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METRICAL THEOREMS ON SYSTEMS OF SMALL INHOMOGENEOUS LINEAR FORMS

MUMTAZ HUSSAIN AND SIMON KRISTENSEN

ABSTRACT. In this paper we establish complete Khintchine–Groshev and Schmidt type theorems for inhomogeneous small linear forms in the so-called doubly metric case, in which the inhomogeneous parameter is not fixed.

1. INTRODUCTION AND BACKGROUND

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a function tending to 0 at infinity referred to as an *approximation* function. Let $\mathbb{I}$ denotes the unit interval $[-1/2, 1/2]$ and $W_\alpha(m, n; \psi)$ denotes the set of points $(X, \alpha) \in \mathbb{I}^{mn} \times \mathbb{I}^n$ for which the systems of inequalities given by

$$\|q_1 x_{1i} + q_2 x_{2i} + \cdots + q_m x_{mi} - \alpha_i\| \leq \psi(|\mathbf{q}|) \quad \text{for} \quad (1 \leq i \leq n), \tag{1}$$

are satisfied for infinitely many $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$. Here and throughout, $\|\cdot\|$ denotes the distance to the nearest integer and the system $q_1 x_{1i} + \cdots + q_m x_{mi}$ of n linear forms in m variables will be written more concisely as $\mathbf{q}X$, where the matrix X is regarded as a point in $\mathbb{I}^{mn}$ and $|\mathbf{q}|$ denotes the supremum norm of the integer vector $\mathbf{q}$.

If the vector α is fixed then the study of the measure and dimension of this set is referred as the *singly metric* and if α is varying then we call this as the *doubly metric*. In general such a study is termed as 'inhomogeneous Diophantine approximation'. The inhomogeneous Diophantine approximation, in some respects, is different from the homogeneous Diophantine approximation which corresponds to the case when $\alpha = \mathbf{0}$. In the former the results are sometimes sharper and easier to prove then the latter due to the extra variable involved which offer extra degree of freedom. This extra variable can however interfere with the homogenous variables and for this reason there is no inhomogeneous counter part to Dirichlet's theorem (see [8, Chapter 3, Theorem III]).

The fundamental result in the metric Diophantine approximation is the Khintchine–Groshev theorem which relates the size of the set to convergence or divergence of the series which in return only depends upon the approximating function. An inhomogeneous version of Khintchine's theorem for $W_\alpha(1, n; \psi)$ (see [8, Theorem II, Chapter 7]) not only require weaker assumptions than the homogeneous case (see [8, Theorem I, Chapter 7]) but is also considerably easier to prove. The same is true for the more general Khintchine–Groshev theorem which requires a little more work than the simultaneous case. This time the proof is a consequence of Schmidt's quantitative theorem [33, Theorem 2] or a special case of [36, Theorem 15, Chapter 1].

Before proceeding with the description of our main results, we introduce some notation. The Vinogradov symbols $\ll$ and $\gg$ will be used to indicate an inequality with an unspecified positive multiplicative constant depending only on m and n. If

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$a \ll b$ and $a \gg b$ we write $a \asymp b$, and say that the quantities a and b are comparable. A *dimension function* is an increasing continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(r) \to 0$ as $r \to 0$. Throughout the paper, $\mathcal{H}^f$ denotes the f–dimensional Hausdorff measure which will be fully defined in section 2.1. Finally, for convenience, for a given approximating function ψ, define the function

$$\Psi(r) := \frac{\psi(r)}{r}.$$

The Hausdorff dimension of a set E will be denoted by $\dim E$.

A natural question which presents itself at this stage is the nature of the the metric structure of $W_\alpha(m,n;\psi)$ if distance to the nearest integer in (1), $\|\cdot\|$ is replaced by absolute value norm $|\cdot|$, *i.e.*, by replacing the nearest integer vector in (1) to with the zero vector and thus considering system of linear forms which are simultaneously close to the origin. The question may be asked in the broader context if the set consists of systems of linear forms in which the forms of approximation are mixed and some forms are required to be small and others close to an integer. More precisely, for $u \in \mathbb{N}$, let $W_{u,\alpha}(m,n;\psi)$ be the set of points $(X,\alpha) \in \mathbb{I}^{mn} \times \mathbb{I}^n$ such that

$$\max\{|\mathbf{q}\cdot\mathbf{x}^{(1)} - \alpha_1 - p_1|, \dots, |\mathbf{q}\cdot\mathbf{x}^{(u)} - \alpha_u - p_u|, \\ |\mathbf{q}\cdot\mathbf{x}^{(u+1)} - \alpha_{u+1}|, \dots, |\mathbf{q}\cdot\mathbf{x}^{(n)} - \alpha_n|\} < \psi(|\mathbf{q}|) \quad (2)$$

for infinitely many integer vectors $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ and $\mathbf{p} \in \mathbb{Z}^u$ for $0 \le u \le n$. Here, $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ are the column vectors of X.

In the case when $u = 0$, $\alpha = \mathbf{0}$, the set $W_{\mathbf{0},\mathbf{0}}(m,n;\psi)$ is well studied by various authors in [11, 13, 14, 24, 27, 28]. The set arose in literature due to its connections with Kolmogrov–Arnold–Moser theory [1, 17], linearization of germs of complex analytic diffeomorphisms of $\mathbb{C}^m$ near a fixed point [19, 20], in operator theory [9, 10] and in recently discovered applications in signal processing [32].

Note that $W_{0,\alpha}(1,n;\psi) = (0,0)$. In this case any $(x_1,\dots,x_n,\alpha_1,\dots,\alpha_n) \in W_{\mathbf{0},\alpha}(1,n;\psi)$ must satisfy the inequalities

$$\max_{1\le i\le n}\left|x_i - \frac{\alpha_i}{q}\right| < \frac{\psi(q)}{q}$$

for infinitely many $q \in \mathbb{Z}\setminus\{0\}$. Since, $\psi(q) \to 0$ as $q \to \infty$ which implies that $\psi(q)/q \to 0$ as $q \to \infty$, and as also $\alpha_i/q \to 0$ as $q \to \infty$, this is possible only when $(x_1,\dots,x_n,\alpha_1,\dots,\alpha_n) = \mathbf{0}$. It can be readily verified that $W_{u,\alpha}(1,n;\psi) = W_\alpha(1,u;\psi) \times \{0\}^{n-u}$ for $u < n$ so that

$$\dim W_{u,\alpha}(1,n;\psi) \;=\; \dim W_\alpha(1,u;\psi).$$

So, throughout this paper, for the set $W_{u,\alpha}(m,n;\psi)$, we assume that $m > 1$.

As in all the absolute value problems, the measure theoretic results for $W_{u,\alpha}$ crucially depend upon the choices of the parameters m, u and n and the reason for such choices should be clear from Lemma 1.5 below.

Theorem 1.1. *Let $m + u > n$ and let ψ be an approximating function. Let f be a dimension function such that $r^{-(m+1)n} f(r)$ and $r^{-(m+u+1-n)n} f(r)$ are monotonic. Then*

$$\mathcal{H}^f(W_{u,\alpha}(m,n;\psi)) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-mn} r^{m+u-1} < \infty, \\ \mathcal{H}^f(\mathbb{I}^{mn}\times\mathbb{I}^n) & \text{if } \sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-mn} r^{m+u-1} = \infty. \end{cases}$$

Note that we have not assumed the approximating function ψ to be monotonic for any m and n. This is possible due to the extra degree of freedom offered by the variable α. This is a doubly metric inhomogeneous version of the set considered in [28], where an analogous result was obtained for the homogeneous fiber obtained by fixing $\alpha = \mathbf{0}$.

As in most of the statements the convergence part is reasonably straightforward to establish and is free from any assumptions on m, u, n and the approximation function. It is the divergence statement which constitutes the main substance and this is where conditions come into play.

The requirement that $r^{-(m+1)n} f(r)$ and other dimension functions are required to be monotonic is a natural and not particularly restrictive condition. Essentially, the condition ensures that the Hausdorff measures cannot be too degenerate compared with the Lebesgue measure at the dimension $(m+1)n$ of the ambient Euclidean space. In the case when $f(r) := r^{(m+1)n}$ the Hausdorff measure $\mathcal{H}^f$ is proportional to the standard $(m+1)n$–dimensional Lebesgue measure supported on $\mathbb{I}^{(m+1)n}$ and the result is the natural analogue of the Khintchine–Groshev theorem for $W_{u,\alpha}(m, n; \psi)$.

