SYMMETRIC WEBS, JONES-WENZL RECURSIONS AND q-HOWE DUALITY

DAVID E. V. ROSE AND DANIEL TUBBENHAUER

ABSTRACT. We define and study the category of symmetric $\mathfrak{sl}_2$-webs. This category is a combinatorial description of the category of all finite dimensional quantum $\mathfrak{sl}_2$-modules. Explicitly, we show that (the additive closure of) the symmetric $\mathfrak{sl}_2$-spider is (braided monoidally) equivalent to the latter. Our main tool is a quantum version of symmetric Howe duality. As a corollary of our construction, we provide new insight into Jones-Wenzl projectors and the colored Jones polynomials.

CONTENTS

1. INTRODUCTION

1.1. **Temperley-Lieb categories and Jones-Wenzl projectors.** A classical result of Rumer, Teller, and Weyl [34], modernly interpreted, states that the *Temperley-Lieb category* $\mathcal{TL}$ describes the full subcategory of quantum $\mathfrak{sl}_2$-modules generated by tensor products of the 2-dimensional vector representation V of quantum $\mathfrak{sl}_2$, which we denote[1] by $\mathfrak{sl}_2$-$\mathbf{Mod}_\wedge$. The former was first introduced in the study of statistical mechanics (as an algebra and also in the non-quantum setting) by Temperley and Lieb in [35] and has played an important role in several areas of mathematics and physics.

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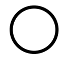
[1]The notation $\mathfrak{sl}_n$-$\mathbf{Mod}_\wedge$ more generally is used below to denote the full subcategory of quantum $\mathfrak{sl}_n$-modules tensor generated by the fundamental representations (which, in the $\mathfrak{sl}_n$ case, are exterior powers of the vector representation). Also, throughout the paper, when we refer to $\mathfrak{sl}_n$-weights, $\mathfrak{sl}_n$-modules, etc. we always mean their quantum versions. Moreover, for the insistent reader, all modules are finite dimensional, left modules.

Explicitly, the objects in $\mathcal{TL}$ are non-negative integers, and the morphisms are given graphically by $\mathbb{Z}[q, q^{-1}]$-linear combinations of non-intersecting tangle diagrams, which we view as mapping from the k_1 boundary points at the bottom of the tangle to the k_2 on the top, modulo boundary preserving isotopy and the local relation for evaluating a circle, that is,

(1)  $= -[2]$

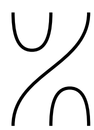
Here, and throughout, $[a]$ for $a \in \mathbb{Z}$ denotes the *quantum integer*, given by

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}} = q^{a-1} + q^{a-3} + \cdots + q^{-a+3} + q^{-a+1} \in \mathbb{Z}[q, q^{-1}]$$

for q a generic parameter. By convention, $[0] = 1$.

The correspondence between $\mathcal{TL}$ and the category $\mathfrak{sl}_2$-$\mathbf{Mod}_\wedge$ associates the $\mathfrak{sl}_2$-module $V^{\otimes k}$ to $k \in \mathbb{Z}_{\geq 0}$, and the morphisms are locally generated (by taking tensor products $\otimes$ and compositions $\circ$ of diagrams[2]) by the basic diagrams

where the first diagram corresponds to the identity, and the latter two correspond to the unique (up to scalar multiplication) $\mathfrak{sl}_2$-intertwiners $V \otimes V \to \mathbb{C}_q = \mathbb{C}(q)$ and $\mathbb{C}_q \to V \otimes V$. For example,



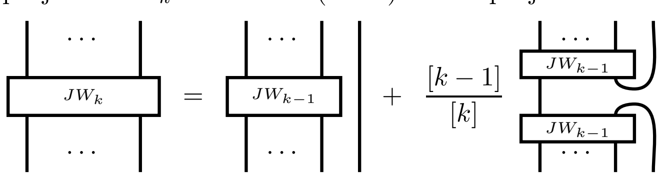
corresponds to a morphism $V \otimes V \otimes V \to V \otimes V \otimes V$. It turns out that the isotopy and circle removal (1) relations are enough. That is, we have the following.

Theorem 1.1. The category $\mathcal{TL}$ and $\mathfrak{sl}_2$-$\mathbf{Mod}_\wedge$ are equivalent (as pivotal) categories.

It is known that every finite dimensional, irreducible quantum $\mathfrak{sl}_2$-module appears as a direct summand of $V^{\otimes k}$ for some big enough k. Thus, we obtain the entire category of finite dimensional quantum $\mathfrak{sl}_2$-modules, denoted by $\mathfrak{sl}_2$-$\mathbf{fdMod}$, by passing to the *Karoubi envelope* $\mathbf{Kar}(\mathcal{TL})$ of $\mathcal{TL}$. Recall that the Karoubi envelope (sometimes also called idempotent completion) is the minimal enlargement of a category in which idempotents split; objects in this category are (roughly) idempotent morphisms, which should be viewed as corresponding to their images.

It is a striking question if one can give a diagrammatic description of $\mathbf{Kar}(\mathcal{TL})$ as well.

A solution to this question is known: an (in principle) explicit description of the entire category $\mathfrak{sl}_2$-$\mathbf{fdMod}$ can be given using the *Jones-Wenzl projectors* (also called Jones-Wenzl idempotents). These were introduced by Jones in [15] and then further studied by Wenzl in [39]. The Jones-Wenzl projectors are morphisms in $\mathcal{TL}$ which correspond to projecting onto, then including from, the highest weight irreducible summand $V_k \subset V^{\otimes k}$. These projectors, which are usually depicted by a box with k incoming and outgoing strands at the top and bottom, admit a recursive definition describing the k-strand Jones-Wenzl projector JW_k in terms of $(k-1)$-strand projector as follows.

(2) 

[2]Let us fix our diagrammatic conventions now: we read from left to right and bottom to top. Tensoring $u \otimes v$ is stacking picture v to the right of u and composition $v \circ u$ is given by stacking picture v on the top of u.

We point out that some authors have a different sign convention here. Our convention comes from the fact that a circle evaluates to $-[2]$ instead of to $[2]$, see (12).

However, working with such projectors in the Karoubi envelope quickly becomes cumbersome and computationally unmanageable due to their recursive definition. In this article, we provide a new, alternative diagrammatic description of the *entire* category $\mathfrak{sl}_2$-**fdMod** of finite dimensional quantum $\mathfrak{sl}_2$-modules.

1.2. **A reminder on $\mathfrak{sl}_n$-webs.** In pioneering work, see [20], Kuperberg extended the diagrammatic description of $\mathfrak{sl}_2$-$\mathbf{Mod}_\wedge$ to the Lie algebra $\mathfrak{sl}_3$ (and the other two rank 2 Lie algebras of type B_2 and G_2 – but we do not use them in this paper). Recall that the question was to find a diagrammatic and combinatorial model for $\mathfrak{sl}_3$-$\mathbf{Mod}_\wedge$, the full subcategory of finite dimensional quantum $\mathfrak{sl}_3$-modules whose objects are finite tensor products of $\bigwedge_q^k\mathbb{C}_q^3$'s, the fundamental $\mathfrak{sl}_3$-modules[3]. Since every irreducible $\mathfrak{sl}_3$-module will appear as a summand of tensor products of $\bigwedge_q^k\mathbb{C}_q^3$'s, we again have that "morally" the study of $\mathfrak{sl}_3$-$\mathbf{Mod}_\wedge$ suffices to understand the entire category of finite dimensional $\mathfrak{sl}_3$-modules.

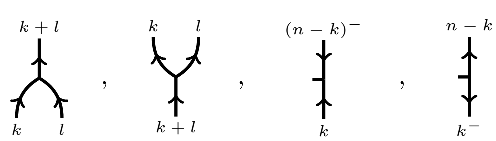
Kuperberg succeeded: he introduced in Section 4 of [20] the $\mathfrak{sl}_3$-*spider*, denoted here by $\mathbf{Sp}(\mathfrak{sl}_3)$. This is a category whose morphisms, called $\mathfrak{sl}_3$-*webs*, are freely generated (via tensoring and composition) by local pieces of certain trivalent, oriented graphs. The category $\mathbf{Sp}(\mathfrak{sl}_3)$ is then obtained by taking a certain quotient, and the main difficulty is to find the "correct" relations such that there is an equivalence of (pivotal) categories $\mathbf{Sp}(\mathfrak{sl}_3) \cong \mathfrak{sl}_3$-$\mathbf{Mod}_\wedge$. Kuperberg gave the relations needed to obtain the aforementioned equivalence. While in the $\mathfrak{sl}_2$ case the circle removal relation (1) suffices, the $\mathfrak{sl}_3$ case requires three local relations (that we do not need and thus, do not explicitly recall here).

It was long an open problem to extend Kuperberg's results to describe $\mathfrak{sl}_n$-$\mathbf{Mod}_\wedge$, the full subcategory of all finite dimensional $\mathfrak{sl}_n$-modules whose objects are finite tensor products of the fundamental $\mathfrak{sl}_n$-representations $\bigwedge_q^k\mathbb{C}_q^n$. As before, by Karoubi completing, it suffices to study $\mathfrak{sl}_n$-$\mathbf{Mod}_\wedge$ to obtain a description of the entire category of finite dimensional modules. A description of this subcategory in terms of $\mathfrak{sl}_n$-webs was realized by Cautis, Kamnitzer and Morrison using the novel method of *quantum skew Howe duality* (for short: q-skew Howe duality), see [5]. Our description of the entire category of finite dimensional quantum $\mathfrak{sl}_2$-modules in this paper is, surprisingly, related to Cautis, Kamnitzer and Morrison's $\mathfrak{sl}_n$-webs, which we briefly recall now. Much more, of course, can be found in their paper.

Cautis, Kamnitzer and Morrison show Theorem 3.3.1 in [5] that $\mathfrak{sl}_n$-$\mathbf{Mod}_\wedge$ is (pivotal) equivalent to the category of $\mathfrak{sl}_n$-webs, a *combinatorially* defined category in which objects are sequences in the symbols $1^\pm, \cdots, (n-1)^\pm$, and morphisms are given by $\mathbb{Z}[q, q^{-1}]$-linear combinations of oriented, trivalent graphs with edges labeled by $1, \cdots, n-1$, such that the sum of the incoming and outgoing labels agree at each vertex. Moreover, by convention, the edges are directed outward at the bottom and inward at the top iff the corresponding boundary number is positive.

The correspondence between this diagrammatic category and the category of $\mathfrak{sl}_n$-modules is given by associating a tensor product of fundamental $\mathfrak{sl}_n$-modules and their duals to each sequence, with k^+ corresponding to $\bigwedge_q^k\mathbb{C}_q^n$ and k^- to its dual $(\bigwedge_q^k\mathbb{C}_q^n)^*$. The generating $\mathfrak{sl}_n$-webs are

(3)

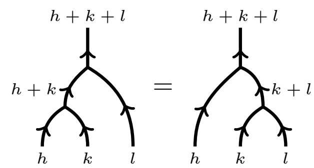


[3]The notation $\bigwedge_q^k$ means the *quantum* alternating tensors. These are roughly the same as the "classical" alternating tensors but with some q's to spice everything up, see for instance Subsection 4.2 in [5].

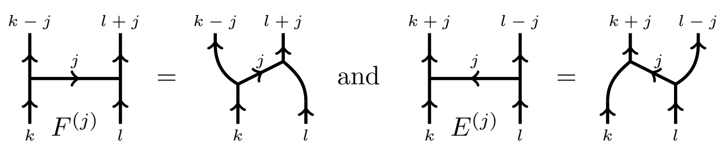
which are called (reading from left to right) *merge*, *split*, *tag in* and *tag out*. These generators correspond to the quantum analog of the unique (up to scalar) $\mathfrak{sl}_n$-intertwiners $\bigwedge_q^k\mathbb{C}_q^n \otimes \bigwedge_q^l\mathbb{C}_q^n \to \bigwedge_q^{k+l}\mathbb{C}_q^n$, $\bigwedge_q^{k+l}\mathbb{C}_q^n \to \bigwedge_q^k\mathbb{C}_q^n \otimes \bigwedge_q^l\mathbb{C}_q^n$, $\bigwedge_q^k\mathbb{C}_q^n \xrightarrow{\cong} (\bigwedge_q^{n-k}\mathbb{C}_q^n)^*$, and $(\bigwedge_q^k\mathbb{C}_q^n)^* \xrightarrow{\cong} \bigwedge_q^{n-k}\mathbb{C}_q^n$, see Section 3.2 in [5].

As before, the main difficulty is deducing the correct collection of relations between these generators, which Cautis, Kamnitzer and Morrison give in Subsection 2.2 of [5]. The subset of their relations consisting of relations between "upward" $\mathfrak{sl}_n$-webs (i.e. those only factoring through tensor products of $\bigwedge_q^k\mathbb{C}_q^n$'s, and not their duals) is of particular relevance to the current work, hence, we recall them now.

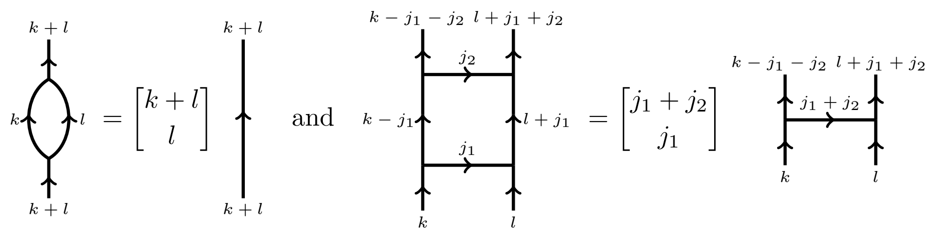
The upward relations are the following, together with their vertical mirror images. First, we have the *Frobenius relations*:

(4) 

To state the remaining relations, define the so-called $F^{(j)}$ *and* $E^{(j)}$*-ladders* as

(5) 

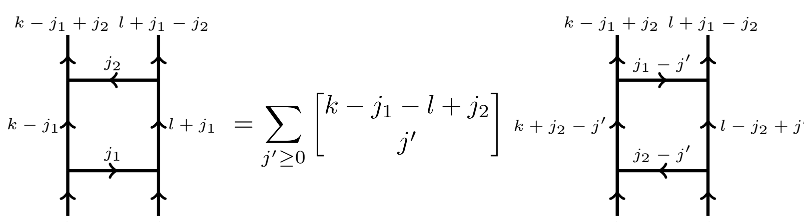
Then the remaining relations are:

(6) 

which are called the *digon removal* and *square removal* relations. In these relations, the *quantum binomial* is given by

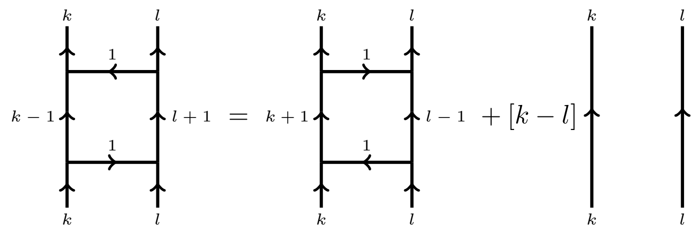
$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{[a][a-1]\cdots[a-b+2][a-b+1]}{[b]!}$$

where $[b]! = [0][1]\cdots[b-1][b]$, $a \in \mathbb{Z}$, $b \in \mathbb{N}$. The final relations:

(7) 

are the (in)famous *square switch* relations. For example, if $j_1 = j_2 = 1$, then the only possible j' values are $j' = 0, 1$ and equation (7) gives[4]:

(8)



The astute reader will recognize the similarity between these final relations and the relations

$$EF1_{(k,l)} = FE1_{(k,l)} + [k-l]1_{(k,l)}$$

in the Beilinson, Lusztig and MacPherson's idempotented quantum group $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ (see [2]) recalled in detail below in Subsection 2.1. Of course, this is no coincidence. One of the main results of [5] is that q-skew Howe duality induces a functor $\Phi_m^n \colon \dot{\mathbf{U}}_q(\mathfrak{gl}_m) \to \dot{\mathbf{U}}_q^n(\mathfrak{gl}_m) \to \mathfrak{sl}_n\text{-}\mathbf{Mod}_\wedge$, where $\dot{\mathbf{U}}_q^n(\mathfrak{gl}_m)$ denotes the quotient of $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ by the ideal generated by $\mathfrak{gl}_m$-weights with entries not in $\{0, \cdots, n\}$.

