$\mathfrak{sl}_3$-web bases, intermediate crystal bases and categorification

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Joint work with Marco Mackaay and Weiwei Pan

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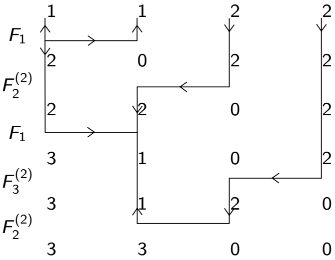


What is categorification?

Forced to reduce this presentation to one sentence, the author would choose:

Interesting integers are shadows of richer structures in categories.

The basic idea can be seen as follows. Take a "set-based" structure S and try to find a "category-based" structure $\mathcal{C}$ such that S is just a shadow of $\mathcal{C}$.

Categorification, which can be seen as "remembering" or "inventing" information, comes with an "inverse" process called decategorification, which is more like "forgetting" or "identifying".

Note that decategorification should be easy.

Exempli gratia

Examples of the pair categorification/decategorification are:

Bettinumbers of a manifold M	$\underset{\text{decat}=\mathrm{rank}(\cdot)}{\overset{\text{categorify}}{\rightleftarrows}}$	Homology groups
Polynomials in $\mathbb{Z}[q, q^{-1}]$	$\underset{\text{decat}=\chi_{\mathrm{gr}}(\cdot)}{\overset{\text{categorify}}{\rightleftarrows}}$	complexes of gr.VS
The integers $\mathbb{Z}$	$\underset{\text{decat}=K_0(\cdot)}{\overset{\text{categorify}}{\rightleftarrows}}$	$K-$ vector spaces
An $A-$ module	$\underset{\text{decat}=K_0^{\oplus}(\cdot)\otimes_{\mathbb{Z}}A}{\overset{\text{categorify}}{\rightleftarrows}}$	additive category

Usually the categorified world is much more interesting.

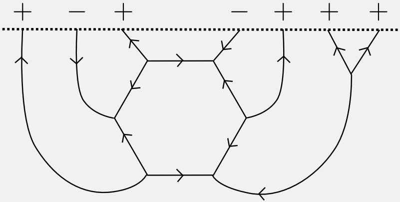
Today decategorification = Grothendieck group!

Kuperberg's $\mathfrak{sl}_3$-webs

Definition(Kuperberg)

A $\mathfrak{sl}_3$-web w is an oriented trivalent graph, such that all vertices are either sinks or sources. The boundary ∂w of w is a sign string $S = (s_1, \ldots, s_n)$ under the convention $s_i = +$ iff the orientation is pointing in and $s_i = -$ iff the orientation is pointing out (we also need $0, 3$ later - but they are not drawn).

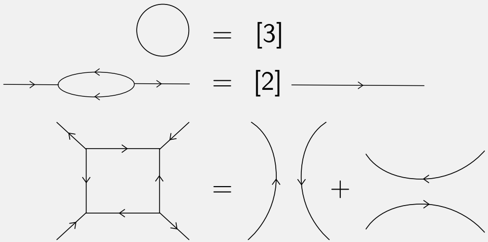
Example



Kuperberg's $\mathfrak{sl}_3$-webs

Definition(Kuperberg)

The $\mathbb{C}(q)$-web space W_S for a given sign string $S = (\pm, \ldots, \pm)$ is generated by $\{w \mid \partial w = S\}$, where w is a web, subject to the relations



Here $[a] = \frac{q^a - q^{-a}}{q - q^{-1}} = q^{a-1} + q^{a-3} + \cdots + q^{-(a-1)}$ is the quantum integer.

Representation theory of $\mathbf{U}_q(\mathfrak{sl}_3)$

A sign string $S = (s_1, \ldots, s_n)$ corresponds to tensors

$$V_S = V_{s_1} \otimes \cdots \otimes V_{s_n},$$

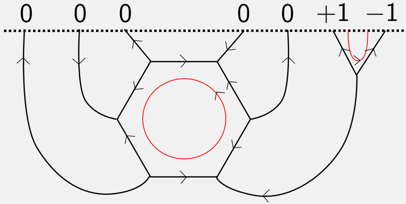
where V_+ is the fundamental $\mathbf{U}_q(\mathfrak{sl}_3)$-representation and V_- is its dual, and webs correspond to intertwiners.