Corollary 1.2. *Let $m + u > n$ and let ψ be an approximating function, then*

$$|W_{u,\alpha}(m, n; \psi)|_{(m+1)n} = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} \Psi(r)^n r^{m+u-1} < \infty, \\ 1 & \text{if } \sum_{r=1}^{\infty} \Psi(r)^n r^{m+u-1} = \infty. \end{cases}$$

From this, it can easily be seen that for $\tau > \frac{m+u}{n} - 1$, $W_{u,\alpha}(m, n; \psi(r \to r^{-\tau})) := W_{u,\alpha}(m, n; \tau)$ is a null set. The following corollary gives the Hausdorff measure and dimension when $f(r) = r^s$, $s > 0$. We refer the reader to [2] and [12] for more details.

Corollary 1.3. *Let $m + u > n$ and ψ be an approximating function. Let s be such that $mn < s \leq (m+1)n$. Then,*

$$\mathcal{H}^s(W_{u,\alpha}(m, n; \psi)) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} \Psi(r)^{s-mn} r^{m+u-1} < \infty, \\ \mathcal{H}^s(\mathbb{I}^{mn} \times \mathbb{I}^n) & \text{if } \sum_{r=1}^{\infty} \Psi(r)^{s-mn} r^{m+u-1} = \infty. \end{cases}$$

Consequently,

$$\dim W_{u,\alpha}(m, n; \psi) = \inf\Big\{ s : \sum_{r=1}^{\infty} \Psi(r)^{s-mn} r^{m+u-1} < \infty \Big\}.$$

Finally, for completeness, the dimension result for $W_{u,\alpha}(m, n; \tau)$ is given for $m + u > n$. This follows directly from the two preceding corollaries.

Corollary 1.4. *For $m + u > n$,*

$$\dim W_{u,\alpha}(m, n; \tau) = \begin{cases} mn + \frac{m+u}{\tau+1} & \text{if } \tau > \frac{m+u}{n} - 1, \\ (m+1)n & \text{if } \tau \leq \frac{m+u}{n} - 1. \end{cases}$$

In fact, the above result gives a lot more than simply the Hausdorff dimension. It also give the Hausdorff measure at the critical exponent which is infinity in this case.

For the case $m + u \leq n$ the conditions on the dimension function in Theorem 1.1 change. This change is due to the fact that if $(X, \alpha) \in W_{u,\alpha}(m, n; \psi)$ and $m + u \leq n$, the affine system with parameters (X, α) is over-determined and its solutions lie in a subset of strictly lower dimension than $(m+1)n$.

To see this consider the case $m = n, \mathbf{u} = 0$ and suppose that $\det X \neq 0$. Multiplying the defining inequalities (2) by X^{-1} implies that we must have

$$|\mathbf{q} - \alpha X^{-1}| \leq C(X)\psi(|\mathbf{q}|),$$

which obviously cannot be true for sufficiently large $\mathbf{q}$, unless the left hand side is equal to zero. The latter can hold true for only a single value of $\mathbf{q}$, and so is not relevant as we require the inequality (2) to hold for infinitely many $\mathbf{q}$. Hence, any solution to infinitely many defining inequalities must lie on the surface defined by $\det X = 0$.

The same logic extends to all other cases. For this we first introduce more notation. For each $m \times n$ matrix $X \in \mathbb{R}^{mn}$ with column vectors $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ define $\tilde{X}$ to be the $m \times (n-u)$ matrix with column vectors $\mathbf{x}^{(u+1)}, \ldots, \mathbf{x}^{(n)}$. The set $\Gamma \subset \mathbb{R}^{mn} \times \mathbb{R}^n$ is the set of $(X, \alpha) \in \mathbb{R}^{mn} \times \mathbb{R}^n$ such that the determinant of each $m \times m$ minor of $\tilde{X}$ is zero. It will now be proved that $W_{u,\alpha}(m, n; \psi) \subset \Gamma$ when $m + u \leq n$.

Lemma 1.5. *For $m + u \leq n$ the set $W_{u,\alpha}(m, n; \psi)$ is contained in Γ and $\dim \Gamma = mn + m + u - 1 < (m+1)n$. Thus*

$$\dim W_{u,\alpha}(m, n; \psi) \leq mn + m + u - 1.$$

Proof. The proof of this lemma is similar to the Lemma 1 of [11]. Therefore we only list the changes here and rest could be worked out easily.

First the dimension of Γ is obtained. Assume that $m = n$, (i.e. $u = 0$). The above simple argument implies that in order for X to be an element of $W_{u,\alpha}(m, n; \psi)$, we must have $\det X = 0$. This gives us $m^2 - 1$ free variables in the X-coordinate and $n = m$ free variables for the α-coordinate. Therefore, the dimension of the set of $(X, \alpha) \in \mathbb{R}^{m^2} \times \mathbb{R}^m$ such that $\det X = 0$ is $m^2 - 1 + m$ as required.

Now assume that $u > 0$ and $m + u = n$. The number of variables in the first u columns of X is mu. In this case, the analogue of the simple argument above shows that in order for (X, α) to be in $W_{u,\alpha}(m, n; \psi)$, we must have $\det \tilde{X} = 0$. Note that $\tilde{X}$ is an $m \times m$ matrix in this case. Again, the number of free variables in $\tilde{X}$ is $m^2 - 1$ as before. Finally, the α-coordinate has $n = m + u$ free variables. Thus the dimension of Γ is $m^2 - 1 + mu + n = mn + m + u - 1$.

Finally, in the case $m + u < n$, if $(X, \alpha) \in W_{u,\alpha}(m, n; \psi)$ so that $|\mathbf{q}X - \alpha| < \psi(|\mathbf{q}|)$ for infinitely many vectors $\mathbf{q} \in \mathbb{Z}^m$, the argument above shows that there must be linear dependence between any m vectors from $\mathbf{x}^{(u+1)}, \ldots, \mathbf{x}^{(n)}$, so that each of the $m \times m$ minors of the matrix $\tilde{X}$ must have determinant zero. Arguing as in [11], we see that there are $(m-1)m + (n - m - u + 1)(m-1)$ degrees of freedom in these coordinates. Adding the mu degrees of freedom from the first m columns of x and the n degrees of freedom from the inhomogeneous part, we see that the dimension of Γ is $(m-1)m + (n - m - u + 1)(m-1) + mu + n = mn + m + u - 1$. □

Theorem 1.6. *Let $m + u \leq n$ and ψ be an approximating function. Let f and $r^{-m(n-m-u+1)} f(r)$ be dimension functions. Assume that $r^{-mn-m-u+1} f(r)$ is monotonic. Then*

$$\mathcal{H}^f(W_{u,\alpha}(m, n; \psi))) = 0 \quad \textit{if} \quad \sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-mn} r^{m+u-1} < \infty.$$

On the other hand, if

$$\sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-mn} r^{m+u-1} = \infty,$$

then

$$\mathcal{H}^f\left(W_{u,\alpha}\left(m,n;\psi\right)\right)=\begin{cases}\infty & \text{if } r^{-mn-m-u+1}f(r)\to\infty \text{ as } r\to 0,\\ K & \text{if } r^{-mn-m-u+1}f(r)\to C \text{ as } r\to 0,\end{cases}$$

for some fixed constant $0\leq C<\infty$*, where* $0<K<\infty$.

Note that if f satisfies $r^{-mn-m-u+1}f(r)\to C>0$ as $r\to 0$ then the Hausdorff measure $\mathcal{H}^f$ is comparable to $(mn+m+u-1)$–dimensional Lebesgue measure. Analogous of the corollaries 1.2 and 1.3 may be stated in the similar way. For the sake of brevity we only state the analogue of Corollary 1.4.