They go on to show in Proposition 5.2.2 of [5] that Φ_m^n factors through $\mathbf{Sp}(\mathfrak{sl}_n)$ and thus, taking the "limit" $m \to \infty$, all the relations in $\mathbf{Sp}(\mathfrak{sl}_n)$ needed for the diagrammatic description of $\mathfrak{sl}_n\text{-}\mathbf{Mod}_\wedge$ follow from relations in $\dot{\mathbf{U}}_q(\mathfrak{gl}_\infty)$. Our main idea in this paper is to adapt Cautis, Kamnitzer and Morrison's approach to quantum *symmetric* Howe duality (for short, q-symmetric Howe duality).

1.3. **Main result.** We now introduce our new description of the representation theory of quantum $\mathfrak{sl}_2$, the category of *symmetric* $\mathfrak{sl}_2$*-webs.*

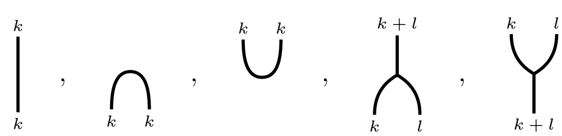
Here a symmetric $\mathfrak{sl}_2$-web u is an equivalence class (modulo boundary preserving planar isotopies) of edge-labeled, trivalent planar graphs with boundary. The labels for the edges of u are numbers from $\mathbb{Z}_{>0}$ such that, at each trivalent vertex, two of the edge labels sum to the third.

We follow Cautis, Kamnitzer and Morrison and first introduce the *free* symmetric $\mathfrak{sl}_2$-spider. Then the *symmetric* $\mathfrak{sl}_2$*-spider* $\boldsymbol{SymSp}(\mathfrak{sl}_2)$ is a certain quotient of it.

Definition 1.2. (**The free symmetric $\mathfrak{sl}_2$-spider**) The *free symmetric* $\mathfrak{sl}_2$*-spider*, which we denote by $\mathbf{SymSp}^f(\mathfrak{sl}_2)$, is the category determined by the following data.

- The objects of $\mathbf{SymSp}^f(\mathfrak{sl}_2)$ are tuples $\vec{k} \in \mathbb{Z}_{>0}^m$ for some $m \in \mathbb{Z}_{\geq 0}$, together with a zero object. We display their entries ordered from left to right according to their appearance in $\vec{k}$. Note that we allow $\varnothing$ as an object (corresponding to the empty sequence in $\mathbb{Z}^0$), which is not to be confused with the zero object.
- The morphisms of $\mathbf{SymSp}^f(\mathfrak{sl}_2)$ from $\vec{k}$ to $\vec{l}$, denoted by $\mathrm{Hom}_{\mathbf{SymSp}^f(\mathfrak{sl}_2)}(\vec{k}, \vec{l})$, are diagrams with bottom boundary $\vec{k}$ and top boundary $\vec{l}$ freely generated as a $\mathbb{C}(q)$-vector space by all symmetric $\mathfrak{sl}_2$-webs that can be obtained by composition $\circ$ (vertical gluing) and tensoring $\otimes$ (horizontal juxtaposition) of the following basic pieces (including the empty diagram $\varnothing$).

(9)

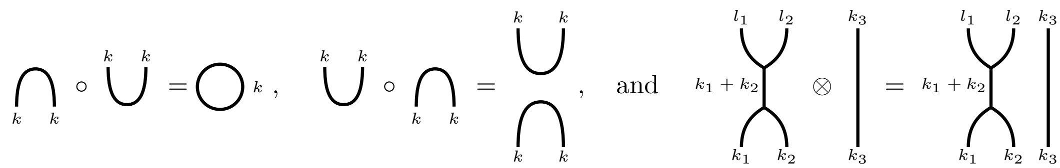


These are called (from left to right) *identity*, *cap*, *cup*, *merge* and *split*.

[4]Note that we do not draw $\mathfrak{sl}_n$-web edges labeled zero.

Remark 1.3. Note the following conventions and properties of $\mathbf{SymSp}^f(\mathfrak{sl}_2)$.

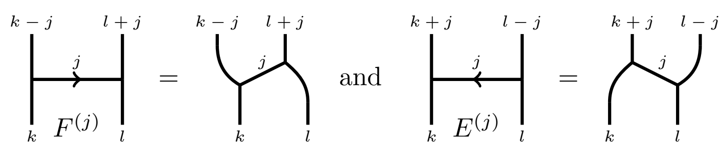
- We consider the (free) symmetric $\mathfrak{sl}_2$-webs up to boundary preserving isotopies. Formally, a (free) symmetric $\mathfrak{sl}_2$-web is an equivalence class, but we abuse language and suppress this technical distinction.
- The category is $\mathbb{C}(q)$-*linear*, i.e. the spaces $\mathrm{Hom}_{\mathbf{SymSp}^f(\mathfrak{sl}_2)}(\vec{k},\vec{l})$ are $\mathbb{C}(q)$-vector spaces and the composition $\circ$ is $\mathbb{C}(q)$-linear. Moreover, the category is *monoidal* by juxtaposition of objects and morphisms, and $\otimes$ is similarly $\mathbb{C}(q)$-linear on morphism spaces.
- The reading conventions for all symmetric $\mathfrak{sl}_2$-webs is from *bottom to top* and *left to right*. That is, given $u, v \in \mathrm{Hom}_{\mathbf{SymSp}^f(\mathfrak{sl}_2)}(\vec{k},\vec{l})$, then $v \circ u$ is obtained by gluing v on top of u and $u \otimes v$ is given by putting v to the right of u. In pictures, e.g. we have

(10) 

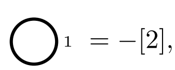
  where in the final equation $k_1 + k_2 = l_1 + l_2$.
- If any of the top boundary labels of the symmetric $\mathfrak{sl}_2$-web u is different from the corresponding bottom boundary component of the symmetric $\mathfrak{sl}_2$-web v, then, by convention, $v \circ u$ is zero.

Definition 1.4. (**The symmetric $\mathfrak{sl}_2$-spider**) The *symmetric $\mathfrak{sl}_2$-spider*, denoted by $\mathbf{SymSp}(\mathfrak{sl}_2)$, is the quotient category obtained from $\mathbf{SymSp}^f(\mathfrak{sl}_2)$ by imposing the following local relations.

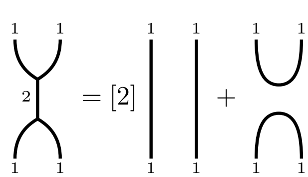
- The *standard relations*, without orientations, that is, Frobenius (4), digon and square removals (6) and the square switches (7). As before, it is convenient to define the $F_i^{(j)}$ and $E_i^{(j)}$-ladders as in (5). In order to keep track of which is which, we (sometimes) add an orientation to the middle edges as a reminder, that is,

(11) 

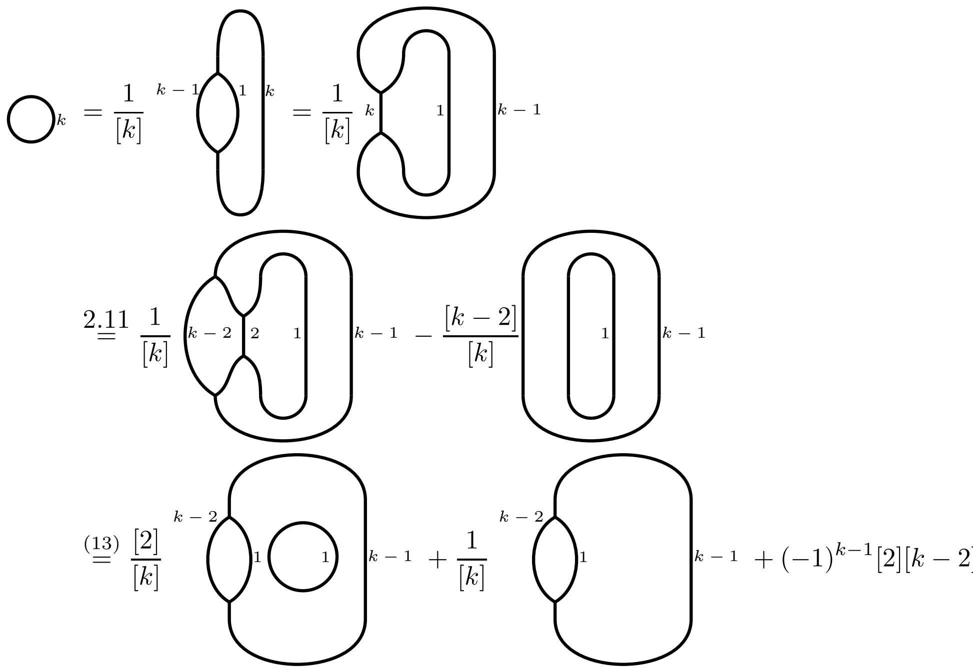
  By convention, if any label appearing in a symmetric $\mathfrak{sl}_2$-web is less than 0, then the corresponding diagram is defined to be the zero morphism.
- The *symmetric relations*, that is, *circle removal*:

(12) 

  and, finally, the *dumbbell relation*:

(13) 

Example 1.5. These relations, together with Lemma 2.11 below, imply that a circle of thickness k evaluates to $(-1)^k[k+1]$. Indeed, we inductively compute:

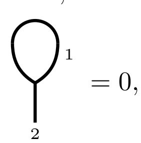


$$= (-1)^k[2]^2[k-1] + (-1)^{k-1}[2][k-1] + (-1)^{k-1}[2][k-2] = (-1)^k[k+1],$$

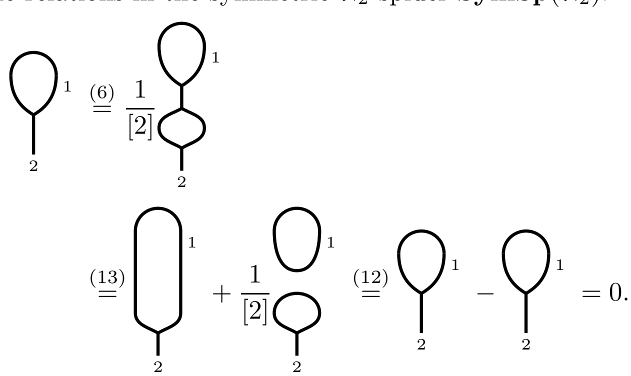
where the last equality follows from $[2][k'] = [k'+1] + [k'-1]$ (for $k' \geq 1$).

Remark 1.6. Equation (12) implies there is a functor $\mathcal{TL} \xrightarrow{\mathcal{I}} \mathbf{SymSp}(\mathfrak{sl}_2)$ given by sending objects $k \in \mathbb{Z}_{\geq 0}$ of $\mathcal{TL}$ to a sequence of 1's of length k, and by viewing morphisms in $\mathcal{TL}$ as symmetric $\mathfrak{sl}_2$-webs. We will show below that this functor is in fact an inclusion of a full subcategory.

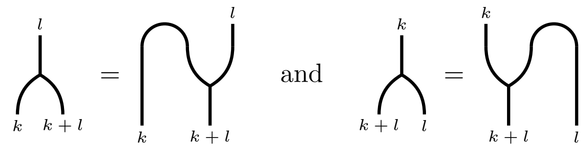
Example 1.7. The so-called the *lollipop relation*, that is,



can be deduced from the relations in the symmetric $\mathfrak{sl}_2$-spider $\mathbf{SymSp}(\mathfrak{sl}_2)$:



Remark 1.8. The following "non-standard" merge and split $\mathfrak{sl}_2$-webs can be defined as composites of the generating morphisms in $\mathbf{SymSp}(\mathfrak{sl}_2)$.



and similarly for rotated versions.

Remark 1.9. Of course, trivalent graphs have previously appeared in the diagrammatic study of quantum $\mathfrak{sl}_2$ under the guise of quantum spin networks, see [16]. The difference in the present work is that we view trivalent vertices as the generators of our category (and deduce all relations between them needed to describe the category of representations) rather than using trivalent vertices as shorthand for Temperley-Lieb diagrams built from Jones-Wenzl projectors.

Recall that $\mathfrak{sl}_2$-**fdMod** denotes the category whose objects are (all) finite dimensional modules of quantum $\mathfrak{sl}_2$, i.e. direct sums of the irreducible $\mathfrak{sl}_2$-modules $\mathrm{Sym}_q^k\mathbb{C}_q^2$ (we explain the quantum symmetric tensors in Subsection 2.1 below), and whose morphisms are $\mathfrak{sl}_2$-intertwiners between these tensor products. Recall that this is a monoidal category where $\otimes$ is the usual tensor product.

Moreover, recall that the *additive closure* of a category $\mathcal{C}$ consist of finite, formal direct sums of objects from $\mathcal{C}$ with morphisms given by matrices whose entries are morphisms from $\mathcal{C}$.

Theorem 1.10. The additive closure[5] of $\mathbf{SymSp}(\mathfrak{sl}_2)$ is monoidally equivalent to $\mathfrak{sl}_2$-**fdMod**.

The functor $\Gamma_{\mathbf{sym}}\colon \mathbf{SymSp}(\mathfrak{sl}_2) \to \mathfrak{sl}_2\text{-}\mathbf{fdMod}$ (see Definition 2.17) inducing this equivalence is given by assigning the irreducible $\mathfrak{sl}_2$-module $\mathrm{Sym}_q^k\mathbb{C}_q^2$ to the label k, and sending the generating morphisms in Equation (9) to the (up to scalar) unique $\mathfrak{sl}_2$-intertwiners between the $\mathfrak{sl}_2$-modules corresponding to their boundaries.

In Section 2, we will prove Theorem 1.10. Of course, there are essentially two things to check: first, that the relations on symmetric $\mathfrak{sl}_2$-webs are satisfied in the category of $\mathfrak{sl}_2$-modules, and second, that we describe all morphisms (and relations between them) in this category. We accomplish the former task using q-symmetric Howe duality, and the latter by noticing the surprising result that the square switch relation (7) gives the Jones-Wenzl recursion formula (2), a result which we think is of independent interest.