Theorem(Kuperberg)

$$W_S \cong \mathrm{Hom}_{\mathbf{U}_q(\mathfrak{sl}_3)}(\mathbb{C}(q), V_S) \cong \mathrm{Inv}_{\mathbf{U}_q(\mathfrak{sl}_3)}(V_S)$$

In fact, the so-called spider category of all webs modulo the Kuperberg relations is equivalent to the representation category of $\mathbf{U}_q(\mathfrak{sl}_3)$.
As a matter of fact, the $\mathfrak{sl}_3$-webs without internal circles, digons and squares form a basis B_S, called web basis, of W_S!

Example



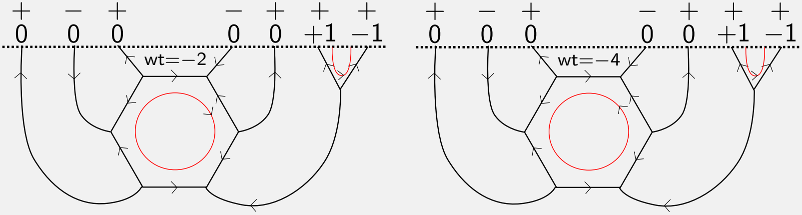
Webs can be coloured with flow lines. At the boundary, the flow lines can be represented by a state string J. By convention, at the i-th boundary edge, we set $j_i = \pm 1$ if the flow line is oriented downward/upward and $j_i = 0$, if there is no flow line. So $J = (0, 0, 0, 0, 0, +1, -1)$ in the example.

Given a web with a flow w_f, attribute a weight to each trivalent vertex and each arc in w_f and take the sum. The weight of the example is -4.

Theorem(Khovanov-Kuperberg)

Pairs of sign S and a state strings J correspond to the coefficients of the web basis relative to tensors of the standard basis $\{e_{\pm}^{-1}, e_{\pm}^{0}, e_{\pm}^{+1}\}$ of $V_{\pm}$.

Example



$$w_S = \cdots + (q^{-2} + q^{-4})(e_{+}^{0} \otimes e_{-}^{0} \otimes e_{+}^{0} \otimes e_{-}^{0} \otimes e_{+}^{0} \otimes e_{+}^{+1} \otimes e_{+}^{-1}) \pm \cdots .$$

What kind of basis is B_S?

Theorem(Khovanov-Kuperberg)

Given (S, J), we have (with $v = -q^{-1}$ and $e_S^J = e_{s_1}^{j_1} \otimes \cdots \otimes e_{s_n}^{j_n}$)

$$w_S^J = e_S^J + \sum_{J' < J} c(S, J, J') e_S^{J'} \quad \text{for} \quad c(S, J, J') \in \mathbb{N}[v, v^{-1}].$$

In general we have $B_S \neq \mathrm{dcan}(W_S)$, but the web basis is bar-invariant.

Theorem(MPT)

We proved, by categorification, that the change-of-basis matrix from Kuperberg's web basis B_S to the dual canonical basis $\mathrm{dcan}(W_S)$ is unitriangular.

Question: The web basis B_S is a somehow "special" basis of W_S. But it is not the dual canonical. So what kind of basis is it?

The quantum algebra $\mathbf{U}_q(\mathfrak{sl}_d)$

Definition

For $d \in \mathbb{N}_{>1}$ the quantum special linear algebra $\mathbf{U}_q(\mathfrak{sl}_d)$ is the associative, unital $\mathbb{C}(q)$-algebra generated by $K_i^{\pm 1}$ and E_i and F_i, for $i = 1, \dots, d-1$, subject to some relations (that we do not need today).

Definition(Beilinson-Lusztig-MacPherson)

For each $\lambda \in \mathbb{Z}^{d-1}$ adjoin an idempotent 1_λ (think: projection to the λ-weight space!) to $\mathbf{U}_q(\mathfrak{sl}_d)$ and add some relations, e.g.

$$1_\lambda 1_\mu = \delta_{\lambda,\nu} 1_\lambda \ \text{ and } \ K_{\pm i} 1_\lambda = q^{\pm \lambda_i} 1_\lambda \ \text{(no } K's \text{ anymore!)}.$$

The idempotented quantum special linear algebra is defined by

$$\dot{\mathbf{U}}(\mathfrak{sl}_d) = \bigoplus_{\lambda,\mu \in \mathbb{Z}^{d-1}} 1_\lambda \mathbf{U}_q(\mathfrak{sl}_d) 1_\mu .$$

Intermediate crystals

Let $d = 3\ell$ and let V_Λ be the irreducible $\dot{\mathbf{U}}(\mathfrak{sl}_d)$-module of highest weight $\Lambda = (3^\ell)$. Kashiwara-Lusztig's lower global crystal (or canonical) basis $\mathrm{can}(V_\Lambda) = \{b_T \mid T \in \mathrm{Std}(3^\ell)\}$ has nice properties, but is in very hard to find.