Corollary 1.7. *For* $m+u\leq n$*,*

$$\dim W_{\mathbf{u},\alpha}(m,n;\tau)=\begin{cases}mn+\frac{m+u}{\tau+1} & \text{if } \tau>\frac{m+u}{m+u-1}-1,\\ mn+m+u-1 & \text{if } \tau\leq\frac{m+u}{m+u-1}-1.\end{cases}$$

Another set of interest is in a sense complementary to $W_{\alpha}\left(m,n;\psi\right)$, namely the set of inhomogeneous badly approximable affine forms. First we define the badly approximable affine forms as

$$\mathrm{Bad}_{\alpha}(m,n)=\{(X,\alpha)\in\mathbb{I}^{mn}\times\mathbb{I}^{n}:\inf_{\mathbf{q}\in\mathbb{Z}^m\setminus\{\mathbf{0}\}}|\mathbf{q}|^{m/n}\|\mathbf{q}X-\alpha\|>0\}.$$

In his seminal paper [30], Kleinbock used ideas and techniques from the theory of dynamical systems to prove a singly metric result, which implies that that $\mathrm{Bad}_{\alpha}(m,n)$ is of maximal Hausdorff dimension. Essentially, his method is based on a deep connection between badly approximable systems of linear forms and orbits of certain lattices in Euclidean space under an appropriate action. One possible strengthening of such dimension results is to show that the sets in question are winning for the so-called Schmidt game. These games were introduced by Schmidt in [34] and later used by himself in [35] to prove that the set $\mathrm{Bad}(m,n)$, obtained by fixing $\alpha=0$ above is winning. It was proved by Einsiedler and Tseng in [22] that each fiber obtained by fixing an α in the above set $\mathrm{Bad}_{\alpha}(m,n)$ is winning, which is a considerable strengthening of Kleinbock's result.

In this paper we consider the variant of the set $\mathrm{Bad}_{\alpha}(m,n)$ complementary to $W_{u,\alpha}\left(m,n;\psi\right)$ obtained by replacing distance to the nearest integer $\|\cdot\|$ for some of the linear forms by absolute value $|\cdot|$ in spirit of the well approximable set discussed above. To be precise, let $\mathrm{Bad}_{u,\alpha}(m,n)$ denote the set of all $(X,\alpha)\in\mathbb{I}^{mn}\times\mathbb{I}^{n}$ for which there exists a constant $C(X)$ such that

$$\max\{|\mathbf{q}\cdot\mathbf{x}^{(\mathbf{1})}-\alpha_1-p_1|,\dots,|\mathbf{q}\cdot\mathbf{x}^{(\mathbf{u})}-\alpha_u-p_u|,\\ |\mathbf{q}\cdot\mathbf{x}^{(\mathbf{u+1})}-\alpha_{u+1}|,\dots,|\mathbf{q}\cdot\mathbf{x}^{(\mathbf{n})}-\alpha_n|\}\geq C(X)(|\mathbf{q}|)^{-(m+1)/n}$$

for all integer vectors $(\mathbf{p},\mathbf{q})\in\mathbb{Z}^u\times\mathbb{Z}^m$ for $0\leq u\leq n$. It is an easy consequence of Theorem 1.1 that for $m+u>n$, $\mathrm{Bad}_{u,\alpha}(m,n)$ is a null–set, i.e. $|\mathrm{Bad}_{u,\alpha}(m,n)|_{(m+1)n}=0$. This raises the natural question of the Hausdorff dimension of $\mathrm{Bad}_{u,\alpha}(m,n)$ for any choices of m,u,n. Considering only the homogeneous fiber $\alpha=0$ and further setting $u=0$, the dimension of the obtained fiber was first studied in [25], wherein it is proved that the set of such badly approximable linear forms in m variables has maximal dimension. Later in [26], this was extended not only to all systems of linear forms but also to the case $0\leq u\leq n$. In this paper, we prove the following doubly metric result.

Theorem 1.8. *The Hausdorff dimension of* $\mathrm{Bad}_{u,\alpha}(m,n)$ *is maximal. If* $m+u\leq n$*, the Lebesgue measure of* $\mathrm{Bad}_{u,\alpha}(m,n)$ *is full.*

As a consequence of our theorem, it is clear that for $m+u \leq n$ the complementary set to $\mathrm{Bad}_{u,\alpha}(m,n)$ should be of zero $(m+1)n$–dimensional Lebesgue measure. Which is indeed a consequence of Theorem 1.6. Also, an obvious corollary of the above theorem is that $\dim \mathrm{Bad}_{u,\alpha}(m,n) = (m+1)n$.

2. Machinery

To kick off this section we first define the basic concepts of Hausdorff measure and dimension.

2.1. Hausdorff Measure and Dimension.

Below is a brief introduction to Hausdorff f–measure and dimension. For further details see [6, 23]. Let $F \subset \mathbb{R}^n$. For any $\rho > 0$, a countable collection $\{B_i\}$ of balls in $\mathbb{R}^n$ with diameters $\mathrm{diam}(B_i) \leq \rho$ such that $F \subset \bigcup_i B_i$ is called a ρ–cover of F. Define for a dimension function f,

$$\mathcal{H}^f_\rho(F) = \inf \sum_i f(\mathrm{diam}(B_i)),$$

where the infimum is taken over all possible ρ–covers of F. The function f is called a dimension function, and the Hausdorff f–measure of F is

$$\mathcal{H}^f(F) = \lim_{\rho \to 0} \mathcal{H}^f_\rho(F).$$

In the particular case when $f(r) = r^s$ with $s > 0$, we write $\mathcal{H}^s$ for $\mathcal{H}^f$ and the measure is referred to as s–dimensional Hausdorff measure. The Hausdorff dimension of F is denoted by $\dim F$ and is defined as

$$\dim F := \inf\{s \in \mathbb{R}^+ \ : \ \mathcal{H}^s(F) = 0\}.$$

2.2. Slicing.

We now state a result which is a key ingredient in the proof of Theorem 1.1. We include the result mainly for completeness, as its application is identical to the one in [27]. Before we state the result it is necessary to introduce a little notation.

Suppose that V is a linear subspace of $\mathbb{R}^k$, $V^\perp$ will be used to denote the linear subspace of $\mathbb{R}^k$ orthogonal to V. Further $V + a := \{v + a : v \in V\}$ for $a \in V^\perp$.

Lemma 2.1. *Let $l, k \in \mathbb{N}$ be such that $l \leq k$ and let f and $g : r \to r^{-l} f(r)$ be dimension functions. Let $B \subset \mathbb{I}^k$ be a* Borel *set and let V be a $(k-l)$–dimensional linear subspace of $\mathbb{R}^k$. If for a subset S of $V^\perp$ of positive $\mathcal{H}^l$ measure*

$$\mathcal{H}^g\left(B \cap (V + b)\right) = \infty \quad \forall\, b \in S,$$

then $\mathcal{H}^f(B) = \infty$.

2.3. Mass Transference Principle.

In this section we describe the mass transference principle for linear forms tailored for our use. The actual framework is broad ranging and deals with the limsup sets defined by a sequence of neighborhoods of 'approximating' planes. The mass transference principle for linear forms naturally enable us to generalize the Lebesgue measure statements of linear forms to the Hausdorff measure statements. It is derived using the 'slicing' technique introduced in [5] and described above.

Let $\mathbf{x}^{(j)}$ denote the jth column vector of X. For $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ the resonant set $R_\mathbf{q}$ is defined by

$$R_\mathbf{q} := \bigcup_{\mathbf{p} \in (\mathbf{q}\mathbb{I}^{mn}) \cap \mathbb{Z}^n} \{(X, \alpha) \in \mathbb{I}^{mn} \times \mathbb{I}^n : \mathbf{q}X - \mathbf{p} - \alpha = \mathbf{0}\} = \bigcup_{\mathbf{p} \in (\mathbf{q}\mathbb{I}^{mn}) \cap \mathbb{Z}^n} R_{\mathbf{q},\mathbf{p}}$$

where each

$$R_{\mathbf{q},\mathbf{p}} = \{(X,\alpha) \in \mathbb{I}^{mn} \times \mathbb{I}^n : \mathbf{q}X - \alpha = \mathbf{p}\} = (R_{\mathbf{q},p_1}, \ldots, R_{\mathbf{q},p_n})$$

and

$$R_{\mathbf{q},p_j} = \left\{(\mathbf{x}^{(j)}, \alpha_j) : \mathbf{q}\cdot\mathbf{x}^{(j)} - \alpha_j = \mathbf{p}_j\right\}.$$

It is then clear that the resonant sets are the hyperplanes of dimension mn, co-dimension n and are contained in $W_\alpha(m,n;\psi)$ for all functions ψ.