Finally, in Section 3 we use symmetric $\mathfrak{sl}_2$-webs to compute the colored Jones polynomial, and discuss some further implications of our construction. To do so, we show that q-symmetric Howe duality induces a *braided monoidal* structure on our diagrammatic category $\mathbf{SymSp}(\mathfrak{sl}_2)$ and conclude that the functor $\Gamma_{\mathbf{sym}}\colon \mathbf{SymSp}(\mathfrak{sl}_2) \to \mathfrak{sl}_2\text{-}\mathbf{fdMod}$ is an equivalence of braided monoidal categories.

We derive some consequences of this in Section 3. For example, in Subsection 3.3 we observe a connection between the $\mathrm{Sym}_q^k\mathbb{C}_q^2$-colored Jones polynomial and the $\bigwedge_q^k\mathbb{C}_q^n$-colored Reshetikhin-Turaev polynomial of a colored, oriented link diagram L_D. For the precise statement see Theorem 3.7.

Acknowledgments: Special thanks to Sabin Cautis for suggesting the study of a "symmetric" presentation of the category of $\mathfrak{sl}_2$-modules.

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[5]We must pass to the additive closure in order to make sense of direct sum decompositions. This is far more satisfying than passing to the Karoubi envelope of $\mathcal{TL}$ since working in the additive closure of a category $\mathcal{C}$ is combinatorially "the same" as working in $\mathcal{C}$.

Veen, Pedro Vaz, Paul Wedrich and Geordie Williamson for helpful discussion, comments, and probing questions.

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2. The proofs

2.1. q-symmetric Howe duality.

In this subsection, we present the requisite material on quantum groups and q-symmetric Howe duality. The main objective is to prove Corollary 2.7, which gives a full functor $\Phi_m \colon \dot{\mathbf{U}}_q(\mathfrak{gl}_m) \to \mathfrak{sl}_n\textbf{-fdMod}$. Along the way, we state q-symmetric Howe duality and deduce its consequences for any $n > 1$ before we specialize to $n = 2$. We use the results in this subsection to demonstrate later in Subsection 2.3 how the relations in the symmetric $\mathfrak{sl}_2$-spider $\mathbf{SymSp}(\mathfrak{sl}_2)$ can be derived from q-symmetric Howe duality.

We begin by recalling the quantum general and special linear algebras, and their idempotented forms. The $\mathfrak{gl}_m$-weight lattice is isomorphic to $\mathbb{Z}^m$. Let $\epsilon_i = (0, \dots, 1, \dots, 0) \in \mathbb{Z}^m$, with 1 being in the i-th coordinate, and $\alpha_i = \epsilon_i - \epsilon_{i+1} = (0, \dots, 1, -1, \dots, 0) \in \mathbb{Z}^m$, for $i = 1, \dots, m-1$. Recall that the Euclidean inner product on $\mathbb{Z}^m$ is defined by $(\epsilon_i, \epsilon_j) = \delta_{i,j}$.

Definition 2.1. For $m \in \mathbb{N}_{>1}$ the *quantum general linear algebra* $\mathbf{U}_q(\mathfrak{gl}_m)$ is the associative, unital $\mathbb{C}(q)$-algebra generated by L_i and L_i^{-1}, for $i = 1, \dots, m$, and E_i, F_i, for $i = 1, \dots, m-1$, subject to the relations (for suitable i, i_1, i_2)

$$L_{i_1} L_{i_2} = L_{i_2} L_{i_1}, \quad L_i L_i^{-1} = L_i^{-1} L_i = 1, \quad L_{i_1} E_{i_2} = q^{(\epsilon_{i_1}, \alpha_{i_2})} E_{i_2} L_{i_1}, \quad L_{i_1} F_{i_2} = q^{-(\epsilon_{i_1}, \alpha_{i_2})} F_{i_2} L_{i_1},$$

$$E_{i_1} F_{i_2} - F_{i_2} E_{i_1} = \delta_{i_1, i_2} \frac{L_{i_1} L_{i_1+1}^{-1} - L_{i_1}^{-1} L_{i_1+1}}{q - q^{-1}},$$

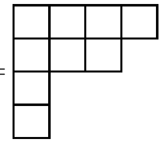
$$E_{i_1}^2 E_{i_2} - [2] E_{i_1} E_{i_2} E_{i_1} + E_{i_2} E_{i_1}^2 = 0 \quad \text{if} \quad |i_1 - i_2| = 1, \qquad E_{i_1} E_{i_2} - E_{i_2} E_{i_1} = 0 \quad \text{else},$$

$$F_{i_1}^2 F_{i_2} - [2] F_{i_1} F_{i_2} F_{i_1} + F_{i_2} F_{i_1}^2 = 0 \quad \text{if} \quad |i_1 - i_2| = 1, \qquad F_{i_1} F_{i_2} - F_{i_2} F_{i_1} = 0 \quad \text{else}.$$

The leftmost relations in the last two lines are the so-called *Serre-relations*.

Definition 2.2. For $m \in \mathbb{N}_{>1}$ the *quantum special linear algebra* $\mathbf{U}_q(\mathfrak{sl}_m)$ is the subalgebra of $\mathbf{U}_q(\mathfrak{gl}_m)$ generated by the elements $E_i, F_i, K_i = L_i L_{i+1}^{-1}$, and $K_i^{-1} = L_{i+1} L_i^{-1}$ for $i = 1, \dots, m-1$.

To distinguish *dominant integral* $\mathfrak{gl}_m$-weights in $\mathbb{Z}^m_{\geq 0}$ (we call these, by abuse of language, just dominant integral $\mathfrak{gl}_m$-weights, although a general dominant integral $\mathfrak{gl}_m$-weight can have negative entries) from general $\mathfrak{gl}_m$-weights, we will denote the former by Greek letters as λ, μ, etc. Recall that such $\mathfrak{gl}_m$-weights $\lambda = (\lambda_1, \dots, \lambda_m)$ with $\lambda_i \geq 0$ can be described by partitions of K where $\sum_{i=1}^m \lambda_i = K$. We denote the set of all partitions of K of length m by $\Lambda^+(m, K)$. Consequently, these dominant integral $\mathfrak{gl}_m$-weights are precisely the elements of $\bigcup_{K \in \mathbb{N}} \Lambda^+(m, K)$. We can picture such λ as a tableaux[6]. For example, if $\lambda = (4, 3, 1, 1) \in \Lambda^+(4, 9)$, then

$\lambda =$ 

where we abuse notation and denote the tableaux and the partition by the same symbol. Thus, in our notation, dominant integral $\mathfrak{gl}_m$-weights λ are in bijective correspondence with tableaux with at most m rows, but with any possible (finite) number of columns.

[6] We use the English convention for tableaux.

Moreover, recall that $\mathbf{U}_q(\mathfrak{gl}_m)$ has a unique highest weight module $V_m(\lambda)$ of highest weight λ for each dominant integral $\mathfrak{gl}_m$-weight λ. We point out that, by taking suitable tensors of the form $V_m(\lambda)\otimes\det^{\otimes -k}$, one can get any finite dimensional, irreducible $\mathbf{U}_q(\mathfrak{gl}_m)$-module. Here $\det^{\otimes -k}$ denotes a tensor product of length k of the dual $\det^* = V_m(-1,\dots,-1)$ of the 1-dimensional $\mathbf{U}_q(\mathfrak{gl}_m)$-module $\det = V_m(1,\dots,1)$ (which is usually called the determinant representation). Thus, it suffices to study the $V_m(\lambda)$ for most purposes, including the remainder of this paper.

It is also worth noting that $\mathbf{U}_q(\mathfrak{gl}_m)$ is a Hopf algebra with coproduct Δ given by

$$\Delta(E_i) = E_i\otimes L_iL_{i+1}^{-1} + 1\otimes E_i,\ \ \Delta(F_i) = F_i\otimes 1 + L_i^{-1}L_{i+1}\otimes F_i \ \text{ and }\ \Delta(L_i) = L_i\otimes L_i.$$

The antipode S and the counit ε are given by

$$S(E_i) = -E_iL_i^{-1}L_{i+1},\ \ S(F_i) = -L_iL_{i+1}^{-1}F_i,\ \ S(L_i) = L_i^{-1},\ \ \varepsilon(E_i)=\varepsilon(F_i)=0 \ \text{ and }\ \varepsilon(L_i)=1.$$

The subalgebra $\mathbf{U}_q(\mathfrak{sl}_m)$ inherits the Hopf algebra structure from $\mathbf{U}_q(\mathfrak{gl}_m)$. We point out, since there are variations in different papers, that we use same conventions as in [5]. The Hopf algebra structure allows to extend actions to tensor products and duals of representations, and gives the existence of a trivial representation (that we simply denote as before by $\mathbb{C}_q = \mathbb{C}(q)$).

Another notion we need in the following is Beilinson, Lusztig and MacPherson's *idempotented form* [2], denoted by $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$. Adjoin an idempotent $1_{\vec{k}}$ for $\mathbf{U}_q(\mathfrak{gl}_m)$ for each $\vec{k}\in\mathbb{Z}^m$ and add the relations

$$\begin{aligned}
&1_{\vec{k}}1_{\vec{l}} = \delta_{\vec{k},\vec{l}}1_{\vec{k}},\\
&E_i1_{\vec{k}} = 1_{\vec{k}+\alpha_i}E_i, \quad \text{with } \alpha_i = (0,\dots,1,-1,\dots,0) \text{ as above},\\
&F_i1_{\vec{k}} = 1_{\vec{k}-\alpha_i}F_i, \quad \text{with } \alpha_i = (0,\dots,1,-1,\dots,0) \text{ as above},\\
&L_i1_{\vec{k}} = q^{k_i}1_{\vec{k}}.
\end{aligned}$$

Definition 2.3. The *idempotented* quantum general linear algebra is defined by

$$\dot{\mathbf{U}}_q(\mathfrak{gl}_m) = \bigoplus_{\vec{k},\vec{l}\in\mathbb{Z}^m} 1_{\vec{l}}\mathbf{U}_q(\mathfrak{gl}_m)1_{\vec{k}}.$$

Remark 2.4. It is convenient to view $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ as generated by the *divided powers*

$$F_i^{(j)} = \frac{F_i^j}{[j]!} \quad\text{and}\quad E_i^{(j)} = \frac{E_i^j}{[j]!}$$

for $i=1,\dots,m-1$. In particular, this point of view is useful if one wishes to work integrally, rather than over a field. In this case, the integral form of $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ is the $\mathbb{Z}[q,q^{-1}]$-subalgebra generated by divided powers and satisfying the following complete list of relations. In the following let $\vec{k}\in\mathbb{Z}^m$ and let all the subscripts be in $\{1,\dots,m-1\}$ and all the superscripts be in $\mathbb{Z}_{\geq 0}$. If some of these indices fall outside of the sets mentioned above, then, by convention, the corresponding element is understood to be zero.

We have *commutation relations* (with the left equations similarly for $E_i^{(j)}$'s)

$$F_{i_1}^{(j_1)}F_{i_2}^{(j_2)}1_{\vec{k}} = F_{i_2}^{(j_2)}F_{i_1}^{(j_1)}1_{\vec{k}} \ \text{ if }\ |i_1-i_2|>1,\quad F_{i_1}^{(j_1)}E_{i_2}^{(j_2)}1_{\vec{k}} = E_{i_2}^{(j_2)}F_{i_1}^{(j_1)}1_{\vec{k}} \ \text{ if }\ |i_1-i_2|>0 \tag{14}$$

the *Serre* and *divided power relations* (with both equations similarly for $E_i^{(j)}$'s)

$$F_{i_1}^2F_{i_2}1_{\vec{k}} - [2]F_{i_1}F_{i_2}F_{i_1}1_{\vec{k}} + F_{i_2}F_{i_1}^21_{\vec{k}} = 0 \ \text{ if }\ |i_1-i_2|=1,\quad F_i^{(j_1)}F_i^{(j_2)}1_{\vec{k}} = \begin{bmatrix} j_1+j_2\\ j_1\end{bmatrix}F_i^{(j_1+j_2)}1_{\vec{k}} \tag{15}$$

and the $EF-FE$*-relations*

$$E_i^{(j_2)}F_i^{(j_1)}1_{\vec{k}} = \sum_{j'}\begin{bmatrix} k_i-j_1-k_{i+1}+j_2\\ j'\end{bmatrix}F_i^{(j_1-j')}E_i^{(j_2-j')}1_{\vec{k}}. \tag{16}$$

Remark 2.5. We will find it convenient to view $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ as a category. Indeed, this is possible for any algebra containing a system of orthogonal idempotents. Explicitly, the objects of $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ are precisely the $\mathfrak{gl}_m$-weights $\vec{k} \in \mathbb{Z}^m$, and $\mathrm{Hom}(\vec{k}, \vec{l}) = 1_{\vec{l}}\mathbf{U}_q(\mathfrak{gl}_m)1_{\vec{k}}$.

We now discuss q-symmetric Howe duality, following the approach of Berenstein and Zwicknagl from [3]. The "classical" symmetric Howe duality can be found in various sources, see [14] and [13] in the algebraic group setting and for example Theorem 5.16 in [6] for the pair $(\mathbf{U}(\mathfrak{gl}_m), \mathbf{U}(\mathfrak{gl}_n))$. Note that Cheng and Wang in Theorem 5.19 and Remark 5.20 of [6] also discuss *super* Howe duality (which is more general and includes skew and symmetric Howe duality as a special case). A slightly stronger result on super Howe duality which, in the non-quantized setting, comes close to what we need can be found in Proposition 2.1 of [33].

Unfortunately, as in the q-skew Howe case, the literature about q-symmetric Howe duality is very limited. We hence adapt Cautis, Kamnitzer and Morrison's results on q-skew Howe duality to our setting, following closely their notation and exposition.

Denote the standard basis of the $\mathbf{U}_q(\mathfrak{gl}_m)$-module $\mathbb{C}_q^m$ by $\{x_1, \dots, x_m\}$, where the action is given via

$$E_i(x_j) = \begin{cases} x_{j-1}, & \text{if } i = j-1, \\ 0, & \text{else,} \end{cases} \quad F_i(x_j) = \begin{cases} x_{j+1}, & \text{if } i = j, \\ 0, & \text{else,} \end{cases} \quad L_i(x_j) = \begin{cases} qx_j, & \text{if } i = j, \\ x_j, & \text{else.} \end{cases} \tag{17}$$

By our conventions, the action of $\mathbf{U}_q(\mathfrak{sl}_m)$ is almost the same as in (17), but the K_i act as q^{+1} on x_i and as q^{-1} on x_{i+1}.

Now fix $m, n > 0$. Then there is an action of $\mathbf{U}_q(\mathfrak{gl}_m) \otimes \mathbf{U}_q(\mathfrak{sl}_n)$ on $\mathbb{C}_q^m \otimes \mathbb{C}_q^n$ and the latter has a basis given by $z_{ij} = x_i \otimes y_j$ for $x_i \in \mathbb{C}_q^m$ and $y_j \in \mathbb{C}_q^n$. The Hopf algebra structures of $\mathbf{U}_q(\mathfrak{gl}_m)$ and $\mathbf{U}_q(\mathfrak{sl}_n)$ induce an action on the tensor algebra $\mathcal{T}(\mathbb{C}_q^m \otimes \mathbb{C}_q^n)$ of $\mathbb{C}_q^m \otimes \mathbb{C}_q^n$.