Leclerc and Toffin have defined an intermediate crystal basis B_Λ of V_Λ by an explicit algorithm that can be used to compute $\mathrm{can}(V_\Lambda)$ inductively, i.e. B_Λ has some nice properties, but is still trackable enough to be written down.

Example (with $\ell = 3$)

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 3 & 5 \\ \hline 4 & 5 & 6 \\ \hline \end{array} \rightsquigarrow \mathrm{LT}(T) = F_1 F_2 F_3^{(2)} F_2 F_1 F_4 F_3 F_2 F_5^{(2)} F_4^{(2)} F_3^{(2)} v_\Lambda.$$

Sitting in-between $\{b_T\}$ and $\{x_{T'}\}$

The intermediate crystal basis sits "in-between" the canonical $\mathrm{can}(V_\Lambda)$ and the tensor basis $\{x_{T'} \in \Lambda_q^\ell(\mathbb{C}_q^d)^{\otimes 3} \supset V_\Lambda \mid T' \in \mathrm{Col}(3^\ell)\}$.

Theorem(Leclerc-Toffin)

We have (for $T' \in \mathrm{Col}(3^\ell)$, $T'' \in \mathrm{Std}(3^\ell)$)

$$\mathrm{LT}(T) = x_T + \sum_{T' \prec T} \alpha_{T'}(v) x_{T'} \quad \text{and} \quad b_T = \mathrm{LT}(T) + \sum_{T'' \prec T} \beta_{T''T}(v)\mathrm{LT}(T'')$$

with certain $\alpha_{T'}(v) \in \mathbb{N}[v, v^{-1}]$ and $\beta_{T''T}(v) \in \mathbb{Z}[v, v^{-1}]$ (with $v = -q^{-1}$). Moreover, the intermediate crystal basis is bar-invariant.

We have seen this before!

An instance of q-skew Howe duality

The commuting actions of $\dot{\mathbf{U}}(\mathfrak{sl}_d)$ and $\dot{\mathbf{U}}(\mathfrak{sl}_3)$ on

$$\Lambda_q^\bullet(\mathbb{C}_q^d)^{\otimes 3} \cong \Lambda_q^\bullet(\mathbb{C}^d \otimes \mathbb{C}^3) \cong \Lambda_q^\bullet(\mathbb{C}_q^3)^{\otimes d}$$

introduce an $\dot{\mathbf{U}}(\mathfrak{sl}_d)$-action on $\Lambda_q^\bullet(\mathbb{C}_q^3)^{\otimes d}$ and an $\dot{\mathbf{U}}(\mathfrak{sl}_3)$-action on $\Lambda_q^\bullet(\mathbb{C}_q^d)^{\otimes 3}$. Here

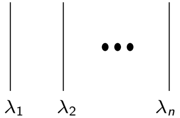
$$\Lambda_q^\bullet(\mathbb{C}_q^l) = \bigoplus_{k=1}^{n} \Lambda_q^k(\mathbb{C}_q^l)$$

and all the $\Lambda_q^k(\mathbb{C}_q^l)$ are irreducible $\dot{\mathbf{U}}(\mathfrak{sl}_l)$-representations.

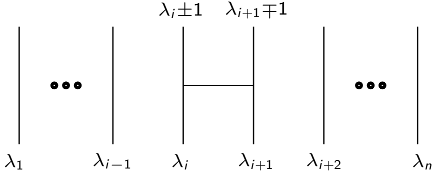
The $\mathfrak{sl}_3$-webs form a $\dot{\mathbf{U}}(\mathfrak{sl}_d)$-module

We defined an action ϕ of $\dot{\mathbf{U}}(\mathfrak{sl}_d)$ on $W_{(3^\ell)} = \bigoplus_{S \in \Lambda(n,n)_3} W_S$ by

$1_\lambda \mapsto$



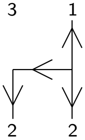
$E_i 1_\lambda,\ F_i 1_\lambda \mapsto$



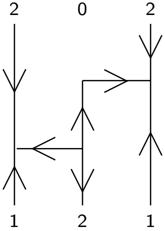
We use the convention that vertical edges labeled 1 are oriented upwards, vertical edges labeled 2 are oriented downwards and edges labeled 0 or 3 are erased.
Think: $0, 3$ indicates the trivial, 1 the V_+ and 2 the V_--representation.

