 6.6. Fix $\kappa > 0$ such that (2.3) holds and pick $\theta \in \mathcal{L}_{\mathcal{E}^{(1)} \setminus \mathcal{T}^{(1)}}(\lambda_{\xi_0})$. Then there exists $r, e > 0$ and an open neighborhood \mathcal{V} of ξ_0 with $\mathcal{V} \subset \mathcal{O}_0$, where \mathcal{O}_0 is defined in Notation 2.6, such that for all $\xi \in \mathcal{V}$ and $|\zeta| < r$

$$\overline{P}\theta(\overline{H}(\xi + \zeta))[\overline{A}, \overline{H}(\xi + \zeta)]\theta(\overline{H}(\xi + \zeta))\overline{P} \geq e\overline{P}\theta(\overline{H}(\xi + \zeta))^2\overline{P}. \quad (6.39)$$

PROOF: We establish the statement for $\xi = \xi_0$ and small $|\zeta|$ first and then show how the neighborhood \mathcal{V} can be picked. Choose $\theta \in \mathcal{L}_{\mathcal{E}^{(1)} \setminus \mathcal{T}^{(1)}}(\lambda_{\xi_0})$. By the Mourre estimate (2.3) there exists $C > 0$ and a compact operator K such that

$$\begin{aligned} \overline{P}\theta(\overline{H}(\xi_0))[\overline{A}, \overline{H}(\xi_0)]\theta(\overline{H}(\xi_0))\overline{P} &= \overline{P}\theta(\overline{H}(\xi_0))[A, \overline{H}(\xi_0)]\theta(\overline{H}(\xi_0))\overline{P} \\ &\geq C\overline{P}\theta(\overline{H}(\xi_0))^2\overline{P} + \overline{P}\theta(\overline{H}(\xi_0))K\theta(\overline{H}(\xi_0))\overline{P}. \end{aligned}$$

Therefore, since $\overline{H}(\xi_0)$ does not have eigenvalues close to λ_{ξ_0} , there is a strict Mourre estimate:

$$\overline{P}\theta(\overline{H}(\xi_0))[\overline{A}, \overline{H}(\xi_0)]\theta(\overline{H}(\xi_0))\overline{P} \geq C\overline{P}\theta(\overline{H}(\xi_0))^2\overline{P}. \quad (6.40)$$

Moreover, due to

$$\begin{aligned} &\overline{P}\theta(\overline{H}(\xi_0 + \zeta))[\overline{A}, \overline{H}(\xi_0 + \zeta)]\theta(\overline{H}(\xi_0 + \zeta))\overline{P} \\ &= \overline{P}\theta(\overline{H}(\xi_0))[\overline{A}, \overline{P}H(\xi_0)\overline{P}]\theta(\overline{H}(\xi_0))\overline{P} \\ &\quad + \overline{P} \left\{ \theta(\overline{H}(\xi_0 + \zeta)) - \theta(\overline{H}(\xi_0)) \right\} [\overline{A}, \overline{H}(\xi_0)]\theta(\overline{H}(\xi_0 + \zeta))\overline{P} \\ &\quad + \overline{P}\theta(\overline{H}(\xi_0 + \zeta))[\overline{A}, \overline{H}(\xi_0)] \left\{ \theta(\overline{H}(\xi_0 + \zeta)) - \theta(\overline{H}(\xi_0)) \right\} \overline{P} \\ &\quad - 2 \sum_{\sigma=1}^{\nu} \zeta_\sigma \overline{P}\theta(\overline{H}(\xi_0 + \zeta))[A, \overline{P} d\Gamma(k_\sigma)\overline{P}]\theta(\overline{H}(\xi_0 + \zeta))\overline{P} \end{aligned}$$

and the Lipschitz continuity of $\zeta \mapsto \theta(\overline{H}(\xi_0 + \zeta))$, the previous equation can be bounded from below as in (6.40) for $|\zeta|$ sufficiently small, say $|\zeta| < r'$ for some $r' > 0$. The choices $r'' < r'$, $r := r' - r''$ and $\mathcal{V} = B_r''(\xi_0)$ then imply the general statement. \square

Proposition 6.7. Let $h \in H_{uv}^3(\mathbb{R}^\nu)$. There exists a neighborhood \mathcal{V} of ξ_0 with $\mathcal{V} \subset \mathcal{O}_0$ and $r > 0$ such that $\overline{H}(\xi + \zeta)$ is of class $C^2(\overline{A})$ for all $\xi \in \mathcal{V}$ and $|\zeta| < r$.

PROOF: Assume $h \in H_{\text{uv}}^3(\mathbb{R}^\nu)$ accordingly. First note that due to $h \in H_{\text{uv}}^3(\mathbb{R}^\nu)$, $\eta \in D(A^2)$ by regularity of eigenstates, see Proposition 5.36. As in the proof of Proposition 6.6 it suffices to check the result for $\xi = \xi_0$ and small $|\zeta|$.

Since at this point it is not clear whether $\overline{R}_z(\xi)$ preserves $D(A)$, we have to be more careful in dealing with commutators of \overline{A} and $\overline{R}_z(\xi)$. We thus introduce a regularization of A by

$$A_\mu := i\mu A(A - i\mu)^{-1}$$

for $\mu \in \mathbb{R}$ and put $\overline{A}_\mu := \overline{P}A_\mu\overline{P}$. Let $\psi, \psi' \in \mathcal{C}_0^\infty$ and calculate

$$\begin{aligned} & \langle \overline{P}\psi, [\overline{A}_\mu, \overline{R}_z(\xi_0 + \zeta)]\overline{P}\psi' \rangle \\ &= -\langle \overline{R}_z(\xi_0 + \zeta)\overline{P}\psi, [\overline{A}_\mu, \overline{P}H(\xi_0 + \zeta)\overline{P}]\overline{R}_z(\xi_0 + \zeta)\overline{P}\psi' \rangle \\ &= \langle \overline{R}_z(\xi_0 + \zeta)\overline{P}\psi, |A_\mu\eta\rangle\langle\eta|2\zeta \cdot d\Gamma(k)\overline{R}_z(\xi_0 + \zeta)\overline{P}\psi' \rangle \\ &\quad + \langle \overline{R}_z(\xi_0 + \zeta)\overline{P}\psi, 2\zeta \cdot d\Gamma(k)|\eta\rangle\langle A_{-\mu}\eta|\overline{R}_z(\xi_0 + \zeta)\overline{P}\psi' \rangle \\ &\quad - \langle \overline{R}_z(\xi_0 + \zeta)\overline{P}\psi, [A_\mu, H(\xi_0)]\overline{R}_z(\xi_0 + \zeta)\overline{P}\psi' \rangle \\ &\quad + \langle \overline{R}_z(\xi_0 + \zeta)\overline{P}\psi, [A_\mu, 2\zeta \cdot d\Gamma(k)]\overline{R}_z(\xi_0 + \zeta)\overline{P}\psi' \rangle, \end{aligned} \quad (6.41)$$

where we have used that \overline{P} commutes with $\overline{R}_z(\xi_0 + \zeta)$ and thus some contributions of rank one operators of the form $|\eta'\rangle\langle\eta|$ and $|\eta\rangle\langle\eta'|$ are annihilated. We now look at the convergence of this expression as $|\mu| \rightarrow \infty$ by studying the individual terms separately.

To treat the first term in (6.41) we note that $A_\mu\eta = i\mu(A - i\mu)^{-1}A\eta \rightarrow A\eta$ as $|\mu| \rightarrow \infty$. Moreover, $\eta \in D(d\Gamma(k_\sigma))$, since $d\Gamma(k_\sigma)$ is $H(\xi_0)$ -bounded. Thus,

$$\begin{aligned} & \lim_{|\mu| \rightarrow \infty} \langle \overline{R}_z(\xi_0 + \zeta)\overline{P}\psi, |A_\mu\eta\rangle\langle\eta|2\zeta \cdot d\Gamma(k)\overline{R}_z(\xi_0 + \zeta)\overline{P}\psi' \rangle \\ &= \lim_{|\mu| \rightarrow \infty} \langle \overline{R}_z(\xi_0 + \zeta)\overline{P}\psi, |A_\mu\eta\rangle\langle\eta|2\zeta \cdot d\Gamma(k)\eta|\overline{R}_z(\xi_0 + \zeta)\overline{P}\psi' \rangle \\ &= \langle \overline{R}_z(\xi_0 + \zeta)\overline{P}\psi, |A\eta\rangle\langle\eta|2\zeta \cdot d\Gamma(k)\eta|\overline{R}_z(\xi_0 + \zeta)\overline{P}\psi' \rangle \end{aligned} \quad (6.42)$$

by symmetry of $d\Gamma(k_\sigma)$. Likewise

$$\begin{aligned} & \lim_{|\mu| \rightarrow \infty} \langle \overline{R}_z(\xi_0 + \zeta)\overline{P}\psi, 2\zeta \cdot d\Gamma(k)|\eta\rangle\langle A_{-\mu}\eta|\overline{R}_z(\xi_0 + \zeta)\overline{P}\psi' \rangle \\ &= \langle \overline{R}_{\xi_0+\zeta}(\overline{z})\overline{P}\psi, |2\zeta \cdot d\Gamma(k)\eta\rangle\langle A\eta|\overline{R}_{\xi_0+\zeta}(z)\overline{P}\psi' \rangle. \end{aligned} \quad (6.43)$$

In order to deal with the third term in (6.41) we let $\overline{P}\chi, \overline{P}\chi' \in \mathcal{C}_0^\infty$ and find $\chi_0, \chi'_0 \in \mathcal{H}$ such that $\overline{P}\chi = (H(\xi_0) - i)^{-1}\chi_0$ and $\overline{P}\chi' = (H(\xi_0) + i)^{-1}\chi'_0$. We make

use of this fact to compute

$$\begin{aligned}
& \lim_{|\mu| \rightarrow \infty} \langle \overline{P}\chi, [H(\xi_0), A_\mu] \overline{P}\chi' \rangle \\
&= \lim_{|\mu| \rightarrow \infty} \langle (H(\xi_0) - i)^{-1} \chi_0, [H(\xi_0), A_\mu] (H(\xi_0) + i)^{-1} \chi'_0 \rangle \\
&= - \lim_{|\mu| \rightarrow \infty} \langle \chi_0, [(H(\xi_0) + i)^{-1}, A_\mu] \chi'_0 \rangle \\
&= - \langle \chi_0, [(H(\xi_0) + i)^{-1}, A] \chi'_0 \rangle \\
&= \langle \overline{P}\chi, [H(\xi_0), A] \overline{P}\chi' \rangle, \tag{6.44}
\end{aligned}$$

where $(H(\xi_0) + i)^{-1} \in C^1(A)$ has been used in the last line. Moreover, this expression clearly extends to $\chi, \chi' \in D(H(\xi_0))$ by continuity and density. It remains to examine (6.41)'s last term. To this end we calculate

$$\begin{aligned}
\lim_{|\mu| \rightarrow \infty} \langle \chi, [d\Gamma(k_\sigma), A_\mu] \chi' \rangle &= \lim_{|\mu| \rightarrow \infty} (\langle d\Gamma(k_\sigma)\psi, A_\mu \chi' \rangle - \langle A_\mu \psi, d\Gamma(k_\sigma)\chi' \rangle) \\
&= \lim_{|\mu| \rightarrow \infty} (\langle d\Gamma(k_\sigma)\chi, A_\mu \chi' \rangle - \langle A_\mu \chi, d\Gamma(k_\sigma)\chi' \rangle) \\
&= \langle \chi, [d\Gamma(k_\sigma), A] \chi' \rangle = \langle \chi, d\Gamma(v_\sigma)\chi' \rangle. \tag{6.45}
\end{aligned}$$

As we have argued before the $H(\xi_0)$ -boundedness of $d\Gamma(v_\sigma)$ and density of \mathcal{C}_0^∞ in $D(H(\xi_0))$ allows us to extend this form expression to $\chi, \chi' \in D(H(\xi_0))$.

Thus, combining (6.42)-(6.45) in their extended versions, we have shown that

$$\begin{aligned}
& \lim_{|\mu| \rightarrow \infty} \langle \overline{P}\psi, [\overline{A}_\mu, \overline{R}_z(\xi_0 + \zeta)] \overline{P}\psi' \rangle \\
&= \langle \overline{R}_z(\xi_0 + \zeta) \overline{P}\psi, |A\eta\rangle \langle 2\zeta \cdot d\Gamma(k)\eta | \overline{R}_z(\xi_0 + \zeta) \overline{P}\psi' \rangle \\
&\quad + \langle \overline{R}_z(\xi_0 + \zeta) \overline{P}\psi, |2\zeta \cdot d\Gamma(k)\eta\rangle \langle A\eta | \overline{R}_z(\xi_0 + \zeta) \overline{P}\psi' \rangle \\
&\quad + \langle \overline{R}_z(\xi_0 + \zeta) \overline{P}\psi, [H(\xi_0), A] \overline{R}_z(\xi_0 + \zeta) \overline{P}\psi' \rangle \\
&\quad + \langle \overline{R}_z(\xi_0 + \zeta) \overline{P}\psi, d\Gamma(v_\sigma) \overline{R}_z(\xi_0 + \zeta) \overline{P}\psi' \rangle. \tag{6.46}
\end{aligned}$$

Since all fiber operators $H(\xi)$ are of class $C^1(A)$, the second term is bounded which proves that $\overline{P}H(\xi_0 + \zeta)\overline{P}$ is of class $C^1(\overline{A})$. Now note that

$$\begin{aligned}
& \langle \overline{P}\psi, \overline{R}_z(\xi_0 + \zeta) [H(\xi_0), A] \overline{R}_z(\xi_0 + \zeta) \overline{P}\psi' \rangle \\
&= \langle \overline{P}\psi, \overline{R}_z(\xi_0 + \zeta) \text{ad}_A(H(\xi_0)) \overline{R}_z(\xi_0 + \zeta) \overline{P}\psi' \rangle,
\end{aligned}$$

since $\overline{R}_{\xi_0 + \zeta}(z) \overline{P} \mathcal{C}_0^\infty \subset D(H(\xi_0)^2)$. As usual we aim to commute a resolvent $\overline{R}_{\xi_0 + \zeta}(z)$ through to the right and use that $\text{ad}_A(H(\xi_0)) \overline{R}_z(\xi_0 + \zeta)^2$ is a bounded operator. As opposed to previous calculations the projections P cause additional terms which have to be dealt with. We calculate

$$\begin{aligned}
& \langle \overline{R}_z(\xi_0 + \zeta) \overline{P}\psi, \text{ad}_A(H(\xi_0)) \overline{R}_z(\xi_0 + \zeta) \overline{P}\psi' \rangle \\
&= - \langle \overline{R}_z(\xi_0 + \zeta) \overline{P}\psi, [\overline{P}H(\xi_0 + \zeta)\overline{P}, \text{ad}_A(H(\xi_0))] \overline{R}_z(\xi_0 + \zeta)^2 \overline{P}\psi' \rangle \\
&\quad + \langle \overline{P}\psi, \text{ad}_A(H(\xi_0)) \overline{R}_z(\xi_0 + \zeta)^2 \overline{P}\psi' \rangle.
\end{aligned}$$

Clearly, the last term in the previous equation is already bounded and it remains to control the first:

$$\begin{aligned}
& \langle \overline{R}_{\bar{z}}(\xi_0 + \zeta) \overline{P} \psi, [\overline{P} H(\xi_0 + \zeta) \overline{P}, \text{ad}_A(H(\xi_0))] \overline{R}_z(\xi_0 + \zeta)^2 \overline{P} \psi' \rangle \\
&= \langle \overline{R}_{\bar{z}}(\xi_0 + \zeta) \overline{P} \psi, [H(\xi_0), \text{ad}_A(H(\xi_0))] \overline{R}_z(\xi_0 + \zeta)^2 \overline{P} \psi' \rangle \\
&\quad + (\zeta^2 + 2\zeta \cdot \xi_0 - \lambda) \langle \overline{R}_{\bar{z}}(\xi_0 + \zeta) \overline{P} \psi, [P, \text{ad}_A(H(\xi_0))] \overline{R}_z(\xi_0 + \zeta)^2 \overline{P} \psi' \rangle \\
&\quad + 2 \sum_{\sigma=1}^{\nu} \zeta_{\sigma} \langle \overline{R}_{\bar{z}}(\xi_0 + \zeta) \overline{P} \psi, [\overline{P} d\Gamma(k_{\sigma}) \overline{P}, \text{ad}_A(H(\xi_0))] \overline{R}_z(\xi_0 + \zeta)^2 \overline{P} \psi' \rangle \\
&= \langle \overline{R}_{\bar{z}}(\xi_0 + \zeta) \overline{P} \psi, \text{ad}_{H(\xi_0)}(\text{ad}_A(H(\xi_0))) \overline{R}_z(\xi_0 + \zeta)^2 \overline{P} \psi' \rangle \\
&\quad + 2 \sum_{\sigma=1}^{\nu} \zeta_{\sigma} \langle \overline{R}_{\bar{z}}(\xi_0 + \zeta) \overline{P} \psi, [\overline{P} d\Gamma(k_{\sigma}) \overline{P}, \text{ad}_A(H(\xi_0))] \overline{R}_z(\xi_0 + \zeta)^2 \overline{P} \psi' \rangle, \quad (6.47)
\end{aligned}$$

where the second term in the second equality vanishes in the last step due to the presence of finite rank operators of the type $|\eta\rangle\langle\eta'|, |\eta'\rangle\langle\eta|$. By the mapping properties of $\text{ad}_{H(\xi_0)}(\cdot)$ the first term in (6.47) is already bounded.

The boundedness of the last term in (6.47) can be seen as follows. Recall that $\text{ad}_A(H(\xi_0))^*$ is $(H(\xi_0) + c)^{3/2}$ -bounded and hence $\eta \in D(\text{ad}_A(H(\xi_0))^*)$. We may thus calculate

$$\begin{aligned}
& \langle \overline{R}_{\bar{z}}(\xi_0 + \zeta) \overline{P} \psi, [\overline{P} d\Gamma(k_{\sigma}) \overline{P}, \text{ad}_A(H(\xi_0))] \overline{R}_z(\xi_0 + \zeta)^2 \overline{P} \psi' \rangle \\
&= \langle \overline{R}_{\bar{z}}(\xi_0 + \zeta) \overline{P} \psi, \overline{P} [d\Gamma(k_{\sigma}), \text{ad}_A(H(\xi_0))] \overline{P} \overline{R}_z(\xi_0 + \zeta)^2 \overline{P} \psi' \rangle \\
&\quad + \langle \overline{R}_{\bar{z}}(\xi_0 + \zeta) \overline{P} \psi, \overline{P} d\Gamma(k_{\sigma}) [P, \text{ad}_A(H(\xi_0))] \overline{R}_z(\xi_0 + \zeta)^2 \overline{P} \psi' \rangle \\
&\quad + \langle \overline{R}_{\bar{z}}(\xi_0 + \zeta) \overline{P} \psi, [P, \text{ad}_A(H(\xi_0))] d\Gamma(k_{\sigma}) \overline{P} \overline{R}_z(\xi_0 + \zeta)^2 \overline{P} \psi' \rangle \\
&= \langle \overline{R}_{\bar{z}}(\xi_0 + \zeta) \overline{P} \psi, |d\Gamma(k_{\sigma}) \eta\rangle \langle \text{ad}_A(H(\xi_0))^* \eta | \overline{R}_z(\xi_0 + \zeta)^2 \overline{P} \psi' \rangle \\
&\quad - \langle \overline{R}_{\bar{z}}(\xi_0 + \zeta) \overline{P} \psi, | \text{ad}_A(H(\xi_0)) \eta\rangle \langle d\Gamma(k_{\sigma}) \eta | \overline{R}_z(\xi_0 + \zeta)^2 \overline{P} \psi' \rangle \\
&\quad + \langle \overline{R}_{\bar{z}}(\xi_0 + \zeta) \overline{P} \psi, \text{ad}_{d\Gamma(k_{\sigma})}(\text{ad}_A(H(\xi_0))) \overline{R}_z(\xi_0 + \zeta)^2 \overline{P} \psi' \rangle. \quad (6.48)
\end{aligned}$$

By Proposition 3.5 $\text{ad}_{d\Gamma(k_{\sigma})}(\text{ad}_A(H(\xi_0)))$ is $(H(\xi_0) + c)^{\frac{3}{2}}$ -bounded and thus, all summands in (6.48) are given by bounded operators. If we combine (6.46), (6.47)

and (6.48), we may conclude that

$$\begin{aligned}
& \langle \bar{P}\psi, [\bar{A}, \bar{R}_z(\xi_0 + \zeta)]\bar{P}\psi' \rangle \\
= & \langle \bar{R}_{\bar{z}}(\xi_0 + \zeta)\bar{P}\psi, 2\text{Re}(|A\eta\rangle\langle 2\zeta \cdot d\Gamma(k)\eta|)\bar{R}_z(\xi_0 + \zeta)\bar{P}\psi' \rangle \\
& + \langle \bar{P}\psi, \text{ad}_A(H(\xi_0))\bar{R}_z(\xi_0 + \zeta)^2\bar{P}\psi' \rangle \\
& + \langle \bar{R}_{\bar{z}}(\xi_0 + \zeta)\bar{P}\psi, \text{ad}_{H(\xi_0)}(\text{ad}_A(H(\xi_0)))\bar{R}_z(\xi_0 + \zeta)^2\bar{P}\psi' \rangle \\
& + \langle \bar{R}_{\bar{z}}(\xi_0 + \zeta)\psi, 2\zeta \cdot d\Gamma(v)\bar{R}_z(\xi_0 + \zeta)\psi' \rangle \\
& + \langle \bar{R}_{\bar{z}}(\xi_0 + \zeta)\bar{P}\psi, |2\zeta \cdot d\Gamma(k)\eta\rangle\langle \text{ad}_A(H(\xi_0))^*\eta|\bar{R}_z(\xi_0 + \zeta)^2\bar{P}\psi' \rangle \\
& - \langle \bar{R}_{\bar{z}}(\xi_0 + \zeta)\bar{P}\psi, |\text{ad}_A(H(\xi_0))\eta\rangle\langle 2\zeta \cdot d\Gamma(k)\eta|\bar{R}_z(\xi_0 + \zeta)^2\bar{P}\psi' \rangle \\
& + \langle \bar{R}_{\bar{z}}(\xi_0 + \zeta)\bar{P}\psi, \zeta \cdot \text{ad}_{d\Gamma(k)}(\text{ad}_A(H(\xi_0)))\bar{R}_z(\xi_0 + \zeta)^2\bar{P}\psi' \rangle. \tag{6.49}
\end{aligned}$$

Since the right hand side of (6.49) clearly is a bounded expression, we have thus shown that $\bar{R}_{\bar{z}}(\xi_0 + \zeta) \in C^1(\bar{A})$.

To show that $\bar{R}_{\bar{z}}(\xi_0 + \zeta)$ is an element of $C^2(\bar{A})$ we have to show that every term in (6.49) is again given by an operator in $C^1(\bar{A})$. Note however that only the first, fifth and sixth term are clearly implemented by such operators. The remaining second, third, fourth and seventh term have to be taken care of separately. We start with the second one:

$$\begin{aligned}
& \langle \bar{P}\psi, [\bar{A}, \text{ad}_A(H(\xi_0))\bar{R}_z(\xi_0 + \zeta)^2]\bar{P}\psi' \rangle \\
= & \langle \bar{P}\psi, [A, \text{ad}_A(H(\xi_0))]\bar{R}_z(\xi_0 + \zeta)^2\bar{P}\psi' \rangle \\
& + \langle \bar{P}\psi, A[P, \text{ad}_A(H(\xi_0))]\bar{R}_z(\xi_0 + \zeta)^2\bar{P}\psi' \rangle \\
& + \langle \bar{P}\psi, [P, \text{ad}_A(H(\xi_0))]A\bar{R}_z(\xi_0 + \zeta)^2\bar{P}\psi' \rangle \\
& - \langle \bar{P}\psi, \text{ad}_A(H(\xi_0))\bar{R}_z(\xi_0 + \zeta)^2\bar{P}[A, (\bar{P}H(\xi_0 + \zeta)\bar{P} - z\bar{P})^2]\bar{P}\bar{R}_z(\xi_0 + \zeta)^2\bar{P}\psi' \rangle.
\end{aligned}$$

The second and third terms of the previous equation can be seen to be bounded expressions in a similar fashion as in the beginning of this section. To deal with the first term, recall that

$$\begin{aligned}
\text{ad}_A(H(\xi)) &= -i\Phi(iag) + d\Gamma(iv \cdot \nabla\omega) + 2\xi \cdot d\Gamma(v) + 2i d\Gamma(v) \cdot d\Gamma(k), \\
\text{ad}_A^2(H(\xi)) &= \Phi(a^2g) + d\Gamma((iv \cdot \nabla)^2\omega) + 2\xi \cdot d\Gamma((iv \cdot \nabla)v) - 2 d\Gamma(v) \cdot d\Gamma(v) \\
&\quad + 2 d\Gamma((iv \cdot \nabla)v) \cdot d\Gamma(k)
\end{aligned}$$

for any $\xi \in \mathbb{R}^\nu$. Thus, $[A, \text{ad}_A(H(\xi_0))]\bar{R}_z(\xi_0 + \zeta)^2$ extends to a bounded form on $\bar{\mathcal{P}}\mathcal{H}$. In order to prove that the second term in (6.49) is bounded, it thus suffices to show that the form

$$\bar{R}_{-c}(\xi_0 + \zeta)^{\frac{1}{2}}[A, (\bar{P}H(\xi_0 + \zeta)\bar{P} - z\bar{P})^2]\bar{R}_z(\xi_0 + \zeta)^2$$

is bounded. We compute

$$\begin{aligned} & (\overline{P}H(\xi_0 + \zeta)\overline{P} - z\overline{P})^2 \\ &= \overline{P}H(\xi_0 + \zeta)\overline{P}H(\xi_0 + \zeta) - \overline{P}H(\xi_0 + \zeta)\overline{P}|V_\zeta\eta\rangle\langle\eta| - z\overline{P}H(\xi_0 + \zeta) \\ & \quad + z\overline{P}|V_\zeta\eta\rangle\langle\eta| \end{aligned}$$

and can now clearly see that only the first and third term are truly problematic, because the other two are given by finite rank operators. If we expand them by writing $\overline{P} = 1 - P$, we can further note that the worst term is coming from the presence of $H(\xi_0 + \zeta)^2$. We thus have to control

$$\begin{aligned} & \langle \overline{P}\psi, \overline{R}_{-c}(\xi_0 + \zeta)^{\frac{1}{2}}[A, H(\xi_0 + \zeta)^2]\overline{R}_z(\xi_0 + \zeta)^2\psi' \rangle \\ &= \langle \overline{P}\psi, \text{ad}_A(H(\xi_0 + \zeta)^2)\overline{R}_{-c}(\xi_0 + \zeta)^{\frac{1}{2}}\overline{R}_z(\xi_0 + \zeta)^2\psi' \rangle \\ & \quad - \langle \overline{P}\psi, [\overline{R}_{-c}(\xi_0 + \zeta)^{\frac{1}{2}}, \text{ad}_A(H(\xi_0 + \zeta)^2)]\overline{R}_z(\xi_0 + \zeta)^2\psi' \rangle. \end{aligned}$$

Define the abbreviation $U := \text{ad}_{\overline{P}H(\xi_0 + \zeta)\overline{P}}(\text{ad}_A(H(\xi_0 + \zeta)^2))\overline{R}_{-c}(\xi_0 + \zeta)^3$ and compute

$$\begin{aligned} & |\langle \overline{P}\psi, [\overline{R}_{-c}(\xi_0 + \zeta)^{\frac{1}{2}}, (\text{ad}_A(H(\xi_0 + \zeta)^2))]\overline{R}_{-c}(\xi_0 + \zeta)^2\overline{P}\psi' \rangle| \\ & \leq C\|\overline{P}\psi\|\|U\| \int_0^\infty t^{-\frac{1}{2}}\|(\overline{P}H(\xi_0 + \zeta)\overline{P} + (c+t)\overline{P})^{-1}\overline{P}\psi'\|dt \\ & \leq \tilde{C}\|\psi\|\|\psi'\|. \end{aligned}$$

Hence, the second term in (6.49) is bounded. It thus remains to check the third, fourth and last term of (6.49). Since these can be dealt with by similar methods, we omit the calculations. \square

6.3 Lipschitz Continuity

Definition 6.8. For $N \in \mathbb{N}$ and a tuple $(\{T_{n,k}\}_{k=1}^{3N}, \{t_{k,n}\}_{k=1}^{3N})$, where $\{T_{n,k}\}_{k=1}^N$ is a family of $H(\xi_0)^n$ -bounded operators, where $T_{n,k} \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ for all k , $\{t_{k,n}\}_{k=1}^N$ is a family of functions on \mathbb{R}^ν Lipschitz continuous in a neighborhood of 0, we define

$$\mathfrak{B}_n(\zeta) := \sum_{k=1}^N t_{k,n}(\zeta)T_{k,n} \quad (6.50)$$

on $D((H(\xi_0) + c)^{n-1/2})$ and

$$\begin{aligned} \mathfrak{F}_n(\zeta) &:= \sum_{k=1}^N t_{k+N,n}(\zeta) |T_{k+N,n}\eta\rangle\langle\eta| \\ & \quad + \sum_{k=1}^N t_{k+2N,n}(\zeta) |\eta\rangle\langle(T_{k+2N,n})^*\eta| \end{aligned} \quad (6.51)$$

for every fiber $\zeta \in \mathbb{R}^\nu$. We call $\mathfrak{B}_n(\zeta)$ the $H(\xi_0)^n$ -bounded operator associated to the tuple $(\{T_{n,k}\}_{k=1}^{3N}, \{t_{k,n}\}_{k=1}^{3N})$ and $\mathfrak{F}_n(\zeta)$ the finite rank operator associated to it.

Proposition 6.9. *Let $z \in \mathbb{C}$. For all $n \in \mathbb{N}$ there exists an integer N_n and a triple $(\{T_{n,k}\}_{k=1}^{3N_n}, \{t_{k,n}\}_{k=1}^{3N_n})$ as in Definition 6.8 such that the $H(\xi_0)^n$ -bounded operator and the finite rank operator associated to it satisfy*

$$(\overline{P}H(\xi_0 + \zeta)\overline{P} - z\overline{P})^n = (H(\xi_0) - z)^n + \mathfrak{B}_n(\zeta) + \mathfrak{F}_n(\zeta). \quad (6.52)$$

Moreover, all $t_{k,n}$ are polynomials of order less than $2n$.

PROOF: The proof is carried out by induction. For $n = 1$ we compute

$$\overline{P}H(\xi_0 + \zeta)\overline{P} - z\overline{P} = H(\xi_0) - z + V_\zeta - (\lambda + c)P - 2\operatorname{Re}\langle V_\zeta \eta | \eta \rangle + \langle V_\zeta \eta, \eta \rangle P.$$

The last three terms are finite rank operators and V_ζ is a $(H(\xi_0) + c)^{1/2}$ -bounded operator. Furthermore, all functions of ζ appearing in this expression are either constants or polynomials of order less than 2. Hence the assertion is true for $n = 1$.