We now consider the *quantum symmetric algebra*

$$\mathrm{Sym}_q^\bullet(\mathbb{C}_q^m \otimes \mathbb{C}_q^n) = \mathcal{T}(\mathbb{C}_q^m \otimes \mathbb{C}_q^n) / {\textstyle\bigwedge}_q^2(\mathbb{C}_q^m \otimes \mathbb{C}_q^n),$$

where ${\textstyle\bigwedge}_q^2(\mathbb{C}_q^m \otimes \mathbb{C}_q^n)$ is the quantum exterior square of $\mathbb{C}_q^m \otimes \mathbb{C}_q^n$. Proposition 2.33 in [3] shows that ${\textstyle\bigwedge}_q^2(\mathbb{C}_q^m \otimes \mathbb{C}_q^n)$ is spanned by the elements

$$z_{ij'} \otimes z_{ij} - qz_{ij} \otimes z_{ij'}, \quad z_{ij'} \otimes z_{i'j} + qz_{i'j'} \otimes z_{ij} - qz_{ij} \otimes z_{i'j'} - q^2 z_{i'j} \otimes z_{ij'},$$
$$z_{i'j} \otimes z_{ij} - qz_{ij} \otimes z_{i'j}, \quad z_{i'j} \otimes z_{ij'} + qz_{i'j'} \otimes z_{ij} - qz_{ij} \otimes z_{i'j'} - q^2 z_{ij'} \otimes z_{i'j},$$

for all $1 \leq i < i' \leq m$ and $1 \leq j < j' \leq n$. The space $\mathrm{Sym}_q^\bullet(\mathbb{C}_q^m \otimes \mathbb{C}_q^n)$ is graded and its k-homogeneous piece, which we denote by $\mathrm{Sym}_q^k(\mathbb{C}_q^m \otimes \mathbb{C}_q^n)$, is the k-th *quantum symmetric tensor* of $\mathbb{C}_q^m \otimes \mathbb{C}_q^n$. By setting $n = 1$, we get the k-th quantum symmetric tensor of $\mathbb{C}_q^m$ denoted by $\mathrm{Sym}_q^k \mathbb{C}_q^m$. Similarly we have the *quantum alternating tensors* ${\textstyle\bigwedge}_q^k \mathbb{C}_q^m$, ${\textstyle\bigwedge}_q^\bullet \mathbb{C}_q^m$, ${\textstyle\bigwedge}_q^k(\mathbb{C}_q^m \otimes \mathbb{C}_q^n)$ and ${\textstyle\bigwedge}_q^\bullet(\mathbb{C}_q^m \otimes \mathbb{C}_q^n)$ (we do not need the quantum alternating tensors much in this paper and refer to Subsection 4.2 in [5] for a more detailed treatment of these).

Our next result is a quantum version of symmetric Howe duality. We point out one crucial difference to the q-skew Howe case is that the direct sum decomposition in (3) of Theorem 2.6 *does not* contain the transpose of λ. To this end, we call a dominant integral $\mathfrak{gl}_m$-weight λ a *n-supported $\mathfrak{gl}_m$-weight* if its tableaux has at most $\min(m, n)$ rows, but still any possible finite number of columns.

Theorem 2.6. (q**-symmetric Howe duality**) We have the following.

(1) For each $K \in \mathbb{Z}_{\geq 0}$, the actions of $\mathbf{U}_q(\mathfrak{gl}_m)$ and $\mathbf{U}_q(\mathfrak{sl}_n)$ on $\mathrm{Sym}_q^K(\mathbb{C}_q^m \otimes \mathbb{C}_q^n)$ commute and generate each others commutant.

(2) There is an isomorphism of $\mathbf{U}_q(\mathfrak{sl}_n)$-modules $\mathrm{Sym}_q^\bullet(\mathbb{C}_q^m \otimes \mathbb{C}_q^n) \cong (\mathrm{Sym}_q^\bullet \mathbb{C}_q^n)^{\otimes m}$ under which the $\vec{k}$-weight space of $\mathrm{Sym}_q^\bullet(\mathbb{C}_q^m \otimes \mathbb{C}_q^n)$ (considered as a $\mathbf{U}_q(\mathfrak{gl}_m)$-module) is identified with $\mathrm{Sym}_q^{k_1}\mathbb{C}_q^n \otimes \cdots \otimes \mathrm{Sym}_q^{k_m}\mathbb{C}_q^n$ (here $\vec{k} = (k_1, \dots, k_m)$).

(3) As a $\mathbf{U}_q(\mathfrak{gl}_m) \otimes \mathbf{U}_q(\mathfrak{sl}_n)$-modules, we have a decomposition for each $K \in \mathbb{Z}_{\geq 0}$ of the form

$$\mathrm{Sym}_q^K(\mathbb{C}_q^m \otimes \mathbb{C}_q^n) \cong \bigoplus_\lambda V_m(\lambda) \otimes V_n(\lambda),$$

where the $\bigoplus$ runs over all n-supported, dominant integral $\mathfrak{gl}_m$-weights $\lambda \in \Lambda^+(m, K)$. Here λ is regarded as a $\mathfrak{sl}_n$-weight[7] for $V_n(\lambda)$. This induces a decomposition

$$\mathrm{Sym}_q^\bullet(\mathbb{C}_q^m \otimes \mathbb{C}_q^n) \cong \bigoplus_\lambda V_m(\lambda) \otimes V_n(\lambda),$$

where the $\bigoplus$ runs over all n-supported, dominant integral $\mathfrak{gl}_m$-weights λ.

Note that, with the exception of the identification of the $\vec{k}$-weight space in item (2), this is essentially the quantum version of the Theorem in Section 2.1.2 in [13].

Proof. The argument is essentially the same as that of Theorem 4.2.2 in [5], with the exception that our task is easier, since from Proposition 2.33 in [3] we already know that $\mathrm{Sym}_q^\bullet(\mathbb{C}_q^m \otimes \mathbb{C}_q^n)$ is flat, i.e. the classical specialization of $\mathrm{Sym}_q^\bullet(\mathbb{C}_q^m \otimes \mathbb{C}_q^n)$ is $\mathrm{Sym}^\bullet(\mathbb{C}^m \otimes \mathbb{C}^n)$. This then allows us to deduce (1) and (3) above as a consequence of the classical result which can be found, for example, in Theorem 2.12 of [13] or in Theorem 5.16 in [6].

The isomorphism (2) is obtained by piecing together results from [3]. Explicitly, this is precisely their Proposition 4.2, using their Lemma 2.32 and Proposition 2.33. To see that the $\vec{k}$-weight space decomposition holds we have to be more explicit. Recall that Berenstein and Zwicknagl show that $\mathrm{Sym}_q^k\mathbb{C}_q^n$ has a basis given by

$$x_{j_1} \otimes \cdots \otimes x_{j_k} \quad \text{for } 1 \leq j_1 \leq \cdots \leq j_k \leq n$$

which we denote by $x_{\underline{j}}$ for $\underline{j} = (j_1, \dots, j_k)$.

Consider

$$T_i \colon \mathrm{Sym}_q^k(\mathbb{C}_q^n) \to \mathrm{Sym}_q^k(\mathbb{C}_q^m \otimes \mathbb{C}_q^n), x_{\underline{j}} \mapsto z_{ij_1} \otimes \cdots \otimes z_{ij_k}.$$

for various $i = 1, \dots, m$. These can be seen as sections of the $\mathbf{U}_q(\mathfrak{sl}_n)$-isomorphism given by Berenstein and Zwicknagl in Proposition 4.2 of [3]. From this, we see that

$$T \colon \bigoplus_{\sum_{i=1}^m k_i = K} \mathrm{Sym}_q^{k_1}(\mathbb{C}_q^n) \otimes \cdots \otimes \mathrm{Sym}_q^{k_m}(\mathbb{C}_q^n) \to \mathrm{Sym}_q^K(\mathbb{C}_q^m \otimes \mathbb{C}_q^n), v_1 \otimes \cdots \otimes v_m \mapsto T_1(v_1) \otimes \cdots \otimes T_m(v_m)$$

is an isomorphism of $\mathbf{U}_q(\mathfrak{sl}_n)$-modules (here $K = k_1 + \cdots + k_m$).

Since the action of $\mathbf{U}_q(\mathfrak{gl}_m)$ on $\mathrm{Sym}_q^\bullet(\mathbb{C}_q^m \otimes \mathbb{C}_q^n)$ is "row-wise," i.e.

$$L_{i'}(z_{ij_1} \otimes \cdots \otimes z_{ij_k}) = L_{i'}(z_{ij_1}) \otimes \cdots \otimes L_{i'}(z_{ij_k}) = \begin{cases} q^k z_{ij_1} \otimes \cdots \otimes z_{ij_k}, & \text{if } i = i', \\ z_{ij_1} \otimes \cdots \otimes z_{ij_k}, & \text{if } i \neq i', \end{cases}$$

the $\vec{k}$-weight space identification follows. □

[7] Recall that any $\mathfrak{gl}_n$-weight $\vec{k} \in \mathbb{Z}^n$ gives a $\mathfrak{sl}_n$-weight in $\mathbb{Z}^{n-1}$ by taking entrywise differences $k_i - k_{i+1}$ and we consider $V_n(\lambda)$ as the irreducible $\mathbf{U}_q(\mathfrak{sl}_n)$-module of highest weight λ obtained by restricting the $\mathbf{U}_q(\mathfrak{gl}_n)$-action on $V_n(\lambda)$.

By Theorem 2.6 part (2), we get linear maps

$$f_{\vec{k}}^{\vec{l}}\colon 1_{\vec{l}}\dot{\mathbf{U}}_q(\mathfrak{gl}_m)1_{\vec{k}} \to \mathrm{Hom}_{\mathbf{U}_q(\mathfrak{sl}_n)}(\mathrm{Sym}_q^{k_1}\mathbb{C}_q^n \otimes \cdots \otimes \mathrm{Sym}_q^{k_m}\mathbb{C}_q^n, \mathrm{Sym}_q^{l_1}\mathbb{C}_q^n \otimes \cdots \otimes \mathrm{Sym}_q^{l_m}\mathbb{C}_q^n) \tag{18}$$

for any two $\vec{k},\vec{l} \in \mathbb{Z}_{>0}^m$ such that $\sum_{i=0}^m k_i = \sum_{i=0}^m l_i$. By part (1) of Theorem 2.6, the homomorphisms $f_{\vec{k}}^{\vec{l}}$ are all surjective, which immediately implies the following result.

Corollary 2.7. There exists a full functor $\Phi_m\colon \dot{\mathbf{U}}_q(\mathfrak{gl}_m) \to \mathfrak{sl}_n$-**fdMod**, which we call the *q-symmetric Howe functor*, which sends all $\mathfrak{gl}_m$-weights[8] of the form $\vec{k} = (k_1,\dots,k_m)$ with each $k_i \geq 0$ to the $\mathbf{U}_q(\mathfrak{sl}_n)$-module $\mathrm{Sym}_q^{k_1}\mathbb{C}_q^n \otimes \cdots \otimes \mathrm{Sym}_q^{k_m}\mathbb{C}_q^n$ and morphisms $X \in 1_{\vec{l}}\dot{\mathbf{U}}_q(\mathfrak{gl}_m)1_{\vec{k}}$ to $f_{\vec{k}}^{\vec{l}}(X)$. □

Denote by $\dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_m)$ the quotient of $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ by the ideal generated by $\mathfrak{gl}_m$-weights $\vec{k}$ such that there exists a negative entry $k_i < 0$. By part (3) of Theorem 2.6, all $\mathfrak{gl}_m$-weights in $\mathrm{Sym}_q^k(\mathbb{C}_q^m \otimes \mathbb{C}_q^n)$ appear as $\mathfrak{gl}_m$-weights appearing in $V_m(\lambda)$ where λ is an n-supported, dominant integral $\mathfrak{gl}_m$-weight. Hence, the functors $(\Phi_m)_{m\in\mathbb{N}}$ induce functors

$$\Phi_m^{\infty}\colon \dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_m) \to \mathfrak{sl}_n\text{-}\mathbf{fdMod} \quad , \quad \Phi_\infty^{\infty}\colon \dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_\infty) = \varinjlim \dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_m) \to \mathfrak{sl}_n\text{-}\mathbf{fdMod}. \tag{19}$$

By part (2) of Theorem 2.6, these functors are full. Since all irreducible $\mathfrak{sl}_2$-modules are of the form $\mathrm{Sym}_q^k\mathbb{C}_q^2$ for some $k \in \mathbb{N}$, we have the more precise statement.

Corollary 2.8. The functor $\Phi_\infty^{\infty}\colon \dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_\infty) \to \mathfrak{sl}_n$-**fdMod** is full. Moreover, for $n = 2$ the induced functor from the additive closure (defined as above before Theorem 1.10) of $\dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_\infty)$, that is,

$$\Phi_\infty^{\infty}\colon \mathbf{Mat}(\dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_\infty)) \to \mathfrak{sl}_2\text{-}\mathbf{fdMod},$$

is essentially surjective. □

Remark 2.9. We point out that this is the place where adapting the approach of Cautis, Kamnitzer and Morrison to the symmetric setting fails, due to the fact that there will be relations in $\mathfrak{sl}_n$-**fdMod** that *do not* come from $\dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_\infty)$.

To this end, recall the dominance order $\trianglelefteq$ for dominant, integral $\mathfrak{gl}_m$-weights, given by setting $\mu \trianglelefteq \lambda$ if and only if $\lambda - \mu$ is a $\mathbb{N}$-linear combination of simple roots α_i. Moreover, a not-necessarily dominant $\mathfrak{gl}_m$-weight $\vec{k}$ is dominated by λ, denoted by $\vec{k} \trianglelefteq \lambda$, if and only if $\vec{k}$ appears in the Weyl group orbit of a dominant integral $\mathfrak{gl}_m$-weight μ with $\mu \trianglelefteq \lambda$.

Let I_λ denote the ideal of $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ generated by all $1_{\vec{k}}$ for $\mathfrak{gl}_m$-weights $\vec{k}$ that are not dominated by λ. Doty shows in Theorem 4.2 of [9] that

$$\dot{\mathbf{U}}_q(\mathfrak{gl}_m)/I_\lambda \cong \bigoplus_{\mu \trianglelefteq \lambda} \mathrm{End}_{\mathbb{C}_q}(V_m(\mu)).$$

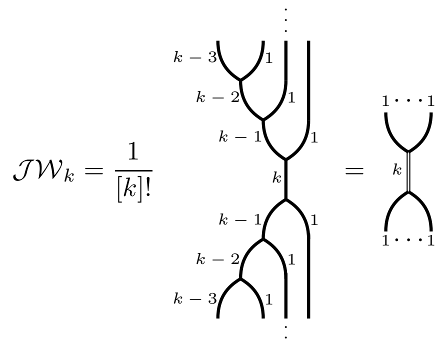
Here comes the catch: in part (3) of Theorem 2.6 we do not have all $V_m(\mu)$ appearing, but only those with n-supported μ. Thus, in order to get faithfulness for the functor Φ_m^{∞}, one has to kill the endomorphism rings of the $V_m(\mu)$'s for non-n-supported μ's. Since this (clearly) depends on n, this introduces new relations which do not come from killing $\mathfrak{gl}_\infty$-weights.