Exempli gratia

$E_1 1_{(22)} \mapsto$



$F_2 E_1 1_{(121)} \mapsto$



An intermediate crystal basis

Proposition(MT)

The Kuperberg web basis B_S is Leclerc-Toffin's intermediate crystal basis under q-skew Howe duality, i.e.

$$\mathrm{LT}(T) = \{F_{i_s}^{(r_s)} \cdots F_{i_1}^{(r_1)} v_{3^\ell} \mid T \in \mathrm{Std}(3^\ell)\} \overset{\mathrm{sHd}}{\longmapsto} w_S^J.$$

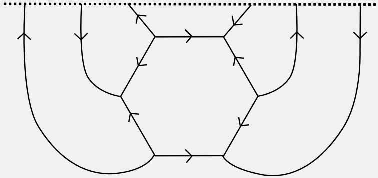
(No K's and E's anymore!)

Corollary

The change-of-basis matrix from Kuperberg's web basis B_S to the dual canonical basis $\mathrm{dcan}(W_S)$ is unitriangular and

$$w_S^J = e_S^J + \sum_{J'<J} c(S,J,J') e_S^{J'} \quad \text{for} \quad c(S,J,J') \in \mathbb{N}[v,v^{-1}].$$

Example

$w =$  $\leftrightsquigarrow$ $T =$

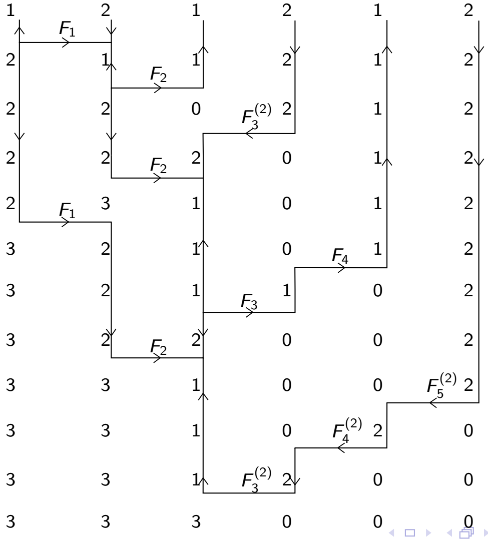
1	2	4
2	3	5
4	6	6

From T we obtain the string

$$LT(T) = F_1F_2F_3^{(2)}F_2F_1F_4F_3F_2F_5^{(2)}F_4^{(2)}F_3^{(2)}.$$

Exempli gratia

$$LT(T)v_{3^3} = F_1F_2F_3^{(2)}F_2F_1F_4F_3F_2F_5^{(2)}F_4^{(2)}F_3^{(2)}v_{3^3}$$



3-multipartitions

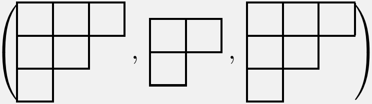
The growth algorithm for webs can be read as webs "are" standard tableaux. So what "are" flows on webs and what "is" the weight of these flows?

Observation

Column-strict tableaux $T \in \mathrm{Col}(3^\ell)$ are 3-multipartitions

2	1	2
4	3	4
6	5	6

$\leftrightarrow$



Question: The flows and their weights encode the coefficients of the web basis B_S in terms of the tensor basis $\{x_{T'} \mid T' \in \mathrm{Col}(3^\ell)\}$. Shouldn't it be possible to encode both using fillings of 3-multipartitions?

3-multitableaux and degrees

Definition

Let $\vec{T} \in \mathrm{Std}(\vec{\lambda})$ be a (filled with numbers from $\{1, \dots, k\}$) 3-multitableau $\vec{T} = (T_1, T_2, T_3)$. For $j \in \{1, \dots, k\}$ let N^j denote the set of all nodes that are filled with the number j and let $\vec{T}^j$ denote the 3-multitableau obtained from $\vec{T}$ by removing all nodes with entries $> j$ and set

$$\deg(\vec{T}^j) = |A^{k \succ N}(\vec{T}^j)| - |R^{k \succ N}(\vec{T}^j)| - a \quad \text{with} \quad a = \begin{cases} 0, & \text{if } |N^j| = 1, \\ 1, & \text{if } |N^j| = 2, \\ 3, & \text{if } |N^j| = 3, \end{cases}$$

Define $\deg(\vec{T})$ by

$$\deg_{\mathrm{BKW}}(\vec{T}) = \sum_{j=1}^{k} \deg(\vec{T}^j).$$

For the experts: This is also known as Brundan-Kleshchev-Wang's degree.

