Rank 2 fusion rings are complete intersections

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Abstract

We give a non-constructive proof that fusion rings attached to a simple complex Lie algebra of rank 2 are complete intersections.

1 Introduction

Attached to a simple complex Lie algebra $\mathfrak{g}$ and a natural number $k \geq 0$ we have the level k fusion ring $\mathcal{F}_k = \mathcal{F}_k(\mathfrak{g})$. As a $\mathbb{Z}$-module it is free and finitely generated over the dominant weights of $\mathfrak{g}$ of level at most k and has a product structure making it a commutative, associative, unital ring. It can be described as a quotient of the representation ring $\mathcal{R}$ by the "fusion ideal" $\mathcal{I}_k$, which has an infinite generating set enumerated by all dominant weights of level strictly higher than k. We consider the problem of finding a minimal generating set of this ideal.

If $\operatorname{rank}(\mathfrak{g}) = r$ then as a commutative, associative $\mathbb{Z}$-algebra $\mathcal{R}$ is isomorphic to the polynomial ring $R = \mathbb{Z}[X_1, \ldots, X_r]$ and $\mathcal{F}_k$ to the quotient R/I_k, where $I_k \subseteq R$ corresponds to the fusion ideal. Then, as $\operatorname{codim}(I_k, R) = r$, the best we can hope for is a generating set of I_k consisting of r elements, which in turn must constitute an R-regular sequence meaning R/I_k is a complete intersection ring.

This goal was achieved for $\mathfrak{g}$ of type $A_r, r \geq 1$ by Gepner (1991) and type $C_r, r \geq 2$ by Bourdeau et al. (1992) by finding an explicit potential function $V_{k+h^\vee}(X_1, \ldots, X_r) \in R$ for each level k whose r partial derivatives generate the fusion ideal. For Lie algebras of the remaining types we will have to use a different approach as it was proven by Bouwknegt and Ridout (2006) that in these cases the fusion ideal cannot be described by analogous potential functions.

In this paper we prove non-constructively that the fusion ideal for a rank 2 fusion ring can always be generated by 2 elements. This settles in affirmative the question above for the remaining rank 2 case of type G_2 though it doesn't give explicit generators.

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2 The setup

Consider a simple complex Lie algebra $\mathfrak{g}$ of rank 2 with root system Φ, Weyl group W and weight lattice P. The irreducible finite-dimensional representations $V(\lambda)$ are characterized by having highest weights in $P_+ = \{\lambda \in P \mid 0 \leq \langle \lambda, \alpha^\vee \rangle$ for all $\alpha \in \Phi_+\}$. The representation ring $\mathcal{R}$ has as elements formal differences of isomorphism classes of finite-dimensional representations of $\mathfrak{g}$ with addition given by direct sums of representations and multiplication given by tensor product over $\mathbb{C}$. We write $[\lambda]$ short for the isomorphism class of the irreducible representation with highest weight $\lambda = a\omega_1 + b\omega_2$, where the ω_i are the fundamental weights. The ring structure is encoded in the structure constants $M^\nu_{\lambda,\mu} = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(V(\lambda) \otimes V(\mu) \otimes V(\nu^*), \mathbb{C})$ for which

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu \in P_+} M^\nu_{\lambda,\mu} V(\nu)$$

where $\nu^* = -w_0(\nu)$ is involution on P given by the longest element $w_0 \in W$.

For a given level $k \geq 0$ we restrict to the alcove $P_k = \{\lambda \in P_+ \mid \langle \lambda, \beta_0^\vee \rangle \leq k\}$, where β_0 is the highest root in Φ. We define the level k fusion ring $\mathcal{F}_k$ to be the quotient of $\mathcal{R}$ by the ideal $\mathcal{I}_k$ generated by all $[\lambda] - \det(w)[w.\lambda]$ with $\lambda, w.\lambda \in P_+$ and $w \in W_k$ an element of the affine Weyl group. Here $w.\lambda = w(\lambda + \rho) - \rho$ denotes the shifted W-action, where $\rho = \omega_1 + \omega_2$. Write $\overline{[\lambda]} = [\lambda] + \mathcal{I}_k$. The ring structure in the fusion ring is given by the truncated tensor product

$$V(\lambda) \overline{\otimes} V(\mu) = \bigoplus_{\nu \in P_k} N^\nu_{\lambda,\mu} V(\nu),$$

$$N^\nu_{\lambda,\mu} = \sum_{w \in W_k} \det(w) M^{w.\nu}_{\lambda,\mu}.$$

Following Beauville (1996) we use the structure constants $N^\nu_{\lambda,\mu}$ to build a fusion rule $N : \mathbb{N}[P_k] \to \mathbb{Z}$ recursively defined by

$$\begin{aligned}
&N(0) = 1, \quad N(\lambda) = 0 \text{ for } \lambda \neq 0 \\
&N(\lambda + \lambda^*) = 1, \quad N(\lambda + \mu) = 0 \text{ for } \lambda^* \neq \mu \\
&N(\lambda + \mu + \nu^*) = N^\nu_{\lambda,\mu} \\
&N(x + y) = \sum_{\lambda \in P_k} N(x + \lambda) N(y + \lambda^*) \text{ for } x, y \in \mathbb{N}[P_k].
\end{aligned}$$

Then $\overline{[\lambda_1]} \cdot \ldots \cdot \overline{[\lambda_s]} = \sum_{\lambda \in P_k} N(\lambda_1 + \cdots + \lambda_s + \lambda^*) \overline{[\lambda]}$ and we get a symmetric bilinear $\mathbb{Z}$-form on $\mathcal{F}_k$ determined by

$$\left(\overline{[\lambda_1]} \cdot \ldots \cdot \overline{[\lambda_s]}, \overline{[\mu_1]} \cdot \ldots \cdot \overline{[\mu_r]} \right) = N(\lambda_1 + \cdots + \lambda_s + \mu_1^* + \cdots + \mu_r^*).$$