Fortunately, in the $\mathfrak{sl}_2$ case, it is easy to identify the missing relations, and in the following sections we show that they are exactly the symmetric relations from Definition 1.4.

[8] It sends all other $\mathfrak{gl}_m$-weights to the zero representation.

2.2. **Jones-Wenzl recursion.** In this subsection we show how the Jones-Wenzl recursion (2) follows from the square switch relations (7) and the dumbbell relation (13).

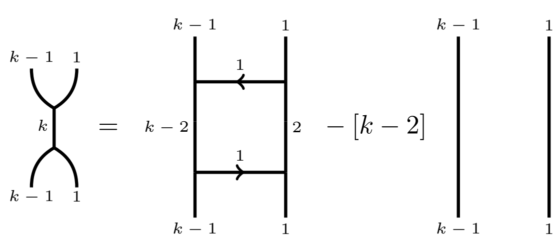
Definition 2.10. (**Symmetric Jones-Wenzl projectors**) For each $k > 0$ we define the k-th *symmetric Jones-Wenzl projectors* $\mathcal{JW}_k$ via



where we repeatedly split a k-labeled edge until all of the top and bottom edges have label 1. The rightmost picture above is a shorthand notation for $\mathcal{JW}_k$ where we view the "doubled" line as encoding the coefficient $\frac{1}{[k]!}$.

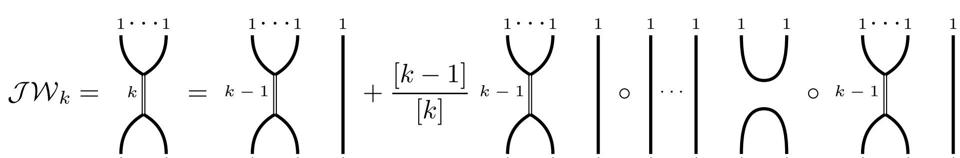
We need the following lemmata.

Lemma 2.11. Let $k > 2$. Then we have



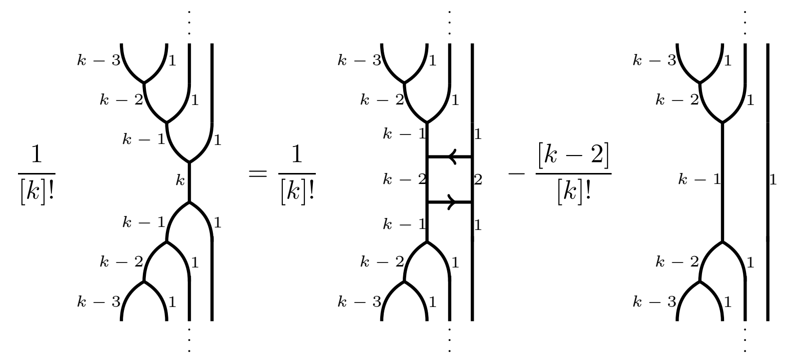
Proof. This is an immediate consequence of equation (8). □

Lemma 2.12. We have

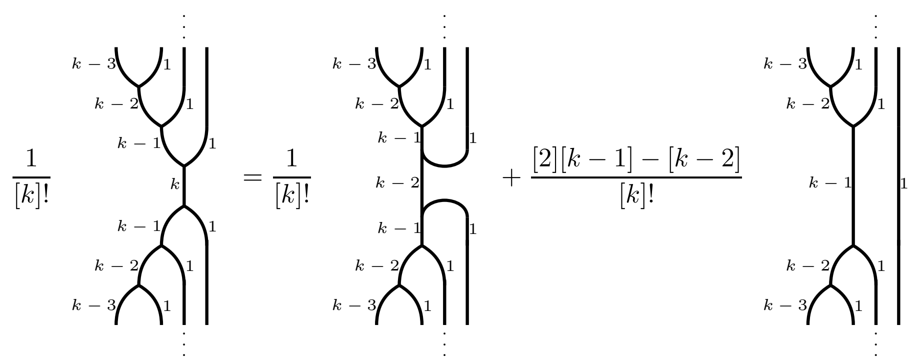


for all $k > 2$.

Proof. Using Lemma 2.11, we find

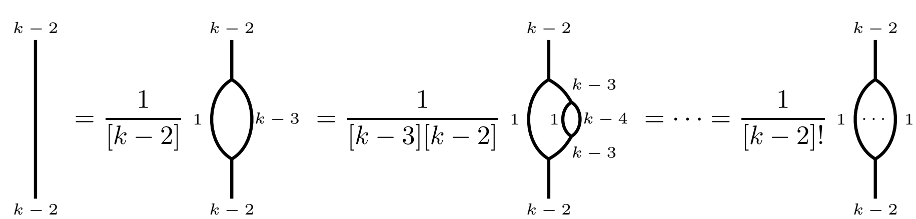


There is now a dumbbell (with edge thickness 2) in the middle picture, and we can use equation (13) to simplify the above to



where we point out that the additional contribution to the rightmost term above results after removing the extra $(k-2,1)$-digon. A straightforward calculation shows that $[k]=[2][k-1]-[k-2]$ and taking this into account, the rightmost term above is $\mathcal{JW}_{k-1}$ with an extra strand on the right.

To see that the other term works out as well, we iteratively "explode" the middle edge of thickness $k-2$ by using the digon removal (6) the other way around, that is



where we continue until all edges are of thickness 1. The diagram now has the desired form in the statement. To see that the coefficient works out, note that

$$\frac{1}{[k]!}\frac{1}{[k-2]!}=\frac{1}{[k-1]!}\frac{1}{[k-1]!}\frac{[k-1]}{[k]}$$

and the two factors $\frac{1}{[k-1]!}$ give the two symmetric Jones-Wenzl projectors $\mathcal{JW}_{k-1}$. $\square$

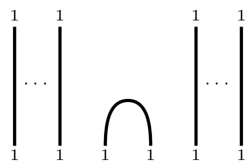
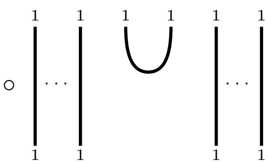
Using these lemmata, we now deduce the main result of this subsection.

Proposition 2.13. The symmetric Jones-Wenzl projectors $\mathcal{JW}_k$ are the images of the Jones-Wenzl projectors JW_k in $\mathcal{TL}$ under the functor $\mathcal{I}\colon \mathcal{TL} \to \mathbf{SymSp}(\mathfrak{sl}_2)$, i.e. $\mathcal{JW}_k = \mathcal{I}(JW_k)$.

Proof. This follows since Lemma 2.12 and equation (13) show that $\mathcal{JW}_k$ satisfy the Jones-Wenzl recursion (2), which uniquely determines JW_k. □

Remark 2.14. This gives the surprising result that, save for the base case, the Jones-Wenzl recursion exactly corresponds to the $\mathfrak{sl}_2$-relations $EF1_{(k,l)} = FE1_{(k,l)} + [k-l]1_{(k,l)}$.

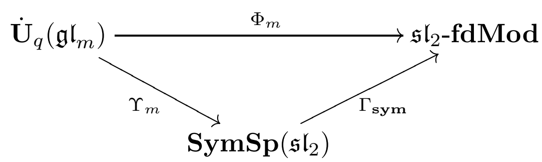
Corollary 2.15. We have

$\mathcal{JW}_k^2 = \mathcal{JW}_k$ and  $\circ\, \mathcal{JW}_k = 0 = \mathcal{JW}_k \circ$ 

Thus, $\mathcal{JW}_k$ is an idempotent which is killed by all possible cap compositions from the top and all possible cup compositions from the bottom.

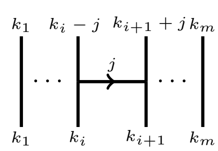
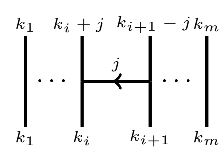
Proof. Since $\mathcal{I}$ is a functor and JW_k are idempotents which annihilate caps/cups, this is an immediate consequence of the previous result. □

2.3. **A diagrammatic description of $\mathfrak{sl}_2$-fdMod.** In this subsection, we prove Theorem 1.10. To do so, we must first deduce the existence of a functor $\Gamma_{\mathbf{sym}}\colon \mathbf{SymSp}(\mathfrak{sl}_2) \to \mathfrak{sl}_2\textbf{-fdMod}$, and then show that $\Gamma_{\mathbf{sym}}$ induces the desired equivalence of categories. The definition of $\Gamma_{\mathbf{sym}}$ is essentially dictated by our desire to have a commutative diagram

(20) 

We will begin by defining the functor Υ_m.

Lemma 2.16. For each $m \geq 0$, there exists a functor $\Upsilon_m\colon \dot{\mathbf{U}}_q(\mathfrak{gl}_m) \to \mathbf{SymSp}(\mathfrak{sl}_2)$ which sends a $\mathfrak{gl}_m$-weight $\vec{k}$ with $k_i \geq 0$ for all i to the sequence obtained by removing all 0's and all other $\mathfrak{gl}_m$-weights to the zero object. This functor is determined on morphisms by the assignment

(21) $\Upsilon_m(F_i^{(j)}1_{\vec{k}}) =$  , $\Upsilon_m(E_i^{(j)}1_{\vec{k}}) =$ 

where we erase any 0-labeled edges in the diagrams depicting the images.

Proof. A straightforward check, using arguments found in Lemma 2.2.1 and Proposition 5.2.1 of [5], shows that the images of relations in $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ are consequences of the standard $\mathfrak{sl}_n$-web relations in equations (4), (6), and (7). □

We now aim to define the functor $\Gamma_{\mathbf{sym}}$. We will first define the images of the generating morphisms in $\mathbf{SymSp}(\mathfrak{sl}_2)$, i.e. define a functor from the free symmetric spider $\mathbf{SymSp}^f(\mathfrak{sl}_2)$, and then check that the relations in $\mathbf{SymSp}(\mathfrak{sl}_2)$ are satisfied. Given a sequence $\underline{i} = (i_1, \ldots, i_m)$ with entries in $\{1,2\}$, we write $x^{\underline{i}}$ as shorthand for

$$x_{i_1} \otimes \cdots \otimes x_{i_m} \in (\mathbb{C}_q^2)^{\otimes m}.$$

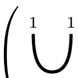
Furthermore, using Lemma 2.32 in [3], we now fix a basis of $\mathrm{Sym}_q^k\mathbb{C}_q^2$ for all k as the one given by the equivalence classes in $\mathrm{Sym}_q^k\mathbb{C}_q^2$ of all $x^{\underline{i}}$ such that $\underline{i}$ is weakly increasing and of length k. We will use the notation $x_{\underline{i}}$ to denote the class of such an element in $\mathrm{Sym}_q^k\mathbb{C}_q^2$.

Definition 2.17. Define a functor $\Gamma_{\mathbf{sym}}\colon \mathbf{SymSp}^f(\mathfrak{sl}_2) \to \mathfrak{sl}_2\text{-}\mathbf{fdMod}$ as follows.

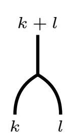
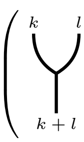
- On objects: the tuples of the form $\vec{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m_{>0}$ are sent to the $\mathbf{U}_q(\mathfrak{sl}_2)$-modules $\mathrm{Sym}_q^{k_1}\mathbb{C}_q^2 \otimes \cdots \otimes \mathrm{Sym}_q^{k_m}\mathbb{C}_q^2$. Moreover, we send, by convention, the empty tuple to the trivial $\mathbf{U}_q(\mathfrak{sl}_2)$-module $\mathbb{C}_q$ and the zero object to the zero representation.
- On morphisms: we send the generators of $\mathbf{SymSp}^f(\mathfrak{sl}_2)$ to the following $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiners, and extend monoidally. We send the thickness k identity strand to $\mathrm{id}_k\colon \mathrm{Sym}_q^k\mathbb{C}_q^2 \to \mathrm{Sym}_q^k\mathbb{C}_q^2$, and define the functor on 1-labeled caps and cups via

(22) $\Gamma_{\mathbf{sym}}\Big($  $\Big) = \mathrm{cap}\colon \mathbb{C}_q^2 \otimes \mathbb{C}_q^2 \to \mathbb{C}_q, \quad \begin{cases} x^{\underline{11}}, x^{\underline{22}} \mapsto 0, \\ x^{\underline{12}} \mapsto -q, \\ x^{\underline{21}} \mapsto 1, \end{cases}$

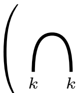
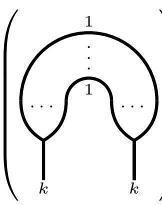
and

(23) $\Gamma_{\mathbf{sym}}\Big($  $\Big) = \mathrm{cup}\colon \mathbb{C}_q \to \mathbb{C}_q^2 \otimes \mathbb{C}_q^2, \quad 1 \mapsto x^{\underline{12}} - q^{-1}x^{\underline{21}}.$

On merge and split generators, we define $\Gamma_{\mathbf{sym}}$ using the functor Φ_2 from Corollary 2.7, that is,

$\Gamma_{\mathbf{sym}}\Big($  $\Big) = \Phi_2(E^{(l)}1_{(k,l)})$, $\Gamma_{\mathbf{sym}}\Big($  $\Big) = \Phi_2(F^{(l)}1_{(k+l,0)})$.

Having defined $\Gamma_{\mathbf{sym}}$ on these generators, we can extend to k-labeled caps via the assignment

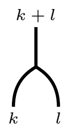
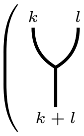
$\Gamma_{\mathbf{sym}}\Big($  $\Big) = \frac{1}{[k]!}\Gamma_{\mathbf{sym}}\Big($  $\Big)$

and similarly for k-labeled cups.

We will denote the images under $\Gamma_{\mathbf{sym}}$ of 1-labeled caps and cups (as above) by cap and cup, and the images of the symmetric $(1,1)$-merge and -split $\mathfrak{sl}_2$-webs by m and s. Moreover, for thickened versions we use the notation cap_k, cup_k, $\mathrm{m}_{k,l}$ and $\mathrm{s}^{k,l}$ in the evident way.

Remark 2.18. The meticulous reader will note that there is an ambiguity in our definition of caps and cups of thickness k, in that we did not choose a particular choice for the symmetric $\mathfrak{sl}_2$-web which splits a k-labeled strand into k strands of thickness 1. Indeed, it follows from the Frobenius relations (4) in $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ that the corresponding morphisms in $\mathfrak{sl}_2$-$\mathbf{fdMod}$ are the same. The concerned reader can use their favorite such symmetric $\mathfrak{sl}_2$-web as the one used in the above definition.