Flows and 3-multitableaux

Theorem(T)

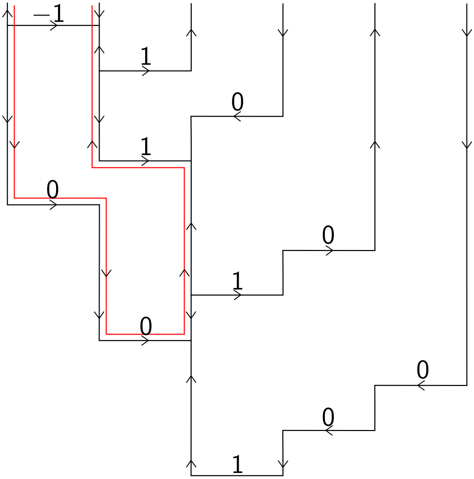
A suitable $\vec{T} \in \mathrm{Std}(\vec{\lambda})$ gives rise to a flow on a web via an extended growth algorithm. Moreover, one can obtain all flows on all webs in this way and $\deg_{\mathrm{BKW}}(\vec{T}) = \sum_{j=1}^{k} \deg(\vec{T}^j)$ is exactly minus the weight of the flow.

Although the algorithm is completely explicit, I do not define it today, since there are several rules that one has to follows. I give an example instead.

Only keep in mind: Webs "are" standard tableaux and flows on webs "are" 3-multitableaux.

Exempli gratia

$$\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 8 & 9 & \\ \cline{1-2} \end{array} , \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 10 & \\ \cline{1-1} 11 & \\ \cline{1-1} \end{array} , \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 9 & \\ \cline{1-2} 7 & & \\ \cline{1-1} \end{array} \right) \rightsquigarrow$$



Please, fasten your seat belts!

Let's categorify everything!

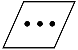
$\mathfrak{sl}_3$-foams

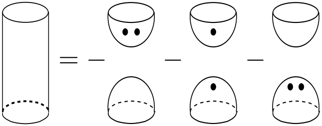
A pre-foam is a cobordism with singular arcs between two webs. Composition consists of placing one pre-foam on top of the other. The following are called the zip and the unzip respectively.

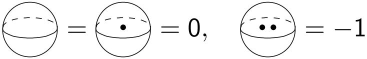
They have dots that can move freely about the facet on which they belong, but we do not allow dot to cross singular arcs.

A foam is a formal $\mathbb{C}$-linear combination of isotopy classes of pre-foams modulo the following relations.

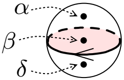
The foam relations $\ell = (3D, NC, S, \Theta)$


$= 0$ (3D)


(NC)



(S)



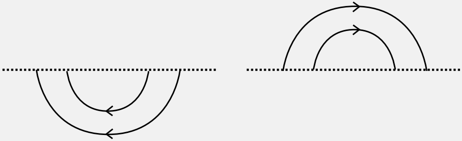
$$= \begin{cases} 1, & (\alpha, \beta, \delta) = (1, 2, 0) \text{ or a cyclic permutation,} \\ -1, & (\alpha, \beta, \delta) = (2, 1, 0) \text{ or a cyclic permutation,} \\ 0, & \text{else.} \end{cases} \qquad (\Theta)$$

Adding a closure relation to ℓ suffice to evaluate foams without boundary!

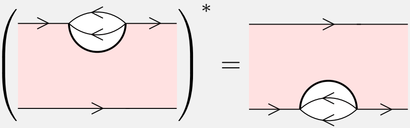
Involution on webs and foams

Definition

There is an involution $*$ on the webs and foams. That is



for webs and for foams



A closed foam is a foam from $\emptyset$ to a closed web $u^* v$.

The $\mathfrak{sl}_3$-foam category

$\mathbf{Foam}_3$ is the category of foams, i.e. objects are webs w and morphisms are foams F between webs. The category is graded by the $q-$degree

$$\deg_q(F) = \chi(\partial F) - 2\chi(F) + 2d + b,$$

where d is the number of dots and b is the number of vertical boundary components. The foam homology of a closed web w is defined by

$$\mathcal{F}(w) = \mathbf{Foam}_3(\emptyset, w).$$

$\mathcal{F}(w)$ is a graded, complex vector space, whose q-dimension can be computed by the Kuperberg bracket (that is counting all flows on w and their weights).