Let us assume that the statement has already been proven for all $m \leq n$. Choose $\psi \in \mathcal{C}_0^\infty$ and compute

$$\begin{aligned} & (\overline{P}H(\xi_0 + \zeta)\overline{P} - z\overline{P})^{n+1}\psi \\ &= ((H(\xi_0) - z)\overline{P} - \overline{P}V_\zeta\overline{P})[(H(\xi_0) - z)^n + \mathfrak{B}_n(\zeta) + \mathfrak{F}_n(\zeta)]\psi. \end{aligned}$$

We study this last equation term by term. Note that

$$(H(\xi_0) - z)\overline{P}(H(\xi_0) - z)^n\psi = (H(\xi_0) - z)^{n+1}\psi - (\lambda - z)^{n+1}P\psi.$$

Hence this contribution is already of the correct form. Using the induction hypothesis and (6.50) we calculate

$$\begin{aligned} & (H(\xi_0) - z)\overline{P}\mathfrak{B}_n(\zeta)\psi \\ &= \sum_{k=1}^N t_{k,n}(\xi)(H(\xi_0) - z)T_{k,n} - (\lambda - z) \sum_{k=1}^N t_{k,n}(\zeta) |\eta\rangle\langle (T_{k,n})^* \eta | \psi. \end{aligned}$$

Note that computations on \mathcal{C}_0^∞ show that $(H(\xi_0) - z)T_{k,n}$ extends to a $H(\xi_0)^{n+1}$ -bounded operator on $D(H(\xi_0)^{n+1})$. Thus, these contributions are also of the cor-

rect type. Similarly,

$$\begin{aligned}
(H(\xi_0) - z)\overline{P}\mathfrak{F}_n(\zeta)\psi &= \sum_{k=1}^{N_n} t_{k+N_n,n}(\zeta) |(H(\xi_0) - z)T_{k+N_n,n}\eta\rangle\langle\eta|\psi \\
&+ \sum_{k=1}^{N_n} t_{k+2N_n,n}(\zeta) (\lambda - z)|\eta\rangle\langle(T_{k+2N_n,n})^*\eta|\psi \\
&- (\lambda - z) \sum_{k=1}^{N_n} t_{k+N_n,n}(\zeta) \langle\eta, T_{k+N_n,n}\eta\rangle P\psi \\
&- (\lambda - z) \sum_{k=1}^{N_n} t_{k+2N_n,n}(\zeta) |\eta\rangle\langle(T_{k+2N_n,n})^*\eta|\psi
\end{aligned}$$

which has the correct form. As a next step we deal with the contributions coming from $\overline{P}V_\zeta\overline{P}$.

$$\begin{aligned}
&\overline{P}V_\zeta\overline{P}(H(\xi_0) - z)^n\psi \\
&= (V_\zeta - 2\text{Re}|V_\zeta\eta\rangle\langle\eta| + \langle V_\zeta\eta, \eta\rangle P)(H(\xi_0) - z)^n\psi \\
&= (H(\xi_0) - z)^nV_\zeta + \sum_{\sigma=1}^{\nu} 2\zeta_\sigma \text{ad}_{\text{d}\Gamma(k_\sigma)}((H(\xi_0) - z)^n)\psi \\
&\quad - (\lambda - z)^n(|V_\zeta\eta\rangle\langle\eta| + \langle V_\zeta\eta, \eta\rangle P) - |\eta\rangle\langle(H(\xi_0) - z)^nV_\zeta\eta|\psi
\end{aligned}$$

which is of the correct type again due to Proposition 3.5. (6.50) implies

$$\begin{aligned}
\overline{P}V_\zeta\overline{P}\mathfrak{B}_n(\zeta)\psi &= (V_\zeta - 2\text{Re}|V_\zeta\eta\rangle\langle\eta| + \langle V_\zeta\eta, \eta\rangle P)\mathfrak{B}_n(\zeta)\psi \\
&= \sum_{k=1}^{N_n} \left(t_{k,n}(\zeta)T_{k,n}V_\zeta\psi - 2 \sum_{\sigma=1}^{\nu} t_{k,n}(\zeta)\xi_\sigma \text{ad}_{\text{d}\Gamma(k_\sigma)}(T_{k,n})\psi \right) \\
&\quad - \sum_{k=1}^{N_n} (t_{k,n}(\zeta)|V_\zeta\eta\rangle\langle T_{k,n}\eta|\psi - t_{k,n}(\zeta)|\eta\rangle\langle(T_{k,n})^*V_\zeta\eta|\psi) \\
&\quad + \sum_{k=1}^{N_n} \langle V_\zeta\eta, \eta\rangle t_{k,n}(\zeta)|\eta\rangle\langle T_{k,n}\eta|\psi. \tag{6.53}
\end{aligned}$$

Note that $T_{k,n}$ being $H(\xi_0)^n$ -bounded implies that $(T_{k,n})^*V_\zeta$ is $H(\xi_0)^{n+1/2}$ -bounded. Thus, the contributions of $\overline{P}V_\zeta\overline{P}\mathfrak{B}_n(\zeta)$ are of the correct form. It remains to check

$\overline{P}V_\zeta\overline{P}\mathfrak{F}_n(\zeta)$. Similar to previous computations, we use (6.51) to calculate

$$\begin{aligned}
\overline{P}V_\zeta\overline{P}\mathfrak{F}_n(\zeta) &= \overline{P}V_\zeta\overline{P}\sum_{k=1}^{N_n} t_{k+N_n,n}(\zeta)|T_{k+N_n,n}\eta\rangle\langle\eta|\psi \\
&= \overline{P}V_\zeta\overline{P}\sum_{k=1}^{N_n} t_{k+N_n,n}(\zeta)|T_{k+N_n,n}\eta\rangle\langle\eta|\psi \\
&= \sum_{k=1}^{N_n} t_{k+N_n,n}(\zeta)|T_{k+N_n,n}V_\zeta\eta\rangle\langle\eta|\psi \\
&\quad - 2\sum_{k=1}^{N_n} t_{k+N_n,n}(\zeta)\sum_{\sigma=1}^{\nu}\zeta_\sigma|\text{ad}_{d\Gamma(k_\sigma)}(T_{k+N_n,n})\eta\rangle\langle\eta|\psi \\
&\quad - \sum_{k=1}^{N_n} t_{k+N_n,n}(\zeta)\langle\eta, T_{k+N_n,n}\eta\rangle|V_\zeta\eta\rangle\langle\eta|\psi \\
&\quad - \sum_{k=1}^{N_n} t_{k+N_n,n}(\zeta)\langle V_\zeta\eta, T_{k+N_n,n}\eta\rangle P\psi \\
&\quad + \sum_{k=1}^{N_n} t_{k+N_n,n}(\zeta)\langle V_\zeta\eta, \eta\rangle\langle\eta, T_{k+N_n,n}\eta\rangle P.
\end{aligned}$$

Clearly all contributions are of the correct type.

Therefore, we see that the induction hypothesis implies (6.52) restricted to \mathcal{C}_0^∞ in the case $n+1$ after relabeling all operators, functions and constants and choosing N_{n+1} sufficiently large. Indeed, we have argued that all contributions are of the correct form and the only problem that we might encounter is that the number of terms is not correct. To this end note that we have the freedom to choose several the operators and functions equal to 0 and N_{n+1} large enough to obtain the correct number of terms. Since all operators are $H(\zeta)^{n+1}$ -bounded the results extend to $D(H(\xi)^{n+1})$ by density of \mathcal{C}_0^∞ w.r.t. respective graph norms.

Finally, it remains to show that the new functions $t_{k,n+1}$ are in fact polynomials of degree less than $2n+2$. This however is true, since all of them are either constant or of the type $p \cdot t_{k,n}$, where p is a polynomial of order less than 2 which implies that all $t_{k,n+1}$ are polynomials of order less than $2n+2$. We illustrate this by an example. For the first term in (6.53) we compute more explicitly:

$$\begin{aligned}
\sum_{k=1}^{N_n} t_{k,n}(\zeta)T_{k,n}V_\zeta &= \sum_{k=1}^{N_n} \xi^2 t_{k,n}(\zeta)T_{k,n} + \sum_{k=1}^{N_n} 2\xi_0 \cdot \zeta t_{k,n}(\zeta)T_{k,n} \\
&\quad - \sum_{k=1}^{N_n} 2\sum_{\sigma=1}^{\nu} \zeta_\sigma t_{k,n}(\zeta)T_{k,n} d\Gamma(k_\sigma).
\end{aligned}$$

A quick look at all calculations carried out in this proof shows that this is the only way the functions $t_{k,n}$ are modified and hence the assertion is correct. \square

The proofs of the next two lemmas are trivial and thus omitted.

Lemma 6.10. *Let $T, T' \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}^{\text{uv}, 0}$ be $(H(\xi_0) + c)^n$ -bounded operators. The following equation extends from a form identity on \mathcal{C}_0^∞ to an operator identity on $D((H(\xi_0) + c)^{n+\frac{1}{2}})$.*

$$\begin{aligned}
& \text{ad}_{\overline{P}H(\xi_0+\zeta)\overline{P}}(T) \\
&= \overline{P} \text{ad}_{H(\xi_0)}(T)\overline{P} - 2 \sum_{\sigma=1}^{\nu} \zeta_\sigma \overline{P} \text{ad}_{\text{d}\Gamma(k_\sigma)}(T)\overline{P} \\
&\quad - \overline{P} \lambda_{\xi_0} |T\eta\rangle\langle\eta| - \overline{P} |\text{ad}_{H(\xi_0)}(T)\eta\rangle\langle\eta| - (\zeta^2 - 2\zeta \cdot \xi_0) \overline{P} |T\eta\rangle\langle\eta| \\
&\quad + \overline{P} \sum_{\sigma=1}^{\nu} \zeta_\sigma (|\text{d}\Gamma(k_\sigma)\eta\rangle\langle T^*\eta| - |T \text{d}\Gamma(k_\sigma)\eta\rangle\langle\eta| - |\text{ad}_{\text{d}\Gamma(k_\sigma)}(T)\eta\rangle\langle\eta|) \\
&\quad - \lambda_{\xi_0} |\eta\rangle\langle T^*\eta| \overline{P} - |\eta\rangle\langle \text{ad}_{H(\xi_0)}(T^*)\eta| \overline{P} - (\zeta^2 - 2\zeta \cdot \xi_0) |\eta\rangle\langle T^*\eta| \overline{P} \\
&\quad + \sum_{\sigma=1}^{\nu} \zeta_\sigma (|T\eta\rangle\langle \text{d}\Gamma(k_\sigma)\eta| - |\eta\rangle\langle T^* \text{d}\Gamma(k_\sigma)\eta| - |\eta\rangle\langle \text{ad}_{\text{d}\Gamma(k_\sigma)}(T^*)\eta|) \overline{P} \quad (6.54)
\end{aligned}$$

Moreover, the following equation extends from a form identity on $D(A)$ to an

operator identity on \mathcal{H} .

$$\begin{aligned}
& \text{ad}_{\overline{P}H(\xi_0+\zeta)\overline{P}}(|T\eta\rangle\langle T'\eta|) \\
&= \overline{P} \left(|\text{ad}_{H(\xi_0)}\eta\rangle\langle T'\eta| - |T\eta\rangle\langle \text{ad}_{H(\xi_0)}(T')\eta| - 2 \sum_{\sigma=1}^{\nu} \zeta_{\sigma} (|T \text{d}\Gamma(k_{\sigma})\eta\rangle\langle T'\eta|) \right) \overline{P} \\
&\quad - 2\overline{P} \sum_{\sigma=1}^{\nu} \zeta_{\sigma} (|\text{ad}_{\text{d}\Gamma(k_{\sigma})}(T)\eta\rangle\langle T'\eta| - |T\eta\rangle\langle T' \text{d}\Gamma(k_{\sigma})\eta| - |T\eta\rangle\langle \text{ad}_{\text{d}\Gamma(k_{\sigma})}(T')\eta|) \overline{P} \\
&\quad + \langle \eta, T\eta \rangle \overline{P} \sum_{\sigma=1}^{\nu} \zeta_{\sigma} |\text{d}\Gamma(k_{\sigma})\eta\rangle\langle T'\eta| - \langle T'\eta, \eta \rangle (\zeta^2 - 2\zeta \cdot \xi_0 + \lambda_{\xi_0}) \overline{P} |T\eta\rangle\langle \eta| \\
&\quad - \langle T'\eta, \eta \rangle \overline{P} |\text{ad}_{H(\xi_0)}(T)\eta\rangle\langle \eta| - 2\langle T'\eta, \eta \rangle \overline{P} \sum_{\sigma=1}^{\nu} \zeta_{\sigma} |T \text{d}\Gamma(k_{\sigma})\eta\rangle\langle \eta| \\
&\quad - \langle T'\eta, \eta \rangle 2\overline{P} \sum_{\sigma=1}^{\nu} \zeta_{\sigma} |\text{ad}_{\text{d}\Gamma(k_{\sigma})}(T)\eta\rangle\langle \eta| - 2\langle T'\eta, \eta \rangle \sum_{\sigma=1}^{\nu} \zeta_{\sigma} |T\eta\rangle\langle \text{d}\Gamma(k_{\sigma})\eta| \overline{P} \\
&\quad + \langle \eta, T\eta \rangle (\zeta^2 - 2\zeta \cdot \xi_0 + \lambda_{\xi_0}) |\eta\rangle\langle T'\eta| \overline{P} + \langle \eta, T\eta \rangle \sum_{\sigma=1}^{\nu} \zeta_{\sigma} |\eta\rangle\langle T' \text{d}\Gamma(k_{\sigma})\eta| \\
&\quad + \langle \eta, T\eta \rangle \sum_{\sigma=1}^{\nu} \zeta_{\sigma} |\eta\rangle\langle \text{ad}_{\text{d}\Gamma(k_{\sigma})}(T')\eta| \tag{6.55}
\end{aligned}$$

Remark 6.11. Let $h \in \mathbb{H}_{\text{uv}}^2(\mathbb{R}^{\nu})$ so that $\eta \in D(A) \cap D(H(\xi_0)^n)$ for all $n \in \mathbb{N}$. Since $H(\xi_0)\eta = \lambda_{\xi_0}\eta$,

$$H(\xi_0)^n A\eta = \lambda_{\xi_0}^n A\eta - \text{ad}_A(H(\xi_0)^n)\eta,$$

where we have used that $\text{ad}_A(H(\xi_0)^m)$ extends to $D(H(\xi_0)^{m+1})$. Hence $A\eta \in D(H(\xi_0)^n)$ for all $n \in \mathbb{N}$ as well. This is used to define $T^*A\eta$, $TA\eta$ etc. in the next lemma.

Lemma 6.12. Let $h \in \mathbb{H}_{\text{uv}}^2(\mathbb{R}^{\nu})$ and $T, T' \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}^{\text{uv}, 1}$ be $(H(\xi_0) + c)^n$ -bounded operators. The following equation extends from a form identity on \mathcal{C}_0^{∞} to an operator identity on $D((H(\xi_0) + c)^{n+\frac{1}{2}})$.

$$\begin{aligned}
& \text{ad}_{\overline{A}}(T) \\
&= \overline{P} \text{ad}_A(T) \overline{P} + \overline{P} (|A\eta\rangle\langle T^*\eta| - |TA\eta\rangle\langle \eta| - |\text{ad}_A(T)\eta\rangle\langle \eta|) \\
&\quad + (|\eta\rangle\langle T^*A\eta| + |\eta\rangle\langle \text{ad}_A(T^*)\eta| - |T\eta\rangle\langle A\eta|) \overline{P}. \tag{6.56}
\end{aligned}$$

Moreover, the following equation extends from a form identity on $D(A)$ to an

operator identity on \mathcal{H} :

$$\begin{aligned}
& \text{ad}_{\bar{A}}(|T\eta\rangle\langle T'\eta|) \\
&= (\langle\eta, T\eta\rangle\{|\eta\rangle\langle T'A\eta| + |\eta\rangle\langle \text{ad}_A(T')\eta|\} - \langle T'\eta, \eta\rangle|T\eta\rangle\langle A\eta|)\bar{P} \\
&+ \bar{P}(\langle\eta, T\eta\rangle|A\eta\rangle\langle T'\eta| - \langle T'\eta, \eta\rangle\{|T'A\eta\rangle\langle\eta| + |\text{ad}_A(T)\eta\rangle\langle\eta|\}) \\
&+ \bar{P}(|T'A\eta\rangle\langle T'\eta| + |\text{ad}_A(T)\eta\rangle\langle T'\eta| - |T\eta\rangle\langle T'A\eta| - |T\eta\rangle\langle \text{ad}_A(T')\eta|)\bar{P}. \quad (6.57)
\end{aligned}$$

In order to prove the Lipschitz property for some operator valued maps later in this section the next result will be of great use.

Lemma 6.13. *Let $\text{Im}(z) \neq 0$. Choose $R > 0$ such that $B_R(\xi_0) \subset \mathcal{O}_0$, let $\zeta \in \mathbb{R}^\nu$ and $\xi \in B_R(\xi_0)$. Then*

$$\forall m \in \mathbb{N} : D_m(\zeta, z) := (H(\xi) + c)^m \bar{R}_z(\xi + \zeta)^m \bar{P} \in \mathcal{B}(\mathcal{H}).$$

Moreover, there exists $r > 0$ such that for all $m \in \mathbb{N}$

$$\forall |\zeta| < r : \|D_m(\zeta, z)\| \leq C_m \frac{p_m(|\zeta|, |z|)}{|\text{Im}(z)|^m}, \quad (6.58)$$

where $p_m : \mathbb{R}^2 \rightarrow \mathbb{R}$, $q_m : \mathbb{R} \rightarrow \mathbb{R}$ is a polynomial of degree less than $2m$ with positive coefficients and $C_m > 0$ is a constant independent of ζ, z and ξ .

PROOF: We prove the statement by induction in m . Clearly, on $\bar{P}\mathcal{H}$ we may compute

$$\begin{aligned}
& (H(\xi) + c)\bar{R}_z(\xi + \zeta)\bar{P} = (H(\xi) + c)\bar{R}_z(\xi + \zeta)\bar{P} \\
&= (H(\xi) + c)\bar{P}\bar{R}_z(\xi + \zeta)\bar{P} = \bar{P}(H(\xi) + c)\bar{P}\bar{R}_z(\xi + \zeta)\bar{P} \\
&= \bar{P}(H(\xi + \zeta) - z + \bar{P}(c + z + V_\zeta)\bar{P})\bar{P}\bar{R}_z(\xi + \zeta)\bar{P} \\
&= \bar{P} + \bar{P}V_\zeta\bar{P}\bar{R}_z(\xi + \zeta)\bar{P} + (c + z)\bar{R}_z(\xi + \zeta)\bar{P}. \quad (6.59)
\end{aligned}$$

All terms except for the second one are easily seen to be bounded operators which depend Lipschitz continuously on the fiber parameter ζ in operator norm. It thus suffices to examine the second term to prove the result in the special case $m = 1$. We calculate

$$\begin{aligned}
\bar{P}V_\zeta\bar{P} &= V_\zeta - 2\text{Re}(|V_\zeta\eta\rangle\langle\eta|) + \langle V_\zeta\eta, \eta\rangle P \\
&= -2\zeta \cdot d\Gamma(k) + \zeta^2 - 2\zeta \cdot \xi - 2\text{Re}(|V_\zeta\eta\rangle\langle\eta|) + \langle V_\zeta\eta, \eta\rangle P
\end{aligned}$$

Since $d\Gamma(k_\sigma)$ is $(H(\xi + \zeta) + c)^{1/2}$ -bounded for all coordinates σ , we may conclude that $\bar{P}V_\zeta\bar{P}\bar{R}_z(\xi + \zeta)\bar{P} \in \mathcal{B}(\mathcal{H})$. By elementary calculations

$$\begin{aligned}
& (1 + 2\zeta \cdot d\Gamma(k)(H(\xi) + c)^{-1})(H(\xi) + c)\bar{R}_z(\xi + \zeta)\bar{P} \\
&= \bar{P} + c\bar{P}\bar{R}_z(\xi + \zeta)\bar{P} - (\zeta^2 - 2\zeta \cdot \xi)\bar{R}_z(\xi + \zeta)\bar{P} - 2\text{Re}(|V_\zeta\eta\rangle\langle\eta|)\bar{R}_z(\xi + \zeta)\bar{P}.
\end{aligned}$$

There exists $R > 0$ such that the operator $1 + 2\zeta \cdot d\Gamma(k)(H(\xi) + c)^{-1}$ becomes invertible for $|\zeta| < R$ and the norm of its inverse can be estimated independently of ζ . The previous equation then implies (6.58). This establishes the validity of the assertion for the case $m = 1$.

Let us assume that the statement has been proven for all $n \leq m - 1$. Then, by (6.59),

$$\begin{aligned}
& (H(\xi) + c)^m \bar{R}_z(\xi + \zeta)^m \bar{P} \\
&= (H(\xi) + c)^{m-1} \bar{R}_z(\xi + \zeta)^{m-1} \bar{P} - (c - z)(H(\xi) + c)^{m-1} \bar{R}_z(\xi + \zeta)^m \bar{P} \\
&\quad - (H(\xi) + c)^{m-1} \bar{P} V_\zeta \bar{P} \bar{R}_z(\xi + \zeta)^m \bar{P} \\
&= D_{m-1}(\zeta) \bar{P} - (c - z + \zeta^2 + 2\xi \cdot \zeta) D_{m-1}(\xi) \bar{R}_z(\xi + \zeta) \bar{P} \\
&\quad + 2 \sum_{\sigma=1}^{\nu} \zeta_\sigma (H(\xi) + c)^{m-1} \bar{P} d\Gamma(k_\sigma) \bar{P} \bar{R}_z(\xi + \zeta)^m \bar{P}
\end{aligned} \tag{6.60}$$

All terms except for the last sum are clearly bounded operators and Lipschitz continuous for $|\zeta| < R$ and can be bounded as in (6.59) in the $m + 1$ case by the induction hypothesis and the usual bounds on the resolvent. It thus suffices to study this remaining term. We recall the definition $B_m^h = (H_0(\xi) + 1)^{m/2} (H(\xi) + c)^{-m/2}$, where h is the coupling function, and compute

$$\begin{aligned}
& (H(\xi) + c)^{m-1} \bar{P} d\Gamma(k_\sigma) \bar{R}_z(\xi + \zeta)^m \bar{P} \\
&= \bar{P} (H(\xi) + c)^{m-1} (H_0(\xi) + 1)^{-m+1} d\Gamma(k_\sigma) (H_0(\xi) + 1)^{-1} B_{2m}^h D_m(\zeta, z),
\end{aligned}$$

where we have used that $H(\xi)$ commutes with \bar{P} and that $H_0(\xi)$ commutes with $d\Gamma(k_\sigma)$. Hence all missing terms are of the correct form. \square

Lemma 6.14. *Let $\text{Im}(z) \neq 0$. Adopt the notation of Lemma 6.13. The map $\zeta \mapsto D_m(\zeta, z)$ is differentiable on $B_r(0)$ for all $m \in \mathbb{N}$. Moreover, there exist a constant $C_m > 0$ independent of ζ, z, ξ and polynomials $p_m : \mathbb{R}^\nu \times \mathbb{R} \rightarrow \mathbb{C}$, $q_m : \mathbb{R} \rightarrow \mathbb{C}$ such that*

$$\|\partial_{\zeta_\sigma} D_m(\zeta, z)\| \leq C_m \frac{|p_m(\zeta, z)|}{|q_m(|\text{Im}(z)|)|}$$

PROOF: Let $m \in \mathbb{N}$ and compute

$$\begin{aligned}
& \partial_{\zeta_\sigma} \overline{R}_z(\xi + \zeta)^m \overline{P} \\
&= \sum_{\ell=0}^{m-1} \overline{R}_z(\xi + \zeta)^\ell [\partial_{\zeta_\sigma} \overline{R}_z(\xi + \zeta)] \overline{R}_z(\xi + \zeta)^{m-1-\ell} \overline{P} \\
&= -2m(\zeta_\sigma + \xi_\sigma) \overline{R}_z(\xi + \zeta)^{m+1} \overline{P} \\
&\quad + 2 \sum_{\ell=0}^{m-1} \overline{R}_z(\xi + \zeta)^{\ell+1} \overline{P} \, d\Gamma(k_\sigma) \overline{P} \overline{R}_z(\xi + \zeta)^{m-\ell} \overline{P} \\
&= -2m(\zeta_\sigma + \xi_\sigma) \overline{R}_z(\xi + \zeta)^{m+1} \overline{P} - 2m \overline{R}_z(\xi + \zeta)^m \overline{P} \, d\Gamma(k_\sigma) \overline{P} \overline{R}_z(\xi + \zeta) \overline{P} \\
&\quad - 2 \sum_{\ell=0}^{m-1} \overline{R}_z(\xi + \zeta)^m \overline{P} L_\sigma(\zeta, \ell, z) D_{m-\ell-1}(\zeta, z) \overline{R}_z(\xi + \zeta) \overline{P},
\end{aligned}$$

where

$$L_\sigma(\zeta, \ell, z) = \text{ad}_{d\Gamma(k_\sigma)}((\overline{P}H(\xi + \zeta)\overline{P} - z\overline{P})^{m-1-\ell})(H(\xi) + c)^{-m+1+\ell}.$$

With the help of Proposition 6.9 we can further calculate

$$\begin{aligned}
& \partial_{\zeta_\sigma} D_m(\zeta, z) \\
&= 2m(\zeta_\sigma + \xi_\sigma) D_m(\zeta, z) \overline{R}_z(\xi + \zeta) \\
&\quad - 2m D_m(\zeta, z) \overline{R}_z(\xi + \zeta) \overline{P} \, d\Gamma(k_\sigma) (H(\xi) + c)^{-1} D_1(\zeta, z) \\
&\quad + 2 \sum_{\ell=0}^{m-1} D_m(\zeta, z) \overline{P} S_{\sigma, m-\ell-1} D_{m-\ell-1}(\zeta, z) \overline{R}_z(\xi + \zeta) \overline{P} \\
&\quad + 2 \sum_{\ell=0}^{m-1} D_m(\zeta, z) \overline{P} S'_{\sigma, m-\ell-1}(\zeta) D_{m-\ell-1}(\zeta, z) \overline{R}_z(\xi + \zeta) \overline{P} \\
&\quad + 2 \sum_{\ell=0}^{m-1} D_m(\zeta, z) \overline{P} \mathfrak{F}(\zeta) \overline{R}_z(\xi + \zeta)^{m-\ell} \overline{P},
\end{aligned}$$

where

$$S_{\sigma, m-1}(z) = \text{ad}_{d\Gamma(k_\sigma)}((H(\xi) - z)^{m-1}) \overline{R}_{-c}(\xi)^{m-1}, \quad (6.61)$$

$$S'_{\sigma, m-1}(\zeta) = \text{ad}_{d\Gamma(k_\sigma)}(\mathfrak{B}_{m-1}(\zeta)) \overline{R}_{-c}(\xi)^{m-1}. \quad (6.62)$$

The statement now follows from Lemma 6.13. \square

Corollary 6.15. *Let $\text{Im}(z) \neq 0$. The map $\zeta \mapsto \overline{R}_z(\xi + \zeta) \overline{P}$ is differentiable on $B_R(0)$, where $R > 0$ is as in Lemma 6.13, and there exist a constant C independent of z, ζ and polynomials $p : \mathbb{R}^\nu \times \mathbb{R} \rightarrow \mathbb{C}$, $q : \mathbb{R} \rightarrow \mathbb{C}$ such that*

$$\|\partial_{\zeta_\sigma} \overline{R}_z(\xi + \zeta) \overline{P}\| \leq C \frac{|p(\zeta, z)|}{|q(\text{Im}(z))|}$$

In particular, $\zeta \mapsto \overline{R}_z(\xi + \zeta)\overline{P}$ is Lipschitz continuous in a neighborhood of 0.

PROOF: We simply note

$$\overline{R}_z(\xi + \zeta)\overline{P} = (H(\xi) + c)^{-1}D_1(\zeta, -c)$$

and apply Lemma 6.14. \square

Lemma 6.16. *Let $k \in C_0^\infty(\mathbb{R})$ and define the operator*

$$k_H(\zeta) := k(\overline{P}H(\xi + \zeta)\overline{P})$$

on $\overline{P}\mathcal{H}$. The map $K : \mathcal{B}(\overline{P}\mathcal{H}) \rightarrow \mathcal{B}(\overline{P}\mathcal{H})$, $\zeta \mapsto k_H(\zeta)$ is differentiable on $B_R(0)$, where $R > 0$ is as in Lemma 6.13. In particular, K is Lipschitz continuous in a neighborhood of 0.

PROOF: By Corollary 6.15 there exist a constant $C > 0$ independent of z, ξ and polynomials $p : \mathbb{R}^\nu \times \mathbb{R} \rightarrow \mathbb{C}$, $q : \mathbb{R} \rightarrow \mathbb{C}$ such that

$$\|\partial_{\zeta_\sigma}\overline{R}_z(\xi + \zeta)\overline{P}\| \leq C \frac{|p(\zeta, z)|}{|q(\text{Im}(z))|}.$$

By choosing an appropriate almost analytic extension \tilde{k} of the function $k \in C_0^\infty(\mathbb{R})$ we can achieve that $\partial_z \tilde{k}(z)p(\zeta, z)q(\text{Im}(z))^{-1}$ is integrable over the support of $\partial_z \tilde{k}$. Hence

$$\partial_{\zeta_\sigma} k_H(\zeta)\overline{P} = -\frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial} \tilde{k}(z) \partial_{\zeta_\sigma} \overline{R}_z(\xi + \zeta)\overline{P} dz$$

which gives rise to a bounded operator by the choice of \tilde{k} . \square

6.4 Hölder Continuity of the Boundary Values of the Resolvent

In Section 6.2 we have seen that $\overline{H}(\xi + \zeta)$ is of class $C^2(\overline{A})$ for sufficiently small $|\zeta|$, or in slightly different words that $\overline{P}H(\xi)\overline{P}$ is of class $C^2(\overline{A})$ for $\xi \in \mathcal{V}'$, where \mathcal{V}' is a neighborhood of ξ_0 . This property can be shown to imply a limiting absorption principle for $\overline{R}_{x+iy}(\xi)$, $\xi \in \mathcal{V}'$ and Hölder continuity w.r.t. x for the respective boundary values, see e.g. [19, 22, 47]. In principle these statements only hold for every fixed value of $\xi \in \mathcal{V}'$ so that all constants appearing in the respective theorems depend on ξ .

It should be noted that $f(H(\xi_0 + \zeta)) \in C^k(A)$ for sufficiently regular coupling functions implies the corresponding statement for the projected Hamiltonian $\overline{H}(\xi + \zeta)$, see Appendix E. Thus limiting absorption principle and Hölder continuity would also follow directly from Sahbani's paper on local regularity, [47]. Note that this result is also used in [19]. However, Sahbani's proof does not cover parameter dependence either and is less explicit when it comes to error estimates than Gérard's proof in [22] which is why we focus on the latter.

More precisely, an inspection of this proof shows that the parameter dependence enters either through the Mourre constants or the iterated commutators $\text{ad}_{\overline{P}A\overline{P}}^k(f(\overline{H}(\xi)))$, where $k = 1, 2$ and $f \in C_0^\infty(\mathbb{R})$. The Mourre constants can be chosen uniformly in ξ by possibly decreasing \mathcal{V}' and all commutators depend Lipschitz continuously on $\xi \in \mathcal{V}'$ by Theorem E.10. Hence we obtain the limiting absorption principle as well as Hölder continuity of the boundary values uniformly for ξ in some neighborhood of \mathcal{V}' of ξ_0 . Without loss of generality we can assume that $\mathcal{V}' \subset \mathcal{O}(\xi_0)$, where the open set $\mathcal{O}(\xi_0)$ denotes those ξ for which A_{ξ_0} is also a conjugate operator of $H(\xi)$. It is defined in Notation 2.6. For the sake of completeness we have included a proof of the uniform limiting absorption principle in Appendix D and a proof of the uniform Hölder continuity w.r.t. the spectral parameter z in Appendix C.