Let $\mathcal{R} = \{R_{\mathbf{q}} : \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$. Given an approximating function Ψ and a resonant set $R_{\mathbf{q}}$, define the Ψ–neighbourhood of $R_{\mathbf{q}}$ as

$$\Delta\left(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)\right) = \left\{(X,\alpha) \in \mathbb{I}^{mn} \times \mathbb{I}^n : \operatorname{dist}\left((X,\alpha), R_{\mathbf{q}}\right) \le \frac{\psi(|\mathbf{q}|)}{|\mathbf{q}|}\right\},$$

where $\operatorname{dist}(X, R_{\mathbf{q}}) := \inf\{|X - Y| : Y \in R_{\mathbf{q}}\}$. Notice that if $m = 1$ then the resonant sets are points and the sets $\Delta\left(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)\right)$ are balls centred at these points.

Let

$$\Lambda_\alpha(m,n;\Psi) = \{(X,\alpha) \in \mathbb{I}^{mn} \times \mathbb{I}^n : X \in \Delta\left(R_{\mathbf{q}}, \Psi(|\mathbf{q}|)\right) \text{for i.m. } \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}\}$$

and define

$$\Delta(\Psi) := \bigcup_{\mathbf{q}\in J} \Delta(R_{\mathbf{q}}, \Psi(|\mathbf{q}|))$$

where

$$J = \{\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\} : |\mathbf{p}| < |\mathbf{q}|\}.$$

Then, $\Lambda_\alpha(m,n;\Psi)$ can be written as a limsup set so that

$$\Lambda_\alpha(m,n;\Psi) = \bigcap_{N=1}^{\infty} \bigcup_{|\mathbf{q}|=N}^{\infty} \Delta(\Psi).$$

Theorem 2.2 (Mass Transference Principle)**.** *Let $\mathcal{R}$ and Ψ as above be given. Let V be a linear subspace of $\mathbb{R}^{(m+1)n}$ such that* $\dim V = n = \operatorname{codim} \mathcal{R}$ *and*

(i) $V \cap R_{\mathbf{q}} \neq \emptyset$ *for all* $\mathbf{q} \in J$,
(ii) $\sup_{\mathbf{q}\in J} \operatorname{diam}\left(V \cap \triangle(R_{\mathbf{q}}, 1)\right) < \infty$.

Let f and $g : r \to g(r) := r^{-mn} f(r)$ be dimension functions such that $r^{-(m+1)n} f(r)$ is monotonic. Suppose for any ball B in $\mathbb{I}^{(m+1)n}$

$$\mathcal{H}^{(m+1)n}\big(B \cap \Lambda_\alpha(m,n; g(\Psi)^{\frac{1}{n}})\big) = \mathcal{H}^{(m+1)n}(B).$$

Then for any ball $B \in \mathbb{I}^{(m+1)n}$

$$\mathcal{H}^f\left(B \cap \Lambda_\alpha(m,n;\Psi)\right) = \mathcal{H}^f(B).$$

For most applications, conditions (i) and (ii) are not particularly restrictive. When $\mathcal{R}$ is a collection of points in $\mathbb{I}^{(m+1)n}$, conditions (i) and (ii) are trivially satisfied and Theorem 2.2 reduces to statement concerning limsup sets arising from the sequence of balls see– [4]. When $\mathcal{R}$ is a collection of mn–dimensional planes in $\mathbb{I}^{(m+1)n}$, condition (i) excludes planes $R_{\mathbf{q}}$ parallel to V and condition (ii) simply means that the angle at which $R_{\mathbf{q}}$ 'hits' V is bounded away from zero by a fixed constant independent of $\mathbf{q} \in J$. This in turn implies that each plane $\mathcal{R}$ intersect V at exactly one point.

2.4. **An interlude on classical Diophantine approximation.** Below is the statement of a generalized form of a Khintchine–Groshev-type theorem for $W_\alpha(m,n;\psi)$ which into our knowledge is a new result and interesting in its own right. We include it here as it can be easily proved by appealing to the mass transference principle stated above.

Theorem 2.3. *Let ψ be an approximating function. Let f and $r^{-mn}f(r)$ be dimension functions such that $r^{-(m+1)n}f(r)$ is monotonic. Then*

$$\mathcal{H}^f(W_\alpha(m,n;\psi)) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-mn}r^{m+n-1} < \infty, \\ \mathcal{H}^f(\mathbb{I}^{mn}\times\mathbb{I}^n) & \text{if } \sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-mn}r^{m+n-1} = \infty. \end{cases}$$

Note that even in the one dimensional settings there is no condition of monotonicity imposed on the approximating function as opposed to the cases when α is zero or when α is non-zero but fixed. In these sets the monotonicity condition on the approximating function cannot be removed due to the Duffin–Schaeffer counter example see, [3, 21].

Note that if the dimension function f is such that $r^{-(m+1)n}f(r)\to\infty$ as $r\to 0$ then $\mathcal{H}^f(\mathbb{I}^{mn}\times\mathbb{I}^n)=\infty$ and Theorem 2.3 is the analogue of the classical result of Jarník (see [12, 29]). In the case when $f(r):=r^{(m+1)n}$ the Hausdorff measure $\mathcal{H}^f$ is proportional to the standard $(m+1)n$–dimensional Lebesgue measure supported on $\mathbb{I}^{(m+1)n}$ and the result is the natural analogue of the Khintchine–Groshev theorem for $W_\alpha(m,n;\psi)$. A singly metric analogue of Theorem 2.3 can be found in Bugeaud's paper [7].

We state the following corollary of Theorem 2.3, which is in fact an original theorem by Cassels [8].

Corollary 2.4. *Let ψ be an approximating function, then*

$$|W_\alpha(m,n;\psi)|_{(m+1)n} = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty}\psi(r)^n r^{m-1} < \infty, \\ 1 & \text{if } \sum_{r=1}^{\infty}\psi(r)^n r^{m-1} = \infty. \end{cases}$$

In fact, we will deduce the divergence case of Theorem 2.3 from this result. It is very similar to the result of Schmidt [33], but in a sense more general as there is no assumption of monotonicity on the approximating function. For the doubly metric case, the Hausdorff dimension for the set $W_\alpha(m,n;\psi)$ was established by Dodson in [18] and, for the singly metric case, by Levesley in [31]. A slightly more general form of Dodson's result was established by Dickinson in [16].

Proof of Theorem 2.3.

The convergence case. The convergence case is straightforward to establish. First notice that the set $W_\alpha(m,n;\psi)$ can be written in the following lim sup form

$$W_\alpha(m,n;\psi) = \bigcap_{N=1}^{\infty}\bigcup_{r>N}\bigcup_{|\mathbf{q}|=r}\Delta\left(R_{\mathbf{q}},\Psi(|\mathbf{q}|)\right).$$

From this it follows that for each $N\in\mathbb{N}$

$$W_\alpha(m,n;\psi) \subset \bigcup_{r>N}\bigcup_{|\mathbf{q}|=r}\Delta\left(R_{\mathbf{q}},\Psi(|\mathbf{q}|)\right).$$

For each resonant set $R_{\mathbf{q}}$ the set $\Delta(R_{\mathbf{q}},\Psi(|\mathbf{q}|))$ can be covered by a collection of $(mn+n)$-dimensional closed hypercubes with disjoint interior and sidelength comparable with $\Psi(|\mathbf{q}|)$. It can be readily verified that the number C of such hypercubes satisfies

$$C \ll \Psi(|\mathbf{q}|)^{-mn}|\mathbf{q}|^{n}.$$

Thus,

$$\begin{aligned}\mathcal{H}^{f}(W_{\alpha}(m,n;\psi)) &\leq \sum_{r>N}\sum_{|\mathbf{q}|=r}\Psi(|\mathbf{q}|)^{-mn}|\mathbf{q}|^{n}f(\Psi(|\mathbf{q}|))\\ &\ll \sum_{r>N} r^{m+n-1}f(\Psi(r))\Psi(r)^{-mn}\to 0\end{aligned}$$

as $N \to \infty$. Thus $\mathcal{H}^{f}(W_{\alpha}(m,n;\psi)) = 0$, as required.