The $\mathfrak{sl}_3$-web algebra

Definition(MPT)

Let $S = (s_1, \dots, s_n)$. The $\mathfrak{sl}_3$-web algebra K_S is defined by

$$K_S = \bigoplus_{u,v \in B_S} {}_uK_v,$$

with

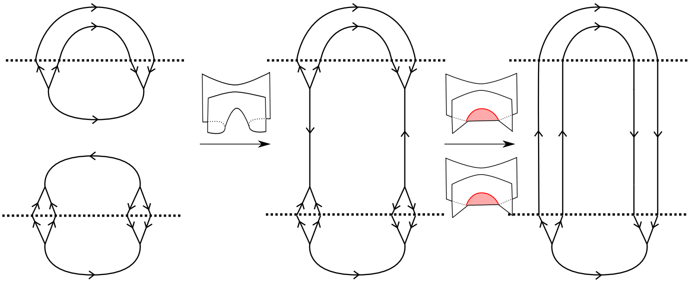
$${}_uK_v = \mathcal{F}(u^* v)\{n\}, \text{ i.e. all foams: } \emptyset \to u^* v.$$

Multiplication is defined as follows.

$${}_uK_{v_1} \otimes {}_{v_2}K_w \to {}_uK_w$$

is zero, if $v_1 \neq v_2$. If $v_1 = v_2$, use the multiplication foam m_v, e.g.

The $\mathfrak{sl}_3$-web algebra



Theorem(s)(MPT)

The multiplication is well-defined, associative and unital. The multiplication foam m_v has q-degree n. Hence, K_S is a finite dimensional, unital and graded algebra. Moreover, it is a graded Frobenius algebra.

Moreover, for $n = d = 3\ell$ we define

$$W_{(3^\ell)} = \bigoplus_{\mu_s \in \Lambda(n,n)_3} W_S$$

on the level of webs and on the level of foams we define

$$\mathcal{W}^{(p)}_{(3^\ell)} = \bigoplus_{\mu_s \in \Lambda(n,n)_3} K_S - (\mathrm{p})\mathbf{Mod}_{gr}\,.$$

With this constructions we obtain our first categorification result.

Theorem(MPT)

$$K_0(\mathcal{W}_{(3^\ell)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \cong W_{(3^\ell)} \text{ and } K_0^{\oplus}(\mathcal{W}^p_{(3^\ell)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \cong W_{(3^\ell)}.$$

Categorification of the LT-algorithm

As a remainder, the LT-algorithm gives

$$\mathrm{LT}(T) = x_T + \sum_{T' \prec T} \alpha_{T'}(v) x_{T'}.$$

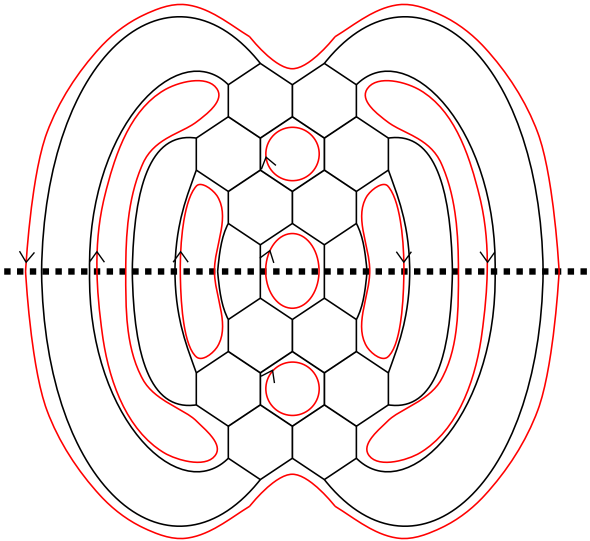
Thus, we need column-strict tableaux and 3-multitableaux (for $x_{T'}$ and $\alpha_{T'}(v)$).

What do we expect to gain? Since Leclerc-Toffin also showed

$$b_T = \mathrm{LT}(T) + \sum_{T'' \prec T} \beta_{T''T}(v) \mathrm{LT}(T'')$$

we expect that we get a method to "compute" the projective indecomposable of K_S, since they should decategorify to the dual canonical basis.

A picture is worth thousand words



$\longleftrightarrow$ 3-multitableau: topology of foam categorifies $\alpha_{T'}(v)$

$\longleftrightarrow$ column-strict tableau: certain idempotent categorifies $x_{T'}$

$\longleftrightarrow$ 3-multitableau: topology of foam categorifies $\alpha_{T'}(v)$

This case is a non-trivial idempotent - the web is not a dual canonical web.