Now note that since \mathcal{V}' is open there exists $r_0 > 0$ such that $\mathcal{V} := B_{r_0}^{\mathbb{R}^{\nu}}(\xi_0) \subset \mathcal{V}'$. By possibly decreasing the size of r_0 we can argue that there exists $r > 0$ such that $\xi + \zeta \in \mathcal{V}'$ for all $\xi \in \mathcal{V}$ and all $|\zeta| < r$. This explains the formulation of the limiting absorption principle given below. The set \mathcal{V} should be thought of as the set from which we draw the base points ξ around which we want to do perturbation theory w.r.t. ζ . The uniformity in these base points is needed to obtain error estimates in the expansion in Theorem 2.12.

Notation 6.17. Throughout this section we denote by I_{ξ_0} the interval on which the Mourre estimate (6.39) holds, that is

$$\overline{P}\theta(\overline{H}(\xi + \zeta))[\overline{A}, \overline{H}(\xi + \zeta)]\theta(\overline{H}(\xi + \zeta))\overline{P} \geq e\overline{P}\theta(\overline{H}(\xi + \zeta))^2\overline{P}.$$

for $\theta \in C_0^\infty(\mathbb{R})$ with $\text{supp}(\theta) \subset I_{\xi_0}$.

Theorem 6.18 (Limiting Absorption Principle). *Let $s \in (1/2, 1)$ and $x + i\epsilon \in \mathbb{C}$, where $\epsilon > 0$ and $x \in I_{\xi_0}$. There exists a neighborhood \mathcal{V} of $\xi_0 \in U$ and $r > 0$ such that the limit*

$$\langle \overline{A} \rangle^{-s} \overline{R}_{x+i0}(\xi + \zeta) \langle \overline{A} \rangle^{-s} := \lim_{\epsilon \rightarrow 0^+} \langle \overline{A} \rangle^{-s} \overline{R}_{x+i\epsilon}(\xi + \zeta) \langle \overline{A} \rangle^{-s} \quad (6.63)$$

exists as a bounded operator for all $\xi \in \mathcal{V}$ and all $|\zeta| < r$. Furthermore, there exists a constant $C > 0$ independent of $x \in I_{\xi_0}$ such that

$$\forall \xi \in \mathcal{V} \forall |\zeta| < r : \|\langle \overline{A} \rangle^{-s} \overline{R}_{x+i0}(\xi + \zeta) \langle \overline{A} \rangle^{-s}\| \leq C.$$

Unlike the previous theorem the next result on Hölder continuity needs a proof. We should note however that Hölder continuity w.r.t. $x \in I_{\xi_0}$ follows immediately from the abstract theory, since $\overline{H}(\xi + \zeta)$ is of class $C^2(\overline{A})$ for $\xi \in \mathcal{V}$ and $|\zeta| < r$. For the sake of completeness a proof is given in Appendix C.

Theorem 6.19 (Hölder Continuity). *Let $s \in (1/2, 1)$. There exists an open subinterval $J \subset I_{\xi_0}$, $\alpha \in (1/2, 1)$ a neighborhood \mathcal{V} of $\xi_0 \in U$ and $C, r > 0$ such that for all $x, x' \in J$ and all $|\zeta|, |\zeta'| < r$:*

$$\begin{aligned} & \| \langle \bar{A} \rangle^{-s} \bar{R}_{x+i0}(\xi + \zeta) \langle \bar{A} \rangle^{-s} - \langle \bar{A} \rangle^{-s} \bar{R}_{x'+i0}(\xi + \zeta') \langle \bar{A} \rangle^{-s} \| \\ & \leq C (|\zeta - \zeta'|^\alpha + |x - x'|^\alpha). \end{aligned} \quad (6.64)$$

In order to prove Hölder continuity w.r.t. the fiber parameter, we need two technical Lemmas. The proof of the first follows along the lines of Mourre's paper and has therefore been moved to the appendix, see Section B.

Notation 6.20. For $\xi \in \mathcal{V}$ and $|\zeta| < r$ we define

$$\theta_H(\zeta) := \theta(\bar{H}(\xi + \zeta)).$$

Since it will be clear from the context which ξ is used in the definition of $\theta_H(\zeta)$, no confusion can arise.

Lemma 6.21 (Mourre's Quadratic Estimate). *Let $\theta \in C_0^\infty(\mathbb{R})$ with $\text{supp}(\theta) \subset J$, where $J \subset I_{\xi_0}$ is chosen such that the strict Mourre estimate (6.39) in Proposition 6.6 holds. Define*

$$\begin{aligned} \bar{B}(\zeta) & := \bar{P} \theta(\bar{H}(\xi + \zeta)) i \text{ad}_{\bar{A}}(\bar{H}(\xi_0 + \zeta)) \theta(\bar{H}(\xi + \zeta)) \bar{P}, \\ \theta^\perp(\bar{H}(\xi + \zeta)) & := 1 - \theta(\bar{H}(\xi + \zeta)) \end{aligned}$$

and

$$\bar{H}(\delta, \zeta) := \bar{H}(\xi + \zeta) + i\delta \bar{B}(\zeta).$$

There exists $\delta_0 > 0$ such that for all $z \in \mathbb{C}$ with $\text{Re}(z) \in J$ and all $|\delta| < \delta_0$ with $\delta \cdot \text{Im}(z) > 0$ the operator $\bar{H}(\delta, \zeta) - z\bar{P}$ is invertible on $\bar{P}\mathcal{H}$ with bounded inverse $G_z(\delta, \zeta)$. Moreover, there exists $C_0 > 0$, a neighborhood \mathcal{V} of ξ_0 and $r > 0$ such that for all $\xi \in \mathcal{V}$ and all $|\zeta| < r$ the following estimates hold:

$$\begin{aligned} \|G_z(\delta, \zeta)\| + \|\bar{H}(\xi + \zeta)G_z(\delta, \zeta)\| & \leq \frac{C_0}{|\delta|}, \\ \|G_z(\delta, \zeta)\langle \bar{A} \rangle^{-s}\| + \|\bar{H}(\xi + \zeta)G_z(\delta, \zeta)\langle \bar{A} \rangle^{-s}\| & \leq \frac{C_0}{|\delta|^{\frac{1}{2}}}, \\ \|\theta_H^\perp(\zeta)G_z(\delta, \zeta)\| + \|\bar{H}(\xi + \zeta)\theta_H^\perp(\zeta)G_z(\delta, \zeta)\| & \leq C_0. \end{aligned}$$

Lemma 6.22. *Adopt the notation and definitions of Lemma 6.21 and choose an arbitrary $\xi \in \mathcal{V}$. There exists $C > 0$ such that*

$$\|B(\zeta) - B(\zeta')\| \leq C|\zeta - \zeta'|$$

for ζ, ζ' in a neighborhood of 0.

PROOF: We calculate $\overline{H}(\xi + \zeta)$

$$\begin{aligned} & \theta_H(\zeta) \text{i ad}_{\overline{A}}(\overline{H}(\xi_0 + \zeta)) \theta_H(\zeta) - \theta_H(\zeta') \text{i ad}_{\overline{A}}(\overline{H}(\xi_0 + \zeta')) \theta_H(\zeta') \\ = & (\theta_H(\zeta) - \theta_H(\zeta')) \text{i ad}_{\overline{A}}(\overline{H}(\xi_0 + \zeta)) \theta_H(\zeta) \\ & + \theta_H(\zeta') (\text{i ad}_{\overline{A}}(\overline{H}(\xi_0 + \zeta)) - \text{i ad}_{\overline{A}}(\overline{H}(\xi_0 + \zeta'))) \theta_H(\zeta) \\ & + \theta_H(\zeta') \text{i ad}_{\overline{A}}(\overline{H}(\xi_0 + \zeta')) (\theta_H(\zeta) - \theta_H(\zeta')) \end{aligned}$$

By Lemma 6.16 the map $\zeta \mapsto \theta_H(\zeta)$ is differentiable and thus locally Lipschitz. Hence there exists $C' > 0$ such that $\|\theta_H(\zeta) - \theta_H(\zeta')\| \leq C'|\zeta - \zeta'|$ for $|\zeta|, |\zeta'| < r$. Thus,

$$\begin{aligned} & \|(\theta_H(\zeta) - \theta_H(\zeta')) \text{i ad}_{\overline{A}}(\overline{H}(\xi_0 + \zeta)) \theta_H(\zeta)\| \\ & \leq C'|\zeta - \zeta'| \|\text{i ad}_{\overline{A}}(\overline{H}(\xi_0 + \zeta)) (\overline{H}(\xi_0 + \zeta) + c\overline{P})^{-2}\| \|\theta_H^{(2)}(\zeta)\|, \end{aligned}$$

where $\theta_H^{(\ell)}(\zeta) = (\overline{H}(\xi_0 + \zeta) + c\overline{P})^\ell \theta_H(\zeta)$ for $\ell \in \mathbb{N}$. Since $(\overline{H}(\xi_0 + \zeta) + c\overline{P})^{-2} \text{i ad}_{\overline{A}}(\overline{H}(\xi_0 + \zeta))$ extends to a bounded operator by the closed graph theorem, we can compute

$$\begin{aligned} & \|\theta_H(\zeta') \text{i ad}_{\overline{A}}(\overline{H}(\xi_0 + \zeta')) (\theta_H(\zeta) - \theta_H(\zeta'))\| \\ & \leq C'|\zeta - \zeta'| \|\overline{H}(\xi_0 + \zeta) + c\overline{P}\|^{-2} \|\text{i ad}_{\overline{A}}(\overline{H}(\xi_0 + \zeta))\| \|\theta_H^{(2)}(\zeta)\|. \end{aligned}$$

Note that $(\overline{H}(\xi_0 + \zeta) + c\overline{P})^{-1} \text{ad}_{\overline{A}}(d\Gamma(k_\sigma)) (\overline{H}(\xi_0 + \zeta) + c\overline{P})^{-1}$ extends to a bounded operator as well which is denoted by $W_\sigma(\zeta)$. Hence,

$$(\text{i ad}_{\overline{A}}(\overline{H}(\xi_0 + \zeta)) - \text{i ad}_{\overline{A}}(\overline{H}(\xi_0 + \zeta'))) = 2 \sum_{\sigma=1}^{\nu} (\zeta_\sigma - \zeta'_\sigma) \text{ad}_{\overline{A}}(d\Gamma(k_\sigma))$$

implies

$$\begin{aligned} & \|\theta_H(\zeta') (\text{i ad}_{\overline{A}}(\overline{H}(\xi_0 + \zeta)) - \text{i ad}_{\overline{A}}(\overline{H}(\xi_0 + \zeta'))) \theta_H(\zeta)\| \\ & \leq 2 \sum_{\sigma=1}^{\nu} |\zeta_\sigma - \zeta'_\sigma| \|\theta_H^{(1)}(\zeta)\|^2 \|W_\sigma(\zeta)\|. \end{aligned}$$

This completes the proof. \square

PROOF OF THEOREM 6.19: Define

$$F_z(\delta, \zeta) := \langle \overline{A} \rangle^{-s} G_z(\delta, \zeta) \langle \overline{A} \rangle^{-s}.$$

For simplicity we suppose that $\delta, \epsilon > 0$. Note that $F_z(\delta, \zeta)$ is bounded uniformly in z, δ, ζ by (B.10). By Mourre's paper there is a constant $C_1 > 0$ such that

$$\|F_z(\delta, \zeta)\| \leq C_1.$$

We compute

$$\begin{aligned}
& \frac{d}{dt} F_z(t, \zeta) \\
&= -i \langle \bar{A} \rangle^{-s} G_z(t, \zeta) \bar{B}(\zeta) G_z(t, \zeta) \langle \bar{A} \rangle^{-s} \\
&= -\langle \bar{A} \rangle^{-s} G_z(t, \zeta) \theta_H^\perp(\zeta) \operatorname{ad}_{\bar{A}}(\bar{H}(\xi_0 + \zeta)) G_z(t, \zeta) \langle \bar{A} \rangle^{-s} \\
&\quad + \langle \bar{A} \rangle^{-s} G_z(t, \zeta) \theta_H(\zeta) \operatorname{ad}_{\bar{A}}(\bar{H}(\xi_0 + \zeta)) \theta_H^\perp(\zeta) G_z(t, \zeta) \langle \bar{A} \rangle^{-s} \\
&\quad - \langle \bar{A} \rangle^{-s} \operatorname{ad}_{\bar{A}}(G_z(t, \zeta)) \langle \bar{A} \rangle^{-s} \\
&\quad + it \langle \bar{A} \rangle^{-s} G_z(t, \zeta) \bar{B}(\zeta) G_z(t, \zeta) \langle \bar{A} \rangle^{-s}.
\end{aligned} \tag{6.65}$$

The last term is bounded uniformly in z, t, ζ due to Lemma 6.21.

For any $s \in \mathbb{R} \setminus \{0\}$ denote by \mathcal{H}_s the Banach space obtained by equipping $D(\langle A \rangle^s)$ with graph norm. We put $\mathcal{H}_0 := \mathcal{H}$ and for a bounded operator B we denote by $\|B\|_{s \rightarrow s'}$ its norm as a map from \mathcal{H}_s to $\mathcal{H}_{s'}$. We have

$$\begin{aligned}
\|G_z(t, \zeta) \langle \bar{A} \rangle^{-s}\| &= \|G_z(t, \zeta)\|_{s \rightarrow 0} \leq \frac{C_0}{|\delta|^{\frac{1}{2}}}, \\
\|\langle \bar{A} \rangle^{-s} G_z(t, \zeta) \langle \bar{A} \rangle^{-s}\| &= \|G_z(t, \zeta)\|_{s \rightarrow -s} \leq C_1.
\end{aligned}$$

By interpolation theory this gives

$$\|\langle \bar{A} \rangle^{-s+1} G_z(t, \zeta) \langle \bar{A} \rangle^{-s}\| = \|G_z(t, \zeta)\|_{s \rightarrow -s+1} \leq \frac{C(s)}{|\delta|^{\frac{1}{2s}}},$$

where $C(s) = C_0^{1/s} C_1^{1-1/s}$. Therefore,

$$\begin{aligned}
\|\langle \bar{A} \rangle^{-s} \operatorname{ad}_{\bar{A}}(G_z(t, \zeta)) \langle \bar{A} \rangle^{-s}\| &\leq \|\bar{A} \langle \bar{A} \rangle^{-1} \langle \bar{A} \rangle^{-s+1} G_z(t, \zeta) \langle \bar{A} \rangle^{-s}\| \\
&\quad + \|\langle \bar{A} \rangle^{-s} G_z(t, \zeta) \langle \bar{A} \rangle^{-s+1} \bar{A} \langle \bar{A} \rangle^{-1}\|, \\
&\leq \frac{C(s)}{|t|^{\frac{1}{2s}}}.
\end{aligned}$$

The first and the second term in (6.65) can be bounded in a similar fashion. Indeed, we can use Lemma 6.21 to estimate

$$\begin{aligned}
& \|\langle \bar{A} \rangle^{-s} G_z(t, \zeta) (1 - \theta_H(\zeta)) \operatorname{ad}_{\bar{A}}(\bar{H}(\xi_0 + \zeta)) G_z(t, \zeta) \langle \bar{A} \rangle^{-s}\| \\
&\leq C \|(\bar{H}(\xi_0 + \zeta) + i)^{-1} \operatorname{ad}_{\bar{A}}(\bar{H}(\xi_0 + \zeta)) (\bar{H}(\xi_0 + \zeta) + i)^{-1}\| \frac{1}{|t|^{\frac{1}{2}}}.
\end{aligned}$$

Due to

$$\begin{aligned}
& (\bar{H}(\xi_0 + \zeta) + i)^{-1} \operatorname{ad}_{\bar{A}}(\bar{H}(\xi_0 + \zeta)) (\bar{H}(\xi_0 + \zeta) + i)^{-1} \\
&= D_1(\zeta)^* (\bar{H}(\xi_0) + i)^{-1} \operatorname{ad}_{\bar{A}}(\bar{H}(\xi_0 + \zeta)) (\bar{H}(\xi_0) + i)^{-1} D_1(\zeta)
\end{aligned}$$

the uniform bound follows from Lemma 6.13, where $z = i$, $m = 1$. Thus, we have shown that (6.65) can be bounded by a constant times $|\delta|^{-1/2s}$. We write

$$F_z(0, \zeta) - F_z(0, \zeta') = [F_z(\delta, \zeta) - F_z(\delta, \zeta')] - \int_0^\delta \left[\frac{d}{dt} F_z(t, \zeta) - \frac{d}{dt} F_z(t, \zeta') \right] dt. \quad (6.66)$$

In order to deal with the first term we calculate

$$\begin{aligned} \|F_z(\delta, \zeta) - F_z(\delta, \zeta')\| &= \delta \|\langle \bar{A} \rangle^{-s} G_z(\delta, \zeta) [\bar{B}(\zeta') - \bar{B}(\zeta)] G_z(\delta, \zeta') \langle \bar{A} \rangle^{-s}\| \\ &\leq \delta |\zeta - \zeta'| C_7 \frac{1}{\delta} = C_7 |\zeta - \zeta'| \end{aligned}$$

By combining (6.65) and (6.66) we can thus show that there exists $C_8 > 0$ independent of z, δ, ζ such that

$$\|F_z(0, \zeta) - F_z(0, \zeta')\| \leq C_8 |\zeta - \zeta'| + C_8 \delta^{1-\frac{1}{2s}}.$$

For $|\zeta - \zeta'|$ sufficiently small we may put $\delta = |\zeta - \zeta'|^\beta$, where $\beta > 0$. Let $\alpha \in (1/2, 1)$. By choosing

$$\beta \left(1 - \frac{1}{2s}\right) = \alpha$$

we obtain

$$\|F_z(\delta, \zeta) - F_z(\delta, \zeta')\| \leq 2C_8 |\zeta - \zeta'|^\alpha.$$

This clearly implies the joint Hölder continuity of $F_z(0, \zeta)$ in z and ζ .

6.5 Feshbach Map and Eigenvalue Equation

We adopt the definitions of \mathcal{V} and $r > 0$ from the previous chapter throughout the rest of this section. For every $\epsilon > 0$ we define the Feshbach map at $\lambda_{\xi+\zeta}$ by

$$\begin{aligned} &F_P(H(\xi + \zeta) - \lambda_{\xi+\zeta} - i\epsilon) \\ &:= P(H(\xi + \zeta) - \lambda_{\xi+\zeta} - i\epsilon)P - PV_\zeta \overline{PR}_{\lambda_{\xi+\zeta} + i\epsilon}(\xi + \zeta) \overline{PV}_\zeta P. \end{aligned}$$

In the non-degenerate case, where $P = |\eta\rangle\langle\eta|$ for the eigenstate η , this map takes the simpler form

$$\begin{aligned} F_P(H(\zeta) - \lambda_{\xi+\zeta} - i\epsilon) &= (\lambda_\xi - \lambda_{\xi+\zeta} - i\epsilon)P + \langle\eta, V_\zeta\eta\rangle P \\ &\quad - \langle\eta, V_\zeta \overline{PR}_{\lambda_{\xi+\zeta} + i\epsilon}(\xi + \zeta) \overline{PV}_\zeta\eta\rangle P \end{aligned} \quad (6.67)$$

and the problem becomes one dimensional.

Proposition 6.23. *The limit $F_P(\zeta) := \lim_{\epsilon \rightarrow 0^+} F_P(H(\zeta) - \lambda_{\xi_0 + \zeta} - i\epsilon)$ exists in norm as a bounded operator on $P\mathcal{H}$ and*

$$F_P(\zeta) = [\lambda_\xi - \lambda_{\xi + \zeta} + \langle \eta, V_\zeta \eta \rangle] P \\ - \langle U(\zeta) \eta, \langle \bar{A} \rangle^{-s} \bar{R}_{\lambda_{\xi + \zeta} + i0}(\xi_0 + \zeta) \langle \bar{A} \rangle^{-s} U(\zeta) \eta \rangle P,$$

where $U(\zeta) := \bar{P} \langle \bar{A} \rangle^s V_\zeta$ and $s \in (1/2, 1)$.

PROOF: We first show that $V_\zeta \eta \in D(\langle \bar{A} \rangle^s)$. Since $\langle \bar{A} \rangle^s (\bar{A} - z)^{-1}$ is a bounded operator, it suffices to show $V_\zeta \eta \in D(\bar{A})$. This however follows directly from Lemma 6.4 by commuting with \bar{A} . Thus, we may calculate

$$\langle \eta, V_\zeta \bar{P} \bar{R}_{\lambda_\xi + i\epsilon}(\xi_0 + \zeta) \bar{P} V_\zeta \eta \rangle = \langle U(\zeta) \eta, \langle \bar{A} \rangle^{-s} \bar{R}_{\lambda_\xi + i\epsilon}(\xi + \zeta) \langle \bar{A} \rangle^{-s} U(\zeta) \eta \rangle,$$

where $U(\zeta) := \langle \bar{A} \rangle^s V_\zeta$. The existence of the limit $\epsilon \rightarrow 0$ is thus a consequence of Theorem 6.18. \square

Proposition 6.24. *$F_P(\zeta)$ is not invertible, that is*

$$\lambda_\xi - \lambda_{\xi + \zeta} + \langle \eta, V_\zeta \eta \rangle - \langle U(\zeta) \eta, \langle \bar{A} \rangle^{-s} \bar{R}_{\lambda_{\xi + \zeta} + i0}(\xi + \zeta) \langle \bar{A} \rangle^{-s} U(\zeta) \eta \rangle = 0.$$

PROOF: Since $F_P(\zeta)$ is a linear operator on a one dimensional space, it is invertible, if and only if

$$\lambda_\xi - \lambda_{\xi + \zeta} + \langle \eta, V_\zeta \eta - \langle U(\zeta) \eta, \langle \bar{A} \rangle^{-s} \bar{R}_{\lambda_{\xi + \zeta} + i0}(\xi + \zeta) \langle \bar{A} \rangle^{-s} U(\zeta) \eta \rangle \neq 0.$$

Suppose for a contradiction that $F_P(\zeta)$ was invertible. Then

$$\lambda_\xi - \lambda_{\xi + \zeta} + \langle \eta, V_\zeta \eta - \langle U(\zeta) \eta, \langle \bar{A} \rangle^{-s} \bar{R}_{\lambda_{\xi + \zeta} + i\epsilon}(\xi + \zeta) \langle \bar{A} \rangle^{-s} U(\zeta) \eta \rangle \\ \rightarrow \lambda_\xi - \lambda_{\xi + \zeta} + \langle \eta, V_\zeta \eta - \langle U(\zeta) \eta, \langle \bar{A} \rangle^{-s} \bar{R}_{\lambda_{\xi + \zeta} + i0}(\xi + \zeta) \langle \bar{A} \rangle^{-s} U(\zeta) \eta \rangle \neq 0$$

implies $F_P(H(\zeta) - \lambda_{\xi + \zeta} - i\epsilon)^{-1} \rightarrow F_P(\zeta)^{-1}$. A look in the proof of Theorem II.1 in [5] reveals that the inverse of $F_P(H(\xi + \zeta) - \lambda_{\xi + \zeta} - i\epsilon)$ determines the inverse $\bar{R}_{\lambda_{\xi + \zeta} + i\epsilon}(\xi + \zeta)$ of $H(\xi + \zeta) - \lambda_{\xi + \zeta} - i\epsilon$ by means of a block decomposition. Our initial assumption would thus imply the existence of the limit $\lim_{\epsilon \rightarrow 0^+} \bar{R}_{\lambda_{\xi + \zeta} + i\epsilon}(\xi + \zeta)$ as a bounded operator on \mathcal{H} . However, this is impossible for $\lambda_{\xi + \zeta}$ sufficiently close to λ_ξ , since $\lambda_\xi \in \sigma_{\text{ess}}(H(\xi))$. \square

PROOF OF THEOREM 2.12: The previous Proposition clearly implies that

$$\lambda_{\xi + \zeta} = \lambda_\xi + \langle \eta, V_\zeta \eta \rangle - \langle U(\zeta) \eta, \langle \bar{A} \rangle^{-s} \bar{R}_{\lambda_{\xi + \zeta} + i0}(\xi + \zeta) \langle \bar{A} \rangle^{-s} U(\zeta) \eta \rangle,$$

where

$$U(\zeta) = \bar{P} \langle \bar{A} \rangle^s V_\zeta = (\zeta^2 + 2\xi \cdot \zeta) \bar{P} \langle \bar{A} \rangle^s - 2\bar{P} \langle \bar{A} \rangle^s \zeta \cdot d\Gamma(k).$$

Hence

$$\begin{aligned}
& \lambda_{\xi+\zeta} \\
&= \lambda_{\xi} + \zeta^2 + 2\xi \cdot \zeta - 2 \sum_{\sigma=1}^{\nu} \zeta_{\sigma} \langle \eta, d\Gamma(k_{\sigma})\eta \rangle \\
&\quad - (\zeta^2 + 2\xi \cdot \zeta)^2 \langle \overline{P}\langle \overline{A} \rangle^s \eta, \langle \overline{A} \rangle^{-s} \overline{R}_{\lambda_{\xi+\zeta}+i0}(\xi + \zeta) \langle \overline{A} \rangle^{-s} \overline{P}\langle \overline{A} \rangle^s \eta \rangle \\
&\quad + 2 \sum_{\sigma=1}^{\nu} \zeta_{\sigma} (\zeta^2 + 2\xi \cdot \zeta) \langle \overline{P}\langle \overline{A} \rangle^s \eta, \langle \overline{A} \rangle^{-s} \overline{R}_{\lambda_{\xi+\zeta}+i0}(\xi + \zeta) \langle \overline{A} \rangle^{-s} U_{\sigma} \eta \rangle \\
&\quad + 2 \sum_{\sigma=1}^{\nu} \zeta_{\sigma} (\xi^2 + 2\xi \cdot \zeta) \langle U_{\sigma} \eta, \langle \overline{A} \rangle^{-s} \overline{R}_{\lambda_{\xi+\zeta}+i0}(\xi + \zeta) \langle \overline{A} \rangle^{-s} \overline{P}\langle \overline{A} \rangle^s \eta \rangle \\
&\quad - 4 \sum_{\sigma, \sigma'=1}^{\nu} \zeta_{\sigma} \zeta_{\sigma'} \langle U_{\sigma'} \eta, \langle \overline{A} \rangle^{-s} \overline{R}_{\lambda_{\xi_0+\zeta}+i0}(\xi + \zeta) \langle \overline{A} \rangle^{-s} U_{\sigma} \eta \rangle, \tag{6.68}
\end{aligned}$$

where $U_{\sigma} := \overline{P}\langle \overline{A} \rangle^s d\Gamma(k_{\sigma})$. Theorem 6.18 implies that

$$\lambda_{\xi+\zeta} = \lambda_{\xi} + \sum_{\sigma=1}^{\nu} (2\xi_{\sigma} - 2\langle \eta, d\Gamma(k_{\sigma})\eta \rangle) \zeta_{\sigma} + \mathcal{O}(|\zeta|^2).$$

In particular, we have that there exists $C > 0$ independent of ζ such that

$$|\lambda_{\xi+\zeta} - \lambda_{\xi}| \leq C|\zeta|. \tag{6.69}$$

Combing the Lipschitz continuity of the map $\zeta \mapsto \lambda_{\xi+\zeta}$ in (6.69) with (6.68) and the joint Lipschitz continuity of the resolvent boundary values in Theorem 6.19, we obtain

$$\lambda_{\xi+\zeta} = \lambda_{\xi} + \sum_{\sigma=1}^{\nu} \zeta_{\sigma} \beta_{\sigma} + \sum_{\sigma, \sigma'=1}^{\nu} \zeta_{\sigma} \zeta_{\sigma'} \beta_{\sigma, \sigma'} + \mathcal{O}(|\zeta|^{2+\alpha}),$$

where

$$\beta_{\sigma} := 2\xi_{\sigma} - 2\langle \eta, d\Gamma(k_{\sigma})\eta \rangle$$

and

$$\begin{aligned}
\beta_{\sigma, \sigma'} &:= \delta_{\sigma, \sigma'} + 4\xi_{\sigma} \xi_{\sigma'} \langle \overline{P}\langle \overline{A} \rangle^s \eta, \langle \overline{A} \rangle^{-s} \overline{R}_{\lambda_{\xi}+i0}(\xi) \langle \overline{A} \rangle^{-s} \overline{P}\langle \overline{A} \rangle^s \eta \rangle \\
&\quad + 4\xi_{\sigma'} \langle \overline{P}\langle \overline{A} \rangle^s \eta, \langle \overline{A} \rangle^{-s} \overline{R}_{\lambda_{\xi}+i0}(\xi) \langle \overline{A} \rangle^{-s} U_{\sigma} \eta \rangle \\
&\quad + 4\xi_{\sigma'} \langle U_{\sigma} \eta, \langle \overline{A} \rangle^{-s} \overline{R}_{\lambda_{\xi}+i0}(\xi_0) \langle \overline{A} \rangle^{-s} \overline{P}\langle \overline{A} \rangle^s \eta \rangle \\
&\quad - 4\langle U_{\sigma} \eta, \langle \overline{A} \rangle^{-s} \overline{R}_{\lambda_{\xi}+i0}(\xi) \langle \overline{A} \rangle^{-s} U_{\sigma'} \eta \rangle.
\end{aligned}$$

A Derivation of Several Commutator Relations

Throughout this section we adapt Notation 4.1. Moreover note that

$$\begin{aligned}\Phi(g)\mathcal{C}_0^\infty &\subset \mathcal{C}_0^\infty, \\ d\Gamma(u)\mathcal{C}_0^\infty &\subset \mathcal{C}_0^\infty, \\ A\mathcal{C}_0^\infty &\subset \mathcal{C}_0^\infty\end{aligned}$$

for all $u, g \in C^\infty(\mathbb{R}^\nu)$. In particular,

$$H_0(\xi)\mathcal{C}_0^\infty \subset \mathcal{C}_0^\infty.$$

These relations enable us to define commutator forms on \mathcal{C}_0^∞ which can then be extended to larger domains by the strategy set out in Section 3.2.

Remark A.1. Let $h \in C^\infty(\mathbb{R})$, $g \in \mathcal{C}_0^\infty(\mathbb{R}^\nu)$. Then

$$\begin{aligned}[A, d\Gamma(h)] &= [d\Gamma(a), d\Gamma(h)] = d\Gamma(iv \cdot \nabla h), \\ [d\Gamma(h) d\Gamma(h), \Phi(g)] &= -\Phi(-h^2g) - 2i\Phi(ihg) d\Gamma(h)\end{aligned}$$

as operators on \mathcal{C}_0^∞ .

Lemma A.2. For any $m \in \mathbb{N}$ the commutator of $\Phi(g)$ with $H_0(\xi)^n$ can be expanded in terms of the iterated commutators $\text{ad}_{H_0(\xi)}^j(\Phi(g))$. In particular, the equation

$$[\Phi(g), H_0(\xi)^n] = \sum_{\ell=0}^{n-1} \binom{n}{\ell} (-1)^{\ell+n} \text{ad}_{H_0(\xi)}^{n-\ell}(\Phi(g)) H_0(\xi)^\ell$$

extends holds in the sense of operators on \mathcal{C}_0^∞ for all $n \in \mathbb{N}$.