The divergence case. The divergence case is an easy consequence of the mass transference principle discussed in the section § 2.3. In view of this we shall use the divergence part of Corollary 2.4 (Cassel's theorem) and the mass transference principle to prove the divergence part of Theorem 2.3. With reference to the framework of §2.3 we note that the set $\Lambda_{\alpha}(m,n;\Psi)$ coincides with the set $W_{\alpha}(m,n;\psi)$. Hence the divergence case follows, provided we can exhibit an n-dimensional subspace V satisfying the assumptions of Theorem 2.2. This is however easily accomplished by splitting up the resonant sets into families each satisfying the two conditions for different subspaces and noting that at least one of the corresponding limsup-sets must have full measure. □

3. Proof of Theorem 1.1

3.1. **The convergence case.** The convergence case is almost identical to the convergence case of Theorem 2.3. We therefore skip the proof.

3.2. **The divergence case.** As discussed earlier the statement of the Theorem essentially reduces to two cases, the finite measure case where $r^{-(m+1)n}f(r) \to C > 0$ as $r \to 0$ and the infinite measure case in which $r^{-(m+1)n}f(r) \to \infty$ as $r \to 0$. Therefore, we split the proof of the Theorem 1.1 into two parts, the finite measure case and the infinite measure case.

Before proceeding, we will need the following key lemma, which will make our proofs work.

Lemma 3.1. *Let $S \subseteq \mathrm{Mat}_{(m+u-n)\times n}(\mathbb{R}) \times \mathbb{R}^{n}$ of full Lebesgue measure. Let $A \subseteq \mathrm{GL}_{n\times n}(\mathbb{R})$ be a subset of a subspace of dimension (n-u)n such that A has positive $(n-u)n$–dimensional Lebesgue measure. Then, the set*

$$\Lambda = \left\{ \left(\begin{pmatrix} X \\ YX \end{pmatrix}, \alpha X \right) \in \mathrm{Mat}_{(m+u)\times n}(\mathbb{R}) \times \mathbb{R}^{n} : X \in A, (Y,\alpha) \in S \right\}$$

has full Lebesgue measure inside $A \times S$, with the product structure being the one implicit in the affine system.

Proof. The proof of the lemma is similar to Lemma 4.1 of [27] with some modifications. Initially, note that the set Λ is contained in a subspace $V \subseteq \mathbb{R}^{(m+u)n+n}$ of dimension

$$(n-u)n + (m+u-n)n + n = (m+1)n,$$

where the first summand comes from the subspace containing A (the upper $n \times n$ part of the matrix), the second summand comes from the lower part of the matrix, and the final n is from the inhomogeneous component. The statement of the lemma translates to this set having full $(m+1)n$-dimensional Lebesgue measure within $A \times S$, interpreted as a subset of V. When speaking of balls and hypercubes below, we will mean balls and hypercubes in V.

Without loss of generality, we will assume that $|A|_{(n-u)n} < \infty$. If this is not the case, we will take a subset of A of positive measure and subsequently apply the standard limiting argument. Suppose now for a contradiction that $|(A \times S) \setminus \Lambda| > 0$. The Lebesgue density theorem states that this set has a point of metric density, say $Z \in (A \times S) \setminus \Lambda$. We will show that the existence of such a point violates the condition that S is full.

Fix an $\epsilon > 0$. There is a $\delta > 0$ such that

$$\frac{|\Lambda \cap B(Z,\delta)|}{|B(Z,\delta)|} < \frac{\epsilon}{2^{(m+1)n}},$$

where $B(Z,\delta)$ denotes the ball centred at Z of radius δ. By definition of the Lebesgue measure, we may take a cover $\mathcal{C}$ of $\Lambda \cap B(Z,\delta)$ by hypercubes in $V \cong \mathbb{R}^{(m+1)n}$ such that

$$\sum_{C \in \mathcal{C}} \operatorname{diam}(C)^{(m+1)n} < \frac{\epsilon}{2^{(m+1)n}} |B(Z,\delta)| = \epsilon \delta^{(m+1)n}.$$

The latter equality follows as we are working in the supremum norm, so that a ball of radius δ is in fact a hypercube of side length 2δ. We let $A_0 \subseteq A$ be the set of those $X \in A$ for which there is a $Y \in S$ such that $\binom{X}{YX} \in B(Z,\delta)$. Note that by Fubini's Theorem A_0 has positive Lebesgue measure. In fact, the measure is equal to $2^{(n-u)n}\delta^{(n-u)n}$ for δ small enough.

For any $X \in A_0$ we define the set

$$B(X) = \left\{ \begin{pmatrix} X \\ YX \end{pmatrix} + \alpha X \in B(Z,\delta) : (Y,\alpha) \in S \right\}.$$

Note that

$$\mathcal{C}(X) = \left\{ \left(\left(\begin{pmatrix} X \\ \operatorname{Mat}_{(m+u-n)\times n}(\mathbb{R}) \end{pmatrix} + \mathbb{R}^n \right) \cap C \right) \in V : C \in \mathcal{C} \right\}.$$

is a cover of $B(X)$ by $((m+u-n)n+n)$-dimensional hypercubes.

As in [27], we define for each $C \in \mathcal{C}$ a function,

$$\lambda_C(X) = \begin{cases} 1 & \text{if } \left(\left(\begin{pmatrix} X \\ \operatorname{Mat}_{(m+u-n)\times n}(\mathbb{R}) \end{pmatrix} + \mathbb{R}^n \right) \cap C \right) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that

$$\int_{A_0} \lambda_C(X) dX \leq \operatorname{diam}(C)^{(n-u)n},$$

where the integral is with respect to the $(n-u)n$-dimensional Lebesgue measure on the subspace of $\mathrm{GL}_{n\times n}(\mathbb{R})$ containing the set A. Also,

$$\sum_{C \in \mathcal{C}(X)} \operatorname{diam}(C)^{(m+u+1-n)n} = \sum_{C \in \mathcal{C}} \lambda_C(X) \operatorname{diam}(C)^{(m+u+1-n)n}.$$

We integrate the latter expression with respect to X to obtain

$$\int_{A_0} \sum_{C \in \mathcal{C}(X)} \operatorname{diam}(C)^{(m+u+1-n)n} dX = \sum_{C \in \mathcal{C}} \int_{A_0} \lambda_C(X) dX \operatorname{diam}(C)^{(m+u+1-n)n}$$
$$\leq \sum_{C \in \mathcal{C}} \operatorname{diam}(C)^{(m+1)n} < \epsilon \delta^{(m+1)n}.$$

Since the right hand side is an integral of a non-negative function over a set of positive measure, there must be an $X_0 \in A_0$ with

$$\sum_{C \in \mathcal{C}(X_0)} \operatorname{diam}(C)^{(m+u+1-n)n} < \frac{\epsilon}{2^{(n-u)n}} \delta^{(m+u-n+1)n}.$$

Indeed, otherwise

$$\int_{A_0} \sum_{C \in \mathcal{C}(X)} \operatorname{diam}(C)^{(m+u+1-n)n} dX \geq |A_0| \frac{\epsilon}{2^{(n-u)n}} \delta^{(m+u-n+1)n)} = \epsilon \delta^{(m+1)n},$$

since $|A_0| = 2^{(n-u)n} \delta^{(n-u)n}$.

We may now estimate the $(m+u+1-n)n$-dimensional measure of $B(X_0)$ from above by this sum. This gives an upper estimate on the measure of $B(I_n)$, as X_0 is invertible. Furthermore, this estimate can be made arbitrarily small. But $B(I_n)$ is a cylinder set over S, so this is a clear contradiction since S was assumed to be full. □

3.3. **Finite measure case.** As in [27], in order to proceed, we will make some restrictions. We will be considering a set for which $\tilde{X}$ is invertible. Evidently, the exceptional set is of measure zero within $\mathbb{R}^{(m+1)n}$. In fact, in order for calculations to work, we will restrict ourselves even further. Let $\epsilon > 0$ and $N > 0$ be fixed but arbitrary. Define,

$$A_{\epsilon,N} = \left\{ (X, \alpha) \in \operatorname{Mat}_{m \times n}(\mathbb{R}) \times \mathbb{R}^n : \epsilon < \det(\tilde{X}) < \epsilon^{-1}, \quad \max_{1 \leq i,j \leq n} |x_{ij}| \leq N \right\}$$

where $\tilde{X}$ denote the $n \times n$-matrix formed by the first n rows of the matrix X. The set is of positive measure for ϵ small enough and N large enough, and as ϵ decreases and N increases, the set fills up $\operatorname{Mat}_{(m+1) \times n}(\mathbb{R})$ with the exception of the null-set of matrices X such that $\tilde{X}$ is singular.