A growth algorithm for foams

Definition(T)

Given a pair of a sign string and a state string (S, J), the corresponding 3-multipartition $\vec{\lambda}$ and two Kuperberg webs $u, v \in B_S$ that extend J to f_u and f_v receptively. We define a foam by

$$\mathcal{F}^{\vec{\lambda}}_{\vec{T}(u_{f_u}),\vec{T}(v_{f_v})} = \underbrace{\mathcal{F}_{\sigma_u}}_{\text{Topology}} \underbrace{e(\vec{\lambda})}_{\text{Idempotent}} \underbrace{d(\vec{\lambda})}_{\text{Dots}} \underbrace{\mathcal{F}^*_{\sigma_v}}_{\text{Topology}} .$$

Theorem(T)

The growth algorithm for foams is well-defined, the only input data are webs and flows on webs, works inductively and gives a graded cellular basis of K_S.

Exempli gratia

We have two webs with flows for the pair $S = (+, -, +, -)$ and $J = (0, 0, 0, 0)$, i.e. two either nested or non-nested circles without flow

In this case the tableaux and the 3-multipartitions are the same, that is

$$T = \begin{array}{|c|c|c|}\hline 2 & 1 & 2 \\ \hline 4 & 3 & 4 \\ \hline \end{array} \leftrightsquigarrow \vec{\lambda} = \left(\begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline \; \\ \hline \end{array}, \begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right)$$

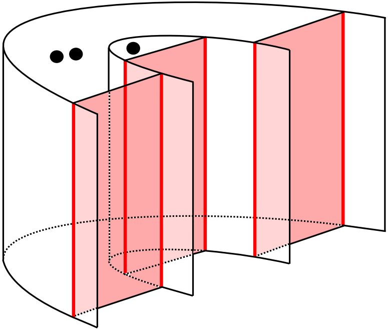
How to assign a dotted idempotent $e(\vec{\lambda})d(\vec{\lambda})$ that is suitable for both webs?

Exempli gratia

$$T_{\vec{\lambda}} = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 7 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\right) \leftrightsquigarrow \mathrm{LT}(\vec{\lambda}) = F_1F_3F_2F_2F_1F_3F_2$$

and define a dotted idempotent $e(\vec{\lambda})d(\vec{\lambda})$ by applying q-skew Howe duality and spread dots based on addable nodes.

$e(\vec{\lambda})d(\vec{\lambda}) =$



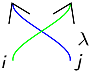
$\mathrm{LT}(\vec{\lambda})v_{3^2}$

$\mathrm{LT}(\vec{\lambda})v_{3^2}$

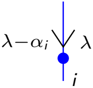
I do not have time but very roughly: In order to define the topology $\mathcal{F}_{\sigma_u}, \mathcal{F}^*_{\sigma_v}$ one let S_n act on $T_{\vec{\lambda}}$ by permuting entries until it looks like the one from the extended growth algorithm. Then use a certain "zipping"-foam for each transposition.

Connection to $\mathbf{U}_q(\mathfrak{sl}_d)$

Khovanov and Lauda's diagrammatic categorification of $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$, denoted $\mathcal{U}(\mathfrak{sl}_d)$, is also related to our framework! Roughly, it consist of string diagrams of the form

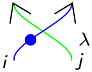


$: \mathcal{E}_i\mathcal{E}_j\mathbf{1}_\lambda \Rightarrow \mathcal{E}_j\mathcal{E}_i\mathbf{1}_\lambda\{(\alpha_i,\alpha_j)\},$



$: \mathcal{F}_i\mathbf{1}_\lambda \Rightarrow \mathcal{F}_i\mathbf{1}_\lambda\{\alpha^{ii}\}$

with a weight $\lambda \in \mathbb{Z}^{n-1}$ and suitable shifts and relations like



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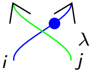
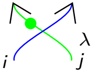


and



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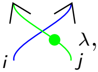


if $i \neq j$.

We define a 2-functor

$$\Psi\colon \mathcal{U}(\mathfrak{sl}_d) \to \mathcal{W}^{(p)}_{(3^\ell)}$$

called foamation, in the following way.