PROOF: For $n = 1$ one can easily check that the statement is true. Next, assume that the formula holds for some $n \in \mathbb{N}$ and calculate

$$\begin{aligned}[\Phi(g), H_0(\xi)^{n+1}] &= [\Phi(g), H_0(\xi)] H_0(\xi)^n + H_0(\xi) [\Phi(g), H_0(\xi)^n] \\ &= -\text{ad}_{H_0(\xi)}(\Phi(g)) H_0(\xi)^n \\ &\quad + \sum_{\ell=0}^{n-1} \binom{n}{\ell} (-1)^{\ell+n} \text{ad}_{H_0(\xi)}^{n-\ell}(\Phi(g)) H_0(\xi)^{\ell+1} \\ &\quad + \sum_{\ell=0}^{n-1} \binom{n}{\ell} (-1)^{\ell+n} \text{ad}_{H_0(\xi)}^{n+1-\ell}(\Phi(g)) H_0(\xi)^\ell \\ &= \sum_{\ell=0}^n \binom{n+1}{\ell} (-1)^\ell \text{ad}_{H_0(\xi)}^{n+1-\ell}(\Phi(g)) H_0(\xi)^\ell\end{aligned}$$

on \mathcal{C}_0^∞ . Thus, the statement is true for all $n \in \mathbb{N}$. \square

Definition A.3. Let $K \subset \mathbb{N}$ with $|K| = n$. For $i \leq n$ we define the set of partitions of length i of K by

$$P_i(K) := \{\{k_1, \dots, k_i\} \mid \{k_1, \dots, k_i\} \subset D, k_1 < k_2 < \dots < k_i\}.$$

Moreover, we introduce the shorthand

$$N_n := \{1, \dots, n\}.$$

Lemma A.4. Let $h_1, \dots, h_n \in C_0^\infty(\mathbb{R}^\nu)$, $g \in C_0^\infty(\mathbb{R}^\nu)$. Then

$$\left[\prod_{i=1}^n d\Gamma(h_i), \Phi(g) \right] = \sum_{i=1}^n \sum_{D \in P_i(N_n)} C_{N_n \setminus D} \Phi(g_D) \prod_{\ell \in N_n \setminus D} d\Gamma(h_\ell), \quad (\text{A.1})$$

where all C_D appearing in the above formula are complex valued constants and

$$|g_D(k)| := |h_{j_1}(k) \cdots h_{j_n}(k)| \cdot |g(k)|$$

for $D = \{j_1, \dots, j_n\}$.

PROOF: The statement can be proven by induction. The case $n = 1$ is clear. Suppose that the statement is true for some $n \in \mathbb{N}$ and calculate

$$\begin{aligned} & \left[\prod_{i=1}^{n+1} d\Gamma(h_i), \Phi(g) \right] \\ &= d\Gamma(h_{n+1}) \left[\prod_{i=1}^n d\Gamma(h_i), \Phi(g) \right] + [d\Gamma(h_{n+1}), \Phi(g)] \prod_{i=1}^n d\Gamma(h_i) \\ &= C_\emptyset d\Gamma(h_{n+1}) \Phi(g_{N_n}) + \sum_{i=1}^{n-1} \sum_{D \in P_i(N_n)} C_{N_n \setminus D} d\Gamma(h_{n+1}) \Phi(g_D) \prod_{\ell \in N_n \setminus D} d\Gamma(h_\ell) \\ & \quad - i\Phi(ih_{n+1}) \prod_{i=1}^n d\Gamma(h_i) \\ &= -iC_\emptyset \Phi(ih_{n+1}g_{N_n}) + C_\emptyset \Phi(g_{N_n}) d\Gamma(h_{n+1}) \\ & \quad + \sum_{i=1}^{n-1} \sum_{D \in P_i(N_n)} C_{N_n \setminus D} \Phi(g_D) \prod_{\ell \in N_{n+1} \setminus D} d\Gamma(h_\ell) \\ & \quad + \sum_{i=1}^{n-1} \sum_{D \in P_i(N_n)} (-iC_{N_n \setminus D}) \Phi(ih_{n+1}g_D) \prod_{\ell \in N_{n+1} \setminus (D \cup \{n+1\})} d\Gamma(h_\ell) \\ & \quad - i\Phi(ih_{n+1}g) \prod_{\ell \in N_{n+1} \setminus \{n+1\}} d\Gamma(h_\ell) \end{aligned}$$

Note that the first sum can be re-written into a sum that runs over $D = \{j_1, \dots, j_i\} \subset N_{n+1}$ for $j_1 < j_2 < \dots < j_i$, where all indices satisfy $j_k \neq n+1$. Similarly, the second sum runs over all $D = \{j_1, \dots, j_i\} \subset N_{n+1}$ with $j_1 < j_2 < \dots < j_i$ and $j_i = n+1$, where $2 \leq i \leq n$. A simple re-labeling of indexes finishes the proof. \square

A similar statement can be proven for the general form of a commutator of a second quantized multiplication operator with a product of field operators.

Lemma A.5. *Let $f \in L^2(\mathbb{R}^\nu)$, $h_1, \dots, h_n \in C_0^\infty$. We then have as an operator identity on \mathcal{C}_0^∞ :*

$$[\mathrm{d}\Gamma(f), \Phi(h_1) \cdots \Phi(h_n)] = \sum_{k=1}^n C_k \Phi(h_1) \cdots \Phi(h_{k-1}) \Phi(\mathrm{i}f h_k) \Phi(h_{k+1}) \cdots \Phi(h_n),$$

where $C_j \in \mathbb{C}$.

PROOF: The proof is a simple application of

$$[\mathrm{d}\Gamma(f), \Phi(h_1) \cdots \Phi(h_n)] = \sum_{k=1}^n \Phi(h_1) \cdots \Phi(h_{k-1}) [\mathrm{d}\Gamma(f), \Phi(h_k)] \Phi(h_{k+1}) \cdots \Phi(h_n)$$

and Remark A.1. \square

Another Lemma regarding commutators of the conjugate operator A with products of second quantized multiplication operators is needed.

Lemma A.6. *Let $n \in \mathbb{N}$, $g_1, \dots, g_n \in C_c^\infty$ and $u_1, \dots, u_n \in C^\infty(\mathbb{R}^\nu)$. Then*

$$\left[A, \prod_{j=1}^n \mathrm{d}\Gamma(u_j) \right] = \sum_{\ell=1}^n \mathrm{d}\Gamma(\mathrm{i}v \cdot \nabla u_\ell) \prod_{\substack{j=1 \\ j \neq \ell}}^n \mathrm{d}\Gamma(u_j)$$

and

$$[A, \Phi(g_1) \cdots \Phi(g_n)] = -\mathrm{i} \sum_{\ell=1}^n \Phi(g_1) \cdots \Phi(g_{\ell-1}) \Phi(\mathrm{i}v \cdot \nabla g_\ell) \Phi(g_{\ell+1}) \cdots \Phi(g_n)$$

hold as operator identities on \mathcal{C}_0^∞ .

PROOF: The assertions can be proven inductively and follow immediately from the formulas

$$\left[A, \prod_{j=1}^n \mathrm{d}\Gamma(u_j) \right] = \sum_{\ell=1}^n \mathrm{d}\Gamma(u_1) \cdots \mathrm{d}\Gamma(u_{\ell-1}) [A, \mathrm{d}\Gamma(u_\ell)] \mathrm{d}\Gamma(u_{\ell+1}) \cdots \mathrm{d}\Gamma(u_n),$$

$$[A, \Phi(g_1) \cdots \Phi(g_n)] = \sum_{\ell=1}^n \Phi(g_1) \cdots \Phi(g_{\ell-1}) [A, \Phi(g_\ell)] \Phi(g_{\ell+1}) \cdots \Phi(g_n)$$

and Remark A.1. \square

Lemma A.7 (Rearrangement Formula). *Suppose that $p \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$. Then*

$$\begin{aligned} & R_{-c}^\ell \operatorname{ad}_A^p((H+c)^m) R_{-c}^m \\ &= (-1)^\ell \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k R_{-c}^{\ell-k} \operatorname{ad}_H^{\ell-k}(\operatorname{ad}_A^p((H+c)^m)) R_{-c}^{m+\ell} \end{aligned}$$

holds for operators on \mathcal{C}_0^∞ and extends to \mathcal{H} , if $\ell \geq p$.

PROOF: Clearly, the assertion is true for $\ell = 0$. The general case can be proven by induction but since the proof is very similar to the proofs of the other form identities in this section, we omit most of the details.

It should be noted however that $\mathcal{C}_0^\infty \subset D(H^n)$ for all $n \in \mathbb{N}$ due to $H_0\mathcal{C}_0^\infty \subset \mathcal{C}_0^\infty$ and Lemma 3.17. Thus, $\mathcal{C}_0^\infty \subset D(\operatorname{ad}_A^p((H+c)^m)R_{-c}^m)$ for all $p \in \mathbb{N}$ and the following calculations are meaningful when carried out on \mathcal{C}_0^∞ .

Assume that the statement holds for some $\ell \in \mathbb{N}$ and calculate

$$\begin{aligned} & R_{-c}^{\ell-1} \operatorname{ad}_A^p((H+c)^m) R_{-c}^m \\ &= (-1)^\ell \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k R_{-c}^{\ell-k} \operatorname{ad}_H^{\ell-k}(\operatorname{ad}_A^p((H+c)^m)) R_{-c}^{m+\ell+1} \\ &\quad + (-1)^\ell \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k R_{-c}^{\ell+1-k} \operatorname{ad}_H^{\ell-k}(\operatorname{ad}_A^p((H+c)^m)) R_{-c}^{m+\ell+1} \\ &= (-1)^{\ell+1} (-1)^{\ell+1} \operatorname{ad}_A^p((H+c)^m) R_{-c}^{m+\ell+1} \\ &\quad + (-1)^{\ell+1} \sum_{k=1}^{\ell} \binom{\ell+1}{k} (-1)^k R_{-c}^{\ell-1-k} \operatorname{ad}_H^{\ell+1-k}(\operatorname{ad}_A^p((H+c)^m)) R_{-c}^{m+\ell+1} \\ &\quad + (-1)^{\ell+1} R_{-c}^{-(\ell+1)} \operatorname{ad}_H^{\ell+1}(\operatorname{ad}_A^p((H+c)^m)) R_{-c}^{m+\ell+1} \\ &= (-1)^{\ell+1} \sum_{k=0}^{\ell+1} \binom{\ell+1}{k} (-1)^k R_{-c}^{\ell-1-k} \operatorname{ad}_H^{\ell+1-k}(\operatorname{ad}_A^p((H+c)^m)) R_{-c}^{m+\ell+1}, \end{aligned}$$

where we have made an index shift to get from the second to the third equation. Finally note that

$$\operatorname{ad}_H^{\ell-k}(\operatorname{ad}_A^p((H+c)^m)) R_{-c}^{m+\ell} \in \mathcal{B}(\mathcal{H}) \Leftrightarrow m + \frac{p}{2} + \frac{\ell}{2} \leq m + \ell \Leftrightarrow \ell \geq p.$$

This proves the last part of the statement. \square

B Proof of Mourre's Quadratic Estimate

The purpose of this section is to prove Lemma 6.21. First of all note that it suffices to show the statement for $\xi = \xi_0$ and $|\zeta| < r'$ for some $r' > 0$. The corresponding

statements with the neighborhood \mathcal{V} of ξ_0 can be obtained in the same way as the corresponding statements in the limiting absorption principle, Theorem 6.18.

In order to establish the invertibility of $\overline{H}(\delta, \xi) - z$ we may simply use Mourre's argument, see [41]. There the estimate

$$\|(\overline{H}(\delta, \zeta) - z)\overline{P}\psi\|^2 \geq (\text{Im}(z))\|\overline{P}\psi\|^2$$

is established by using the assumption $\text{Im}(z)\delta > 0$. This estimate then implies the bounded invertibility of $\overline{H}(\delta, \zeta) - z\overline{P}$. Moreover, Mourre proves that for all bounded and self-adjoint operators \overline{T} on $\overline{P}\mathcal{H}$ and all bounded operators \overline{B}' satisfying $\overline{B}'\overline{B}'^* \leq \overline{B}(\zeta)$ the estimate

$$\|\overline{B}'G_z(\delta, \zeta)\overline{T}\| \leq \frac{\|\overline{T}G_z(\delta, \zeta)\overline{T}\|^{\frac{1}{2}}}{|\delta|^{\frac{1}{2}}}$$

holds. However, all of Mourre's arguments can only be carried out for fixed ζ and thus the constants may be ζ -dependent as well. Therefore, we have to go through the proof more carefully to guarantee that all constants can be chosen independently of z, ζ, δ . The choice $\overline{B}' = \theta_H(\zeta)$ and $\overline{T} = \overline{P} = \text{id}_{\overline{P}\mathcal{H}}$ gives

$$\|\theta_H(\zeta)G_z(\delta, \zeta)\| \leq \frac{1}{|\delta|^{\frac{1}{2}}} \|G_z(\delta, \zeta)\|^{\frac{1}{2}}. \quad (\text{B.1})$$

Likewise the choice $\overline{B}' = \theta_H(\zeta)$ and $\overline{T} = \langle \overline{A} \rangle^{-s}$ yields

$$\|\theta_H(\zeta)G_z(\delta, \zeta)\langle \overline{A} \rangle^{-s}\| \leq \frac{1}{|\delta|^{\frac{1}{2}}} \|\langle \overline{A} \rangle^{-s}G_z(\delta, \zeta)\langle \overline{A} \rangle^{-s}\|^{\frac{1}{2}}. \quad (\text{B.2})$$

To finish the proof we have to prove the inequalities stated in the lemma. First note that

$$-\theta_H(\xi)G_z(\delta, \zeta) + \theta_H^\perp(\zeta)G_z(0, \zeta) = [1 + i\delta G_z(0, \zeta)\theta_H^\perp(\zeta)\overline{B}(\zeta)]G_z(\delta, \zeta). \quad (\text{B.3})$$

By definition of the cutoff function θ we have that

$$\|G_z(0, \zeta)\theta_H^\perp(\zeta)\| \leq \sup_{x \in \mathbb{R}} \frac{1 - \theta(x)}{|x - \text{Re}(z)|} \leq C_1(J, \theta),$$

where $C_1(J, \theta) > 0$ is independent of ζ, z and δ . Moreover, it is clear that there exists a constant $C > 0$ independent of ζ, z and δ such that $\|\overline{B}(\xi)\| \leq C$. Hence

$$\|G_z(0, \zeta)\theta_H^\perp(\zeta)\overline{B}(\zeta)\| \leq C_1(J, \theta)C. \quad (\text{B.4})$$

By choosing

$$\delta_0 < \frac{1}{2C_1(J, \theta)C}$$

we have thus shown that $T_z(\zeta) := 1 + \delta G_z(0, \zeta) \theta_H^\perp(\zeta) \overline{B}(\zeta)$ is invertible for all $|\delta| < \delta_0$ with norm $\|T_z(\zeta)^{-1}\| \leq 2$. (B.3) now implies

$$G_z(\delta, \zeta) = -T_z(\zeta)^{-1} \theta_H(\zeta) G_z(\delta, \zeta) + T_z(\zeta)^{-1} \theta_H^\perp(\zeta) G_z(0, \zeta). \quad (\text{B.5})$$

By (B.1), we may estimate

$$\|\theta_H(\zeta) G_z(\delta, \zeta)\|^2 \leq \frac{1}{|\delta|} \|G_z(\delta, \zeta)\| \leq \frac{\alpha^2}{2} \|G_z(\delta, \zeta)\|^2 + \frac{1}{2|\delta|^2 \alpha^2} \quad (\text{B.6})$$

for any $\alpha > 0$. Note that we may choose $\delta_0 < 1$ and $0 < \alpha$ to be small. Hence (B.4), (B.5) and (B.6) imply that there exists a constant $C_1 > 0$ independent of ζ, z and δ such that

$$\|G_z(\delta, \zeta)\| \leq \frac{C}{|\delta|} \quad (\text{B.7})$$

With (B.7) we can estimate

$$\begin{aligned} \|\theta_H^\perp(\zeta) G_z(\delta, \zeta)\| &= \|\theta_H^\perp(\zeta) G_z(0, \zeta)\| \|1 - i\delta \overline{B}(\zeta) G_z(\delta, \zeta)\| \\ &\leq \|\theta_H^\perp(\zeta) G_z(0, \zeta)\| \left(1 + |\delta| C \frac{1}{|\delta|}\right) \leq C_2(J, \theta), \end{aligned} \quad (\text{B.8})$$

where $C_2(J, \theta)$ does not depend on z, ξ, δ . By a similar trick we obtain

$$\begin{aligned} \|\overline{H}(\xi + \zeta) \theta_H^\perp(\zeta) G_z(\delta, \zeta)\| &\leq \|\overline{H}(\xi + \zeta) \theta_H^\perp(\zeta) G_z(0, \zeta)\| \|1 - i\delta \overline{B}(\zeta) G_z(\delta, \zeta)\| \\ &\leq C_3(J, \theta), \end{aligned} \quad (\text{B.9})$$

where we have used that the function $x(1 - \theta(x))/(x - \text{Re}(z))$ is bounded uniformly on \mathbb{R} . Again the constant $C_3(J, \theta)$ does not depend on z, ζ, δ .

Clearly, these bounds show that

$$\begin{aligned} \|\overline{H}(\xi + \zeta) G_z(\delta, \zeta)\| &\leq \|\overline{H}(\xi + \zeta) \theta_H(\zeta) G_z(\delta, \zeta)\| + \|\overline{H}(\xi + \zeta) \theta_H^\perp(\zeta) G_z(\delta, \zeta)\| \\ &\leq C_4(\theta) \|G_z(\delta, \zeta)\| + C_3(J, \theta) \leq \frac{1}{|\delta|} C_5(J, \theta). \end{aligned}$$

We have thus established the first and the last of the claimed estimates on $G_z(\delta, \zeta)$. In order to prove the second estimate, we refer to Mourre's paper again, see [41]. There it is established that the estimates (B.7)-(B.9) together with (B.1) and (B.2) imply that $\|\langle \overline{A} \rangle^{-s} G_z(\delta, \zeta) \langle \overline{A} \rangle^{-s}\| \leq C'$, where $C' > 0$ is independent of z, δ . This result is proven via a differential inequality technique and the uniform constant C' is obtained from the uniform constants in (B.7)-(B.9).

Since we have established that the constants in (B.7)-(B.9) are independent of z, δ as well as ξ , we may conclude that

$$\|\langle \overline{A} \rangle^{-s} G_z(\delta, \zeta) \langle \overline{A} \rangle^{-s}\| \leq C_6(J, \theta), \quad (\text{B.10})$$

where $C_6(J, \theta)$ is independent of z, ξ, δ . With this bound it clearly follows from (B.2) that

$$\|\theta_H(\xi)G_z(\delta, \zeta)\langle\bar{A}\rangle^{-s}\| \leq \frac{C_6(J, \theta)}{|\delta|^{\frac{1}{2}}} \quad (\text{B.11})$$

The relation $G_z(\delta, \zeta) = G_z(0, \delta)(1 - i\bar{B}(\zeta)G_z(\delta, \zeta))$, (B.7)-(B.9) and (B.11) let us obtain the desired inequality

$$\|G_z(\delta, \zeta)\langle\bar{A}\rangle^{-s}\| + \|\bar{H}(\xi + \zeta)G_z(\delta, \zeta)\langle\bar{A}\rangle^{-s}\| \leq \frac{C}{|\delta|^{\frac{1}{2}}}$$

after inserting the localizations $\theta_H(\zeta)$ and $\theta_H^\perp(\zeta)$. Since the details are similar to previous calculations, we omit them.

C Hölder-Continuity in the Limiting Absorption Principle

Let $\epsilon, \delta > 0$. In order to establish Hölder-continuity of the limit, we write

$$\begin{aligned} F_z(0, \zeta) - F_{z'}(0, \zeta) &= -[F_z(\delta, \zeta) - F_{z'}(\delta, \zeta)] \\ &\quad + \int_0^\delta \left[\frac{d}{dt} F_z(t, \zeta) - \frac{d}{dt} F_{z'}(t, \zeta) \right] dt. \end{aligned} \quad (\text{C.1})$$

Note that

$$\begin{aligned} \|F_z(\delta, \zeta) - F_{z'}(\delta, \zeta)\| &= |z - z'| \|\langle\bar{A}\rangle^{-s} G_z(\delta, \zeta) G_{z'}(\delta, \zeta) \langle\bar{A}\rangle^{-s}\| \\ &\leq |z - z'| C \frac{1}{|\delta|}, \end{aligned} \quad (\text{C.2})$$

where $C_1 > 0$ is independent of z, δ, ζ . Hence the first term in (C.1) is even Lipschitz continuous. It thus remains to study the integral. The discussion after (6.65) implies the estimate

$$\int_0^{|\delta|} \left\| \frac{d}{dt} F_z(t, \zeta) - \frac{d}{dt} F_{z'}(t, \zeta) \right\| dt \leq C_3 \int_0^{|\delta|} \frac{1}{|t|^{\frac{1}{2}}} dt = 2C_3 |\delta|^{\frac{1}{2}}. \quad (\text{C.3})$$

Now choose $\delta = |z - z'|^{2/3}$ for $|z - z'|$ sufficiently small. If we combine (C.2) and (C.3), we may conclude that there exists a constant $C_6 > 0$ independent of ζ, z, δ such that

$$\|F_z(0, \zeta) - F_{z'}(0, \zeta)\| \leq C_6 |z - z'| \frac{1}{\delta} + C_6 \delta^{\frac{1}{2}} = 2C_6 |z - z'|^{\frac{1}{3}}$$

which proves the claimed Hölder continuity w.r.t. the spectral parameter.

D Adding a Parameter to a Result by Gérard

In [22] the limiting absorption principle (LAP) is derived via a chain of implications starting from certain estimates of (energy) localizations $\chi(H)$ of a Hamiltonian H and weights $\langle A \rangle$ of the conjugate operator. The crucial technical assumption is a strict Mourre estimate in the energy region of interest and that H is of class $C^2(A)$. Most likely to avoid assumptions on H -boundedness of the commutators $\text{ad}_A(H)$ and $\text{ad}_A^2(H)$ the Hamiltonian is replaced by $H_\tau = H\tau(H)$, where $\tau \in \mathbb{R}^\nu$ is a plateau function equaling 1 in a neighborhood of the region of interest. To apply his result in our setup we have to investigate how the bounds appearing in the LAP will depend on a parameter κ , if H is replaced by a family of operators $H(\kappa)$ and the purpose of this section is to carry out this investigation. More precisely, we are concerned with the question whether the bounds in the LAP can be formulated to hold uniformly on a (sufficiently small) neighborhood U of a fixed parameter value κ_0 .

As a first step we replace the single operator H by a family of self adjoint operators $H(\kappa)$ for $\kappa \in \mathbb{R}^d$ with common domain $D(H(\kappa)) = D$. In view of our results on Lipschitz continuity of the maps $\zeta \mapsto \text{ad}_A^k(f(\overline{P}H(\xi + \zeta)\overline{P}))$ established in Theorem E.10 we will assume that

$$\exists C > 0 \exists r > 0 \forall \kappa \in B_r(\kappa_0) : \|\text{ad}_A^k(f(H(\kappa)))\| \leq C. \quad (\text{D.1})$$

provided that $f(H(\kappa)) \in C^k(A)$ for the self-adjoint operator A . Let A be self-adjoint and $H(\kappa)$ be of class $C^2(A)$ for $\kappa \in U$. Put $U := B_r(\kappa_0)$, where r is given as above. Suppose that there is a constant c_0 and an interval I independent of $\kappa \in U$ such that

$$\mathbf{1}_I(H(\kappa))[iA, H(\kappa)]\mathbf{1}_I(H(\kappa)) \geq c_0\mathbf{1}_I(H(\kappa)).$$

This is a needed generalization of Gérard's estimate (1.1) in a κ -uniform version. Lemma 2.1 in Gérard's paper establishes that $(z - H)^{-1}$ and $\chi(H)$, where $\chi \in C_0^\infty(\mathbb{R})$, are bounded operators on $D(\langle A \rangle^s)$ for $s \in [0, 1]$. It can immediately be proven for all $H(\kappa)$, since they satisfy the same conditions as the single operator H . The same is true for Lemma 2.2, that is all $\chi(H(\kappa)) \in C^k(A)$, if all $H(\kappa)$ are of class $C^k(A)$.

Let $J \subset I$ be a compact interval and define the set $J^+ := \{z \in \mathbb{C} \mid \text{Re}(z) \in J, \text{Im}(z) > 0\}$. Let $\tau, \chi_1 \in C_0^\infty(\mathbb{R})$ such that $\tau(x) = \chi_1(x) = 1$ for all $x \in I$ and $\tau(x) = 1$ for all $x \in \text{supp}(\chi_1)$. Put $H_\tau(\kappa) = H(\kappa)\tau(H(\kappa))$. The commutator expansion on $D(\langle A \rangle^k)$ for $k \in \mathbb{N} \setminus \{1\}$ and $f \in C_0^\infty(\mathbb{R})$ in Lemma 2.3,

$$[f(A), H_\tau(\kappa)] = \sum_{j=1}^{k-1} \frac{1}{j!} f^{(j)}(A) \text{ad}_A^j(H_\tau(\kappa)) + R_k(f, A, H_\tau(\kappa)),$$

can also be generalized provided $H_\tau \in C^k(A)$, since it is a structural result. The important estimate

$$\|\langle A \rangle^s R_k(f, A, H_\tau(\kappa)) \langle A \rangle^{s'}\| \leq C(f) \|\text{ad}_A^k(H_\tau(\kappa))\|$$

valid for all $s, s' \geq 0$ with $s + s' < k$ can now be bounded uniformly for $\kappa \in U$, due to (D.1). This leads to similar uniform estimates of the error terms R_1 and R_2 in Gérard's Proposition 2.4. One of the crucial ideas in the paper is that a chain of implications in Gérard's Lemma 3.1 can be used to show the limiting absorption principle. The needed generalization is

Lemma D.1. *Let $s \in (0, 1]$ and consider the following statements:*

1. $\exists C > 0 \forall \kappa \in U : \sup_{z \in J^+} \|\langle A \rangle^{-s} (H(\kappa) - z)^{-1} \langle A \rangle^{-s}\| \leq C < \infty$.
2. *There exists $C > 0$ such that for all $\kappa \in U$, all $z \in J^+$ and all $u \in (H(\kappa_0) + i)^{-1} D(\langle A \rangle^s)$ we have that*

$$\|\langle A \rangle^{-s}\| \leq C \|\langle A \rangle^s (H(\kappa) - z) u\|.$$

3. *There exists $C > 0$ such that for all $\kappa \in U$, all $z \in J^+$ and all $u \in D(\langle A \rangle^s)$ we have that*

$$\|\langle A \rangle^{-s} \chi_1(H) u\| \leq C \|\langle A \rangle^s (H_\tau(\kappa) - z) \chi_1(H(\kappa)) u\|.$$

Then 3. \Rightarrow 2. \Rightarrow 1.

The proof is a word by word copy of Gérard's proof. The uniformity w.r.t. $\kappa \in U$ in every implication comes from the built in uniformity in the modified statements 3. and 2. In order to prove LAP Gérard establishes 3. in his Lemma 3.1. The proof of this statement hinges on Proposition 2.4 and Lemma 2.3 as well as the Mourre estimate. A crucial role is played by the Mourre constant c_0 and the error estimates in Proposition 2.4. Since both terms, called R_1 and R_2 by Gérard, are shown to be estimated uniformly w.r.t. $\kappa \in U$, the resulting constant in the desired estimate 3. in Lemma D.1 in our modified version of Gérard's proof is also uniform w.r.t. $\kappa \in U$.

Another technicality in Gérard's derivation of his version of the estimate in 3. is to rescale the conjugate operator A , that is to consider $A_a = a^{-1}A$ for $a > 0$. For a sufficiently large a an error estimate in Gérard's proof improves significantly. In the parameter dependent case we need to be able to make this choice for a independent of κ . This however is a consequence of the uniform estimates on $\|R_1\|$ and $\|R_2\|$.

E Regularity of the Functional Calculus of the Projected Hamiltonian

In complete analogy to Definition 3.21 we define mixed commutators of $\overline{P}H(\xi_0 + \xi)\overline{P}$ and \overline{A} .

Definition E.1 (Projected, Iterated Commutators). We introduce the abbreviation $\mathcal{Q}_{\overline{P}} := \{\overline{P}T_1\overline{P}, \overline{P}T_2\overline{P}\}$, where $T_1 := H(\xi_0 + \xi)$ and $T_2 := A$. Moreover, let $c > 0$ be large enough such that $-c \in \rho(H)$. For $n \in \mathbb{N}$, define

$$\mathfrak{I}_n := \{\underline{w} = (\underline{w}(1), \dots, \underline{w}(n)) \in \{1, 2\}^n \mid \underline{w}(1) = 2\}.$$

For $\underline{w} \in \mathfrak{I}_n$ we define its ℓ -th truncation $\underline{w}^{(\ell)}$ by

$$\underline{w}^{(\ell)} := (\underline{w}(1), \dots, \underline{w}(\ell)) \in \{1, 2\}^\ell.$$

Let $T \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}^{\text{uv}, k}$ be a $H(\xi_0)^m$ -bounded operator, where $k = \mathfrak{n}_{\underline{w}}^A$. Suppose $h \in \mathbb{H}_{\text{uv}}^{k+1}(\mathbb{R}^\nu)$. The ℓ -th truncation $\text{ad}_{\overline{\mathcal{T}}}^{\underline{w}^{(\ell)}}(T)$ of the mixed commutator corresponding to \underline{w} on \mathcal{C}_0^∞ is iteratively defined by

$$\langle \psi, \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}^{(1)}}(T)\psi' \rangle := \langle \psi, [\overline{A}, T]\psi \rangle$$

for $\ell = 1$ and

$$\langle \psi, \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}^{(\ell)}}(T)\psi' \rangle := \langle \psi, [\overline{P}T_{\underline{w}^{(\ell)}}\overline{P}, \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}^{(\ell-1)}}(T)]\psi' \rangle$$

for $\ell \geq 2$. The mixed commutator corresponding to $\underline{w} \in \mathfrak{I}_n$ is then defined as

$$\text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(T) := \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}^{(n)}}(T).$$

This operator is sometimes simply referred to as a mixed or iterated commutator. It is convenient to define $\mathfrak{I}_0 := \{0\}$ and $\text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(T) := S$ for $\underline{w} \in \mathfrak{I}_0$.

Remark E.2. As before the mixed commutators extend from \mathcal{C}_0^∞ to $D((H(\xi_0) + c)^{m+n/2})$ provided that T is $(H(\xi_0) + c)^m$ -bounded. Moreover, note that instead of T we could have used the corresponding projected operator $\overline{P}T\overline{P}$.

In the next lemma we establish a connection between the projected mixed commutators and the unprojected ones.

Lemma E.3. *Let $n, m \in \mathbb{N}$ and $\psi \in D(H(\xi_0)^{m+n/2})$. For every $\underline{w} \in \{1, 2\}^n$ there exists a tuple $(\{T_{n,k}\}_{k=1}^{2N}, \{t_{k,n}\}_{k=1}^{2N})$ such that*

$$\begin{aligned} \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}((\overline{P}H(\xi_0 + \zeta)\overline{P} - z\overline{P})^m)\psi &= \overline{P} \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}((H(\xi_0) - z)^m)\overline{P}\psi + \overline{P} \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(\mathfrak{B}_m(\zeta))\overline{P}\psi \\ &\quad + \overline{P} \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(\mathfrak{F}_m(\zeta))\overline{P}\psi + \overline{P}\mathfrak{G}_{m, \underline{w}}(\zeta)\overline{P}, \end{aligned}$$

where $\mathfrak{B}_m(\zeta)$ is the $(H(\xi_0) + c)^m$ -bounded operator associated to the tuple, $\mathfrak{F}_m(\zeta)$ is the finite rank operator associated to it (see Definition 6.8). $\mathfrak{G}_{m, \underline{w}}(\zeta)$ is a finite rank operator which depends Lipschitz continuously on ζ .