The reader may also be curious about our choices in the definition of $\Gamma_{\mathbf{sym}}$ on split and merge morphisms, i.e. why not set

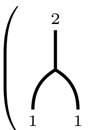
$\Gamma_{\mathbf{sym}}\Bigg($  $\Bigg) = \Phi_2(F^{(k)}1_{(k,l)})$, $\Gamma_{\mathbf{sym}}\Bigg($  $\Bigg) = \Phi_2(E^{(k)}1_{(0,k+l)})$?

Indeed, this will lead to the same definition, following from the equalities

$$E^{(k+l)}F^{(k)}1_{(k,l)} = E^{(l)}1_{(k,l)} \quad , \quad E^{(k)}F^{(k+l)}1_{(k+l,0)} = F^{(l)}1_{(k+l,0)}$$

in $\dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_2)$ and the fact that $\Phi_2(E^{(k)}_{(0,k)})$ and $\Phi_2(F^{(k)}_{(k,0)})$ are both the identity morphism of $\mathrm{Sym}_q^k(\mathbb{C}_q^2)$.

Example 2.19. Since we will need these explicitly later, we now record the $(1,1)$-merge and the $(1,1)$-split morphisms. They are given by

(24) $\Gamma_{\mathbf{sym}}\Bigg($  $\Bigg) = \mathrm{m}\colon \mathbb{C}_q^2 \otimes \mathbb{C}_q^2 \to \mathrm{Sym}_q^2\mathbb{C}_q^2, \quad \begin{cases} x^{\underline{11}} \mapsto x_{\underline{11}},\ x^{\underline{12}} \mapsto x_{\underline{12}}, \\ x^{\underline{21}} \mapsto qx_{\underline{12}},\ x^{\underline{22}} \mapsto x_{\underline{22}}, \end{cases}$

and

(25) $\Gamma_{\mathbf{sym}}\Bigg($  $\Bigg) = \mathrm{s}\colon \mathrm{Sym}_q^2\mathbb{C}_q^2 \to \mathbb{C}_q^2 \otimes \mathbb{C}_q^2, \quad \begin{cases} x_{\underline{11}} \mapsto [2]x^{\underline{11}},\ x_{\underline{22}} \mapsto [2]x^{\underline{22}}, \\ x_{\underline{12}} \mapsto q^{-1}x^{\underline{12}} + x^{\underline{21}}. \end{cases}$

Moreover, the 2-labeled cap is given by

(26) $\Gamma_{\mathbf{sym}}\Bigg($  $\Bigg) = \mathrm{cap}_2\colon \mathrm{Sym}_q^2\mathbb{C}_q^2 \otimes \mathrm{Sym}_q^2\mathbb{C}_q^2 \to \mathbb{C}_q, \quad \begin{cases} x_{\underline{11}} \otimes x_{\underline{22}} \mapsto q^2[2],\ x_{\underline{12}} \otimes x_{\underline{22}} \mapsto -1, \\ x_{\underline{22}} \otimes x_{\underline{11}} \mapsto [2],\ \text{rest} \mapsto 0. \end{cases}$

We encourage the reader to work out cup_2, which we will use in algebraic form below as well.

Lemma 2.20. $\Gamma_{\mathbf{sym}}$ descends to give a monoidal functor $\Gamma_{\mathbf{sym}}\colon \mathbf{SymSp}(\mathfrak{sl}_2) \to \mathfrak{sl}_2\text{-}\mathbf{fdMod}$.

Proof. It is clear that, if $\Gamma_{\mathbf{sym}}$ is well-defined, then it also preserves the monoidal structure (which is given by placing diagrams next to each other). To check that $\Gamma_{\mathbf{sym}}$ is well-defined, it suffices to show that the relations of the symmetric $\mathfrak{sl}_2$-spider $\mathbf{SymSp}(\mathfrak{sl}_2)$ hold in $\mathfrak{sl}_2\text{-}\mathbf{fdMod}$.

The "standard" $\mathfrak{sl}_n$-web relations – Frobenius (4), digon and square removals (6) and the square switches (7) – follow from Corollary 2.7, since these are all induced by relations in $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$.

Here we have to utilize the property that the images of the divided powers $F_i^{(j)}1_{\vec{k}}$ and $E_i^{(j)}1_{\vec{k}}$ under $\Phi_m\colon \dot{\mathbf{U}}_q(\mathfrak{gl}_m) \to \mathfrak{sl}_2\text{-}\mathbf{fdMod}$ coincide with the images of the general symmetric ladders in equation (21) under $\Gamma_{\mathbf{sym}}\colon \mathbf{SymSp}(\mathfrak{sl}_2) \to \mathfrak{sl}_2\text{-}\mathbf{fdMod}$. This follows from our definition of $\Gamma_{\mathbf{sym}}$ on symmetric merge and split $\mathfrak{sl}_2$-webs, the $\dot{\mathbf{U}}_q(\mathfrak{gl}_3)$ equalities

$$E_1^{(j)}1_{(k,l,0)} = E_2^{(l-j)}E_1^{(j)}F_2^{(l-j)}1_{(k,l,0)} \quad , \quad F_1^{(j)}1_{(k,l,0)} = E_2^{(l)}F_1^{(j)}F_2^{(l)}1_{(k,l,0)}$$

and the fact that the diagram

$$\begin{array}{ccc} \dot{\mathbf{U}}_q(\mathfrak{gl}_m) & \hookrightarrow & \dot{\mathbf{U}}_q(\mathfrak{gl}_{m+1}) \\ & \searrow^{\Phi_m} & \downarrow \Phi_{m+1} \\ & & \mathfrak{sl}_2\text{-}\mathbf{fdMod} \end{array}$$

commutes for any of the standard inclusions $\dot{\mathbf{U}}_q(\mathfrak{gl}_m) \hookrightarrow \dot{\mathbf{U}}_q(\mathfrak{gl}_{m+1})$.

It now remains to check that the additional symmetric and isotopy relations are satisfied.

Circle removal: The circle removal follows from the computation

$$(\mathrm{cap} \circ \mathrm{cup})(1) = \mathrm{cap}(x^{\underline{12}}) - q^{-1}\mathrm{cap}(x^{\underline{21}}) = -q - q^{-1} = -[2]$$

where we point out the negative sign to the reader. As is known to experts, this is unavoidable if one wishes to have isotopy invariance in an unoriented model.

Dumbbell relation: This can again be directly verified. For example, we have

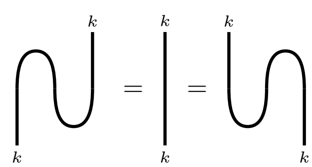
$$(\mathrm{s} \circ \mathrm{m})(x^{\underline{21}}) = x^{\underline{12}} + qx^{\underline{21}}$$

and

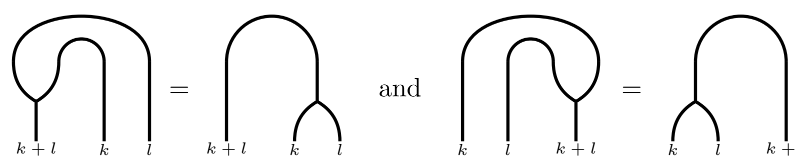
$$([2]\mathrm{id} + \mathrm{cup} \circ \mathrm{cap})(x^{\underline{21}}) = [2]x^{\underline{21}} + x^{\underline{12}} - q^{-1}x^{\underline{21}} = x^{\underline{12}} + qx^{\underline{21}}.$$

The remainder of the computations follow similarly.

Isotopy relations: The remaining isotopy relations locally reduce to the following relations:

(27) 

and

(28) 

and versions of the ones from (28) involving cups.

We start with (28), first noting that it suffices to verify the case where either $k = 1$ or $l = 1$. Indeed, assuming that the relation is known in these cases, we can repeatedly use the first relation in equation (6) to explode the k- and l-labeled strands into 1-labeled strands. We can then pull each of the split and merge vertices (which all have at least one 1-labeled strand) around the cap, and use (6) to reassemble the exploded strand.

For the remaining cases, we can then use a similar argument to verify the relation (doubly) inductively. All together, we see that it suffices to prove the relations explicitly when $k = 1 = l$. We hence compute that the lower part of the lefthand side of the first equation in (28) is given by

$$\mathrm{s} \otimes \mathrm{id} \otimes \mathrm{id} \colon \mathrm{Sym}^2_q\mathbb{C}^2_q \otimes \mathbb{C}^2_q \otimes \mathbb{C}^2_q \to \mathbb{C}^2_q \otimes \mathbb{C}^2_q \otimes \mathbb{C}^2_q \otimes \mathbb{C}^2_q, \quad \begin{cases} x_{\underline{11}} \otimes x^{\underline{ij}} \mapsto [2]x^{\underline{11ij}}, \\ x_{\underline{12}} \otimes x^{\underline{ij}} \mapsto q^{-1}x^{\underline{12ij}} + x^{\underline{21ij}}, \\ x_{\underline{22}} \otimes x^{\underline{ij}} \mapsto [2]x^{\underline{22ij}}, \end{cases}$$

for all choices of $i, j \in \{1, 2\}$. Most of these terms will be sent to zero after composing with the top, and the only surviving terms are

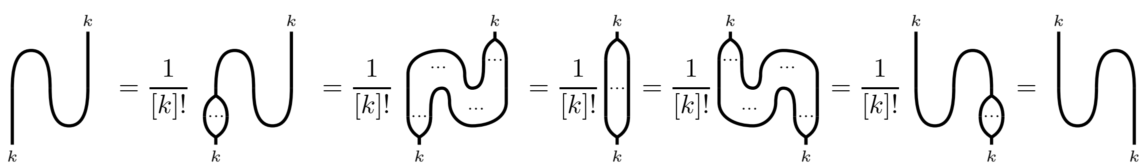
$$\mathrm{cap} \circ (\mathrm{id} \otimes \mathrm{cap} \otimes \mathrm{id}) \circ (\mathrm{s} \otimes \mathrm{id} \otimes \mathrm{id}) \colon \mathrm{Sym}^2_q \mathbb{C}^2_q \otimes \mathbb{C}^2_q \otimes \mathbb{C}^2_q \to \mathbb{C}_q, \quad \begin{cases} x_{\underline{11}} \otimes x^{\underline{22}} \mapsto q^2[2], \\ x_{\underline{12}} \otimes x^{\underline{12}} \mapsto -1, \\ x_{\underline{12}} \otimes x^{\underline{21}} \mapsto -q, \\ x_{\underline{22}} \otimes x^{\underline{11}} \mapsto [2]. \end{cases}$$

The bottom part of the righthand side is given (for all $1 \leq i \leq j \leq 2$) by

$$\mathrm{id} \otimes \mathrm{m} \colon \mathrm{Sym}^2_q \mathbb{C}^2_q \otimes \mathbb{C}^2_q \otimes \mathbb{C}^2_q \to \mathrm{Sym}^2_q \mathbb{C}^2_q \otimes \mathrm{Sym}^2_q \mathbb{C}^2_q, \quad \begin{cases} x_{\underline{ij}} \otimes x^{\underline{11}} \mapsto x_{\underline{ij}} \otimes x_{\underline{11}}, \\ x_{\underline{ij}} \otimes x^{\underline{12}} \mapsto x_{\underline{ij}} \otimes x_{\underline{12}}, \\ x_{\underline{ij}} \otimes x^{\underline{21}} \mapsto q x_{\underline{ij}} \otimes x_{\underline{12}}, \\ x_{\underline{ij}} \otimes x^{\underline{22}} \mapsto x_{\underline{ij}} \otimes x_{\underline{22}}, \end{cases}$$

which composes with the map in equation (26) to give the correct result. The check of the second equation in (28) is similar, as are the checks of the versions of this relation involving cups.

We can deduce the general form of (27) from the $k = 1$ case and the relations in (28) (and their analogs) using the following diagrammatic argument:



Here the middle equalities follow from repeated application of the case $k = 1$, and the diagram in the middle is, by digon removal (6), the identity.

The $k = 1$ case follows by combining the computation

$$(\mathrm{id} \otimes \mathrm{cup})(x_i) = x^{\underline{i12}} - q^{-1} x^{\underline{i21}}$$

with

$$(\mathrm{cap} \otimes \mathrm{id})(x^{\underline{i12}}) = \begin{cases} 0, & \text{if } i = 1, \\ +1, & \text{if } i = 2, \end{cases} \quad , \quad (\mathrm{cap} \otimes \mathrm{id})(x^{\underline{i21}}) = \begin{cases} 0, & \text{if } i = 2, \\ -q, & \text{if } i = 1, \end{cases}$$

for the left diagram and

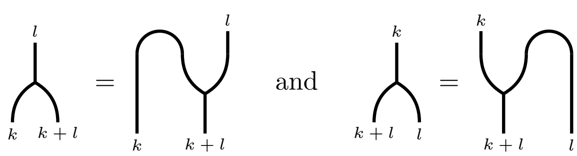
$$(\mathrm{cup} \otimes \mathrm{id})(x_i) = x^{\underline{12i}} - q^{-1} x^{\underline{21i}}$$

with

$$(\mathrm{id} \otimes \mathrm{cap})(x^{\underline{12i}}) = \begin{cases} 0, & \text{if } i = 2, \\ +1, & \text{if } i = 1, \end{cases} \quad , \quad (\mathrm{id} \otimes \mathrm{cap})(x^{\underline{21i}}) = \begin{cases} 0, & \text{if } i = 1, \\ -q, & \text{if } i = 2, \end{cases}$$

for the right. We point out that the signs work out as they should.

Finally, we point out that all isotopies similar to



are not relations, but rather definitions of the elements on the left-hand sides. □

As a consequence of this proof, we immediately observe the following.

Corollary 2.21. The diagram from (20) commutes. □

Remark 2.22. We can extend $\Gamma_{\mathbf{sym}}$ additively to a functor

$$\Gamma_{\mathbf{sym}} \colon \mathbf{Mat}(\mathbf{SymSp}(\mathfrak{sl}_2)) \to \mathfrak{sl}_2\textbf{-fdMod}$$

that we, by abuse of notation, denote using the same symbol. Here $\mathbf{Mat}(\mathbf{SymSp}(\mathfrak{sl}_2))$ is the additive closure of the symmetric $\mathfrak{sl}_2$-spider. As we recalled above before Theorem 1.10, this means that objects of $\mathbf{Mat}(\mathbf{SymSp}(\mathfrak{sl}_2))$ are finite, formal direct sums of the objects of $\mathbf{SymSp}(\mathfrak{sl}_2)$ and morphisms are matrices (whose entries are morphisms from $\mathbf{SymSp}(\mathfrak{sl}_2)$) between these sums. Note that this category is again entirely diagrammatic.

We are now ready to prove Theorem 1.10.

Proof (of Theorem 1.10). We have a well-defined functor $\Gamma_{\mathbf{sym}} \colon \mathbf{Mat}(\mathbf{SymSp}(\mathfrak{sl}_2)) \to \mathfrak{sl}_2\textbf{-fdMod}$ that preserves the monoidal structure. It only remains to show that $\Gamma_{\mathbf{sym}}$ is essentially surjective, full and faithful.

Essentially surjective: This follows directly from the definition of $\Gamma_{\mathbf{sym}}$, since every finite dimensional $\mathbf{U}_q(\mathfrak{sl}_2)$-module is isomorphic to a direct sum of copies of $\mathrm{Sym}_q^k \mathbb{C}_q^2$.