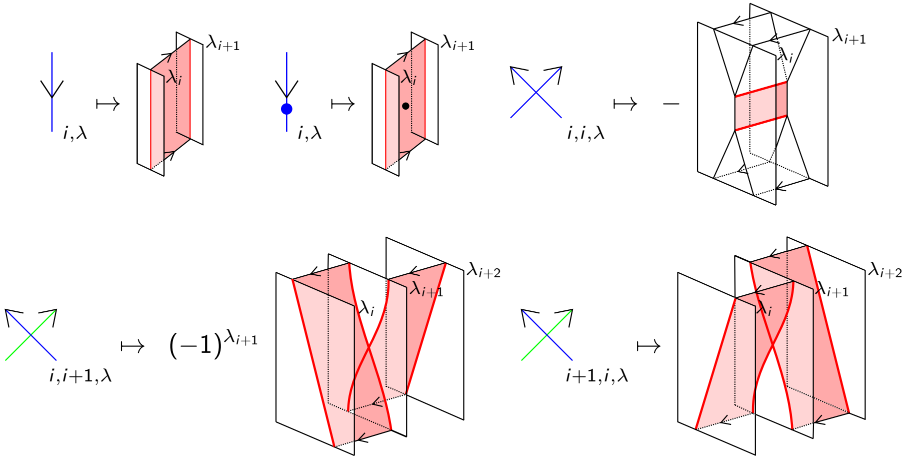
On objects: The functor is defined by sending an $\mathfrak{sl}_d$-weight $\lambda = (\lambda_1, \dots, \lambda_{d-1})$ to an object $\Psi(\lambda)$ of $\mathcal{W}^{(p)}_{(3^\ell)}$ by

$$\Psi(\lambda) = S,\ S = (a_1, \dots, a_\ell),\ a_i \in \{0,1,2,3\},\ \lambda_i = a_{i+1} - a_i,\ \sum_{i=1}^{\ell} a_i = 3^\ell.$$

On morphisms: The functor on morphisms is by glueing the ladder webs from before on top of the $\mathfrak{sl}_3$-webs in $W_{(3^\ell)}$.

$\mathfrak{sl}_3$-foamation

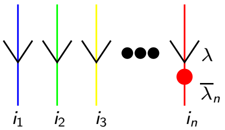
On 2-**cells:** We define



And some others.

The category $\mathcal{U}(\mathfrak{sl}_d)$ has a certain subquotient, called the cyclotomic Khovanov-Lauda and Rouquier (KL-R) algebra R_Λ, that categorifies the irreducible $\dot{\mathbf{U}}_q(\mathfrak{sl}_d)$-representation of highest weight Λ.

It is defined by taking only downwards pointing arrow's (only $\mathcal{F}$'s) and mod out by the so-called cyclotomic relation



Based on work of Brundan, Kleshchev and Wang, Hu and Mathas have defined a graded cellular basis for R_Λ. We also have only $\mathcal{F}$'s - could this be a coincidence?

The growth algorithm for foams gives the HM-basis

Theorem(T)

The growth algorithm for foams gives the HM-basis under categorified q-skew Howe duality.

Corollary("Almost" directly)

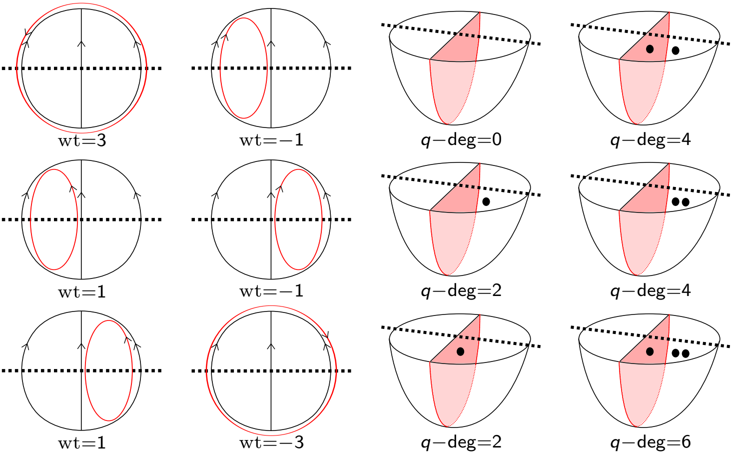
We have $\psi_p([D_p^\lambda]) = b^\lambda$ under the isometry

$$\psi_p \colon K_0^{\oplus}(\mathcal{W}^p_{(3^\ell)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \to W_{(3^\ell)},$$

that is projective covers D_p^λ (who give a complete list of all projective, irreducible K_S-modules) of the simple heads D^λ of the cell modules C^λ categorify the upper global crystal basis b^λ of $W_{(3^\ell)}$. In principle, the D_p^λ are computable from the extended growth algorithm.

Exempli gratia

Every web has a graded cellular basis parametrised by flow lines.



That these foams are really a graded cellular basis follows from our theorem. Note that the Kuperberg bracket gives $[2][3] = q^{-3} + 2q^{-1} + 2q + q^{3}$.

There is still much to do...

Thanks for your attention!