PROOF: We prove the statement by induction. The case $n = 1$ follows from $[\overline{P}T\overline{P}, (\overline{P}H(\xi_0 + \zeta)\overline{P} - z\overline{P})^m] = \overline{P}[T, (\overline{P}H(\xi_0 + \zeta)\overline{P} - z\overline{P})^m]\overline{P}$ and (6.52). We thus assume the statement holds for some $n \in \mathbb{N}$. Let $\underline{w} \in \{1, 2\}^n$ and abbreviate

$$S_1 := \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}((H(\xi_0) - z)^m), \quad S_2 := \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(\mathfrak{B}_{\underline{w}}(\zeta)), \quad S_3 := \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(\mathfrak{B}_{\underline{w}}(\zeta))$$

where we have used the notation of (6.56). Let $\psi, \psi' \in \mathcal{C}_0^\infty$. We first assume that $\underline{w}' = (2, \underline{w})$. Using the induction hypothesis we compute:

$$\begin{aligned} & \langle \psi, [\overline{A}, \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}((\overline{P}(H(\xi_0 + \zeta) - z\overline{P})^m))] \psi' \rangle \\ &= \langle \psi, \overline{P} \text{ad}_A(\text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}((H(\xi_0) - z)^m)) \overline{P} \psi' \rangle + \langle \psi, \overline{P} \text{ad}_A(\text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(\mathfrak{B}_m(\zeta))) \overline{P} \psi' \rangle \\ & \quad + \langle \psi, \overline{P} \text{ad}_A(\text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(\mathfrak{F}_m(\zeta))) \overline{P} \psi' \rangle \\ & \quad + \sum_{j=1}^3 \langle \psi, \overline{P} |A\eta\rangle \langle S_j^* \eta | \overline{P} \psi' \rangle - \sum_{j=1}^3 \langle \psi, \overline{P} |S_j \eta\rangle \langle A\eta | \overline{P} \psi' \rangle \end{aligned}$$

This proves the assumption in this case. The proof that

$$\begin{aligned} & \text{ad}_{\overline{P}(H(\xi_0 + \zeta)\overline{P})}^{\underline{w}}(\text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}((\overline{P}H(\xi_0 + \zeta) - z\overline{P})^m)) \\ &= \text{ad}_{H(\xi_0)}^{\underline{w}}(\text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}((H(\xi_0) - z)^m)) + \overline{P} \text{ad}_{\overline{\mathcal{T}}}^{(1, \underline{w})}(\mathfrak{B}_m(\zeta)) \overline{P} + \overline{P} \mathfrak{F}_m^{(1, \underline{w})}(\zeta) \overline{P} \\ & \quad + \sum_{j=1}^3 \overline{P} |A\eta\rangle \langle S_j^* \eta | - \sum_{j=1}^3 \overline{P} |S_j \eta\rangle \langle A\eta | \overline{P} \end{aligned}$$

in the case $\underline{w}' = (1, \underline{w})$ is similar. \square

Remark E.4. A similar statement holds, if $(H(\xi_0 + \zeta) - z)^m$ is replaced by a $(H(\xi_0) + c)^m$ -bounded operator T which is given on \mathcal{C}_0^∞ in terms of the sets $\mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ and the ξ dependence is dropped. (The ζ dependent functions can either be chosen equal to zero or constant)

We are now in a position to prove the analogue of Lemma 4.2 in the case of the projected operators. Since the presence of the projections causes the appearance of several additional terms, we introduce some more notation to keep the formulas in a manageable length.

Let $n, m \in \mathbb{N}$, $\underline{w} \in \{1, 2\}^n$ and define $M_{\underline{w}} \geq m + \text{Int}(n/2)$, where $\text{Int}(r)$ denotes the smallest integer that is bigger than $r \in \mathbb{R}$ and $T \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ be $H(\xi_0)^{M_{\underline{w}}+1}$ -bounded. Define

$$\begin{aligned} \mathfrak{f}_1(T, \zeta) &:= -\overline{P}R_z^{M_{\underline{w}}}\overline{P} \text{ad}_A(T)\overline{P}R_z^{M_{\underline{w}}+2}\overline{P} \\ & \quad + \overline{P}R_z^{M_{\underline{w}}+1}\overline{P} \text{ad}_{H(\xi_0)}(\text{ad}_A(T))\overline{P}R_z^{M_{\underline{w}}+2}\overline{P} \\ & \quad + 2 \sum_{\sigma=1}^{\nu} \zeta_\sigma \overline{P}R_z^{M_{\underline{w}}+1}\overline{P} \text{ad}_{\text{d}\Gamma(k_\sigma)}(\text{ad}_A(T))\overline{P}R_z^{M_{\underline{w}}+2}\overline{P}. \end{aligned} \quad (\text{E.1})$$

Similarly, for an $H(\xi_0)^{M_{\underline{w}}}$ -bounded $T' \sim \mathfrak{C}_{\alpha'_1, \alpha'_2, \alpha'_3}$ we define

$$\mathfrak{f}_2(T', \zeta) := \overline{P} \operatorname{ad}_A(T') \overline{P} \overline{R}_z^{M_{\underline{w}+1}} \overline{P}, \quad (\text{E.2})$$

and

$$\mathfrak{R}_2(T', \zeta) := \overline{P} |A\eta\rangle \langle (T')^* \eta | \overline{P} \overline{R}_z^{M_{\underline{w}+1}} \overline{P} + |T' \eta\rangle \langle A\eta | \overline{P} \overline{R}_z^{M_{\underline{w}+1}} \overline{P} \quad (\text{E.3})$$

Moreover, we put

$$\begin{aligned} \mathfrak{R}_1(\zeta) &:= \overline{R}_z^{M_{\underline{w}+1}} \overline{P} (|\mathfrak{B}_{M_{\underline{w}+1}}(\zeta)\eta\rangle \langle A\eta| - |A\eta\rangle \langle (\mathfrak{B}_{M_{\underline{w}+1}}(\zeta))^* \eta|) \overline{R}_z^{M_{\underline{w}+1}} \overline{P} \\ &\quad + \overline{R}_z^{M_{\underline{w}+1}} \overline{P} (|\mathfrak{F}_{M_{\underline{w}+1}}(\zeta)\eta\rangle \langle A\eta| - |A\eta\rangle \langle (\mathfrak{F}_{M_{\underline{w}+1}}(\zeta))^* \eta|) \overline{R}_z^{M_{\underline{w}+1}} \overline{P} \\ &\quad + \overline{R}_z^{M_{\underline{w}+1}} \overline{P} \operatorname{ad}_A(\mathfrak{F}_{M_{\underline{w}+1}}(\zeta)) \overline{R}_z^{M_{\underline{w}+1}} \overline{P}, \end{aligned} \quad (\text{E.4})$$

where $\mathfrak{B}_{M_{\underline{w}+1}}(\zeta)$ and $\mathfrak{F}_{M_{\underline{w}+1}}(\zeta)$ are the $(H(\xi_0) + c)^{M_{\underline{w}+1}}$ -bounded and the finite rank operator appearing in (6.52) in the case $m = M_{\underline{w}} + 1$.

Lemma E.5. *Let $n, m \in \mathbb{N}$, $\underline{w} \in \{1, 2\}^n$ and define $M_{\underline{w}} \geq m + \operatorname{Int}(n/2)$, where $\operatorname{Int}(r)$ denotes the smallest integer that is bigger than $r \in \mathbb{R}$. Put $k = \mathbf{n}_{\underline{w}}^A$ and suppose that $h \in \mathbb{H}_{\text{uv}}^{k+1}(\mathbb{R}^\nu)$. Then*

$$\operatorname{ad}_{\overline{\mathcal{T}}_P}^{\underline{w}}((\overline{H}(\xi_0 + \zeta) - z)^m) \overline{R}_z(\xi_0 + \zeta)^{M_{\underline{w}+1}} \in \mathbb{C}^1(\overline{A}),$$

that is $J_{\underline{w}}(\zeta) := \operatorname{ad}_{\overline{A}}^{\underline{w}}\left(\operatorname{ad}_{\overline{\mathcal{T}}_P}^{\underline{w}}((\overline{H}(\xi_0 + \zeta) - z)^m) \overline{R}_z(\xi_0 + \zeta)^{M_{\underline{w}+1}}\right) \in \mathfrak{B}(\overline{P}\mathcal{H})$. Moreover, there exists a neighborhood \mathcal{O} of ξ_0 such that the map $\zeta \mapsto J_{\underline{w}}(\zeta)$ is Lipschitz continuous in operator norm. In particular,

$$\begin{aligned} &\operatorname{ad}_{\overline{A}}^{\underline{w}}\left(\operatorname{ad}_{\overline{\mathcal{T}}_P}^{\underline{w}}(\overline{R}_z) \overline{R}_z^{M_{\underline{w}+1}}\right) \overline{P} \\ &= \left[\operatorname{ad}_{\overline{\mathcal{T}}}^{\underline{w}}((H(\xi_0) - z)^m) + \operatorname{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(\mathfrak{B}_m(\xi)) + \operatorname{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(\mathfrak{F}_m(\zeta)) \right] \mathfrak{f}_1((H(\xi_0) - z)^{M_{\underline{w}+1}}, \zeta) \\ &\quad + \left[\operatorname{ad}_{\overline{\mathcal{T}}}^{\underline{w}}((H(\xi_0) - z)^m) + \operatorname{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(\mathfrak{B}_m(\zeta)) + \operatorname{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(\mathfrak{F}_m(\zeta)) \right] \mathfrak{f}_1(\mathfrak{B}_{M_{\underline{w}+1}}(\zeta), \zeta) \\ &\quad + \mathfrak{f}_2(\operatorname{ad}_{\overline{\mathcal{T}}}^{\underline{w}}((H(\xi_0) - z)^m), \zeta) + \mathfrak{f}_2(\operatorname{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(\mathfrak{B}_m(\zeta)), \zeta) + \mathfrak{f}_2(\operatorname{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(\mathfrak{F}_m(\zeta)), \zeta) \\ &\quad + \mathfrak{R}(\zeta), \end{aligned} \quad (\text{E.5})$$

where

$$\begin{aligned} \mathfrak{R}(\zeta) &= \mathfrak{R}_1(\zeta) + \mathfrak{R}_2(\operatorname{ad}_{\overline{\mathcal{T}}}^{\underline{w}}((H(\xi_0) - z)^m), \zeta) + \mathfrak{R}_2(\operatorname{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(\mathfrak{B}_m(\zeta)), \zeta) \\ &\quad + \mathfrak{R}_2(\operatorname{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(\mathfrak{F}_m(\zeta)), \zeta). \end{aligned}$$

PROOF: For $\ell \in \mathbb{N}$ define

$$S_{\underline{w}}^\ell = \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(H(\xi_0) - z)^\ell, \quad S_A^\ell = \text{ad}_A((H(\xi_0) - z)^\ell)$$

and abbreviate

$$\overline{H} \equiv \overline{P}H(\xi_0 + \zeta)\overline{P}, \quad \overline{R}_z \equiv \overline{R}_z(\xi_0 + \zeta).$$

Let $\psi, \psi' \in \mathcal{C}_0^\infty$. We use Lemma E.3 in order to calculate

$$\begin{aligned} & \langle \overline{P}\psi, [\overline{A}, \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}((\overline{H} - z)^m)\overline{R}_z^{M_{\underline{w}}+1}]\overline{P}\psi' \rangle \\ &= \langle \overline{P}\psi, \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}((H(\xi_0) - z)^m)\overline{P}[\overline{A}, \overline{R}_z^{M_{\underline{w}}+1}]\overline{P}\psi' \rangle \\ &+ \langle \overline{P}\psi, \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(\mathfrak{B}_m(\zeta))\overline{P}[\overline{A}, \overline{R}_z^{M_{\underline{w}}+1}]\overline{P}\psi' \rangle \\ &+ \langle \overline{P}\psi, \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(\mathfrak{F}_m(\zeta))\overline{P}[\overline{A}, \overline{R}_z^{M_{\underline{w}}+1}]\overline{P}\psi' \rangle \\ &+ \langle \overline{P}\psi, [\overline{A}, \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}((H(\xi_0) - z)^m)]\overline{P}\overline{R}_z^{M_{\underline{w}}+1}\overline{P}\psi' \rangle \\ &+ \langle \overline{P}\psi, [\overline{A}, \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(\mathfrak{B}_m(\zeta))]\overline{P}\overline{R}_z^{M_{\underline{w}}+1}\overline{P}\psi' \rangle \\ &+ \langle \overline{P}\psi, [\overline{A}, \text{ad}_{\overline{\mathcal{T}}}^{\underline{w}}(\mathfrak{F}_m(\zeta))]\overline{P}\overline{R}_z^{M_{\underline{w}}+1}\overline{P}\psi' \rangle. \end{aligned} \tag{E.6}$$

Let $\Psi, \Psi' \in D(A)$. Making use of (6.52) we compute

$$\begin{aligned} & \langle \overline{P}\Psi, [\overline{A}, \overline{R}_z^{M_{\underline{w}}+1}]\overline{P}\Psi' \rangle \\ &= -\langle \overline{P}\Psi, \overline{R}_z^{M_{\underline{w}}+1}\overline{P}[A, (H(\xi_0) - z)^{M_{\underline{w}}+1}]\overline{P}\overline{R}_z^{M_{\underline{w}}+1}\overline{P}\Psi' \rangle \\ &- \langle \overline{P}\Psi, \overline{R}_z^{M_{\underline{w}}+1}\overline{P}[A, \mathfrak{B}_{M_{\underline{w}}+1}(\zeta)]\overline{P}\overline{R}_z^{M_{\underline{w}}+1}\overline{P}\Psi' \rangle \\ &+ \langle \overline{P}\Psi, \overline{R}_z^{M_{\underline{w}}+1}\overline{P}|\mathfrak{B}_{M_{\underline{w}}+1}(\zeta)\eta\rangle\langle A\eta|\overline{P}\overline{R}_z^{M_{\underline{w}}+1}\overline{P}\Psi' \rangle \\ &- \langle \overline{P}\Psi, \overline{R}_z^{M_{\underline{w}}+1}\overline{P}|A\eta\rangle\langle (\mathfrak{B}_{M_{\underline{w}}+1}(\zeta))^*\eta|\overline{P}\overline{R}_z^{M_{\underline{w}}+1}\overline{P}\Psi' \rangle \\ &- \langle \overline{P}\Psi, \overline{R}_z^{M_{\underline{w}}+1}\overline{P}[A, \mathfrak{F}_{M_{\underline{w}}+1}(\zeta)]\overline{P}\overline{R}_z^{M_{\underline{w}}+1}\overline{P}\Psi' \rangle \\ &+ \langle \overline{P}\Psi, \overline{R}_z^{M_{\underline{w}}+1}\overline{P}|\mathfrak{F}_{M_{\underline{w}}+1}(\zeta)\eta\rangle\langle A\eta|\overline{P}\overline{R}_z^{M_{\underline{w}}+1}\overline{P}\Psi' \rangle \\ &- \langle \overline{P}\Psi, \overline{R}_z^{M_{\underline{w}}+1}\overline{P}|A\eta\rangle\langle (\mathfrak{F}_{M_{\underline{w}}+1}(\zeta))^*\eta|\overline{P}\overline{R}_z^{M_{\underline{w}}+1}\overline{P}\Psi' \rangle. \end{aligned}$$

Let $T \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ be $H(\xi_0)^{M_{\underline{w}}+1}$ -bounded. It is easy to check that

$$-\langle \overline{P}\Psi, \overline{R}_z^{M_{\underline{w}}+1}\overline{P}[A, T]\overline{P}\overline{R}_z^{M_{\underline{w}}+1}\overline{P}\Psi' \rangle = \langle \overline{P}\Psi, \mathfrak{f}_1(T, \xi)\overline{P}\Psi' \rangle,$$

where $\mathfrak{f}_1(T, \xi)$ is defined in (E.1). For an $H(\xi_0)^{M_{\underline{w}}}$ -bounded $T' \sim \mathfrak{C}_{\alpha'_1, \alpha'_2, \alpha'_3}$ we compute

$$\langle \overline{P}\psi, [\overline{A}T']\overline{P}\overline{R}_z^{M_{\underline{w}}+1}\overline{P}\psi' \rangle = \langle \overline{P}\psi, \mathfrak{f}_2(T', \zeta)\overline{P}\psi' \rangle + \langle \overline{P}\psi, \mathfrak{R}_2(T', \zeta)\overline{P}\psi' \rangle,$$

where $\mathfrak{f}_2(T', \zeta)$ is defined in (E.2) and $\mathfrak{R}_2(T', \zeta)$ in (E.3). Moreover,

$$\langle \overline{P}\Psi, [\overline{A}, \mathfrak{F}_{\underline{w}}(\zeta)]\overline{P}\Psi' \rangle = \langle \overline{P}\Psi, \text{ad}_A(\mathfrak{F}_{\underline{w}}(\zeta))\overline{P}\Psi' \rangle.$$

Since $(H(\xi_0) - z)^\ell \sim \mathfrak{C}_{\alpha_1, \alpha_2, \alpha_3}$ for some α_i and is $(H(\xi_0) - z)^\ell$ -bounded and $\mathfrak{B}_{M_{\underline{w}+1}}(\zeta) \sim \mathfrak{C}_{\alpha'_1, \alpha'_2, \alpha'_3}$ for some α'_i is $(H(\xi_0) - z)^{M_{\underline{w}+1}}$ -bounded, (E.6) is a bounded form. Lipschitz continuity follows from

$$\begin{aligned} & (H(\xi_0) + c)^{M_{\underline{w}}} \mathfrak{f}_1(T, \xi) \\ &= -\overline{P} D_{M_{\underline{w}}}(\zeta, z) \overline{P} \operatorname{ad}_A(T) (H(\xi_0) + c)^{-M_{\underline{w}-2}} \overline{P} D_{M_{\underline{w}+2}}(\zeta) \overline{P} \\ & \quad + \overline{P} D_{M_{\underline{w}}}(\xi, z) \overline{R}_z \overline{P} \operatorname{ad}_{H(\xi_0)}(\operatorname{ad}_A(T)) (H(\xi_0) + c)^{-M_{\underline{w}-2}} \overline{P} D_{M_{\underline{w}+2}}(\zeta, z) \overline{P} \\ & \quad + 2 \sum_{\sigma=1}^{\nu} \zeta_\sigma \overline{P} D_{M_{\underline{w}}}(\zeta, z) \overline{R}_z \overline{P} \operatorname{ad}_{\operatorname{d}\Gamma(k_\sigma)}(\operatorname{ad}_A(T)) (H(\xi_0) + c)^{-M_{\underline{w}-2}} \overline{P} D_{M_{\underline{w}+2}}(\zeta, z) \overline{P} \end{aligned}$$

and

$$\begin{aligned} \mathfrak{f}_2(T', \zeta) &= \overline{P} \operatorname{ad}_A(T') (H(\xi_0) + c)^{-M_{\underline{w}-1}} \overline{P} D_{M_{\underline{w}+1}}(\zeta, z) \overline{P} \\ & \quad + \overline{P} |A\eta\rangle \langle (T')^* \eta | \overline{P} \overline{R}_z^{M_{\underline{w}+1}} \overline{P} + |T' \eta\rangle \langle A\eta | \overline{P} \overline{R}_z^{M_{\underline{w}+1}} \overline{P}. \end{aligned}$$

This completes the proof by Lemma 6.13. \square

In analogy to (4.20) we define

$$\prod_{p=1}^i S_{\underline{w}_p}(\zeta) \overline{R}_{-c}^{Q_p} := S_{\underline{w}_i}(\zeta) \overline{R}_{-c}^{Q_i} \cdots S_{\underline{w}_1}(\zeta) \overline{R}_{-c}^{Q_1}, \quad (\text{E.7})$$

where

$$S_{\underline{w}_p}(\zeta) \in \left\{ \operatorname{ad}_{\overline{T}}^{\underline{w}_p}((H(\xi_0) - z)^{M_p}), \operatorname{ad}_{\overline{T}}^{\underline{w}_p}(\mathfrak{B}_{M_p}(\zeta)), \operatorname{ad}_{\overline{T}}^{\underline{w}_p}(\mathfrak{F}_{M_p}(\zeta)) \cdot \mathfrak{G}_{M_p, \underline{w}_p}(\zeta) \right\},$$

Here, $\underline{w}_p \in \mathfrak{I}_n$ for some $n \in \mathbb{N}$, $M_p \in \mathbb{N}$, $Q_p \geq M_p + \operatorname{Int}(n/2)$ and $\mathfrak{B}_{\underline{w}_p}(\zeta)$, $\mathfrak{F}_{\underline{w}_p}(\zeta)$ are as in Lemma E.3 and $\mathfrak{G}_{M_p, \underline{w}_p}(\xi)$ is as in Lemma E.5. Moreover, we define

$$\begin{aligned} \overline{\mathfrak{U}}_{i,k,b}(\zeta) &:= \operatorname{Span} \left\{ \overline{R}_z^B \prod_{p=1}^j S_{\underline{w}_p}(\xi) \overline{R}_{-c}^{Q_p} \mid B \geq b, 1 \leq j \leq i, \sum_{p=1}^i \mathfrak{n}_{\underline{w}_p}^A = k, \right. \\ & \quad \left. Q_p \geq M_p + \operatorname{Int}(|\underline{w}_p|/2) \right\}. \quad (\text{E.8}) \end{aligned}$$

By Lemma E.5 $\operatorname{ad}_{\overline{A}}(U_{i,k,b}(\zeta))$, where $U_{i,k,b}(\zeta) \in \overline{\mathfrak{U}}_{i,k,b}(\zeta)$, can be expressed as a linear combination of operators $U_{i',k',b'}(\zeta)$ as in (E.7), where some of the Q'_p are possibly not large enough for these operators to be elements of some $\overline{\mathfrak{U}}_{i',k',b'}(\zeta)$.

In order to resolve this issue we begin to redistribute resolvents \overline{R}_z from left to right. Due to Lemma E.3 this redistribution process is going to preserve the structure of the spans $\overline{\mathfrak{U}}_{i',k',b'}(\zeta)$. Hence, by a word-by-word copy of the proof of Lemma 4.4, we can prove the following statement:

Lemma E.6. *Let $i, k, b \in \mathbb{N}$ and suppose $b \geq 3 \cdot 2^i$. Then*

$$\text{ad}_{\bar{A}}(\bar{\mathfrak{U}}_{i,k,b}(\zeta)) \subset \bar{\mathfrak{U}}_{i+1,k+1,b-3 \cdot 2^i}(\zeta).$$

Once we have proven this statement however, we can proceed as in the case without projections and prove the analog of Proposition 4.5.

Proposition E.7. *Let $k \in \mathbb{N}$. There exists $m_k \in \mathbb{N}$ so that $\bar{R}_z^m \in C^k(\bar{A})$ for all $m \geq m_k$. Moreover,*

$$\text{ad}_{\bar{A}}^k(\bar{R}_z(\xi_0 + \zeta)^m) \in \mathfrak{U}_{k,k,m-m_k}(\zeta). \quad (\text{E.9})$$

As a direct consequence of Lemma 6.13 we obtain that the map

$$\zeta \mapsto \text{ad}_{\bar{A}}^k(\bar{R}_z(\xi_0 + \zeta)^m)$$

is Lipschitz continuous on any bounded neighborhood of 0. Clearly, we may carry on with this type of argument and develop an analog of the strategy used to prove Theorem 4.6 in a step by step adaptation of every tool needed along the way. We begin with adapting (4.26).

$$\prod_{p=1}^i \bar{R}_z^\theta S_{\underline{w}_p} R_{z_0}^{Q_p} := \bar{R}_z^\theta S_{\underline{w}_i} \bar{R}_{z_0}^{Q_i} \cdots \bar{R}_z^\theta S_{\underline{w}_1} \bar{R}_{-c}^{Q_1}, \quad (\text{E.10})$$

where the constants are as in (4.26) and the operators $S_{\underline{w}_p}$ are as in (E.7). As above we proceed by defining

$$\begin{aligned} \bar{\mathfrak{V}}_{i,k,b}(\xi) := \text{Span}\left\{ \bar{R}_z^B \prod_{p=1}^j S_{\underline{w}_p}(\xi) \bar{R}_{-c}^{Q_p} \bar{R}_z \mid B \geq b, 1 \leq j \leq i, \sum_{p=1}^j \mathbf{n}_{\underline{w}_p}^A = k, \right. \\ \left. \theta \in \{0, 1\}, Q_p \geq M_p + \text{Int}(|\underline{w}_p|/2) \right\}. \quad (\text{E.11}) \end{aligned}$$

Once more we have all necessary tools at our disposal to prove

Lemma E.8. *Let $i, k, b \in \mathbb{N}$ and suppose $b \geq 5 \cdot 2^i$. Then*

$$\text{ad}_{\bar{A}}(\bar{\mathfrak{V}}_{i,k,b}(\zeta)) \subset \bar{\mathfrak{V}}_{i+1,k+1,b-5 \cdot 2^i}(\zeta).$$

and

Lemma E.9. *Let $k \in \mathbb{N}$. There exists $m_k \in \mathbb{N}$ such that $R_{-c} R_z \in C(\bar{A})$ for all $m \geq m_k$ and*

$$\text{ad}_{\bar{A}}(\bar{R}_{-c}(\xi_0 + \zeta)^m \bar{R}_z(\xi_0 + \zeta)) \in \bar{\mathfrak{V}}_{i+1,k+1,b-5 \cdot 2^i}(\zeta).$$

In particular, the map $\zeta \mapsto \text{ad}_{\bar{A}}(\bar{R}_{-c}(\xi_0 + \zeta)^m \bar{R}_z(\xi_0 + \zeta))$ is Lipschitz continuous on any bounded neighborhood of 0.

by literally copying the proofs of Lemma 4.9 and Lemma 4.10. With these two structural results we are then in a position to invoke a word by word copy of the proof of Theorem 4.6 and obtain that the following similar result holds.

Theorem E.10. *Let $f \in C_0^\infty(\mathbb{R})$. For all $k \in \mathbb{N}$ we have that*

$$f(\overline{PH}(\xi_0 + \zeta)\overline{P}) \in C^k(\overline{A}).$$

Moreover, the map $\zeta \mapsto \text{ad}_A^k(f(\overline{PH}(\xi_0 + \zeta)\overline{P}))$ is Lipschitz continuous w.r.t. operator norm on every bounded neighborhood of 0.

CHAPTER 3: Generalized Dilation Analyticity

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1 Introduction

The investigation of the essential spectrum of a self-adjoint operator via spectral deformation techniques goes back to two papers by Aguilar and Combes and Balslev and Combes, see [3] and [6]. The starting point of the whole theory is the behavior of the Laplace operator under certain unitary transformations. In particular, we define the unitary group of dilations on $L^2(\mathbb{R}^n)$ by

$$U(\theta)\psi(x) = e^{\frac{n}{2}\theta}\psi(e^\theta x).$$

Under conjugation with $U(\theta)$ the Laplace operator transforms into

$$U(\theta)\Delta U(\theta)^{-1} = e^{-2\theta}\Delta.$$

Thus, the spectrum of $e^{-2\theta}\Delta$ is a half-line starting at 0 which has an angle of $-2\text{Im}(\theta)$ to the real line. The situation can be interpreted in the following way: by conjugating Δ with $U(\theta)$ the spectrum of the transformed operator swings out into the complex plane with an angle dependent on size and sign of the imaginary part of θ . The observation by Aguilar and Combes was that for certain potentials V , the Schrödinger operator $H = -\Delta + V$ exhibits a similar behavior when conjugated with $U(\theta)$. This idea is generalized by Balslev and Combes to the situation of many-body Hamilton operators. The class of potentials for which this strategy works are called *dilation analytic*. The theory of dilation analytic potentials and its application to quantum mechanics is summed up in [45]. However, it should be noted that its applicability is limited. The reason being that many physically relevant potentials are not dilation analytic. An example of this is the Hamiltonian describing several electrons in the external field generated by several nucleons, the interaction terms of the type $|x - X|^{-1}$, where x is an electron position and X the position of a nucleon, are not dilation analytic, see [30]. Several authors have then begun to further develop an analogue of the theory in the non dilation analytic case, see [8, 30, 48, 51].

One of the great successes of the theory lies in the examination of so-called *resonances*. A resonance can be defined as the poles of the meromorphic continuation of certain matrix elements. More precisely, suppose that for a self adjoint operator H on \mathcal{H} the matrix elements

$$M(\psi, \psi', z) := \langle \psi, (H - z)^{-1}\psi' \rangle,$$

have a meromorphic continuation into an open subset $U \subset \mathbb{C}$ for certain vectors ψ, ψ' . The poles of this continuation are called resonances of the operator H . It should be stressed that typically real poles correspond to eigenvalues of H . We will now sketch how this continuation can be constructed in the context of dilation analyticity. Put $H_\theta = U(\theta)HU(\theta)^{-1}$ for $\theta \in \mathbb{R}$. One first needs to rewrite $M(\psi, \psi', z)$ as

$$M_\theta(\psi, \psi', z) = \langle U(-\bar{\theta})\psi, (H_\theta - z)^{-1}U(\theta)\psi' \rangle$$

and then analytically extend to complex θ in a certain region Ω in the complex plane for fixed z . For this expression to be meaningful the vectors ψ, ψ' are chosen in the set of analytic vectors of the generator A of the unitary group $U(\theta)$. Then we fix θ_0 in Ω and extend $M_{\theta_0}(\psi, \psi', z)$ meromorphically in z . The study of resonances plays a huge role in quantum physics and in particular in perturbation theory, for an application in the context of the quantum N -body problem see [31]. A broader discussion of the topic not just aiming at applications in physics can be found in [55]. For a textbook discussion and overview, see [26, 27]. It should be noted however, that the definition of resonances is a subtle business, since our definition depends on the set of vectors from which ψ, ψ' are drawn as well as the unitary group. We will not address this issue here, however a discussion can be found in an overview article by Simon, [52], and the textbook [27] by Hislop and Sigal.

It is interesting to note that the generator A of the unitary group of dilations is given by $A = \frac{1}{2}(p \cdot x + x \cdot p)$, where $p = i\nabla$. This operator appears in the context of Mourre theory as the conjugate operator in spectral theory. It is therefore natural to ask whether there is a connection between the two notions. Formally, the transformed operator $H\theta = e^{-i\theta A} H e^{i\theta A}$ is given as the power series $H - i\theta \text{ad}_A(H) + \frac{\theta^2}{2!} i^2 \text{ad}_A^2(H) - \dots$, where $i^k \text{ad}_A^k(H)$ are the self adjoint operators implementing the k -th commutator form $[iA, [iA, \dots [iA, H]] \dots]$. In the context of Mourre theory a key role is played by the so-called limiting absorption principle in which certain boundary values of the resolvent are constructed as a limit between weighted spaces controlled by the conjugate operator A :

$$\langle A \rangle^{-s} (H - \lambda \mp i0)^{-1} \langle A \rangle^{-s} := \lim_{\epsilon \searrow 0} \langle A \rangle^{-s} (H - \lambda \mp i\epsilon)^{-1} \langle A \rangle^{-s}. \quad (1.1)$$

The concept of conjugate operators and their applicability to spectral problems in quantum mechanics goes back to Mourre in his paper [41]. Maybe one of the first results which actively connects Mourre theory to the mentioned formal power series goes back to the paper [32] by Jensen, Perry and Mourre. There, n times differentiability of the limiting operator in (1.1) is established, if the map $\theta \mapsto H_\theta$ is n times differentiable as a map between \mathbb{C} and $\mathcal{B}(\mathcal{H}, D(H))$, where $D(H)$ is the Banach space obtained by equipping $D(H)$ with the graph norm of H . Their result is then applied to scattering theory of Schrödinger operators.