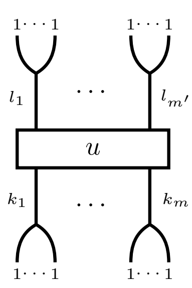
Full and faithful: By additivity, we can verify everything on objects of the form $\vec{k} \in \mathbb{Z}_{>0}^m$. Given $\vec{k} \in \mathbb{Z}_{>0}^m, \vec{l} \in \mathbb{Z}_{>0}^{m'}$, we have to show that

$$\mathrm{Hom}_{\mathbf{SymSp}(\mathfrak{sl}_2)}(\vec{k}, \vec{l}) \cong \mathrm{Hom}_{\mathfrak{sl}_2\textbf{-fdMod}}(\Gamma_{\mathbf{sym}}(\vec{k}), \Gamma_{\mathbf{sym}}(\vec{l})) \tag{29}$$

as $\mathbb{C}(q)$-vector spaces. Surjectivity in (29) follows[9] from Corollary 2.7 and Corollary 2.21.

To see injectivity in (29) above, we start by considering the case when $k_i = 1$ and $l_j = 1$ for all $1 \le i \le m$ and $1 \le j \le m'$. Then, given two symmetric $\mathfrak{sl}_2$-webs $u, v \in \mathrm{Hom}_{\mathbf{SymSp}(\mathfrak{sl}_2)}(\vec{k}, \vec{l})$, we can use Proposition 2.13 and the Jones-Wenzl recursion from Lemma 2.12 to express these two symmetric $\mathfrak{sl}_2$-webs in terms of Temperley-Lieb diagrams. Since the only non-isotopy Temperley-Lieb relation (that is, equation (12)) is a subset of the symmetric $\mathfrak{sl}_2$-web relations, distinct symmetric $\mathfrak{sl}_2$-webs give distinct elements in $\mathcal{TL}$. Injectivity then follows from Theorem 1.1 since distinct elements in $\mathcal{TL}$ give distinct $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners. As a consequence of this argument, we see that the functor $\mathcal{I} \colon \mathcal{TL} \to \mathbf{SymSp}(\mathfrak{sl}_2)$ is an inclusion of a full subcategory.

The general case follows from this. Given any symmetric $\mathfrak{sl}_2$-web $u \in \mathrm{Hom}_{\mathbf{SymSp}(\mathfrak{sl}_2)}(\vec{k}, \vec{l})$ we can compose with split and merge morphisms to obtain



$$\in \mathrm{Hom}_{\mathbf{SymSp}(\mathfrak{sl}_2)}(\underbrace{(1,\ldots,1)}_{k_1+\cdots+k_m}, \underbrace{(1,\ldots,1)}_{l_1+\cdots+l_{m'}})$$

where we indicate with dots compositions of merge and split morphisms, the order of which do not matter due to the Frobenius relations (4).

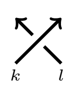
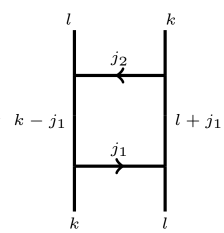
[9] We point out that this shows that all symmetric $\mathfrak{sl}_2$-webs that contain cups and caps can be expressed as linear combinations of compositions of $F_i^{(j)}$ and $E_i^{(j)}$-ladders.

The above argument, together with the digon removals from (6), shows that the images of two distinct symmetric $\mathfrak{sl}_2$-webs $u, v \in \mathrm{Hom}_{\mathbf{SymSp}(\mathfrak{sl}_2)}(\vec{k}, \vec{l})$ have to be distinct. Explicitly, the digon relations show that the splitting procedure is invertible while the argument above shows that the images of their "enlargements" are distinct. $\square$

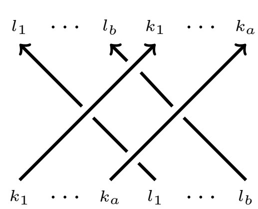
Remark 2.23. We do not state and prove Theorem 1.10 (following history) in terms of a *pivotal* equivalence between $\mathbf{SymSp}(\mathfrak{sl}_2)$ and $\mathfrak{sl}_2$-**fdMod** due to an unavoidable sign issue coming from the use of unoriented diagrams. In our case, this arises since the vector representation of quantum $\mathfrak{sl}_2$ is *anti*-symmetrically self-dual. In order to incorporate this, we would have to make the diagrammatic calculus more sophisticate by introducing extra orientations and tag morphisms (as in [5]). Since these issues are usually not relevant to topological applications before categorifying or passing to the $\mathfrak{sl}_n$ case, we avoid them for the time being and stay closer to the "traditional" Temperley-Lieb calculus.

3. The colored Jones polynomial via symmetric $\mathfrak{sl}_2$-webs

3.1. **Braiding via quantum Weyl group elements.** In this subsection, we extend Theorem 1.10 to incorporate the braided structure on $\mathfrak{sl}_2$-**fdMod**. We begin by defining the following morphisms in $\mathrm{Hom}_{\mathbf{SymSp}(\mathfrak{sl}_2)}\big((k,l),(l,k)\big)$.

(30) $\beta^{\mathrm{Sym}}_{k,l} =$  $= (-1)^k q^{-k-\frac{kl}{2}} \sum_{\substack{j_1,j_2 \geq 0 \\ j_1 - j_2 = k-l}} (-q)^{j_1}$ 

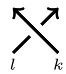
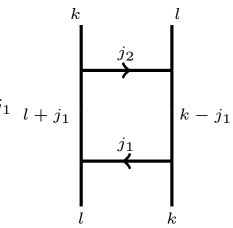
which give rise to the braiding. More generally, for any two objects $\vec{k}, \vec{l}$ in $\mathbf{SymSp}(\mathfrak{sl}_2)$ define

$\beta^{\mathrm{Sym}}_{\vec{k},\vec{l}} =$  $\in \mathrm{Hom}_{\mathbf{SymSp}(\mathfrak{sl}_2)}\big((k_1,\dots,k_a,l_1,\dots,l_b),(l_1,\dots,l_b,k_1,\dots,k_a)\big)$

by taking tensor products of compositions of the morphisms $\beta^{\mathrm{Sym}}_{k,l}$. We now aim to show the following result. To understand it recall that $\mathfrak{sl}_2$-**fdMod** is a *braided* monoidal category where the braiding is induced via the $\mathfrak{sl}_2$-R-matrix (the explicit construction of the braided monoidal structure on the category $\mathfrak{sl}_2$-**fdMod** can be found in many sources, e.g. Chapter XI, Section 2 and Section 7 in [37]).

Theorem 3.1. The morphisms $\beta^{\mathrm{Sym}}_{\vec{k},\vec{l}}$ define a braiding on $\mathbf{SymSp}(\mathfrak{sl}_2)$ and the additive closure of $\mathbf{SymSp}(\mathfrak{sl}_2)$ is braided monoidally equivalent to $\mathfrak{sl}_2$-**fdMod**.

In particular, $\beta^{\mathrm{Sym}}_{k,l}$ is invertible, with inverse explicitly given by

(31) $$\left(\beta^{\mathrm{Sym}}_{k,l}\right)^{-1} = $$  $$ = (-1)^k q^{k+\frac{kl}{2}} \sum_{\substack{j_1,j_2\geq 0\\ j_1-j_2=k-l}} (-q)^{-j_1}$$ 

as can be verified via a direct computation (compare also to Proposition 5.2.3 in [23]).

To prove Theorem 3.1, we will again follow Cautis, Kamnitzer and Morrison (who in turn follow Lusztig [23] and Chuang and Rouquier [7]) by defining the operator[10]

(32) $$T_i 1_{\vec{k}} = (-1)^{k_i} q^{-k_i - \frac{k_i k_{i+1}}{2}} \sum_{\substack{j_1,j_2\geq 0\\ j_1-j_2=k_i-k_{i+1}}} (-q)^{j_1} E_i^{(j_2)} F_i^{(j_1)} 1_{\vec{k}}$$

for any $\mathfrak{gl}_m$-weight $\vec{k} \in \mathbb{Z}^m_{\geq 0}$ and any $i = 1, \ldots, m-1$, called *Lusztig's i-th braiding operator.*

Remark 3.2. These operators specify elements in $\dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_\infty)$, since the sum in (32) truncates to one which is finite. This is due to the fact that sufficiently high powers of $F_i^{(j_1)} 1_{\vec{k}}$ map to $\mathfrak{gl}_m$-weights with negative entries, and hence, are zero in $\dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_\infty)$.

We point out that the elements $T_i 1_{\vec{k}} \in \dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_\infty)$ differ from the corresponding elements of Cautis, Kamnitzer and Morrison, both in that we work with (multiples of) Lusztig's $T''_{i,+1}$ (instead of $T''_{i,-1}$) and since in their setting they kill all $\mathfrak{gl}_m$-weights whose entries do not lie in $\{0,\ldots,n\}$. Fortunately, most of their calculations follow from those of Lusztig in Subsection 5.1.1 of [23]. Thus, we can adopt most of Cautis, Kamnitzer, and Morrison's calculations without further issue.

Lemma 3.3. The $T_i 1_{\vec{k}}$ (viewed as elements of $\dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_\infty)$) are invertible and satisfy the braid relations

$$T_{i+1}T_iT_{i+1}1_{\vec{k}} = T_iT_{i+1}T_i1_{\vec{k}} \quad \text{and} \quad T_iT_{i'}1_{\vec{k}} = T_{i'}T_i1_{\vec{k}} \quad \text{if } |i-i'|$$

for all $\mathfrak{gl}_m$-weights $\vec{k} \in \mathbb{Z}^m_{\geq 0}$ and all $i, i' = 1, \ldots, m-1$ (and all $m \in \mathbb{N}$).

Proof. Almost word-for-word as in Lemma 6.1.1 and Lemma 6.1.2 from [5]. □

We now proceed with the proof of Theorem 3.1.

Proof (of Theorem 3.1). The one-line explanation is that both $\beta^{\mathrm{Sym}}_{\vec{k},\vec{l}}$ and the braiding on $\mathfrak{sl}_2$-**fdMod** come from Lusztig's braiding operator from (32) above.

To be more thorough, we first introduce an analog of $\dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_\infty)$ akin to the category studied by Cautis, Kamnitzer and Morrison. Let

$$\dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_\bullet) = \bigoplus_{m>0} \dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_m)$$

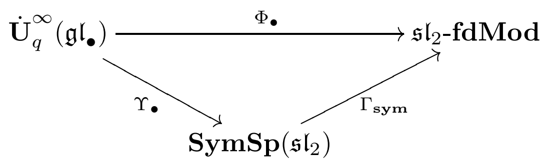
which is in fact a monoidal category. For example, the tensor product is given on objects by concatenating a $\mathfrak{gl}_{m_1}$-weight with a $\mathfrak{gl}_{m_2}$-weight to obtain a $\mathfrak{gl}_{m_1+m_2}$-weight (see Section 6 of [5] for more details). Given a $\mathfrak{gl}_{m_1}$-weight $\vec{k}$ and a $\mathfrak{gl}_{m_2}$-weight $\vec{l}$, define the braiding operator

(33) $$\beta^{\infty}_{\vec{k}\vec{l}} = T_w 1_{\vec{k}\otimes\vec{l}}, \quad \text{where } w \text{ is the permutation } w(i) = \begin{cases} m_2 + i, & \text{if } i \leq m_1, \\ i - m_1, & \text{if } i > m_1, \end{cases}$$

[10] Formally, we must work over $\mathbb{C}(q^{\frac{1}{2}})$ to define these, hence we pass to these coefficients.

and $T_w = T_{i_1} \cdots T_{i_r}$ when $w = s_{i_1} \cdots s_{i_r}$ is a reduced expression (the choice of reduced expression does not matter by Lemma 3.3). A direct adaptation of Theorem 6.1.4 in [5] shows that these elements endow $\dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_\bullet)$ with the structure of a braided monoidal category (this uses again Lemma 3.3 which, as mentioned in Remark 3.2, is based on calculations by Lusztig).

We now claim that the functors in the triangle



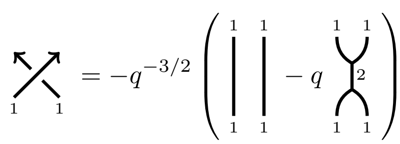
induced by the functors in the commuting diagram from (20) are braided, which suffices to prove the result. The fact that $\mathbf{SymSp}(\mathfrak{sl}_2)$ is braided and that $\Upsilon_\bullet$ preserves the braiding follows directly by comparing equations (30) and (32) (and the fact that this functor is full and essentially surjective).

It finally suffices to show that $\Phi_\bullet$ is braided. Explicitly, we must check that

$$\Phi_\bullet(\beta^{\infty}_{\vec{k},\vec{l}}) = \beta^R_{\Phi_\bullet(\vec{k}),\Phi_\bullet(\vec{l})} = \beta^R_{\mathrm{Sym}_q^{k_1}\mathbb{C}_q^n \otimes \cdots \otimes \mathrm{Sym}_q^{k_m}\mathbb{C}_q^n, \mathrm{Sym}_q^{l_1}\mathbb{C}_q^n \otimes \cdots \otimes \mathrm{Sym}_q^{l_{m'}}\mathbb{C}_q^n}, \tag{34}$$

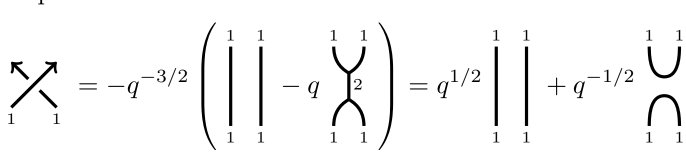
where β^R denotes the braiding coming from the $\mathfrak{sl}_2$-R-matrix (as mentioned above). To see this, we note that all of the steps used to prove Theorem 6.2.1 in [5] carry directly over to the symmetric case. Their arguments reduce to showing that $\Phi_\bullet(\beta^{\infty}_{1,1}) = \beta^R_{\mathbb{C}_q^2,\mathbb{C}_q^2}$, where the latter denotes the standard braiding on $\mathbb{C}_q^2 \otimes \mathbb{C}_q^2$ given by the $\mathfrak{sl}_2$-R-matrix.

To check this final equality, it suffices to show that when $k = 1 = l$, equation (30) maps under $\Gamma_{\mathbf{sym}}$ to the braiding on $\mathbb{C}_q^2 \otimes \mathbb{C}_q^2$. As mentioned in Lemma 6.2.2 of [5], β^R is determined on this by the fact that it acts by $q^{1/2}$ on $\mathrm{Sym}_q^2(\mathbb{C}_q^2)$ and by $-q^{-3/2}$ on $\bigwedge_q^2(\mathbb{C}_q^2)$. In this case equation (30) is given by



and since the second term in $\Gamma_{\mathbf{sym}}(\beta^{\mathrm{Sym}}_{k,l})$ factors through $\mathrm{Sym}_q^2(\mathbb{C}_q^2)$, this acts by $-q^{-3/2}$ on $\bigwedge_q^2(\mathbb{C}_q^2)$. Similarly, equation (13) shows that the dumbbell acts on $\mathrm{Sym}_q^2(\mathbb{C}_q^2)$ by multiplying with [2]. From this we see that $\Gamma_{\mathbf{sym}}(\beta^{\mathrm{Sym}}_{k,l})$ acts by $-q^{-3/2}(1 - q[2]) = q^{1/2}$ as desired.