We will prove that the divergence assumption implies that the set $W_{u,\alpha}(m, n; \psi)$ is full in $A_{\epsilon,N}$. We will translate the statements about $W_{u,\alpha}(m, n; \psi)$ to one about usual Diophantine approximation i.e. to the set $W_\alpha(m + u - n, n; \psi)$.

Consider the set of n affine forms in $m + u - n$ variables defined by the matrix $(\hat{X}, \hat{\alpha})$. Suppose furthermore that these linear forms satisfy the inequalities

$$\|\mathbf{r}\hat{X} - \hat{\alpha}\|_i \leq \frac{\psi(|\mathbf{r}|)}{nN}, \quad 1 \leq i \leq n, \tag{3}$$

for infinitely many $\mathbf{r} \in \mathbb{Z}^{m+u-n} \setminus \{\mathbf{0}\}$, where $\|\mathbf{x}\|_i$ denotes the distance from the i'th coordinate of $\mathbf{x}$ to the nearest integer. A special case of Corollary 2.4 states, that the divergence condition of our theorem implies that the set of such affine forms $(\hat{X}, \hat{\alpha})$ is full inside the set $\operatorname{Mat}_{(m+u-n) \times n}(\mathbb{R}) \times \mathbb{R}^n$, and hence in particular also in the image of $A_{\epsilon,N}$ under the map sending (X, α) to $(\hat{X}, \hat{\alpha})$.

Now, suppose that $(X, \alpha) \in A_{\epsilon,N}$ is such that $(\hat{X}, \hat{\alpha})$ is in the set defined by (3). We claim that X is in $W_{u,\alpha}(m, n; \psi)$. Indeed, let $\mathbf{r}_k$ be an infinite sequence of integer vector such that the inequalities (3) are satisfied for each k, and let $\mathbf{p}_k$ be the nearest integer vector to $\mathbf{r}_k\hat{X}$. Now define $\mathbf{q}_k = (\mathbf{p}_u, \mathbf{p}_k, \mathbf{r}_k)$. The inequalities defining $W_{u,\alpha}(m,n;\psi)$ will be satisfied for these values of $\mathbf{q}_k$, since for any $(X,\alpha) \in W_{u,\alpha}(m,n;\psi)$, we have

$$\left|\mathbf{q}_k \begin{pmatrix} I_u & 0 \\ X_u & X' \end{pmatrix} - \alpha\right| \asymp \left|\mathbf{q}_k \begin{pmatrix} I_n \\ \hat{X} \end{pmatrix} \tilde{X} - \alpha\right|$$
$$\asymp_\epsilon \left|(\pm\|\mathbf{r}_k \cdot \mathbf{x}^{(1)} + \hat{\alpha}_1\|_1, \ldots, \pm\|\mathbf{r}_k \cdot \mathbf{x}^{(n)} + \hat{\alpha}_n\|_n)\tilde{X}\right|.$$

Where $\hat{X}$ splits into n column vectors $(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)})$ and $X = (X_u\ X')$. The i'th coordinate of the first vector is at most $\psi(|q|)/nN$, so carrying out the matrix multiplication, using the triangle inequality and the fact that $|x_{ij}| \leq N$ for $1 \leq i,j \leq n$ shows that

$$\left|\mathbf{q}_k \begin{pmatrix} I_u & 0 \\ X_u & X' \end{pmatrix} - \alpha\right|_i < \psi(|\mathbf{q}_k|).$$

Applying Lemma 3.1, the divergence part of Theorem 1.1 follows in the case of Lebesgue measure.

3.4. **Infinite measure.** The infinite measure case of the Theorem 1.1 can be easily deduced from the following lemma.

Lemma 3.2. *Let ψ be an approximating function and let f and $g : r \to r^{-n(n-u)}f(r)$ be dimension functions with $r^{-(m+1)n}f(r) \to \infty$ as $r \to 0$. Further, let $r^{-(m+u-n)n}g(r)$ be a dimension function and $r^{-(m+u+1-n)n}g(r)$ be monotonic. If*

$$\sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-mn} r^{m+u-1} = \infty,$$

then

$$\mathcal{H}^f(W_{u,\alpha}(m,n;\psi)) = \infty.$$

Proof. The proof is almost identical to Lemma 4.2 of [27] with the obvious modifications. Again, it relies on a Lipschitz injective map which preserves measure. As in the finite measure case, we fix $\epsilon > 0$, $N \geq 1$ and let

$$A_{\epsilon,N} = \left\{(X,\alpha) \in \mathrm{Mat}_{m\times n}(\mathbb{R}) \times \mathbb{R}^n : \epsilon < \det(\tilde{X}) < \epsilon^{-1}, \quad \max_{1\leq i,j\leq n} |x_{ij}| \leq N\right\},$$

where $\tilde{X}$ denotes the $n \times n$-matrix formed by the first n rows. We also define the set

$$\tilde{A}_{\epsilon,N} = \left\{\tilde{X} \in \mathrm{GL}_n(\mathbb{R}) : \epsilon < \det(\tilde{X}) < \epsilon^{-1}, \quad \max_{1\leq i,j\leq n} |x_{ij}| \leq N\right\}.$$

Consider a subspace

$$A_0 = \left\{X \in \tilde{A}_{\epsilon,N} : X = \begin{pmatrix} I_u & 0 \\ X_u & X' \end{pmatrix}\right\},$$

of $\tilde{A}_{\epsilon,N}$, which is of $(n-u)n-$dimensional positive Lebesgue measure. For an appropriately chosen constant $c > 0$ depending only on m, n, ϵ and N, we find that the map

$$\eta : W_\alpha(m+u-n, n, c\psi) \times A_0 \to W_{u,\alpha}(m,n;\psi), \quad (Y,X) \mapsto \begin{pmatrix} X \\ YX \end{pmatrix},$$

is a Lipschitz embedding. Indeed, it is evidently injective as X is invertible for all elements of the domain.

We have,

$$\begin{aligned}\mathcal{H}^f(W_{u,\alpha}(m,n;\psi)) &\geq \mathcal{H}^f\left(\eta\left(W_\alpha(m+u-n,n,c\psi)\times A_0\right)\right)\\ &\asymp \mathcal{H}^f\left(W_\alpha(m+u-n,n,c\psi)\times A_0\right).\end{aligned}$$

Now after taking into account the conditions on the dimension functions as stated in the lemma along with the divergence part of Theorem 2.3 for $W_\alpha(m+u-n,n,c\psi)$, we proceed exactly as in [27]. We apply the slicing Lemma 2.1 across A_0 to get the infinite measure under the appropriate assumption of divergence. The details are identical to those in [27], and we omit them. □

4. Proof of Theorem 1.6

As in the preceding results, the convergence case is a rather tedious affair, and boils down to estimating volumes and applying the Borel–Cantelli Lemma. We omit the details and proceed with the divergence case.

Following an argument from [11, 15], for each $(X,\alpha)\in W_{u,\alpha}(m,n;\psi)$ let $\tilde{X} = (\mathbf{x}^{(u+1)}\dots\mathbf{x}^{(n)})$. From the proof of Lemma 1.5 the rank of $\tilde{X}$ is at most $m-1$. Now, we restrict ourselves to those $(X,\alpha)\in W_{u,\alpha}(m,n;\psi)$ for which $\tilde{X}$ has rank $m-1$. It can be readily verified that the set of (X,α) for which $\tilde{X}$ has lower rank is of strictly lower dimension. Define a subset $\widehat{W}_{\mathbf{u},\alpha}(m,n;\psi)$ of $W_{u,\alpha}(m,n;\psi)$ so that the column vectors $\mathbf{x}^{(u+1)},\dots,\mathbf{x}^{(m+u-1)}$ are linearly independent. Let $\Gamma'\subset\Gamma$ be the set of points $(X,\alpha)\in\Gamma$ such that $\tilde{X}$ has rank $m-1$ and $\mathbf{x}^{(u+1)},\dots,\mathbf{x}^{(m+u-1)}$ are the linearly independent vectors.