In the present paper this idea is developed in a systematic and abstract fashion. Starting from a self-adjoint operator (Hamiltonian) and an arbitrary unitary group $U(\theta)$ generated by a self adjoint operator A the equivalence of the existence of an analytic continuation of the map $\theta \mapsto H_\theta$ to the existence of *all* commutators $i^k \text{ad}_A^k(H)$ as H -bounded operators satisfying the estimate $\|\text{ad}_A^k(H)(H + i)^{-1}\| \leq C^k k!$ for some $C > 0$ is shown. As a consequence the power series $H\psi - i\theta \text{ad}_A(H)\psi + \frac{\theta^2}{2!} i^2 \text{ad}_A^2(H)\psi - \dots$ is summable for $|\theta| < C^{-1}$ and all $\psi \in D(H)$. The corresponding operator is closed and densely defined. If for $\kappa > 0$ one further assumes a Mourre estimate

$$[H, iA] \geq C - C' E(|H - \lambda| \geq \kappa) \langle H \rangle + K$$

around a point $\lambda \in \sigma(H)$, absence of essential spectrum of H_θ in the set $Y_\theta = \{z \mid \operatorname{Re}(z) \in (\lambda - \kappa/2, \lambda + \kappa/2), \operatorname{Im}(z) > -C\operatorname{Im}(\theta)\}$ can be shown, where $|\theta| < R$ and $\operatorname{Im}(\theta) > 0$. Consequently, the operator H_θ only has discrete spectrum in Y_θ . The rough idea behind this argument can be easily explained. For a normalized sequence $\overline{P}\psi_n$ such that $(\overline{P}H_\theta\overline{P} - \mu\overline{P})\psi_n \rightarrow 0$, where $\mu \in \sigma(\overline{P}H_\theta\overline{P})$, we compute for $\operatorname{Im}(\theta) > 0$

$$\operatorname{Im}\mu = -\langle \overline{P}\psi_n, (\overline{P}H_\theta\overline{P} - \mu\overline{P})\psi_n \rangle - \operatorname{Im}\theta \langle \overline{P}\psi_n, \operatorname{ad}_A(H)\overline{P}\psi_n \rangle + \mathcal{O}(|\theta|^2).$$

Ignoring the error terms coming from $-C'E(|H - \lambda| \geq \kappa)\langle H \rangle + K$ in the Mourre estimate for the commutator we obtain

$$\operatorname{Im}\mu \leq -C\operatorname{Im}\theta - \langle \overline{P}\psi_n, (\overline{P}H_\theta\overline{P} - \mu\overline{P})\psi_n \rangle + \mathcal{O}(|\theta|^2) + \text{remainder}$$

If the remaining error terms are well behaved in the limit $n \rightarrow \infty$, we have reached the conclusion that there cannot be any spectrum of $\overline{P}H_\theta\overline{P}$ with imaginary part strictly larger than $-C\operatorname{Im}\theta$. The corresponding statement for the essential spectrum of H_θ follows from the Feshbach method. It is tempting to think of the $\overline{P}\psi_n$ as a Weyl sequence. Such a sequence however need not exist, because it is not clear whether $\overline{P}H_\theta\overline{P}$ is a normal operator. It should further be noted that the width of the set Y_θ is needed to control the remainder terms.

Finally, if H is substituted by an analytic family $\{H(\xi)\}_{\xi \in U}$, where $U \subset \mathbb{C}^n$, of type A , this deformation technique is then used to apply Kato's perturbation theory (which clearly holds at least in the case $n = 1$) to eigenvalues of $H_\theta(\xi) = U(\theta)H(\xi)U(\theta)^{-1}$ by ensuring that all results in the single operator case come with a certain uniformity in ξ . If the family $H_\theta(\xi)$ happens to have real eigenvalues, these have to be eigenvalues of the operators $H(\xi)$. Thus, our result allows an analytic perturbation theory of eigenvalues of H even if these are embedded in the essential spectrum.

A trivial case of an operator which admits a band of embedded eigenvalues depending real-analytically on a parameter is provided in the following example.

Example 1.1 ([12]). Let $H_0 = \Delta^2$ as an operator on $H^4(\mathbb{R}^d)$. Let $f \in C_c^\infty(\mathbb{R}^d)$ be nonnegative $f \geq 0$.

Then, for any $\xi > 0$ and since $(-\Delta + \xi)^{-1}$ is positivity improving, we have $u(\xi) = (-\Delta + \xi)^{-1}f$ is Schwarz class and strictly positive everywhere.

Put

$$V(\xi) = -\frac{1}{u(\xi)}(-\Delta - \xi)f \in C_c^\infty(\mathbb{R}^d).$$

Then

$$V(\xi)u(\xi) = -(-\Delta - \xi)f = -(-\Delta - \xi)(-\Delta + \xi)u(\xi) = -\Delta^2u(\xi) + \xi^2u(\xi).$$

Hence $(H_0 + V(\xi))u(\xi) = \xi^2u(\xi)$ and consequently, $H(\xi) = H_0 + V(\xi)$ has an embedded eigenvalue at the energy $E = \xi^2$.

One can choose to read $V(\xi)$ as a function of $\xi > 0$. The associated family of operators $H(\xi) = \Delta^2 + V(\xi)$ will now have a persistent (real analytic) band of embedded eigenvalues $E(\xi) = \xi^2$.

Since the strategy of the paper is roughly speaking to transform the Hamiltonian into a non self adjoint operator with receding essential spectrum in the area of interest, the question whether or not our assumptions are too strong arises. In particular, one could be tempted to hope that the minimal requirements of Kato's theory are sufficient. This however is not the case, since the following example illustrates that one cannot expect the usual conclusions of Kato to hold true, when one considers analytic perturbations of self-adjoint operators with embedded eigenvalues.

Example 1.2. Let $\mathcal{H} = L^2(\mathbb{R}^2) \oplus \mathbb{C}$ and define

$$H(\xi) = \begin{pmatrix} -\Delta - \xi^2 \mathbf{1}_{[|x| \leq 1]} & 0 \\ 0 & 0 \end{pmatrix}$$

with domain $H^2(\mathbb{R}^2) \oplus \mathbb{C}$. There exist $\rho > 0$, such that For $\xi \in \mathbb{R}$ with $0 < |\xi| < \rho$, the operator $-\Delta - \xi^2 \mathbf{1}_{[|x| \leq 1]}$ has a unique eigenvalue $\lambda(\xi)$, which is simple and depends real analytically on $0 < |\xi| < \rho$. See [?]. We may extend λ to a continuous function on $(-\rho, \rho)$ by setting $\lambda(0) = 0$. Hence, for $\xi \in (-\rho, \rho)$, $\sigma_{\text{pp}}(H(\xi)) = \{\lambda(\xi), 0\}$ and $\xi \rightarrow H(\xi)$ is clearly analytic of Type (A). We observe two things: **(I)** At $\xi = 0$, there is a single simple eigenvalue $\lambda = 0$. But we have two branches of eigenvalues coming out for $\xi \neq 0$. That is, the total multiplicity of the eigenvalue cluster is not upper semi continuous. **(II)** The lower of the two eigenvalue branches $\xi \rightarrow \lambda(\xi)$ does not have an algebraic singularity at $\xi = 0$, more precisely; $|\lambda(\xi)| \leq e^{-(a\xi^2)^{-1}}$ for some $a > 0$, cf. [50]. For a closely related example, see [24, p. 585].

2 General Theory

2.1 Generalized Dilations

In this subsection H and A denotes two self-adjoint operators on a complex separable Hilbert space \mathcal{H} . The inner product $\langle \cdot, \cdot \rangle$ is assumed linear in the second variable and conjugate linear in the first variable. We associate to an (unbounded) operator T with domain $D(T)$, its graph norm $\|\psi\|_T = \|T\psi\| + \|\psi\|$, as a norm on the subspace $D(T)$. We shall frequently, for self-adjoint T , exploit the easy estimate $\frac{1}{2}\|\psi\|_T \leq \|(T + i)\psi\| \leq \|\psi\|_T$.

We work throughout Section 2 under the following condition:

Condition 2.1.

1. Abbreviating $U(t) = e^{itA}$ for $t \in \mathbb{R}$, we assume $U(t)D(H) \subset D(H)$ for all $t \in \mathbb{R}$ and

$$\forall \psi \in D(H) : \sup_{|t| \leq 1} \|HU(t)\psi\| < \infty.$$

2. The quadratic form on $D(H) \cap D(A) \times D(H) \cap D(A)$ given by

$$(\psi, \varphi) \mapsto \langle H\psi, A\varphi \rangle - \langle A\psi, H\varphi \rangle$$

is continuous w.r.t. the norm $\|(\psi, \varphi)\|_H := \|\psi\|_H + \|\varphi\|_H$.

3. There exists $R > 0$ such that for any $\psi \in D(H)$, the map

$$\mathbb{R} \ni t \mapsto H_t\psi := U(t)HU(-t)\psi$$

extends to a strongly analytic \mathcal{H} -valued function $\{H_\theta\psi\}_{\theta \in S_R}$, where

$$S_R := \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < R\}.$$

This defines a collection of linear operators $\{H_\theta\}_{\theta \in S_R}$ with domain $D(H)$.

4. For H_θ defined above, note that $H_\theta(H + i)^{-1} \in \mathcal{B}(\mathcal{H})$ by the closed graph theorem.² We suppose that

$$M := \sup_{\theta \in B_R^{\mathbb{C}}(0)} \|H_\theta(H + i)^{-1}\| < \infty.$$

- Remarks 2.2.**
1. The Conditions 2.1.1 and 2.1.2 go back to Mourre [41] and are equivalent to saying that H is of class $C^1(A)$ with commutator $[H, A]^\circ$ bounded as an operator from $D(H)$ into \mathcal{H} . See [38, Prop. B.11].
 2. Another consequence of Conditions 2.1.1 and 2.1.2 is the density of $D(H) \cap D(A)$ in both $D(H)$ and $D(A)$, equipped with their respective graph norms. See [38, Lemma. B.10].
 3. It suffices that the map $\theta \mapsto H_\theta\psi$ extends from $(-R, R)$ to $B_R^{\mathbb{C}}(0)$ in order to obtain an extension into S_R . Indeed, since we assume that $U(t)D(H) \subset D(H)$, the composition $H_\theta U(t)$ makes sense on $D(H)$ for all $\theta \in B_R^{\mathbb{C}}(0)$. Let $t \in \mathbb{R}$ and $\theta \in (t - R, t + R)$, then

$$H_\theta\psi = U(t)H_{\theta-t}U(-t)\psi$$

extends from $(t - R, t + R)$ to an analytic function on $B_R^{\mathbb{C}}(t)$ for all $\psi \in D(H)$. Sliding t along the real axis produces an analytic continuation of $H_\theta\psi$ to the whole strip S_R .

² H_θ is closable, since $H_\theta \subset H_\theta^*$.

We recall from [33] that if $U \subset \mathbb{C}$ is open then a family $\{T_\theta\}_{\theta \in U}$ of closed operators is said to be *analytic of Type (A)* if the domain of T_θ does not depend on θ and the map $U \ni \theta \rightarrow T_\theta \psi$ is analytic for any ψ in the common domain. If $U \subset \mathbb{C}^d$, then $\{T_\theta\}_{\theta \in U}$ is said to be analytic of Type (A), if it is separately analytic of Type (A) in each of its d variables.

Lemma 2.3. *Assume Conditions 2.1.1 and 2.1.2. We have*

1. *For any $\psi \in D(H)$, $\theta \in \mathbb{C}$ and $m \in \mathbb{N}$, we have $\psi_m(\theta) := e^{-A^2/(2m)+i\theta A}\psi \in D(H)$.*
2. *If $\theta \in \mathbb{R}$, we have $\lim_{m \rightarrow \infty} \psi_m(\theta) = U(\theta)\psi$ in the topology of $D(H)$. In particular ($\theta = 0$), the set of vectors in $D(H)$ that are analytic vectors for A are dense in $D(H)$.*
3. *For all $\psi \in D(H)$ and $m \in \mathbb{N}$, the map $\theta \rightarrow H\psi_m(\theta)$ is entire.*

Proof. Put $\psi_m(\theta) = e^{-A^2/(2m)+i\theta A}\psi$. Using the Fourier transform, we may write

$$\psi_m(\theta) = \sqrt{\frac{2\pi}{m}} e^{-m\theta^2/2} \int_{\mathbb{R}} e^{-mt^2/2+m\theta t} U(t)\psi dt.$$

Note that for any $m \in \mathbb{N}$, the integral converges absolutely in $D(H)$, since $\|U(t)\psi\|_{D(H)} \leq e^{c|t|}$, for all $t \in \mathbb{R}$, where $c > 0$ is some constant. This is a consequence of Condition 2.1.1 and implies 1.

Let $\theta \in \mathbb{R}$. To show that $\psi_m(\theta) \rightarrow U(\theta)\psi$ in $D(H)$, it suffices to argue that $H\psi_m(\theta) \rightarrow HU(\theta)\psi$ in \mathcal{H} . Since $U(\theta)\psi \in D(H)$ for real θ , it suffices to prove this with $\theta = 0$. Here we observe that

$$\int_{|t| \geq 1} e^{-mt^2/2} U(t)\psi dt \rightarrow 0,$$

due to the estimate $\|U(t)\psi\|_{D(H)} \leq e^{c|t|}$ from before. Furthermore, the estimate

$$\|(HU(t) - U(t)H)\psi\| \leq C|t|, \tag{2.2}$$

valid for $|t| \leq 1$ with some $C > 0$, cf. [?, Props. 2.29 and 2.34] (applied with $S = H$, $\mathcal{H}_1 = D(H)$, $\mathcal{H}_2 = \mathcal{H}$, $W_1(t) = U(t)|_{D(H)}$ and $W_2(t) = U(t)$), finally yields 2.

We now establish 3. Since the map $\theta \rightarrow \psi_m(\theta)$ is entire it suffices, by Vitali-Porter's theorem, to show that $n\|H(H+in)^{-1}\psi_m(\theta)\|$ is bounded locally uniformly in $\theta \in \mathbb{C}$. But this follows easily from the estimates already invoked above. \square

It turns out that under the assumption in Condition 2.1.1, the remaining three items are equivalent to the statement that all iterated commutators of H with A are H -bounded and satisfy a certain growth bound. If these bounds are satisfied the analytic continuation of the family H_θ can be written as a power series in a neighborhood of 0. More precisely, we can prove

Proposition 2.4. *Assume Condition 2.1.1. Then the following two properties are equivalent:*

1. *Conditions 2.1.2, 2.1.3 and 2.1.4.*
2. *There exists a constant $C > 0$ such that: the iterated commutators $\text{ad}_A^k(H)$ exist as H -bounded operators for all $k \in \mathbb{N}$ and*

$$\|\text{ad}_A^k(H)(H + i)^{-1}\| \leq C^k k!. \quad (2.3)$$

In the confirming case, $\{H_\theta\}_{\theta \in B_{(3C)^{-1}}^{\mathbb{C}}(0)}$ is an analytic family of Type (A), and for all $\theta \in B_{(3C)^{-1}}^{\mathbb{C}}(0)$ and $\psi \in D(H)$, we have

$$H_\theta \psi := \sum_{k=0}^{\infty} \frac{(-\theta)^k}{k!} i^k \text{ad}_A^k(H) \psi \quad (2.4)$$

and

$$\frac{1}{2} \|\psi\|_H \leq \|\psi\|_{H_\theta} \leq 2 \|\psi\|_H. \quad (2.5)$$

Remark 2.5. If one supposes 1 with a given R and M coming from Condition 2.1.3 and 2.1.4, respectively, then one may choose $C = \max\{1, M\}/R$ in (2.3).

Conversely, if one assumes 2 with a given C , then one may choose $R = (3C)^{-1}$ and $M = 3$.

Since we have elected to state our assumptions in terms of an analytic extension of H , we shall below employ the estimate (2.3) with

$$C = \frac{\max\{1, M\}}{R}. \quad (2.6)$$

The expansion (2.4) of H_θ and the relative bounds (2.5) will then hold true for $\theta \in B_{R'}^{\mathbb{C}}(0)$, where

$$R' = \frac{1}{3C} = \frac{R}{3 \max\{1, M\}}. \quad (2.7)$$

Proof. We begin with $2 \Rightarrow 1$. Therefore, we assume that for all k , the iterated commutators exist as H -bounded operators $\text{ad}_A^k(H)$ and that (2.3) holds.

That Condition 2.1.2 follows is obvious (take $k = 1$).

Note that Condition 2.1.1 ensures that H_θ is well-defined for real θ as an operator with domain $D(H)$.

Exploiting (2.3), we may for $\psi \in D(H)$ and $|\theta| < 1/C$ estimate

$$\sum_{k=0}^{\infty} \left\| \frac{\theta^k}{k!} \text{ad}_A^k(H) \psi \right\| \leq \frac{\|(H + i)\psi\|}{1 - C|\theta|}. \quad (2.8)$$

where $\|\cdot\|_H$ denotes the graph norm of H . Hence, the prescription

$$S_\theta\psi := \sum_{k=0}^{\infty} \frac{(-\theta)^k}{k!} i^k \operatorname{ad}_A^k(H)\psi \quad (2.9)$$

defines an analytic \mathcal{H} -valued function defined in the disc $B_{1/C}^{\mathbb{C}}(0)$. It is now easy to check that the map $\psi \mapsto S_\theta\psi$ defines – for each $\theta \in B_{1/C}^{\mathbb{C}}(0)$ a linear operator with domain $D(H)$.

The estimate (2.8) implies that

$$\forall \psi \in D(H) : \quad \|S_\theta\psi\| \leq (1 - C|\theta|)^{-1} \|\psi\|_H, \quad (2.10)$$

and in particular that S_θ is H -bounded.

To show that S_θ is closed, i.e., $D(S_\theta) = D(H)$, we establish that the two corresponding graph norms are equivalent for θ in a sufficiently small disc centered at 0. Note that the (2.10) already establishes that there exists a constant $C_1 > 0$ independent of $\theta \in B_{1/(3C)}^{\mathbb{C}}(0)$ such that

$$\|\psi\|_{S_\theta} \leq C_1 \|\psi\|_H.$$

In complete analogy to the first estimate, we estimate for $\psi \in D(H)$ and $\theta \in B_{1/C}^{\mathbb{C}}(0)$:

$$\begin{aligned} \|\psi\|_H &= \|\psi\| + \|H\psi\| \leq \|\psi\|_{S_\theta} + \sum_{k=1}^{\infty} (C|\theta|)^k \|\psi\|_H \\ &= \|\psi\|_{S_\theta} + \frac{C|\theta|}{1 - C|\theta|} \|\psi\|_H. \end{aligned} \quad (2.11)$$

Hence, for $\theta \in B_{1/(3C)}^{\mathbb{C}}(0)$ we have

$$\|\psi\|_H \leq 2\|\psi\|_{S_\theta}.$$

This proves the claimed equivalence of graph norms and thus that S_θ is closed as an operator with domain $D(H)$ for all $\theta \in B_{1/(3C)}^{\mathbb{C}}(0)$. Abbreviating $R = 1/(3C)$, this proves that $\{S_\theta\}_{\theta \in B_R^{\mathbb{C}}(0)}$ is an analytic family of Type (A). (Note that redoing the estimate (2.8) using $|\theta| \leq 1/(3C)$ yields $\|\psi\|_{S_\theta} \leq 2\|\psi\|_H$ as well.)

It remains, recalling Remark 2.2 3, to argue that $S_\theta = H_\theta$ for $\theta \in (-R, R)$. Let $\psi, \phi \in D(H)$ and put $\psi_m = e^{-A^2/(2m)}\psi$ and $\phi_m = e^{-A^2/(2m)}\phi$. Then, with the notation of Lemma 2.3, we have

$$f_m(\theta) = \langle \psi_m, H_\theta \phi_m \rangle = \langle \psi_m(\bar{\theta}), H\phi_m(\theta) \rangle$$

a priori for real θ , but extending to an entire function of θ . Here we used Lemma 2.3 3.

We may use the assumption on the existence of iterated H -bounded commutators $\text{ad}_A^k(H)$ to compute

$$\frac{d^k f_m}{d\theta^k} \Big|_{\theta=0} = \langle \psi_m, (-i)^k \text{ad}_A^k(H) \phi_m \rangle.$$

Since analytic functions in $B_R^{\mathbb{C}}(0)$ are determined by their derivatives at zero, we may conclude that

$$\langle \psi_m, H_\theta \phi_m \rangle = \langle \psi_m, S_\theta \phi_m \rangle$$

for all $\theta \in B_R^{\mathbb{C}}(0)$. Finally, we exploit Lemma 2.3 once more to compute the limit $m \rightarrow \infty$ in the above identity and conclude that for all $\theta \in (-R, R)$ and $\psi, \phi \in D(H) \cap D(A)$, we have $\langle \psi, H_\theta \phi \rangle = \langle \psi, S_\theta \phi \rangle$. By density of $D(H) \cap D(A)$ in $D(H)$, we conclude that $H_\theta = S_\theta$ for $\theta \in (-R, R)$ as desired. It now follows from (2.10) that we may choose $M = 3$ in Condition 2.1.4.

In order to prove that $1 \Rightarrow 2$, we assume that Conditions 2.1.2– 2.1.4 holds true. Let $\eta, \psi \in D(H)$. By Condition 2.1.1 and the analyticity of $\theta \mapsto H_\theta \psi$, we may use [40, Prop. 2.2] to argue that all iterated commutators of A with H exists and are implemented by H -bounded operators, provided we can establish that for every $j \in \mathbb{N}$ there exist H -bounded operators $H_0^{(j)}$, such that

$$\forall \theta \in (-R, R) : \quad \frac{d^j}{d\theta^j} \langle \eta, H_\theta \psi \rangle \Big|_{\theta=0} = \langle \eta, H_0^{(j)} \psi \rangle.$$

As a starting point we use the analyticity of $\theta \mapsto H_\theta \psi$ to obtain a power series expansion for $|\theta| < r < R$, that is

$$\langle \eta, H_\theta \psi \rangle = \sum_{k=0}^{\infty} \theta^k b_k(\eta, \psi), \quad b_k(\eta, \psi) = \frac{1}{2\pi i} \int_{\Gamma_r} \theta^{-k-1} \langle \eta, H_\theta \psi \rangle d\theta, \quad (2.12)$$

where $\eta \in \mathcal{H}$ and Γ_r is the circle in the complex plane with radius r centered at 0. Observe that the $b_k(\eta, \psi)$'s define sesquilinear forms.

Using Condition 2.1.4, we get an $M > 0$ such that

$$|b_k(\eta, \psi)| \leq \|\eta\| \|\psi\|_H \frac{M}{R^k},$$

where we also took the limit $r \rightarrow R$. For every $\psi \in D(H)$ (and $k \in \mathbb{N}$) there thus exists a vector $\tilde{\psi}$ such that $b_k(\eta, \psi) = \langle \eta, \tilde{\psi} \rangle$ for all $\eta \in D(H)$. It follows that the assignment $B_k \psi := \tilde{\psi}$ defines an H -bounded linear operator on $D(H)$. With this construction, we have

$$\frac{d^j}{d\theta^j} \langle \eta, H_\theta \psi \rangle \Big|_{\theta=0} = \langle \eta, k! B_k \psi \rangle$$

and [40, Prop. 2.2] now implies that (2.3) holds with $C := \max\{1, M\}/R$. \square

In the following we abbreviate

$$W_\theta := H_\theta - H = \sum_{k=1}^{\infty} \frac{(-\theta)^k}{k!} i^k \operatorname{ad}_A^k(H) \quad (2.13)$$

as an operator with domain $D(H)$. Observe for $\theta \in B_{R'}^{\mathbb{C}}(0)$ the estimate

$$\|W_\theta(H + i)^{-1}\| \leq \frac{C|\theta|}{1 - C|\theta|} \leq \frac{3C}{2}|\theta|, \quad (2.14)$$

We have the following, rough but sufficient, spectral localization result.

Proposition 2.6. *Assume Condition 2.1. Then*

$$\forall \theta \in B_{R'}^{\mathbb{C}}(0) : \quad \sigma(H_\theta) \subset \{x + iy \mid |y| \leq 4C|\theta|(|x| + 1)\}.$$

Proof. Let $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$ and compute on $D(H)$:

$$H_\theta - z = [1 + W_\theta(H - z)^{-1}] (H - z)$$

Hence, $H_\theta - z$ is invertible if $\|W_\theta(H - z)^{-1}\| < 1$ due to the Neumann series. The norm appearing in the previous inequality can be estimated trivially by

$$\|W_\theta(H - z)^{-1}\| \leq \|W_\theta(H + i)^{-1}\| \sup_{p \in \mathbb{R}} \frac{|p + i|}{|p - z|}.$$

Let $c > 0$. Suppose $z = x + iy$ with $|y| \geq c(|x| + 1)$. Then $|p + i|^2/|p - z|^2 \leq (p^2 + 1)/((p - x)^2 + c^2x^2 + c^2) \leq 4/c^2$ uniformly in p , x and y . Using (2.14), we have:

$$\|W_\theta(H - z)^{-1}\| \leq \frac{3C|\theta|}{c}.$$

for $z = x + iy$ with $|y| \geq c|x|$. The choice $c = 4C|\theta|$ ensures convergence of the Neumann series. \square

Lemma 2.7. *Assume Condition 2.1 and let $\theta \in B_{R'}^{\mathbb{C}}(0)$. We have*

$$D(H_\theta^*) = D(H) \quad \text{and} \quad H_\theta^* = H_{\bar{\theta}}.$$

Proof. Let $\psi, \phi \in D(H)$. We compute

$$\begin{aligned} \langle \psi, H_\theta \phi \rangle &= \sum_{k=0}^{\infty} \left\langle \psi, \frac{(-\theta)^k}{k!} i^k \operatorname{ad}_A^k(H) \phi \right\rangle \\ &= \sum_{k=0}^{\infty} \left\langle \frac{(-\bar{\theta})^k}{k!} i^k \operatorname{ad}_A^k(H) \psi, \phi \right\rangle = \langle H_{\bar{\theta}} \psi, \phi \rangle. \end{aligned}$$

Hence $H_{\bar{\theta}} \subset H_{\bar{\theta}}^*$. Conversely, let $\phi \in D(H)$, $\psi \in D(H_{\bar{\theta}}^*)$ and set

$$y = \max\{1, 8CR'\}. \quad (2.15)$$

Observe that $iy \in \rho(H_{\bar{\theta}}) \setminus \mathbb{R}$, due to Proposition 2.6. We compute, using the notation from (2.13)

$$\begin{aligned} |\langle \psi, H\phi \rangle| &\leq |\langle \psi, H_{\bar{\theta}}\phi \rangle| + |\langle \psi, W_{\bar{\theta}}\phi \rangle| \\ &= |\langle \psi, H_{\bar{\theta}}\phi \rangle| + |\langle \psi, (H_{\bar{\theta}} - iy)(H_{\bar{\theta}} - iy)^{-1}W_{\bar{\theta}}\phi \rangle| \\ &\leq \|H_{\bar{\theta}}^*\psi\| \|\phi\| + \|(H_{\bar{\theta}}^* + iy)\psi\| \|(H_{\bar{\theta}} - iy)^{-1}W_{\bar{\theta}}\phi\|. \end{aligned}$$

Note that

$$H_{\bar{\theta}} - iy = (H - iy)(1 + (H - iy)^{-1}W_{\bar{\theta}})$$

and that, recalling (2.7), (2.14) and (2.15),

$$\|(H - iy)^{-1}W_{\bar{\theta}}\| \leq \left(\sup_{x \in \mathbb{R}} \frac{x^2 + 1}{x^2 + y^2} \right)^{1/2} \|(H - i)^{-1}W_{\bar{\theta}}\| \leq \frac{1}{2}.$$

Abbreviating $B_{\bar{\theta}} = (1 + (H - iy)^{-1}W_{\bar{\theta}})^{-1}$, we may estimate

$$\|(H_{\bar{\theta}} - iy)^{-1}W_{\bar{\theta}}\phi\| \leq \|B_{\bar{\theta}}\| \|(H - iy)^{-1}W_{\bar{\theta}}\phi\| \leq C\|\phi\|.$$

Hence, there exists a $C_{\psi} > 0$ such that

$$\forall \phi \in D(H) : \quad |\langle \psi, H\phi \rangle| \leq C_{\psi}\|\phi\|,$$

and therefore we may conclude that $\psi \in D(H^*) = D(H)$, exploiting the self-adjointness of H . This shows that $D(H_{\bar{\theta}}^*) = D(H)$ and that $H_{\bar{\theta}}^* = H_{\bar{\theta}}$. \square

2.2 The Mourre Estimate

At this stage we will single out a specific energy $\lambda_0 \in \mathbb{R}$, where we shall assume that H has an eigenvalue. In order for the dilated Hamiltonian to have its essential spectrum out of the way of the eigenvalue, we shall impose a Mourre estimate locally around λ_0 . To formulate the requirement, we need the notation $E_H(B)$ for the spectral projection associated with a Borel set $B \subset \mathbb{R}$ and the self-adjoint operator H .

Condition 2.8. Let $\lambda_0 \in \mathbb{R}$. For the pair of self-adjoint operators H and A satisfying Condition 2.1, we further assume:

1. $\lambda_0 \in \sigma_{\text{pp}}(H)$.
2. There exist $e, C, \kappa > 0$ and a compact operator K , such that

$$\text{i ad}_A(H) \geq e - CE_H(\mathbb{R} \setminus [\lambda_0 - \kappa, \lambda_0 + \kappa])\langle H \rangle - K \quad (2.16)$$

in the sense of quadratic forms on $D(H)$.

3. We suppose that there exists a conjugation³ \mathcal{C} on \mathcal{H} satisfying $\mathcal{C}D(H) \subset D(H)$, $\mathcal{C}D(A) \subset D(A)$,

$$\mathcal{C}H = H\mathcal{C} \quad \text{and} \quad \mathcal{C}A = -A\mathcal{C}.$$

Notation 2.9. We write $P_0 = E_H(\{\lambda_0\})$ for the orthogonal projection onto the eigenspace of H associated with the eigenvalue λ_0 . Furthermore, we abbreviate $\overline{P}_0 = 1 - P_0$ for the projection onto the orthogonal complement of the eigenspace.

Remarks 2.10. 1. Observe that it is a consequence of Conditions 2.1 2, 2.8 2 and the Virial Theorem [20] that P_0 is a finite rank projection.

2. Choosing κ possibly smaller, one may replace the compact operator K in (2.16) with a positive multiple of the eigenprojection P_0 and obtain

$$\text{iad}_A(H) \geq e' - C'(E_H(\mathbb{R} \setminus [\lambda_0 - \kappa', \lambda_0 + \kappa'])(H) + P_0), \quad (2.17)$$

for suitably chosen constants $e' \in (0, e]$, $\kappa' \in (0, \kappa]$ and $C' \geq C$. It is in this form that we shall use the Mourre estimate, and for convenience we assume $\kappa' \leq \sqrt{3}$.

As a preparation for a Feshbach analysis, we have:

Lemma 2.11. *Assume Conditions 2.1, 2.8.1 and 2.8.2. The following three statements are true for all $\theta \in B_{R'}^{\mathbb{C}}(0)$:*

1. $\overline{P}_0 H_\theta \overline{P}_0$ is a closed operator with domain $\overline{P}_0 D(H)$.
2. $[\overline{P}_0 H_\theta \overline{P}_0]^* = \overline{P}_0 H_{\overline{\theta}} \overline{P}_0$ on $\overline{P}_0 D(H)$.
3. For all $\theta \in B_{R'}^{\mathbb{C}}(0)$: $\sigma(\overline{P}_0 H_\theta \overline{P}_0) \subset \{x + iy \mid |y| \leq 4C|\theta|(|x| + 1)\}$.