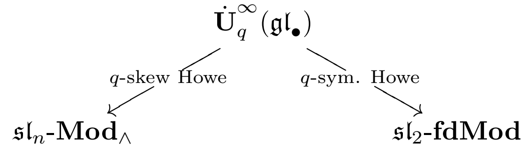
Alternatively, we can check graphically that this agrees with the standard formula for a (positive) crossing in $\mathcal{TL}$. We compute that



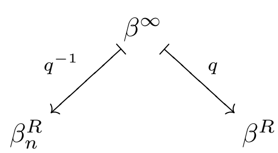
Here we remind the reader that the dumbbell can be replaced by [2] times the identity plus a cap-cup. This is the Kauffman bracket formula for the braiding on $\mathfrak{sl}_2$-**fdMod** (which is known to give the same result as the one coming from the $\mathfrak{sl}_2$-R-matrix braiding). □

Remark 3.4. More generally, the above argument extends without difficulties to show that the braiding in $\mathfrak{sl}_n$-**fdMod** between tensor products of the $\mathfrak{sl}_n$-modules $\mathrm{Sym}_q^k(\mathbb{C}_q^n)$ coming from the $\mathfrak{sl}_n$-R-matrix is given as the image of the braiding β^{∞} of $\dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_\bullet)$ under the functor $\Phi_\bullet^n \colon \dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_\bullet) \to \mathfrak{sl}_n$-**fdMod** induced by the functors from (19).

Remark 3.5. In [5], they show that the braiding between tensor products of fundamental representations in $\mathfrak{sl}_n$-$\mathbf{Mod}_\wedge$ is similarly given by Lusztig's braiding operators $T_i 1_{\vec{k}} \in \dot{\mathbf{U}}_q^n(\mathfrak{gl}_\bullet)$, where $\dot{\mathbf{U}}_q^n(\mathfrak{gl}_\bullet)$ is the quotient of $\dot{\mathbf{U}}_q^\infty(\mathfrak{gl}_\bullet)$ by all $\mathfrak{gl}_\bullet$-weights containing an entry strictly lower than 0 or strictly larger than n. In addition, they show that q-skew Howe duality gives a braided monoidal functor $\dot{\mathbf{U}}_q^n(\mathfrak{gl}_\bullet) \to \mathfrak{sl}_n$-$\mathbf{Mod}_\wedge$. Since we have maps $\dot{\mathbf{U}}_q^\infty(\mathfrak{gl}_\bullet) \to \dot{\mathbf{U}}_q^n(\mathfrak{gl}_\bullet)$, this gives the following diagram of braided monoidal functors



We again point out that there is a slight difference between the q-symmetric Howe duality and the q-skew Howe duality cases coming from the fact that we need to use Lusztig's $T''_{i,+1}$ instead of $T''_{i,-1}$, which is utilized by Cautis, Kamnitzer, and Morrison. Since $T''_{i,+1}$ and $T''_{i,-1}$ only differ by a substitution of $q \leftrightarrow q^{-1}$, this gives the schematic



where we point out that β_n^R is the braiding of $\mathfrak{sl}_n$-$\mathbf{Mod}_\wedge$ coming from the $\mathfrak{sl}_n$-R-matrix while β^R is the braiding of $\mathfrak{sl}_2$-$\mathbf{fdMod}$ coming from the $\mathfrak{sl}_2$-R-matrix

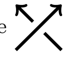
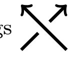
This observation appears related to the decategorification of the "mirror symmetry" between colored HOMFLY-PT homology conjectured in [11] (e.g. in (5-17) in their paper). See Section 3.3 below for a more precise discussion.

3.2. **The colored Jones polynomials via "MOY"-graphs.** In this subsection we explore how the braiding from Subsection 3.1 on the symmetric $\mathfrak{sl}_2$-spider can be used to study the *colored Jones polynomials* of colored, oriented links L, which we denote by $\mathcal{J}_{\vec{c}}(L_D)$. Here $\vec{c} = (c_1, \dots, c_N)$ denotes the colors of the N-component, oriented link L and L_D is a colored, oriented diagram for L. In the interest of brevity, we refer the reader to the wide literature on the subject, in particular Chapter XI, Section 7 of [37], for the definition of this invariant and a thorough treatment of its properties. We only comment that it can be computed by associating a morphism between trivial representations in $\mathfrak{sl}_2$-$\mathbf{fdMod}$ to any colored, oriented link diagram L_D of a colored, oriented link L (and rescaling to get an invariant which is not framing-dependent).

This translates to using equations (30) and (31) to view a colored, oriented link diagram for L_D as a morphism in $\mathbf{SymSp}(\mathfrak{sl}_2)$, which necessarily evaluates to an element in $\mathbb{C}(q^{\frac{1}{2}})$ (in fact, one can show that one always gets an element in $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$), and multiplying by a certain normalization factor which can be computed directly from the crossing data of the diagram. For the 1-colored case this factor is $(-q^{\frac{3}{2}})^{-\omega(L_D)}$, where $\omega(L_D)$ is the writhe[11] of L_D. More generally, if K_D is a colored, oriented knot diagram colored with the color c, then we rescale via

$$\mathcal{J}_{\vec{c}}(K_D) = -q^{-C\omega}\overline{\mathcal{J}}_{\vec{c}}(K_D). \tag{35}$$

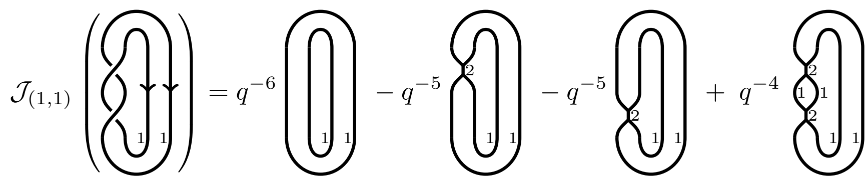
Here $\overline{\mathcal{J}}_{\vec{c}}(K_D)$ is the *framing depended*, colored Jones polynomial and $C = \frac{c^2+2c}{2}$ is the so-called *quadratic Casimir number* of the color $\mathrm{Sym}_q^c\mathbb{C}_q^2$. For a colored, oriented link diagram L_D one normalizes

[11]The writhe is the difference between the number of positive  and negative crossings  in the diagram.

as for colored, oriented knot diagrams K_D, multiplying by the different normalization factors for each component.

We note that this approach is similar in the 1-colored case to computing the Jones polynomial using the Kauffman bracket, but in the colored case avoids the use of cabling and Jones-Wenzl projectors, trading them instead for our "symmetric version" of the MOY-calculus [28] typically used to compute the $\bigwedge_q^k \mathbb{C}_q^n$-colored $\mathfrak{sl}_n$-link invariant.

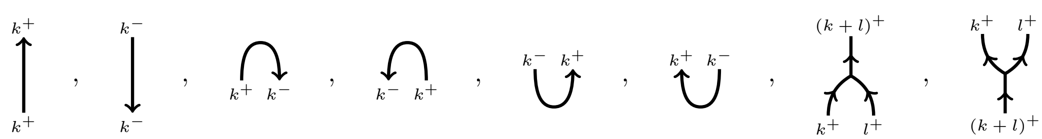
Example 3.6. As an example, we compute the (1-colored) Jones polynomial of the Hopf link using symmetric $\mathfrak{sl}_2$-webs.



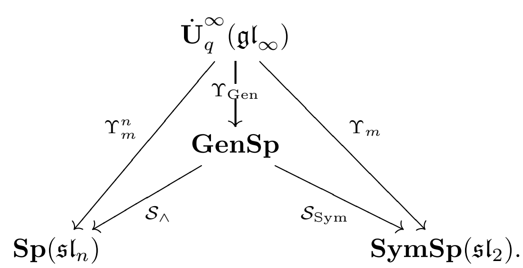
$$= q^{-6}[2]^2 - 2q^{-5}[2][3] + q^{-4}[2]^2[3] = [2](q^{-1} + q^{-5})$$

This is (up to a normalization factor $-[2]$ and possible different conventions for q) the known formula for the Jones polynomial of the Hopf link.

3.3. **A remark on "mirror symmetry".** We now aim to give a slightly more precise formulation of the "mirror symmetry" phenomena mentioned in Remark 3.5. Consider the *generic spider* **GenSp**, the category whose objects are tuples in the symbols $k^{\pm}$ for $k \in \mathbb{Z}_{>0}$, and whose morphisms are $\mathbb{C}[q, q^{-1}]$-linear combinations of *generic webs*, that is, oriented, trivalent graphs generated by

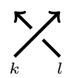


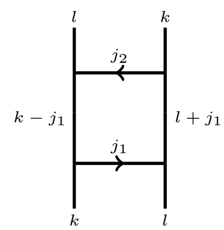
modulo planar isotopy and the (oriented) standard $\mathfrak{sl}_n$-web relations from equations (4), (6) and (7). The oriented version of Lemma 2.16 gives a functor $\Upsilon_{\mathrm{Gen}} \colon \dot{\mathbf{U}}_q^{\infty}(\mathfrak{gl}_\infty) \to \mathbf{GenSp}$. Since **GenSp** clearly admits specialization functors to $\mathbf{Sp}(\mathfrak{sl}_n)$ and $\mathbf{SymSp}(\mathfrak{sl}_2)$, which we denote by $\mathcal{S}_\wedge$ and $\mathcal{S}_{\mathrm{Sym}}$ respectively, we have the following commuting diagram:



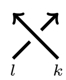
Here Υ_m^n is the functor from Subsection 5.2. in [5].

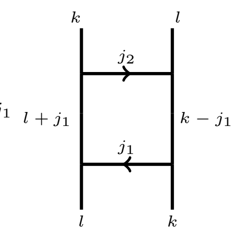
Given a colored, oriented braid B, the non-rescaled crossing formulae

(36) $\beta^{\mathrm{Gen}}_{k,l} =$ 

$= \sum_{\substack{j_1,j_2\geq 0\\ j_1-j_2=k-l}} (-q)^{j_1}$ 

and its inverse

(37) $\left(\beta^{\mathrm{Gen}}_{k,l}\right)^{-1} =$ 

$= \sum_{\substack{j_1,j_2\geq 0\\ j_1-j_2=k-l}} (-q)^{-j_1}$ 

assign a morphism in **GenSp** to its closure $\mathrm{cl}(B)$, which maps to a multiple of the colored Jones polynomial of $\mathrm{cl}(B)$ under the functor $\mathcal{S}_{\mathrm{Sym}}\colon \mathbf{GenSp} \to \mathbf{SymSp}(\mathfrak{sl}_2)$.

Note that this element also maps to a multiple of the $\bigwedge_q^k \mathbb{C}_q^n$-colored $\mathfrak{sl}_n$-link polynomial by first making the substitution $q \leftrightarrow q^{-1}$ and then applying the functor $\mathcal{S}_{\wedge}\colon \mathbf{GenSp} \to \mathbf{Sp}(\mathfrak{sl}_n)$, since the braiding in $\mathbf{Sp}(\mathfrak{sl}_n)$ is given by a multiple of the image of equations (36) and (37) after making this substitution.

The relations in **GenSp** suffice[12] to express any closure of a generic web appearing in the morphisms assigned to a (colored, oriented) braid in terms of colored circles. Viewing these colored circles as parameters $\{\xi_i\}_{i=1}^{\infty}$, we arrive at the following result.

Theorem 3.7. There exists an invariant of (colored, oriented) braid conjugacy classes

$$P(B) \in \mathbb{Z}[q, q^{-1}, \xi_1, \xi_2, \ldots]/\mathcal{I},$$

where $\mathcal{I}$ is the (possibly empty) ideal of relations between colored circles in **GenSp**. The specialization $\xi_k = [k+1]$ gives a multiple of the colored Jones polynomial of the closure $\mathrm{cl}(B)$ of B, and the substitution $q \leftrightarrow q^{-1}$ and subsequent specialization $\xi_k = \begin{bmatrix} n \\ k \end{bmatrix}$ gives a multiple of the colored $\mathfrak{sl}_n$-link polynomial of the closure $\mathrm{cl}(B)$ of B. □

The following conjecture is equivalent to the symmetric-skew "mirror symmetry" conjecture of Gukov and Stošić, see e.g. (5-17) in [11].

Conjecture 3.8. There exists a specialization of $P(B)$ which gives (a multiple of) the Sym_q^k-colored HOMFLY-PT polynomial. Applying the substitution $q \leftrightarrow q^{-1}$ yields the $\bigwedge_q^k$-colored HOMFLY-PT polynomial.

A proof of Conjecture 3.8 using our methods would yield a new proof of the known "mirror-symmetry" between colored HOMFLY-PT polynomials which follows (by using completely different methods) from Lemma 4.2 in [22]. One could hope that our approach is more conceptual and leads to new insights on the categorified level (that is, for colored HOMFLY-PT homology) as well.

[12] This fact was observed in joint between the first author and Queffelec at the categorical level [29]. See also Queffelec's recent preprint with Sartori [31] which utilizes and outlines the decategorified statement.

3.4. **And the categorified story?** Khovanov's construction of link homology categorifying the Jones polynomial [17] can be viewed as a categorification of the Temperley-Lieb category $\mathcal{TL}$, as made precise in the work of Bar-Natan [1]. One hence expects that a categorification of our symmetric $\mathfrak{sl}_2$-web category will be the natural setting for a categorification of the colored Jones polynomial. We plan to explore exactly this issue in subsequent work, constructing a 2-category of symmetric $\mathfrak{sl}_2$-foams, akin to previous work by Khovanov [19], Mackaay, Stošić and Vaz [25], Morrison and Nieh [27] and Queffelec and the first author [30].

Such a categorification should give a colored $\mathfrak{sl}_2$-link homology theory which avoids the use of infinite complexes categorifying Jones-Wenzl projectors as in [8], [10] or [32], and hence, will be manifestly finite dimensional (in contrast to those mentioned above, as well as Webster's approach [38]). We point out that work of Hogancamp [12] has shown how to extract a finite dimensional colored $\mathfrak{sl}_2$-link homology theory from these infinite dimensional theories.

We expect the category of symmetric $\mathfrak{sl}_2$-foams to be related to categorified quantum groups, via a symmetric analog of the categorical skew Howe duality pioneered by Cautis, Kamnitzer, and Licata [4] and utilized recently in a large body of work by several researchers (including the authors of this paper), see [21], [24], [26], [30] and [36]. Finally, we suspect that a duality between symmetric and traditional foams will lead to a precise formulation of "mirror symmetry" between (symmetric or skew) colored $\mathfrak{sl}_n$-link homologies.

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Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA
E-mail address: `davidero@usc.edu`

Centre for Quantum Geometry of Moduli Spaces, University Aarhus, Ny Munkegade 118, building 1530, room 316, 8000 Aarhus C, Denmark
E-mail address: `dtubben@qgm.au.dk`