For clarity we only discuss the case $m+u=n$. Let $\epsilon>0$ be fixed and let G be the set of points of the form

$$\Big((\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m+u-1)},\sum_{j=1}^{m-1}a_j\mathbf{x}^{(u+j)}),\big(\tfrac{\alpha_m}{a_1(m-1)},\dots,\tfrac{\alpha_m}{a_{m-1}(m-1)},\alpha_m\big)\Big)$$

where

$$\big((\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m+u-1)}),\big(\tfrac{\alpha_m}{a_1(m-1)},\dots,\tfrac{\alpha_m}{a_{m-1}(m-1)}\big)\big)\in\widehat{W}_{\mathbf{u},\alpha}(m,m+u-1;\psi) \qquad (4)$$

and $|a_j|\in\left(\epsilon,\frac{1}{m-1}\right)$. Note that by the triangle inequality,

$$\begin{aligned}\Big|\mathbf{q}\cdot\sum_{j=1}^{m-1}a_{u+j}\mathbf{x}^{(u+j)}-\alpha_m\Big| &= \Big|\sum_{j=1}^{m-1}a_{u+j}\mathbf{q}\cdot\mathbf{x}^{(u+j)}-\alpha_m\Big|\\ &= \Big|a_{u+1}\Big(\mathbf{q}\cdot\mathbf{x}^{(u+1)}-\tfrac{\alpha_m}{a_{u+1}(m-1)}\Big)\\ &\quad+\dots+a_{u+m-1}\Big(\mathbf{q}\cdot\mathbf{x}^{(u+m-1)}-\tfrac{\alpha_m}{a_{u+m-1}(m-1)}\Big)\Big|\\ &\leq \Big(\sum_{j=1}^{m-1}|a_j|\Big)\psi(|\mathbf{q}|)\leq\psi(|\mathbf{q}|),\end{aligned}$$

for the infinitely many integer vectors $\mathbf{q}$ we obtain from condition (4). It follows that $G\subseteq W_{u,\alpha}(m,n;\psi)$. For the cases $m+u\leq n$, we need a slightly more complicated expression for G with more linearly dependent columns in the matrix. However, the ideas should be clear from the above simpler case.

The upshot of the above calculation is that it suffices to prove that $\mathcal{H}^f(G)=\infty$ under the given assumptions in the infinite measure case and that $\mathcal{H}^f(G)>0$ in the finite measure case. In both cases, the required result follows on considering a bi-Lipschitz embedding of a set of known measure into G. This will work due to the following lemma, which follows directly from Corollary 2.4 in [23].

Lemma 4.1. *Let f be a dimension function. Suppose that $L \subset \mathbb{R}^l$, $M \subset \mathbb{R}^k$ and $\eta : L \to M$ is a bijection satisfying a bi–Lipschitz condition. Then $\mathcal{H}^f(C) \asymp \mathcal{H}^f(\eta(C))$.*

Initially, we define the set

$$\begin{aligned}&\widetilde{W}_{u,\alpha}(m, m+u-1; \psi)\\ &\qquad = \left\{(X, \alpha) \in \widehat{W}_{u,\alpha}(m, m+u-1; \psi) : \alpha_i = \tfrac{\alpha_m}{a_i(m-1)} \text{ for } |a_i| \in (\epsilon, \tfrac{1}{m-1})\right\}.\end{aligned}$$

We parametrise the elements of $\widetilde{W}_{u,\alpha}(m, m+u-1; \psi)$ by the coordinates $(X, a_1, \ldots, a_{m-1}, \alpha_m)$ and define the map

$$\eta : \widetilde{W}_{u,\alpha}(m, m+u-1; \psi) \to G$$

by

$$\begin{aligned}&\eta\left((\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m+u-1)}), a_1, \ldots, a_{m-1}, \alpha_m\right)\\ &\qquad = \left((\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m+u-1)}, \sum_{j=1}^{m-1} a_j \mathbf{x}^{(u+j)}), (\tfrac{\alpha_m}{a_1(m-1)}, \ldots, \tfrac{\alpha_m}{a_{m-1}(m-1)}, \alpha_m)\right)\end{aligned}$$

Agruing as in [15], we find that η is a bi-Lipschitz map.

We are now in position to prove the infinite measure case of Theorem 1.6.

Lemma 4.2. *Let $m + u \leq n$. Let ψ be an approximating function and let f and $g : r \to r^{-m(n-(m+u-1))} f(r)$ be dimension functions with $r^{-mn-m-u+1} f(r) \to \infty$ as $r \to 0$. If*

$$\sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-mn} r^{m+u-1} = \infty,$$

then

$$\mathcal{H}^f(G) = \infty.$$

Proof. Initially, note that the set $\widehat{W}_{u,\alpha}(m, m+u-1; \psi)$ has infinite Hausdorff f-measure if and only if this is the case for $\widetilde{W}_{u,\alpha}(m, m+u-1; \psi)$. Hence, by Lemma 4.1 it suffices to prove that $\mathcal{H}^f(\widehat{W}_{u,\alpha}(m, m+u-1; \psi)) = \infty$.

The remainder of the proof is now similar to Lemma 7 in [11] and is an application of the slicing Lemma 2.1. The only change in the argument is that we slice the set $\widehat{W}_{u,\alpha}(m, m+u-1; \psi)$ across the subspace V consisting of the elements for which the linearly dependent part of the matrix X is equal to zero, i.e. in our restricted setup where the last column of X is 0. The conditions required for the slicing lemma to yield the desired conclusion are guaranteed by Theorem 1.1. □

Clearly, since $G \subseteq W_{u,\alpha}(m, n; \psi)$, this implies that $\mathcal{H}^f(W_{u,\alpha}(m, n; \psi)) = \infty$ and completes the proof of the theorem.

We now deal with the case where $r^{-mn-m-u+1} f(r) \to C$ as $r \to 0$ and $C > 0$ is finite. In this case $\mathcal{H}^f$ is comparable to $mn+m+u-1$–dimensional Lebesgue measure and

$$\sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-mn} r^{m+u-1} = \sum_{r=1}^{\infty} \psi^{m+u-1}(r).$$

From Lemma 4.1,

$$|\widehat{W}_{u,\alpha}(m, m+u-1; \psi) \times (-1/(m-1), 1/(m-1))^{m(n-m-u+1)}|_{mn+m+u-1} \asymp |G|_{mn+m+u-1}.$$

From Corollary 1.2, $|\widehat{W}_{u,\alpha}(m, m+u-1; \psi)|_{(m+1)(m+u-1)} = 1$. The set of $(X, \alpha) \in \widehat{W}_{u,\alpha}(m, m+u-1; \psi)$ for which $\tilde{X}$ has rank at most $m-2$ has strictly smaller dimension than $\widehat{W}_{u,\alpha}(m, m+u-1; \psi)$. Thus $|\widehat{W}_{u,\alpha}(m, m+u-1; \psi)|_{(m+1)(m+u-1)} = 1$. Hence the $mn+m+u-1$–dimensional Lebesgue measure of G is positive which further implies that the $mn+m+u-1$–dimensional Lebesgue measure of $W_{u,\alpha}(m, n; \psi)$ is also positive and finite as required.

5. Proof of Theorem 1.8

Before we prove the theorem we note that the defining inequalities take a particularly pleasing form. Namely, $(X, \alpha) \in \mathrm{Bad}_{u,\alpha}(m, n)$ if and only if

$$\left|(\mathbf{p}, \mathbf{q}) \begin{pmatrix} I_u & 0 \\ X_u & \tilde{X} \end{pmatrix} - \alpha\right| \geq C(X, \alpha)|\mathbf{q}|^{-\frac{m+u}{n}+1}, \tag{5}$$

for all $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^u \times \mathbb{Z}^m \setminus \{\mathbf{0}\}$, where X is the matrix $(X_u,\ \tilde{X})$. Put differently, we want an appropriate lower bound on the system of linear forms

$$\max\left(\|\mathbf{q} \cdot \mathbf{x}^{(\mathbf{1})} - \alpha_1\|, \ldots, \|\mathbf{q} \cdot \mathbf{x}^{(\mathbf{u})} - \alpha_u\|, |\mathbf{q} \cdot \mathbf{x}^{(\mathbf{u+1})} - \alpha_{u+1}|, \ldots, |\mathbf{q} \cdot \mathbf{x}^{(\mathbf{n})} - \alpha_n|\right),$$

where $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^u \times \mathbb{Z}^m \setminus \{\mathbf{0}\}$.

We now split the proof into two parts depending upon the choices of m, u and n.