Proof. As for 1, note first that $H_\theta \overline{P}_0$ with domain $D(H)$ is closed, since $\overline{P}_0 D(H) \subset D(H)$ and H_θ with domain $D(H)$ is a closed operator (Proposition 2.4). To conclude, observe that the graph of $\overline{P}_0 H_\theta \overline{P}_0$ is the range of the open map $\mathcal{H} \oplus \mathcal{H} \ni (\psi, \varphi) \rightarrow (\overline{P}_0 \psi, \overline{P}_0 \varphi) \in \overline{P}_0 \mathcal{H} \oplus \overline{P}_0 \mathcal{H}$ applied to the graph of $H_\theta \overline{P}_0$.

We turn to the claim 2. Clearly, $\overline{P}_0 H_\theta \overline{P}_0 \subset [\overline{P}_0 H_\theta \overline{P}_0]^*$. Let $\varphi \in D([\overline{P}_0 H_\theta \overline{P}_0]^*)$ viewed as an element of $\overline{P}_0 \mathcal{H} \subset \mathcal{H}$, and compute for $\psi \in D(H)$:

$$\begin{aligned} \langle \varphi, H_\theta \psi \rangle &= \langle \overline{P}_0 \varphi, H_\theta (\overline{P}_0 + P_0) \psi \rangle \\ &= \langle \varphi, \overline{P}_0 H_\theta \overline{P}_0 \psi \rangle + \langle \overline{P}_0 \varphi, H_\theta P_0 \psi \rangle. \end{aligned}$$

Since P_0 is finite rank operator and H_θ is closed, it follows from the Closed Graph Theorem, that $H_\theta P_0$ is bounded. Hence, there exists $C > 0$ such that

$$|\langle \varphi, H_\theta \psi \rangle| \leq C \|\psi\|,$$

³see Definition A.1.

which implies that $\varphi \in D((H_\theta)^*) = D(H_{\bar{\theta}}) = D(H)$. Here we used Lemma 2.7. Since $\overline{P_0}\mathcal{H} \cap D(H) = \overline{P_0}D(H)$, we are done.

The last claim 3 may be established by repeating the proof of Proposition 2.6. \square

Remarks 2.12. We make some remarks regarding the conjugation \mathcal{C} from Condition 2.8 3:

1. Since $\mathcal{C}D(H) \subset D(H)$ and $\mathcal{C}H = H\mathcal{C}$, we readily conclude that $\mathcal{C}P_0 = P_0\mathcal{C}$ and $\mathcal{C}\overline{P_0} = \overline{P_0}\mathcal{C}$. We write \mathcal{C} also for the conjugation acting in $\overline{P_0}\mathcal{H}$.
2. As an operator identity on $D(H)$, we have

$$\mathcal{C}H_\theta = H_{\bar{\theta}}\mathcal{C}. \quad (2.18)$$

3. Finally, invoking Lemma 2.11 as well, as an operator identity on $\overline{P_0}D(H)$

$$\mathcal{C}\overline{P_0}H_\theta\overline{P_0} = \overline{P_0}H_{\bar{\theta}}\overline{P_0}\mathcal{C}. \quad (2.19)$$

In formulating the following proposition, we make use of the eigenvalue λ_0 from Condition 2.8 and the constants e' and κ' from (2.17). The radius R' was defined in (2.7).

Proposition 2.13. *Assume Conditions 2.1 and 2.8. Abbreviate for $\sigma, \rho > 0$ and $\theta \in \mathbb{C}^+ = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$:*

$$\mathcal{R}_\theta(\sigma, \rho) = \{z \in \mathbb{C} \mid \text{Re}(z) \in (\lambda_0 - \rho, \lambda_0 + \rho), \text{Im}(z) \in (-\sigma \text{Im}(\theta), \infty)\}, \quad (2.20)$$

There exist constants $R'', \rho > 0$ with $R'' \leq R'$, such that

$$\forall \theta \in B_{R''}^{\mathbb{C}}(0) \cap \mathbb{C}^+: \quad \mathcal{R}_\theta(e'/2, \rho) \cap \sigma(\overline{P_0}H_\theta\overline{P_0}) = \emptyset. \quad (2.21)$$

Proof. Using the constants from (2.17), we define a bounded operator

$$L := C'E_H(\mathbb{R} \setminus [\lambda_0 - \kappa', \lambda_0 + \kappa'])\langle H \rangle(H - \lambda_0)^{-1}. \quad (2.22)$$

Note that $\|L\| \leq 2C'\langle \lambda_0 \rangle / \kappa'$, where we used $\kappa' \leq \sqrt{3}$. We claim suitable choices

$$\begin{aligned} \rho &= \min\left\{1, \frac{e'}{8C'\langle \lambda_0 \rangle / \kappa'}\right\}, \\ R'' &= \min\left\{R', \frac{e'}{12C'(6|\lambda_0| + 14)(2C'\langle \lambda_0 \rangle / \kappa' + 2C)}\right\}, \end{aligned} \quad (2.23)$$

where C and R' were defined in (2.6) and (2.7), respectively. Recall that $R''C \leq R'C \leq 1/3$.

Let $\theta \in \mathbb{C}^+ \cap B_{R''}^{\mathbb{C}}(0)$ and $\mu \in \mathcal{R}_\theta(e'/2, \rho) \cap \sigma(\overline{P_0}H_\theta\overline{P_0})$. Note that due to Lemma 2.11 3, we may estimate

$$|\mu| \leq (|\lambda_0| + \rho + 1)(1 + 16R''^2C^2)^{1/2} \leq 2|\lambda_0| + 4. \quad (2.24)$$

By Remark 2.12 3 and Lemma 2.11 1 and 2, we may apply Corollary A.3 to the operator $\overline{P}_0(H_\theta - \mu)\overline{P}_0$. Thus there exists a sequence $\psi_n \in \overline{P}_0D(H)$ with $\|\psi_n\| = 1$, such that

$$o_n := \|\overline{P}_0(H_\theta - \mu)\overline{P}_0\psi_n\| \rightarrow 0 \quad \text{for } n \rightarrow \infty. \quad (2.25)$$

We estimate for all n using (2.5)

$$\begin{aligned} \|\overline{P}_0\psi_n\|_H &\leq 2\|\overline{P}_0\psi_n\|_{H_\theta} \\ &\leq 2(\|\overline{P}_0H_\theta\overline{P}_0\psi_n\| + \|P_0H_\theta\overline{P}_0\psi_n\|) \\ &\leq 2(o_n + |\mu| + 2(|\mu| + 1)) \\ &= 2o_n + 6|\mu| + 4. \end{aligned} \quad (2.26)$$

Exploiting the power series expansion (2.4) of H_θ , the Mourre estimate (2.17) and simplifying for real expectation values, we obtain for any n

$$\begin{aligned} \text{Im}(\mu) &= \text{Im}\langle \overline{P}_0\psi_n, (\mu - H_\theta)\overline{P}_0\psi_n \rangle + \text{Im}\langle \overline{P}_0\psi_n, H_\theta\overline{P}_0\psi_n \rangle \\ &= \text{Im}\langle \overline{P}_0\psi_n, (\mu - H_\theta)\overline{P}_0\psi_n \rangle - \text{Im}\langle \overline{P}_0\psi_n, \theta i \text{ad}_A(H)\overline{P}_0\psi_n \rangle \\ &\quad - \text{Im}\langle \overline{P}_0\psi_n, \sum_{k=2}^{\infty} \frac{(-\theta)^k}{k!} i^k \text{ad}_A^k(H)\overline{P}_0\psi_n \rangle \\ &= \text{Im}\langle \overline{P}_0\psi_n, (\mu - H_\theta)\overline{P}_0\psi_n \rangle - \text{Im}(\theta)\langle \overline{P}_0\psi_n, i \text{ad}_A(H)\overline{P}_0\psi_n \rangle \\ &\quad - \sum_{k=2}^{\infty} \frac{\text{Im}((-\theta)^k)}{k!} \langle \overline{P}_0\psi_n, i^k \text{ad}_A^k(H)\overline{P}_0\psi_n \rangle \\ &\leq o_n - \text{Im}(\theta) [e' - C'\langle \overline{P}_0\psi_n, E(|H - \lambda| \geq \kappa')\langle H \rangle \overline{P}_0\psi_n \rangle] \\ &\quad - \sum_{k=2}^{\infty} \frac{\text{Im}((-\theta)^k)}{k!} \langle \overline{P}_0\psi_n, i^k \text{ad}_A^k(H)\overline{P}_0\psi_n \rangle. \end{aligned} \quad (2.27)$$

Note that for all k , we have $|\text{Im}((-\theta)^k)| \leq 2^k |\text{Im}(\theta)| |\theta|^{k-1}$. Therefore,

$$\begin{aligned} \left| \sum_{k=2}^{\infty} \frac{\text{Im}((-\theta)^k)}{k!} \langle \overline{P}_0\psi_n, i^k \text{ad}_A^k(H)\overline{P}_0\psi_n \rangle \right| &\leq \sum_{k=2}^{\infty} |\text{Im}((-\theta)^k)| C^k \|\overline{P}_0\psi_n\|_H \\ &\leq C |\text{Im}(\theta)| \sum_{k=2}^{\infty} 2^{k-1} |\theta|^{k-1} C^{k-1} \|\overline{P}_0\psi_n\|_H \\ &= C |\text{Im}(\theta)| \frac{2C|\theta|}{1 - 2C|\theta|} \|\overline{P}_0\psi_n\|_H \\ &\leq 6 |\text{Im}(\theta)| R'' C^2 (o_n + 3|\mu| + 2), \end{aligned} \quad (2.28)$$

where we used (2.26) and that $C|\theta| \leq 1/3$ in the last step. We estimate using (2.14), recalling the definition (2.22) of the bounded operator L ,

$$\begin{aligned}
& C' \left| \langle \bar{P}_0 \psi_n, E_H(\mathbb{R} \setminus [\lambda_0 - \kappa', \lambda_0 + \kappa']) \langle H \rangle \bar{P}_0 \psi_n \rangle \right| \\
& \leq \|L \bar{P}_0 (H - \lambda_0) \bar{P}_0 \psi_n\| \\
& \leq \|L\| \| \bar{P}_0 (H_\theta - \mu) \bar{P}_0 \psi_n \| + \|L\| \|W_\theta \bar{P}_0 \psi_n\| + |\lambda_0 - \mu| \|L\| \\
& \leq \|L\| o_n + \|L\| \|W_\theta (H + i)^{-1}\| \| \bar{P}_0 \psi_n \|_H + |\lambda_0 - \mu| \|L\| \\
& \leq \|L\| o_n + \frac{3}{2} C R'' \|L\| \| \bar{P}_0 \psi_n \|_H + \kappa'' \|L\| \\
& \leq \|L\| (1 + 3C R'') o_n + 3(3|\mu| + 2) C R'' \|L\| + \rho \|L\|, \tag{2.29}
\end{aligned}$$

where we used (2.26) in the final step.

Combining (2.27), (2.28) and (2.29) we obtain

$$\begin{aligned}
\operatorname{Im}(\mu) & \leq -\operatorname{Im}(\theta) (e' - 3R'' C(3|\mu| + 2)(\|L\| + 2C) - \rho \|L\|) \\
& \quad + (1 + 6|\operatorname{Im}(\theta)| (R'' C^2 + \|L\|(1 + 3C R''))) o_n.
\end{aligned}$$

By the choices of ρ and R'' from (2.23), (2.24) and $\|L\| \leq 2C' \langle \lambda_0 \rangle / \kappa'$ to estimate

$$3R'' C(3|\mu| + 2)(\|L\| + 2C) + \rho \|L\| \leq \frac{e'}{2}$$

and thus, taking the limit $n \rightarrow \infty$ using (2.25), we arrive at

$$\operatorname{Im}(\mu) \leq -\operatorname{Im}(\theta) \frac{e'}{2}.$$

This completes the proof. \square

The following theorem is proven using the Feshbach reduction method, for which Proposition 2.13 above is an essential prerequisite.

Theorem 2.14. *Assume Conditions 2.1 and 2.8. Then*

$$\forall \theta \in B_{R''}^{\mathbb{C}}(0) \cap \mathbb{C}^+ : \quad \sigma_{\text{ess}}(H_\theta) \cap \mathcal{R}_\theta(e'/2, \rho) = \emptyset.$$

The constants ρ, R'' and the sets \mathcal{R}_θ come from Proposition 2.13.

Proof. By Proposition 2.13 there exist $R'', \rho > 0$ such that for all $|\theta| < R''$ the closed operator $\bar{P}_0 H_\theta \bar{P}_0 - z \bar{P}_0$ is invertible on $\bar{P}_0 \mathcal{H}$ for all $z \in \mathcal{R} := \mathcal{R}_\theta(e'/2, \rho)$. Define reduced resolvents

$$\bar{R}_\theta(z) := (\bar{P}_0 H_\theta \bar{P}_0 - z \bar{P}_0)^{-1}$$

on $\overline{P_0}\mathcal{H}$ for $z \in \mathcal{R}$. Recall that W_θ is defined in (2.13). For $z \in \mathcal{R}$ we can construct the Feshbach map on the finite dimensional subspace $P_0\mathcal{H}$:

$$\begin{aligned} F_{P_0}(z) &= P_0(H_\theta - z)P_0 - P_0H_\theta\overline{P_0}\overline{R_\theta}(z)\overline{P_0}H_\theta P_0 \\ &= P_0(W_\theta + \lambda_0 - z)P_0 - P_0W_\theta\overline{P_0}\overline{R_\theta}(z)\overline{P_0}W_\theta P_0. \end{aligned}$$

Clearly, $F_{P_0}(z)$ is a finite rank operator, which can be interpreted as a matrix, and hence; by isospectrality of the Feshbach reduction [11, 25],

$$\mu \in \sigma(H_\theta) \cap \mathcal{R} \Leftrightarrow \det(F_{P_0}(\mu)) = 0.$$

By the Unique Continuation Theorem for holomorphic functions, the set $\sigma(H_\theta) \cap \mathcal{R}$ are locally finite. Note that $\mu \in \sigma(H_\theta) \cap \mathcal{R}$ is necessarily an eigenvalue for H_θ . In order to establish the theorem, it remains to prove that the Riesz projections pertaining to the eigenvalues in \mathcal{R} are of finite rank. Let $\mu \in \mathcal{R} \cap \sigma(H_\theta)$ and choose $r > 0$, such that $D \subset \mathcal{R} \setminus \sigma(H_\theta)$, where $D = \{z \in \mathbb{C} \mid 0 < |z - \mu| \leq r\}$ denotes a closed punctured disc.

The inverse of $F_{P_0}(z)$ for $z \in D$ has a Laurent expansion

$$F_{P_0}(z)^{-1} = \sum_{k=1}^N B_{-k}(z - \mu)^{-k} + \sum_{k=0}^{\infty} B_k(z - \mu)^k$$

convergent in the punctured disc D . Here $N \geq 1$ and $\{B_k\}_{k=-N}^{\infty}$ denote linear operators on $P_0\mathcal{H}$. See [43, Sect. 6.1]. Note that the inverse has no essential singularities since we are in finite dimension.

By [11, 25], for $z \in \mathcal{R} \setminus \sigma(H_\theta)$, the inverse $R_\theta(z)$ of $H_\theta - z$ can be recovered from the inverse Feshbach operator and the reduced resolvent via the block decomposition

$$\begin{aligned} P_0R_\theta(z)P_0 &= F_{P_0}(z)^{-1}, \\ P_0R_\theta(z)\overline{P_0} &= -F_{P_0}(z)^{-1}P_0W_\theta\overline{P_0}\overline{R_\theta}(z), \\ \overline{P_0}R_\theta(z)P_0 &= -\overline{R_\theta}(z)\overline{P_0}W_\theta P_0F_{P_0}(z)^{-1}, \\ \overline{P_0}R_\theta(z)\overline{P_0} &= \overline{R_\theta}(z) + \overline{R_\theta}(z)\overline{P_0}W_\theta P_0F_{P_0}(z)^{-1}P_0W_\theta\overline{P_0}\overline{R_\theta}(z). \end{aligned}$$

Note that the map $z \mapsto \overline{R_\theta}(z)$ is analytic in \mathcal{R} , so the only singularities are those in $\sigma(H_\theta)$, coming from the inverse Feshbach operator.

Let $\gamma: [0, 2\pi] \rightarrow \mathbb{C}$ be the closed curve $\gamma(t) = \mu + re^{it}$ parametrizing the (outer) boundary of D , encircling μ . Recall the construction of the Riesz projections

$$P_\theta(\mu) = \frac{1}{2\pi i} \int_\gamma R_\theta(z) dz$$

associated with the eigenvalue μ . The block decomposition of $R_\theta(z)$, induces a block decomposition of $P_\theta(\mu)$ and the Riesz projection have finite rank, provided $\overline{P}_0 P_\theta(\mu) \overline{P}_0$ are of finite rank. To check this, we compute

$$\begin{aligned} & \overline{P}_0 P_\theta(\mu) \overline{P}_0 \\ &= \frac{1}{2\pi i} \int_\gamma [\overline{R}_\theta(z) + \overline{R}_\theta(z) \overline{P}_0 W_\theta P_0 F_{P_0}(z)^{-1} P_0 W_\theta \overline{P}_0 \overline{R}_\theta(z)] dz \\ &= \sum_{k=1}^N \frac{1}{2\pi i} \int_\gamma (z - \mu)^{-k} \overline{R}_\theta(z) \overline{P}_0 W_\theta P_0 B_{-k} P_0 W_\theta \overline{P}_0 \overline{R}_\theta(z) dz \\ & \quad + \frac{1}{2\pi i} \int_\gamma \sum_{k=0}^{\infty} (z - \mu)^k \overline{R}_\theta(z) \overline{P}_0 W_\theta P_0 B_k P_0 W_\theta \overline{P}_0 \overline{R}_\theta(z) dz, \end{aligned}$$

where we have used that the function $\mathcal{R} \ni z \mapsto \overline{R}_\theta(z)$ is analytic. Moreover, the integral in the last line of the equation above is carried out over an analytic function, once again, and thus equals 0. The remaining N singular integrals can be evaluated by Cauchy's Integral Formula:

$$\begin{aligned} & \frac{1}{2\pi i} \int_\gamma (z - \mu)^{-k} \overline{R}_\theta(z) \overline{P}_0 W_\theta P_0 B_{-k} P_0 W_\theta \overline{P}_0 \overline{R}_\theta(z) dz \\ &= \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \overline{R}_\theta(z) \overline{P}_0 W_\theta P_0 B_{-k} P_0 W_\theta \overline{P}_0 \overline{R}_\theta(z) \Big|_{z=\mu} \\ &= \frac{1}{(k-1)!} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^{k-1-j} j! (k-1-j)! \\ & \quad \times \overline{R}_\theta(\mu)^{1+j} \overline{P}_0 W_\theta P_0 B_{-k} P_0 W_\theta \overline{P}_0 \overline{R}_\theta(\mu)^{k-j} \\ &= (-1)^{k-1} \sum_{j=0}^{k-1} \overline{R}_\theta(\mu)^{j+1} \overline{P}_0 W_\theta P_0 B_{-k} P_0 W_\theta \overline{P}_0 \overline{R}_\theta(\mu)^{k-j}. \end{aligned}$$

Since each term in the sum above is a finite rank operator, we conclude that $\overline{P}_0 P_\theta(\mu) \overline{P}_0$ is of finite rank.

Since $\sigma(H) \cap \mathcal{R}$ is locally finite and all the associated Riesz projections have finite rank, we have shown that $\sigma_{\text{ess}}(H_\theta) \cap \mathcal{R} = \emptyset$. This completes the proof. \square

Note that

$$D(U(\theta)) = \left\{ \psi \in \mathcal{H} \mid \int_{\mathbb{R}} e^{2\text{Im}(\theta)x} dE_\psi(x) < \infty \right\},$$

where E_ψ is the spectral measure for A associated with the state ψ . Motivated by this we abbreviate for $r \geq 0$:

$$D_r(A) = \left\{ \psi \in \mathcal{H} \mid \int_{\mathbb{R}} e^{2r|x|} dE_\psi(x) < \infty \right\}.$$

Having established Theorem 2.14, we may conclude the following theorem by invoking a general result of Hunziker and Sigal [31, Theorem 5.2].

Theorem 2.15. *Assume Conditions 2.1 and 2.8. Let $\theta \in B_{R''}^{\mathbb{C}}(0)$ with $\theta \neq 0$. Then the dilated Hamiltonian H_θ have an isolated eigenvalue at λ_0 . Denote by P_θ the associated Riesz projection. The following statements hold true:*

1. $\text{Range}(P_\theta)$ is the eigenspace of H_θ pertaining to the eigenvalue λ_0 .
2. $P_0 = U(-\theta)P_\theta U(\theta)$ as a form identity $D_{|\text{Im}(\theta)|}(A)$.
3. $\text{Rank}(P_0) = \text{Rank}(P_\theta)$.
4. Let $r < R''$. Then $\text{Range}(P_0) \subset D_r(A)$.

Remark 2.16. The above theorem implies that eigenfunctions pertaining to the eigenvalue λ_0 are analytic vectors for the operator A . A result previously established by brute force in [40]. Here, however, we needed Condition 2.8.3, which played no role for the method employed in [40].

2.3 Analytic Perturbation Theory

Condition 2.17. Let $\xi_0 \in \mathbb{R}^d$ and $U \subset \mathbb{R}^d$ an open (connected) neighborhood of ξ_0 , A a self-adjoint operator on \mathcal{H} and $\{H(\xi)\}_{\xi \in U}$ a family of self-adjoint operators on \mathcal{H} .

1. $D(H(\xi)) = D(H(\xi_0)) =: \mathcal{D}$ for all $\xi \in U$.
2. For all ξ in U , the operator $H(\xi)$ satisfies Condition 2.1 with the same constants R and M .
3. The pair A and $H(\xi_0)$ satisfies Condition 2.8.
4. There exists $\theta_0 \in B_R^{\mathbb{C}}(0)$ with $\text{Im}(\theta_0) \neq 0$, such that the map $\xi \rightarrow H_{\theta_0}(\xi)$ extends from U to an analytic family of Type (A) defined for $\xi \in U_{\mathbb{C}} \subset \mathbb{C}^d$, an open (connected) set with $U \subset U_{\mathbb{C}} \cap \mathbb{R}$.

Remark 2.18. Suppose one strengthens Condition 2.17 and assumes that $\xi \rightarrow H_\theta(\xi)$ extends to an analytic family of Type (A) not just for one θ_0 but for all θ in a complex disc of radius $\Theta < R'$ around 0. Then one may use Morera's theorem to conclude that for any $\psi \in \mathcal{D}$ and n , we have

$$i^n \text{ad}_A^n(H(\xi))\psi = \frac{(-1)^n}{2\pi i} \int_{|\theta|=\Theta/2} n! \theta^{-n-1} H_\theta(\xi) \psi \, d\theta,$$

a priori for real ξ , but since the right-hand side extends analytically to ξ in a complex neighborhood of ξ_0 , so does the left-hand side. This will in particular permit one to conclude that also for complex ξ does the closed operator $H(\xi)$ iteratively admit

commutators with A of arbitrary order. (Note that $H(\bar{\xi}) \subset H(\xi)^*$, by unique continuation.) Furthermore, the iterated commutators must coincide (strongly) with the analytic extension from real ξ of $\text{ad}_A^n(H(\xi))\psi$, obtained above.

Recall the notation λ_0 for the eigenvalue of $H(\xi_0)$ with eigenprojection P_0 . By Theorem 2.15, we know that λ_0 is an isolated eigenvalue of $H_{\theta_0}(\xi_0)$ with finite rank eigenprojection P_{θ_0} . Denote by n_0 the common rank of P_0 and P_{θ_0} .

Fix $0 < \rho' < \rho$ and $0 < e'' < e'/2$, such that $\sigma(H_{\theta_0}(\xi_0)) \cap B_{2\rho'}^{\mathbb{C}}(\lambda_0) = \{\lambda_0\}$.

Remark 2.19. We may choose $r' > 0$, such that for all $\xi \in B_{r'}^{\mathbb{C}^d}(\xi_0)$, we have

$$\sigma(H_{\theta_0}(\xi)) \cap B_{\rho'}^{\mathbb{C}}(\lambda_0) = \sigma_{\text{pp}}(H_{\theta_0}(\xi)) \cap B_{\rho'}^{\mathbb{C}}(\lambda_0)$$

and the total multiplicity of the eigenvalues in $B_{\rho'}^{\mathbb{C}}(\lambda_0)$ equals n_0 ([33, Sect. IV.4]).

By [31, Theorem 5.2], we may now conclude – just as we did with Theorem 2.15 – that

$$\sigma_{\text{pp}}(H(\xi)) \cap (\lambda_0 - \rho', \lambda_0 + \rho') = \sigma(H_{\theta_0}(\xi)) \cap (\lambda_0 - \rho', \lambda_0 + \rho'). \quad (2.30)$$

If the perturbation parameter ξ is one-dimensional, we may in light of Theorem 2.14 and Condition 2.17, invoke Kato, in the form of [33, Theorem VII.1.8], and conclude the following theorem.

Theorem 2.20. *Suppose Condition 2.17 and that $d = 1$. There exist*

- $r > 0$ with $(\xi_0 - r, \xi_0 + r) \subset U$,
- natural numbers $0 \leq m_{\pm} \leq n_0$ and $n_1^{\pm}, \dots, n_{m_{\pm}}^{\pm} \geq 1$ with $n_1^{\pm} + \dots + n_{m_{\pm}}^{\pm} \leq n_0$,
- real analytic functions $\lambda_1^{\pm}, \dots, \lambda_{m_{\pm}}^{\pm} : I_{\pm} \rightarrow \mathbb{R}$, where $I_- = (\xi_0 - r, \xi_0)$ and $I_+ = (\xi_0, \xi_0 + r)$,

such that

1. for any $\xi \in I_{\pm}$, we have $\sigma_{\text{pp}}(H(\xi)) \cap (\lambda_0 - \rho', \lambda_0 + \rho') = \{\lambda_1^{\pm}(\xi), \dots, \lambda_{m_{\pm}}^{\pm}(\xi)\}$,
2. The eigenvalue branches $I_{\pm} \ni \xi \rightarrow \lambda_j^{\pm}(\xi)$ have constant multiplicity n_j^{\pm} .

In the case of multiple parameters, the structure of the point spectrum becomes more complicated, and we need the notion of semi-analytic sets, which we recall from [35] in the following definition.

Definition 2.21. 1. Let $W \subset \mathbb{R}^{\nu}$ be an open set. We write $\mathcal{O}(W)$ for the smallest ring⁴ of subsets of W containing sets of the form $\{y \in W \mid f(y) > 0\}$ and $\{y \in W \mid f(y) = 0\}$ where f ranges over real analytic functions $f : W \rightarrow \mathbb{R}$.

⁴collection of sets stable under complement as well as under finite intersections and unions.

2. Let $M \subset \mathbb{R}^\nu$ be an open set. A subset $\Sigma \subset M$ is called a semi-analytic subset of M if: for any $x \in M$, there exists an open neighborhood $W \subset M$ of x , such that $\Sigma \cap W \in \mathcal{O}(W)$.

We may now formulate our main theorem:

Theorem 2.22. *Suppose Condition 2.17. There exists $r > 0$ and $\rho > 0$, such that with $M = B_r^{\mathbb{R}^d}(\xi_0) \times (\lambda_0 - \rho', \lambda_0 + \rho')$, we have that $\Sigma = \{(\xi, \lambda) \in M \mid \lambda \in \sigma_{\text{pp}}(H(\xi))\}$ is a semi-analytic subset of M .*

Proof. Let r' be chosen as in Remark 2.19. The projection onto the total eigenspace is the Riesz projection,

$$Q(\xi) = \frac{1}{2\pi i} \int_{|z-\lambda_0|=\rho'} (H_{\theta_0}(\xi) - z)^{-1} dz,$$

which depends analytically on $\xi \in B_{r'}^{\mathbb{C}}(\xi_0)$. Write $V(\xi) = \text{Range}(Q(\xi))$ for the total eigenspace of dimension n_0 and $\Pi: \mathbb{C}^{n_0} \rightarrow \Pi(\xi_0)$ a linear isomorphism identifying the unperturbed eigenspace with \mathbb{C}^{n_0} . Following [23], we choose r' such that $\|Q(\xi) - Q(\xi_0)\| \leq 1/2$ for $|\xi - \xi_0| \leq r'$. Then $\Theta(\xi) := Q(\xi)|_{V(\xi_0)}$ defines a linear isomorphism from $V(\xi_0)$ onto $V(\xi)$ and

$$\forall \xi \in B_r^{\mathbb{C}^d}(\xi_0) : T(\xi) = \Pi^* \Theta(\xi)^* H_{\theta_0}(\xi) \Theta(\xi) \Pi$$

defines a family of linear operators on \mathbb{C}^d depending analytically on ξ and satisfying that $\sigma(T(\xi)) = \sigma(H_{\theta_0}(\xi)) \cap B_{\rho'}^{\mathbb{C}}(\lambda_0)$. Hence, recalling (2.30),

$$\Sigma = \{(\xi, \lambda) \in M \mid \det(T(\xi) - \lambda) = 0\}.$$

Split into real and imaginary parts $\det(T(\xi) - \lambda) = u(\xi, \lambda) + iv(\xi, \lambda)$, to obtain two real analytic real-valued functions. Define semi-analytic sets for $i = 1, \dots, n_0$ by setting

$$\Sigma_i^{\text{R}} = \{(\xi, \lambda) \in M \mid \forall j = 0, \dots, i-1 : \partial_\lambda^j u(\lambda, \xi) = \partial_\lambda^j v(\lambda, \xi) = 0 \text{ and } \partial_\lambda^i u(\xi, \lambda) \neq 0\}$$

and

$$\Sigma_i^{\text{I}} = \{(\xi, \lambda) \in M \mid \forall j = 0, \dots, i-1 : \partial_\lambda^j u(\lambda, \xi) = \partial_\lambda^j v(\lambda, \xi) = 0 \text{ and } \partial_\lambda^i v(\xi, \lambda) \neq 0\}$$

Then

$$\Sigma = \bigcup_{i=1}^{n_0} (\Sigma_i^{\text{R}} \cup \Sigma_i^{\text{I}})$$

is a semi-analytic subset of M . □

3 Example

We introduce the two-particle Hamiltonian on $L^2(\mathbb{R}^d)$ by

$$H'_V = \omega_1(p_1) + \omega_2(p_2) + V(x_1 - x_2),$$

where $p_i = -i\nabla_{x_i}$, $x_i \in \mathbb{R}^d$ and the operator V is multiplication by $V \in L^1(\mathbb{R}^d)$. The functions ω_i should be thought of as effective dispersion relations. By taking the Fourier transform of this operator we see that it is unitarily equivalent to

$$H_V = \omega_1(k_1) + \omega_2(k_2) + t_V,$$

where

$$(t_V f)(k_1, k_2) := \int \hat{V}(u) f(k_1 - u, k_2 + u) du.$$

Note that, since $L^2(\mathbb{R}^d \times \mathbb{R}^d) = \int_{\mathbb{R}^d}^{\oplus} L^2(\mathbb{R}^d)$, this operator can be fibered w.r.t. $\xi \in \mathbb{R}^d$ by the unitary transform $f(\xi, k) \mapsto f_\xi(k) = f(-\xi - k, k)$, where all fibers are equipped with the Hilbert space $\mathcal{H}_\xi = L^2(\mathbb{R}^d)$. The fiber operators take the form

$$H(\xi) = \omega_\xi(k) + T_V,$$

where

$$\begin{aligned} (\omega_\xi f)(k) &= (\omega_1(\xi - k) + \omega_2(k))f(k), \\ (T_V f)(k) &= (\bar{V} * f)(k) \end{aligned}$$

and $\bar{V} = \hat{V}(-k)$. Furthermore, we define a self-adjoint operator for every fiber ξ by

$$A_\xi = \frac{i}{2} (v_\xi \cdot \nabla_k + \nabla_k \cdot v_\xi), \quad (3.31)$$

where the vector field v_ξ is given by

$$v_\xi(k) = (\nabla_k \omega_\xi)(k) e^{-k^2}. \quad (3.32)$$

Condition 3.1 (Properties of ω_1, ω_2 and V).