5.1. **The case $m+u \leq n$.** First, by way of motivating our proof when $m+u \leq n$, we discuss the case when $u = 0$ and $m = n$. In this case, we consider the inequalities

$$|\mathbf{q}X - \alpha| \geq C(X)$$

for all $\mathbf{q} \in \mathbb{Z}^m$. It is readily verified that unless there is linear dependence among the columns of X, this is trivially satisfied. Hence, in this simple sub-case,

$$\left(\mathbb{R}^{m^2} \setminus \{X \in \mathbb{R}^{m^2} : \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)} \text{ are linearly dependent}\}\right) \times \mathbb{R}^m \subseteq \mathrm{Bad}_{0,\alpha}(m, m).$$

In other words, the set $\mathrm{Bad}_{0,\alpha}(m, m)$ contains the set of matrices of full rank, which is full with respect to Lebesgue measure. Hence, $\mathrm{Bad}_{0,\alpha}(m, m)$ is full. This clearly implies that $\dim \mathrm{Bad}_{0,\alpha}(m, m) = m^2 + m$, which proves our main theorem in the particular case $m = n, u = 0$.

Note that in fact we get the stronger inequality

$$|\mathbf{q}X - \alpha| > C(X)|\mathbf{q}| \quad \forall\ \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\} \tag{6}$$

in the set considered. This follows as an invertible matrix can only distort the unit cube by a specified amount and the fact that the inhomogeneous variables are full in the interval. This is much stronger than the defining inequality of the set $\mathrm{Bad}_{0,\alpha}(m, m)$ in this special case. This feature also applies to the more general setting when $m+u \leq n$ and further underlines the quantitative difference between the case $m+u \leq n$ and the converse $m+u > n$.

We now give a full proof in the case $m+u \leq n$. We will argue much in the spirit of the above. For a generic $X \in \mathbb{R}^{mn}$, the matrix $\tilde{X}$ in (5) has full rank.

Performing Gaussian elimination on the columns of a matrix of the form of (5) implies the existence of an invertible $(n \times n)$-matrix $E(X)$ such that

$$\begin{pmatrix} I_u & 0 \\ X_u & \tilde{X} \end{pmatrix} = \begin{pmatrix} I_u & 0 & 0 \\ \hat{X} & I_m & 0 \end{pmatrix} E(X).$$

Applying this matrix from the right to a vector $(\mathbf{p}, \mathbf{q})$, we see that

$$(\mathbf{p}, \mathbf{q}) \begin{pmatrix} I_u & 0 \\ X_u & \tilde{X} \end{pmatrix} = (\mathbf{p}, \mathbf{q}) \begin{pmatrix} I_u & 0 & 0 \\ \hat{X} & I_m & 0 \end{pmatrix} E(X) = \begin{pmatrix} \mathbf{p} + \mathbf{q}\hat{X} \\ \mathbf{q} \\ \mathbf{0} \end{pmatrix}^T E(X).$$

Multiplication by the matrix E on the right hand side only serves to distort the unit cube in the absolute value to a different parallelipiped depending on X. This induces a different norm on the image, but by equivalence of norms on Euclidean spaces, this distortion can be absorbed in a positive constant. In other words,

$$\left| (\mathbf{p}, \mathbf{q}) \begin{pmatrix} I_u & 0 \\ X_u & \tilde{X} \end{pmatrix} - \alpha \right| \geq C(X) \left| \begin{pmatrix} \mathbf{p} + \mathbf{q}\hat{X} \\ \mathbf{q} \\ \mathbf{0} \end{pmatrix}^T - \tilde{\alpha}^T \right| \tag{7}$$

for all $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^u \times \mathbb{Z}^m \setminus \{\mathbf{0}\}$, where $\tilde{\alpha} = \alpha E(X)^{-1}$.

Finally, since the norm is the *supremum* norm, we get

$$\left| \begin{pmatrix} \mathbf{p} + \mathbf{q}\hat{X} \\ \mathbf{q} \\ \mathbf{0} \end{pmatrix}^T - \tilde{\alpha}^T \right| \geq C(\tilde{\alpha}) |\mathbf{q}|,$$

for sufficiently large $\mathbf{q}$. By (7), almost every (X, α) is in $\mathrm{Bad}_{u,\alpha}(m, n)$, and in fact with the stronger requirement from (6). This completes the proof of Theorem 1.8 in the case when $m + u \leq n$.

5.2. **The case when m+u>n.** We will apply the following lemma, which is obtained using the techniques used in the proof of Lemma 3.1, although the proof is a little simpler. The method of proof is found in [26], where the corresponding result is Lemma 2.

Lemma 5.1. *Let $S \subseteq \mathrm{Mat}_{(m+u-n)\times n}(\mathbb{R}) \times \mathbb{R}^n$ of Hausdorff dimension $(m+u+1-n)n$. Let $A \subseteq \mathrm{GL}_{n\times n}(\mathbb{R})$ be a subset of a subspace of dimension (n-u)n such that A has positive $(n-u)n$–dimensional Lebesgue measure. Then, the set*

$$\Lambda = \left\{ \left(\begin{pmatrix} X \\ YX \end{pmatrix}, \alpha X \right) \in \mathrm{Mat}_{(m+u)\times n}(\mathbb{R}) \times \mathbb{R}^n : X \in A, (Y, \alpha) \in S \right\}$$

has Hausdorff dimension $(m+1)n$.

Proving the main theorem is now easy in this case. Let

$$A = \left\{ X \in \mathrm{GL}_n(\mathbb{R}) : X = \begin{pmatrix} I_u & 0 \\ X' & X'' \end{pmatrix} \right\}.$$

Evidently, the $(n-u)n$-dimensional Lebesgue measure of A is positive. Let $S = \mathrm{Bad}_\alpha(m + u - n, n)$ be the usual set of badly approximable systems of n affine forms in $m + u - n$ variables considered by Kleinbock [30]. Kleinbock's result tells us that $\dim S = (m + u - n)n + n$.

The set Λ of Lemma 5.1 now consists of affine forms of the form

$$\left(\begin{pmatrix} X \\ YX \end{pmatrix}, \alpha X\right) = \left(\begin{pmatrix} I_u & 0 \\ X' & X'' \\ Y\begin{pmatrix} I_u & 0 \\ X' & X'' \end{pmatrix} & \end{pmatrix}, \alpha\begin{pmatrix} I_u & 0 \\ X' & X'' \end{pmatrix}\right),$$

with $(Y, \alpha) \in \mathrm{Bad}_\alpha(m+u-n, n)$ and $X'' \in \mathrm{GL}_{n-u}(\mathbb{R})$. Evidently, the linear part of these forms have a the matrix representation of the form

$$\begin{pmatrix} I_u & 0 \\ X_u & \tilde{X} \end{pmatrix}.$$

We will show that the lower sub-matrix $(X_u, \tilde{X})$ together with the affine part αX is an element of $\mathrm{Bad}_{u,\alpha}(m, n)$. This trivially implies Theorem 1.8 via Lemma 5.1.

However, this is easy. For $(Y, \alpha) \in S$, there is a constant $C(Y) > 0$ such that for any $\mathbf{r} \in \mathbb{Z}^{m+u-n} \setminus \{\mathbf{0}\}$ and any $\mathbf{p} \in \mathbb{Z}^n$,

$$\left|(\mathbf{p}, \mathbf{r})\begin{pmatrix} I_n \\ Y \end{pmatrix} - \alpha\right| \geq C(Y)|\mathbf{r}|^{-\frac{m+u}{n}+1}.$$

On the other hand, for $\mathbf{p} \in \mathbb{Z}^u$ and $\mathbf{q} \in \mathbb{Z}^m$,

$$\left|(\mathbf{q})\begin{pmatrix} X \\ YX \end{pmatrix} - \alpha X\right| = \left|\left((\mathbf{p}, \mathbf{q})\begin{pmatrix} I_n \\ Y \end{pmatrix} - \alpha\right)X\right| \geq C(X, Y)|\mathbf{q}|^{-\frac{m+u}{n}+1},$$

with the last inequality following by multiplying the defining inequalities with X^{-1}. In other words, the set Λ arising from Lemma 5.1 is a subset of $\mathrm{Bad}_{u,\alpha}(m, n)$ with some additional 'dummy' coordinates attached in the first u rows. It follows that $\dim \mathrm{Bad}_{u,\alpha}(m, n) \geq \dim \Lambda = (m+1)n$, which completes the proof.

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