1. Let $j \in \{1, 2\}$. ω_j is real analytic on \mathbb{R}^d . Moreover, there exists $p_j \in \mathbb{N}$ and $C, B, B' > 0$ such that

$$|\partial^\alpha \omega_j(k)| \leq C \langle k \rangle^{p_j}, \quad |\omega_j(k)| \geq B \langle k \rangle^{p_j} - B'. \quad (3.33)$$

for every multi-index $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq 1$.

2. Assume further that there exists an $R > 0$ such that ω_1 and ω_2 extend to analytic functions on the d -dimensional strip $\mathcal{S}_R := \{(z_1, \dots, z_d) \in \mathbb{C}^d \mid |\operatorname{Im}(z_i)| \leq R, i = 1, \dots, d\}$. We denote the analytic continuations of these functions by the same symbols.
3. The analytic continuations of the ω_j still satisfy the bounds (3.33) on \mathcal{S}_R .
4. For all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq 1$ we have that $\widehat{x^\alpha V} \in L^2(\mathbb{R}^d)$.
5. There exists $a \geq MR$ and $n \in \mathbb{N}, n \geq d/2$ such that $e^{b|\cdot|} \partial^\alpha V(\cdot) \in L^1(\mathbb{R}^d)$ for all $b < a$ and all $\alpha \in \mathbb{N}_0^d, |\alpha| \leq n$.

Remark 3.2.

1. By Condition 3.1.1 $B\langle k \rangle^{p_j} - B' \leq |\omega_j(k)| \leq C\langle k \rangle^{p_j}$ and thus $D(M_{\omega_j}) = D(M_{\langle \cdot \rangle^{p_j}})$. Consequently, $D(M_{\omega_\xi}) = D(M_{\langle \cdot \rangle^p}) =: \mathcal{D}$, where $p = \max\{p_1, p_2\}$. Thus all operators $H(\xi)$ have the common domain \mathcal{D} .
2. The two estimates in Condition 3.1.1 are clearly satisfied by the real analytic functions $f(k) = (|k|^2)^p$. These have an analytic continuation into \mathcal{S}_R for any $R > 0$ and the corresponding estimates are still satisfied by the analytic continuations.
3. Condition 3.1.2 implies that ω_ξ as well as every component of $\nabla_k \omega_\xi$ extend to analytic functions in the strip \mathcal{S}_R . Since $|e^{-(x+iy)^2}| \leq e^{-x^2} e^{+y^2}$, the same is true for the vector field v_ξ .
4. Condition 3.1.1 and Condition 3.1.3 imply

$$\sup_{z \in \mathcal{S}_R} |v_\xi(z)| \leq M < \infty. \quad (3.34)$$

and

$$\sup_{z \in \mathcal{S}_R} |\nabla_z \omega_\xi(z) \cdot v_\xi(z)| \leq M' < \infty. \quad (3.35)$$

The next Lemma connects the action of the unitary group generated by A_ξ to objects related to the solution of the ODE defined by the vector field v_ξ . This result has already been proven in the PhD thesis of one of the authors, see [44].

Lemma 3.3. *Let γ_t be the solution of the ODE*

$$\frac{d}{dt} \gamma_t(k) = v_\xi(\gamma_t(k)), \quad \gamma_0(k) = k. \quad (3.36)$$

and define

$$J(t, k) = e^{\int_0^t \nabla \cdot v_\xi(\gamma_s(k)) ds}. \quad (3.37)$$

Then

$$(e^{itA_\xi} f)(k) = \sqrt{J(t, k)} f(\gamma_t(k)). \quad (3.38)$$

It is easily checked that the useful relation

$$J(t, \gamma_{-t}(k)) = \frac{1}{J(-t, k)} \quad (3.39)$$

holds.

Lemma 3.4. *Assume Conditions 3.1.2 and 3.1.3. Then the solution of (3.36), γ_t , admits an analytic continuation into the d -dimensional strip \mathcal{S}_r , where $M'r < R$ and $M' = \max\{M, 1\}$. Moreover, this continuation maps \mathcal{S}_r into $\mathcal{S}_{M'r} \subset \mathcal{S}_R$.*

PROOF. By definition γ_t solves the ODE

$$\frac{d}{dt} \gamma_t(k) = v_\xi(\gamma_t(k)), \quad \gamma_0(k) = k.$$

Since v is real-analytic, so is $\gamma_t(k)$ and hence it admits an analytic continuation into some region $G \subset \mathbb{C}$. We do not introduce a new symbol for this continuation. Without loss of generality we may assume that $G = B_r^{\mathbb{C}}(0)$ for some $r > 0$. By possibly decreasing $r > 0$, we can assume that $M'r < R$, where $M' = \max\{1, M\}$ with R, M are as in Condition 3.1.2 and Condition 3.1.3.

For $z \in B_r^{\mathbb{C}}(t)$, where $t \in \mathbb{R}^d$, we clearly have that $z - t \in B_r^{\mathbb{C}}(0)$ and hence

$$\tilde{\gamma}_z(k) := \gamma_{z-t}(\gamma_t(k))$$

is analytic on $B_r^{\mathbb{C}}(t)$. Choose t such that $V := B_r^{\mathbb{C}}(t) \cap B_r^{\mathbb{C}}(0) \neq \emptyset$. Let $z \in V \cap \mathbb{R}$ and note that $\tilde{\gamma}_z(k) = \gamma_z(k)$. Hence analyticity of γ_z and $\tilde{\gamma}_z$ implies that $\tilde{\gamma}_z = \gamma_z$ for all $z \in V$ due to the identity theorem. This in turn implies that $\tilde{\gamma}_z$ is an analytic continuation of γ_z to $B_r^{\mathbb{C}}(t)$. It is clear that this way of extending γ_z can be used to obtain an analytic extension to the strip \mathcal{S}_r .

Let $I \subset \mathbb{R}$ be a compact interval and pick an open subinterval $J \subset I$. By continuity of $z \mapsto \gamma_z(k)$ we have that for every $x \in I$ there exists $\delta_x \in (0, r)$ such that $\gamma_{x+it}(k) \in B_r^{\mathbb{C}^d}(\gamma_x(k))$ for $|t| < \delta_x$. By compactness we deduce that there exists $\delta > 0$ such that for all $x \in I$ $\gamma_{x+it}(k) \in B_\delta^{\mathbb{C}^d}(\gamma_x(k)) \subset \mathcal{S}_R$ provided that $|t| < \delta$. Hence the composition $v_\xi(\gamma_z(k))$ extends analytically from J to the complex rectangle $R_\delta := \{z \mid \operatorname{Re}(z) \in J, |\operatorname{Im}(z)| < \delta\} \subset \mathcal{S}_R$. Since for $z \in J$ the ODE in (3.36) is satisfied, it carries over to $z \in R_\delta$ by uniqueness of analytic continuation.

For $j = 1, \dots, d$ we denote by γ_t^j and v_j the j -th component of γ_t and v_ξ^j respectively. Let $x + it \in R_\delta$. The conclusion of the preceding paragraph now allows us to calculate

$$\frac{d}{dt} \operatorname{Im}[\gamma_{x+it}^j(k)] = \operatorname{Im}[i v_\xi^j(\gamma_{x+it}(k))] = \operatorname{Re}[v_\xi^j(\gamma_{x+it}(k))] \leq M,$$

where we have used the uniform bound on v_ξ in Condition 3.1.3. This inequality implies

$$|\operatorname{Im}[\gamma_{x+it}^j(k)]| = |\operatorname{Im}[\gamma_{x+it}^j(k)] - \operatorname{Im}[\gamma_x^j(k)]| \leq M|t| \leq Mr,$$

since γ_z maps into \mathbb{R}^d for real values of z . Hence $\gamma_{x+it}(k) \in \mathcal{S}_{Mr}$ for all $x \in J$. Covering \mathbb{R} with compact intervals I_n we obtain that $\gamma_{x+it}(k) \in \mathcal{S}_{Mr}$ for any $x \in \mathbb{R}$ and $|t| < r$. Hence the analytic continuation of γ_t maps \mathcal{S}_r into \mathcal{S}_{Mr} . \square

Remark 3.5.

1. This result implies that the ODE in (3.36) extends to $z \in \mathcal{S}_r$. Indeed, we simply copy the argument given for the extension to R_δ in the previous proof.
2. The above remark allows us to considering $\frac{d}{da}(\gamma_{ae^{it}}^j(k) - k_j)$ for $a < r$ instead of $\frac{d}{dt}\operatorname{Im}[\gamma_{x+it}^j(k)y]$ and we can prove $|\gamma_{ae^{it}} - k| \leq Ca$. Hence,

$$|\gamma_z(k) - k| \leq C|z| \tag{3.40}$$

in complete analogy to the preceding proof.

The previous two Lemmas allow us to explicitly compute how conjugation by the unitary group generated by A effects the fiber Hamiltonians and argue that the so obtained expressions admit analytic continuations.

Lemma 3.6. *Assume Condition 3.1. Then the map*

$$t \mapsto e^{-itA_\xi} T_V e^{itA_\xi} := T_V^t$$

extends to an analytic $B(\mathcal{H})$ -valued function on \mathcal{S}_r .

PROOF. Note that

$$(e^{-iA_\xi t} T_V e^{iA_\xi t} g)(k) = \sqrt{J(-t, k)} \int_{\mathbb{R}^d} \hat{V}(\gamma_{-t}(k') - \gamma_{-t}(k)) \sqrt{J(-t, k')} g(k') dk'.$$

By the assumptions $\hat{V}(\gamma_{-t}(k') - \gamma_{-t}(k))$ extends to a function on the strip \mathcal{S}_R for $|t| < R$. It remains to show that, if this extension is substituted in the above equation, it still yields a function in $L^2(\mathbb{R}^d)$. Let $n \in \mathbb{N}$ denote the unique integer satisfying $2n > d$. Then $j_n(k) := (1 + |k|^{2n})^{-1} \in L^1(\mathbb{R}^d)$ and hence $j_n * g \in L^2(\mathbb{R}^d)$ for all $g \in L^2(\mathbb{R}^d)$. Note that there exists a constant $C_t > 0$ independent of k such that $J(-t, k) \leq C_t$. We define $\beta_t(k, k') := \gamma_{-t}(k') - \gamma_{-t}(k)$ and compute

$$\begin{aligned} & \int_{\mathbb{R}^d} J(-t, k) \left| \int_{\mathbb{R}^d} \hat{V}(\beta_t(k, k')) \sqrt{J(-t, k')} g(k') dk' \right|^2 dk \\ &= C_t^2 \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |(1 + |k - k'|^{2n}) \hat{V}(\beta_t(k, k'))| |g(k')| j_n(k - k') dk' \right)^2 dk. \end{aligned} \tag{3.41}$$

Now we define $\alpha_t(k, k') = \gamma_{-t}(k) - k - (\gamma_{-t}(k') - k')$ and compute $|\alpha_t(k, k')| \leq 2M|t|$

$$\begin{aligned}
& |(1 + |k - k'|^{2n})\hat{V}(\beta_t(k, k'))| \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \int_{\mathbb{R}^d} (1 + |k - k'|^{2n}) e^{i(k-k') \cdot x} e^{i\alpha_t(k, k') \cdot x} V(x) dx \right| \\
&\leq \frac{1}{(2\pi)^{\frac{d}{2}}} \|e^{2M|t|\cdot} V(\cdot)\|_{L^1(\mathbb{R}^d)} + \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \int_{\mathbb{R}^d} (\Delta_x^n e^{i(k-k') \cdot x}) e^{i\alpha_t(k, k') \cdot x} V(x) dx \right| \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \|e^{2M|t|\cdot} V(\cdot)\|_{L^1(\mathbb{R}^d)} \\
&\quad + \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \int_{\mathbb{R}^d} e^{i(k-k') \cdot x} \Delta_x^n \left(e^{i\alpha_t(k, k') \cdot x} V(x) \right) dx \right|, \tag{3.42}
\end{aligned}$$

where Δ_x^n denotes the n -th product of the Laplace operator w.r.t. the variable $x \in \mathbb{R}^d$. Let $\kappa \in \mathbb{N}_0^d$, define $D_x^\kappa = \partial_{x_1}^{\kappa_1} \dots \partial_{x_d}^{\kappa_d}$ and calculate

$$\begin{aligned}
& \left| \int_{\mathbb{R}^d} e^{i(k-k') \cdot x} D_x^\kappa \left(e^{i\alpha_t(k, k') \cdot x} V(x) \right) dx \right| \\
&\leq \sum_{\substack{\kappa' \leq \kappa \\ \kappa' \in \mathbb{N}_0^d}} \binom{\kappa}{\kappa'} \int_{\mathbb{R}^d} \left| (D_x^{\kappa-\kappa'} e^{i\alpha_t(k, k') \cdot x}) (D_x^{\kappa'} V(x)) \right| dx \\
&\leq \sum_{\substack{\kappa' \leq \kappa \\ \kappa' \in \mathbb{N}_0^d}} \binom{\kappa}{\kappa'} (2M|t|)^{|\kappa-\kappa'|} \int_{\mathbb{R}^d} e^{M|t||x|} \left| (D_x^{\kappa'} V(x)) \right| dx \tag{3.43}
\end{aligned}$$

Combining (3.42) and (3.43) we argue that there is a constant $C_{n,t} > 0$ such that

$$|(1 + |k - k'|^{2n})\hat{V}(\beta_t(k, k'))| \leq C_{n,t} \sum_{\substack{\kappa \in \mathbb{N}_0^d, \\ |\kappa| \leq n}} \|e^{M|t|\cdot} (D_x^\kappa V)(\cdot)\|. \tag{3.44}$$

Finally, (3.41) and (3.44) yield

$$\begin{aligned}
& \|e^{-iA_\xi t} T_V e^{iA_\xi t} g\|_{L^2(\mathbb{R}^d)}^2 \\
&\leq C_t^2 C_{n,t} \sum_{\substack{\kappa \in \mathbb{N}_0^d, \\ |\kappa| \leq n}} \|e^{M|t|\cdot} (D_x^\kappa V)(\cdot)\| \|j_n * |g|\|_{L^2(\mathbb{R}^d)}^2 \\
&\leq C_t^2 C_{n,t} \sum_{\substack{\kappa \in \mathbb{N}_0^d, \\ |\kappa| \leq n}} \|e^{M|t|\cdot} (D_x^\kappa V)(\cdot)\| \|j_n\|_{L^1(\mathbb{R}^d)}^2 \|g\|_{L^2(\mathbb{R}^d)}^2. \tag{3.45}
\end{aligned}$$

Thus, the analytic continuation of $e^{-iA_\xi t} T_V e^{iA_\xi t}$ to $B_r^{\mathbb{C}}(0)$ is a bounded operator. The continuation to the strip \mathcal{S}_r can be constructed as in the abstract case by

translating the continuation to $B_r^{\mathbb{C}}(0)$ along the real axis by conjugation with the unitary group $e^{iA_\xi t}$ for $t \in \mathbb{R}$. \square

Lemma 3.7. *For all $t \in \mathbb{R}$ we have that*

$$e^{-itA_\xi} M_{\omega_\xi} e^{itA_\xi} = M_{\omega_\xi \circ \gamma_{-t}}$$

on $D(M_{\omega_\xi})$, where γ_t is the solution to (3.36).

PROOF. Let $\psi \in D(M_{\omega_\xi})$. Since the action of e^{itA} is defined by (3.38), we use (3.39) and the group property of the flow γ_t to compute

$$\begin{aligned} [e^{-itA} M_{\omega_\xi} e^{itA} f](k) &= \omega_\xi(\gamma_{-t}(k)) \sqrt{J(t, \gamma_{-t}(k))} \sqrt{J(t, k)} f(k) \\ &= \omega_\xi(\gamma_{-t}(k)) f(k). \end{aligned}$$

This proves the statement. \square

Proposition 3.8. *Assume Condition 3.1. Then the family of operators $H_V(\xi)$ satisfies Condition 2.1.*

PROOF. We begin by establishing Condition 2.1.1 in our example. Let $t \in [0, 1]$, $\psi \in \mathcal{D}$ and compute

$$\|H(\xi)U(t)\psi\| = \|H_t(\xi)\psi\| \leq \|M_{\omega_\xi \circ \gamma_{-t}}\psi\| + \|T_V\|\|\psi\|. \quad (3.46)$$

The first term on the right hand side of the previous estimate can be treated as follows.

$$\begin{aligned} \|M_{\omega_\xi \circ \gamma_{-t}}\psi\|^2 &= \int_{\mathbb{R}^d} |\omega_\xi(\gamma_{-t}(k))|^2 |\psi(k)|^2 dk \\ &\leq \int_{\mathbb{R}} |\omega_2(\gamma_{-t}(k))|^2 |\psi(k)|^2 dk + \int_{\mathbb{R}} |\omega_1(\xi - \gamma_{-t}(k))|^2 |\psi(k)|^2 dk \\ &\leq C \int_{\mathbb{R}} \langle \gamma_{-t}(k) \rangle^{2p_2} |\psi(k)|^2 dk + C \int_{\mathbb{R}} \langle \xi - \gamma_{-t}(k) \rangle^{2p_1} |\psi(k)|^2 dk. \end{aligned}$$

Let $q \in \{p_1, p_2\}$. A two-fold application of equation (3.40) implies

$$\begin{aligned} \langle \xi - \gamma_{-t}(k) \rangle^{2p} &= (1 + |\xi - \gamma_{-t}(k)|^2)^q \leq (1 + |\xi|^2 + 2|\xi||\gamma_{-t}(k)| + |\gamma_{-t}(k)|^2)^q \\ &\leq \left(1 + \frac{(|\xi| + C|t|)^2}{1 + |k|^2} + 2(|\xi| + C|t|) \frac{|k|}{1 + |k|^2} \right)^q \langle k \rangle^{2q} \\ &\leq (1 + (|\xi| + C|t|)^2 + 2(|\xi| + C|t|))^q \langle k \rangle^{2q}. \end{aligned}$$

Define $d_q(\xi, t) := (1 + (|\xi| + C|t|)^2 + 2(|\xi| + C|t|))^q$. Then

$$\begin{aligned} \|M_{\omega_\xi \circ \gamma_{-t}}\psi\|^2 &\leq C d_{p_2}(0, |t|) \int_{\mathbb{R}} |\langle k \rangle^{p_2} \psi(k)|^2 dk \\ &\quad + C d_{p_1}(\xi, |t|) \int_{\mathbb{R}} |\langle k \rangle^{p_1} \psi(k)|^2 dk \end{aligned} \quad (3.47)$$

which implies that

$$\sup_{t \in [0,1]} \|H(\xi)U(t)\psi\| \leq Cd_{p_2}(0,1)\|\langle k \rangle^{p_2}\psi\| + Cd_{p_1}(\xi,1)\|\langle k \rangle^{p_1}\psi\| + \|T_V\|\|\psi\|$$

which is a finite upper bound, since $\psi \in D(\langle k \rangle^{p_1}) \cap D(\langle k \rangle^{p_2})$.

In order to verify Condition 2.1.2 we define $w_\xi := \frac{1}{2}\nabla \cdot v_\xi$ and compute

$$\begin{aligned} [H(\xi), iA_\xi] &= v_\xi \cdot \nabla_k \omega_\xi + M_{w_\xi} T_{\hat{V}} + T_{\hat{V}} M_{w_\xi} \\ &\quad + \sum_{\sigma=1}^d \left(M_{(v_\xi)_\sigma} T_{(i^d x^{\delta_\sigma} \bar{V})^\wedge} + T_{(i^d x^{\delta_\sigma} \bar{V})^\wedge} M_{(v_\xi)_\sigma} \right), \end{aligned}$$

where the components of $\delta_\sigma \in \mathbb{R}^d$ are given by $(\delta_\sigma)_{\sigma'} = \delta_{\sigma, \sigma'}$. Since w_ξ and $v_\xi \cdot \nabla_k \omega_\xi$ are bounded functions by Condition 3.1.3 and (3.35) and $V(x)e^{b|x|}$ is assumed to be in $L^1(\mathbb{R}^d)$ by Condition 3.1.5 for some $b > 0$, the expression on the right of the above equation is even a bounded operator.

By Lemma 3.4 and Lemma 3.6 we may conclude that the map $t \mapsto H_t(\xi)$ admits a strongly analytic continuation into a d -dimensional strip \mathcal{S}_r of the complex plane. This establishes Condition 2.1.3.

It thus remains to examine whether or not Condition 2.1.4 holds. A look at (3.45) shows that the operator norm of the analytic continuation of $e^{itA_\xi} T_V e^{-itA_\xi}$ is uniformly bounded on $B_r^{\mathbb{C}}(0)$. Denote this upper bound by $C_V > 0$. Substituting C_V for $\|T_V\|$ in (3.46) and then use (3.47) we obtain the estimate

$$\begin{aligned} &\sup_{t \in B_r^{\mathbb{C}}(0)} \|e^{itA_\xi} H(\xi) e^{-itA_\xi} (H(\xi) + i)^{-1} \psi\| \\ &\leq C_V \|(H(\xi) + i)^{-1} \psi\| + Cd_{p_2}(0,r) \|M_{\langle k \rangle^{p_2}} (H(\xi) + i)^{-1} \psi\| \\ &\quad + d_{p_1}(\xi, r) \|M_{\langle k \rangle^{p_1}} (H(\xi) + i)^{-1} \psi\|. \end{aligned} \tag{3.48}$$

Since $\mathcal{D} = D(\omega_1) \cap D(\omega_2) = D(M_{\langle k \rangle^{p'}}$), where $p' = \max\{p_1, p_2\}$, $M_{\langle k \rangle^{p_j}} (H(\xi) + i)^{-1}$, where $j = 1, 2$, extend to bounded operators by the closed graph theorem. This completes the proof. \square

Proposition 3.9. *Assume Condition 3.1. The family of operators $H(\xi)$ satisfies Condition 2.8 and Condition 2.17.*

Proof. The first part of Condition 2.17 has already been shown. Now note that $\omega_\xi(k)$ extends to an analytic function in a strip \mathcal{S}_R . Recall that the analytic extension of the flow $\gamma_t(k)$ to the strip \mathcal{S}_r maps into the (possibly bigger) strip $\mathcal{S}_{M'r}$, where $M' = \max\{1, M\}$ and M is given by (3.34). Fix $t_0 \in B_r^{\mathbb{C}}(0)$ and let $k \in \mathbb{R}^d$. Since $M'r < R$, the map $\xi \mapsto \omega_\xi \circ \gamma_{-t_0}(k) = \omega_1(\xi - \gamma_{-t_0}(k)) + \omega_2(k)$ extends to complex ξ provided $|\text{Im}(\xi)| < R' := R - M'r$. The analyticity of the maps $\xi \mapsto H_{t_0}(\xi)\psi$ on $B_{R'}^{\mathbb{C}^d}(0)$ for every $\psi \in \mathcal{D}$ now follows. Moreover, $R'' > 0$ can be chosen independently of ξ and the upper bound in (3.48) can be chosen

independently of $\xi \in B_{R''}^{\mathbb{C}}(0)$. Hence the second and fourth part of Condition 2.17 are satisfied. The third part as well as Condition 2.8 will be established in the remaining part of the proof.

The conjugation \mathcal{C} in Condition 2.8 is easily seen to be given by complex conjugation on $L^2(\mathbb{R}^d)$. Put $w = \frac{1}{2}\nabla \cdot v_\xi$. Recall that the commutator $[H(\xi), iA_\xi]$ is bounded and given by

$$\begin{aligned} [H(\xi), iA_\xi] &= |\nabla_k \omega_\xi(k)|^2 e^{-k^2} + M_w T_{\hat{V}} + T_{\hat{V}} M_w \\ &\quad + \sum_{\sigma=1}^d (M_{(v_\xi)_\sigma} T_{(i^d x^\delta \sigma \hat{V})^\wedge} + T_{(i^d x^\delta \sigma \hat{V})^\wedge} M_{(v_\xi)_\sigma}), \end{aligned}$$

Let $K = [H(\xi), iA_\xi] - v_\xi \cdot \nabla_k \omega_\xi$. It is easy to see that K extends to a compact operator. Note that for $f \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned} f(H(\xi_0)) - f(M_{\omega_{\xi_0}}) &= \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z) ((H(\xi_0) - z)^{-1} - (M_{\omega_{\xi_0}} - z)^{-1}) \frac{d}{dz} \\ &= \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z) (H(\xi_0) - z)^{-1} T_V (M_{\omega_{\xi_0}} - z)^{-1} \frac{d}{dz} \end{aligned}$$

which is compact as $\omega_\xi(k) \rightarrow \infty$ when $|k| \rightarrow \infty$. Choosing f with support in $[\lambda_0 - 2\kappa, \lambda_0 + 2\kappa]$ and $f \equiv 1$ on $[\lambda_0 - \kappa, \lambda_0 + \kappa]$ and multiplying from the left and the right by $E_{H(\xi_0)}([\lambda_0 - \kappa, \lambda_0 + \kappa])$ thus shows that $E_{H(\xi_0)}([\lambda_0 - \kappa, \lambda_0 + \kappa])$ may be replaced by $E_{M_{\omega_{\xi_0}}}([\lambda_0 - \kappa, \lambda_0 + \kappa])$ at the cost of a compact error. As

$$e^{-k^2} |\nabla \omega_{\xi_0}|^2 E_{M_{\omega_0}}([\lambda_0 - \kappa, \lambda_0 + \kappa]) \geq e E_{M_{\omega_0}}([\lambda_0 - \kappa, \lambda_0 + \kappa])$$

by the choice of κ and

$$e^{-k^2} |\nabla \omega_{\xi_0}|^2 \leq C$$

for some constant $C > 0$, we may now conclude the proof by collecting all estimates. \square

A Intertwining Conjugations

Recall that a *conjugation* is an anti-linear operator \mathcal{C} that satisfies $\mathcal{C}^2 = 1$ and $\langle \psi, \psi' \rangle = \langle \mathcal{C}\psi, \mathcal{C}\psi' \rangle$.

Definition A.1. Let T be a closed and densely defined operator with domain $D(T)$. A conjugation \mathcal{C} is said to intertwine T and T^* , if $\mathcal{C}D(T) \subset D(T^*)$ and $\mathcal{C}T = T^*\mathcal{C}$ on $D(T)$.

Note that, if the conjugation \mathcal{C} intertwines T and T^* it also intertwines T^* and T . Indeed, suppose that \mathcal{C} intertwines T and T^* and let $\psi \in D(T^*)$. Then there exists a $\psi' \in \mathcal{H}$ such that

$$\langle \psi', \phi \rangle = \langle \psi, T\phi \rangle = \langle \mathcal{C}\psi, \mathcal{C}T\phi \rangle = \langle \mathcal{C}\psi, T^*\mathcal{C}\phi \rangle$$

for all $\phi \in D(T)$. We can thus conclude that $\mathcal{C}\psi \in D(T^{**}) = D(T)$, since T is closed. $\mathcal{C}T^* = T\mathcal{C}$ on $D(T^*)$ follows trivially from $\mathcal{C}D(T^*) \subset D(T)$, $\mathcal{C}^2 = 1$ and $\mathcal{C}T = T^*\mathcal{C}$. In particular, if \mathcal{C} intertwines T and T^* , we have the inclusions

$$\mathcal{C}D(T) \subset D(T^*) = \mathcal{C}^2D(T^*) \subset \mathcal{C}D(T)$$

and hence

$$\mathcal{C}D(T) = D(T^*).$$

Lemma A.2. *Let T be a closed, densely defined operator. Suppose there exists a conjugation \mathcal{C} which intertwines T and T^* . Then*

$$\lambda \in \rho(T) \quad \Leftrightarrow \quad \exists c > 0 \forall \psi \in D(T) : \|(T - \lambda)\psi\| \geq c\|\psi\|.$$

Proof. Suppose that $\lambda \in \rho(T)$. Then there exists a bounded inverse B such that $B(T - \lambda) = 1$. Thus,

$$\|\psi\| = \|B(T - \lambda)\psi\| \leq \|B\|\|(T - \lambda)\psi\|$$

and $\|(T - \lambda)\psi\| \geq c\|\psi\|$ follows from $\|B\| \neq 0$.

Conversely, let us assume that there exists $c > 0$ such that $\|(T - \lambda)\psi\| \geq c\|\psi\|$. Note that $V := \text{Ran}(T - \lambda)$ is a closed subspace of \mathcal{H} . Choose $\phi \perp V$ and compute

$$|\langle \phi, T\psi \rangle| \leq |\langle \phi, (T - \lambda)\psi \rangle| + |\lambda|\langle \phi, \psi \rangle| \leq |\lambda|\|\phi\|\|\psi\|$$

for every $\psi \in D(T)$. This implies that $\phi \in D(T^*)$ and we can thus calculate

$$\langle (T^* - \bar{\lambda})\phi, \psi \rangle = \langle \phi, (T - \lambda)\psi \rangle = 0$$

whenever $\psi \in D(T)$. Since $D(T)$ is dense, we have now established that $\bar{\lambda} \in \sigma_{\text{pp}}(T^*)$. The intertwining relation $T\mathcal{C} = \mathcal{C}T^*$ then gives the contradiction that $\lambda \in \sigma_{\text{pp}}(T)$. Therefore, we must have $V = \mathcal{H}$. The preceding argument shows that $T - \lambda$ is bijective and we can conclude that it has a left inverse B . Since

$$\|B(T - \lambda)\psi\| = \|\psi\| \leq \frac{1}{c}\|(T - \lambda)\psi\|,$$

B is bounded.

Due to $\mathcal{C}^2 = 1$, the anti-linearity of \mathcal{C} and the intertwining property, the estimate $\|(T - \lambda)\psi\| \geq c\|\psi\|$ on $D(T)$ can be re-written as

$$\forall \psi \in D(T^*) : \|(T^* - \bar{\lambda})\psi\| = \|(T - \lambda)\mathcal{C}\psi\| \geq c\|\mathcal{C}\psi\| = c\|\psi\|.$$

Therefore, the operator T^* satisfies the same conditions as T and we can construct a left inverse $B' \in \mathcal{B}(\mathcal{H})$ for $T^* - \bar{\lambda}$ in a similar fashion. Hence $B'(T^* - \bar{\lambda}) = 1$ which in turn implies $(T - \lambda)B'^* = 1$. (Here we used that $\text{Range}(B'^*) \subset D(T)$.) Therefore, $T - \lambda$ has a bounded right inverse as well, and we can conclude that $\lambda \in \rho(T)$. \square

Negation of the assertion in Lemma A.2, immediately implies the following useful corollary:

Corollary A.3. *In the situation of Lemma A.2, we have*

$$\lambda \in \sigma(T) \quad \Leftrightarrow \quad \exists \{\psi_n\}_{n \in \mathbb{N}} \subset D(T), \|\psi_n\| = 1 : \|(T - \lambda)\psi_n\| \rightarrow 0.$$

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