Since $\left(\overline{[\lambda]}, \overline{[\mu]} \right) = N(\lambda + \mu^*) = \delta_{\lambda,\mu}$ it is positive definite. It satisfies

$$\left(\overline{[\lambda]} \cdot \overline{[\nu]}, \overline{[\mu]} \right) = \left(\overline{[\lambda]}, \overline{[\nu^*]} \cdot \overline{[\mu]} \right). \tag{1}$$

As a commutative, associative $\mathbb{Z}$-algebra the representation ring $\mathcal{R}$ is freely generated by the two fundamental representations $X = V(\omega_1)$ and $Y = V(\omega_2)$, i.e. it may be presented as the polynomial ring $R = \mathbb{Z}[X, Y]$ and the fusion ring is presented by a corresponding quotient R/I_k. Since R/I_k is free and finitely generated as a $\mathbb{Z}$-module the bilinear form gives an isomorphism to its $\mathbb{Z}$-dual $\mathrm{Hom}_{\mathbb{Z}}(R/I_k, \mathbb{Z})$. The property (1) implies that this is actually an isomorphism of R-modules.

The exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ induces to

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(R/I_k, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(R/I_k, \mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(R/I_k, \mathbb{Q}/\mathbb{Z}) \to 0.$$

Since $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules $\mathrm{Hom}_{\mathbb{Z}}(R/I_k, \mathbb{Q})$ and $\mathrm{Hom}_{\mathbb{Z}}(R/I_k, \mathbb{Q}/\mathbb{Z})$ are injective R/I_k-modules. Then $R/I_k \simeq \mathrm{Hom}_{\mathbb{Z}}(R/I_k, \mathbb{Z})$ has finite injective dimension, i.e. it is Gorenstein.

The set of integers $\mathbb{Z}$ is a 1-dimensional discrete valuation ring, so it is a regular ring. Then also $\mathbb{Z}[X, Y]$ is a regular ring. The goal is to show that I_k can be generated over R by two elements. Then since I_k has codimension 2 in R these two generators must constitute a regular R-sequence, i.e. R/I_k is a complete intersection ring.

3 Main result

We first prove a technical lemma.

Lemma 1. *Let A be a noetherian ring, let B be a ring that is a finitely generated A-module and let $b \in B$. Consider B as an $A[X]$-module by mapping $X \mapsto b$. Then there is an isomorphism of $A[X]$-modules*

$$Ext^i_A(B, A) \simeq Ext^{i+1}_{A[X]}(B, A[X])$$

for all $i \geq 0$.

Proof. Set $B[X] = A[X] \otimes_A B$ with trivial $A[X]$-action. Multiplication on $B[X]$ by $X - b$ fits in to a short exact sequence

$$0 \to B[X] \stackrel{X-b}{\to} B[X] \to B \to 0.$$

Apply the left exact functor $\mathrm{Hom}_{A[X]}(-, A[X])$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathrm{Ext}^i_{A[X]}(B[X], A[X]) & \longrightarrow & \mathrm{Ext}^i_{A[X]}(B[X], A[X]) & \longrightarrow & \mathrm{Ext}^{i+1}_{A[X]}(B, A[X]) & \longrightarrow & \cdots \\
 & & \downarrow \simeq & & \downarrow \simeq & & & & \\
 & & A[X] \otimes_A \mathrm{Ext}^i_A(B, A) & \longrightarrow & A[X] \otimes_A \mathrm{Ext}^i_A(B, A) & \longrightarrow & \mathrm{Coker} & \longrightarrow & 0
\end{array}$$

where the vertical isomorphisms come from the fact that $A \to A[X]$ is flat. The lower homomorphism makes the diagram commutative so by naturality of the isomorphisms it is multiplication by $X - b$ identifying the cokernel with $\mathrm{Ext}^i_A(B, A)$. Diagram chasing gives us a map

$$\mathrm{Ext}^i_A(B, A) \to \mathrm{Ext}^{i+1}_{A[X]}(B, A[X])$$

which is an isomorphism by the Five Lemma. □

Theorem 2. *Let $I \subseteq R = \mathbb{Z}[X,Y]$ be an ideal such that we have an isomorphism of R-modules*

$$Hom_{\mathbb{Z}}(R/I, \mathbb{Z}) \simeq R/I. \tag{2}$$

Then I is generated by an R-regular sequence of length 2.

Proof. Necessarily from the duality (2) R/I is a finitely generated $\mathbb{Z}$-module, so Lemma 1 applied to the ring $\mathbb{Z}$, the $\mathbb{Z}$-module R/I and distinguished element $x = X + I \in R/I$ gives us

$$\mathrm{Ext}^i_{\mathbb{Z}}(R/I, \mathbb{Z}) \simeq \mathrm{Ext}^{i+1}_{\mathbb{Z}[X]}(R/I, \mathbb{Z}[X]).$$

Now R/I is still finitely generated as an $\mathbb{Z}[X]$-module so the lemma applied once more with the element $y = Y + I \in R/I$ gives us

$$\mathrm{Ext}^{i+1}_{\mathbb{Z}[X]}(R/I, \mathbb{Z}[X]) \simeq \mathrm{Ext}^{i+2}_{\mathbb{Z}[X,Y]}(R/I, \mathbb{Z}[X,Y]).$$

With $i = 0$ and the assumption we get

$$\mathrm{Ext}^2_R(R/I, R) \simeq \mathrm{Hom}_{\mathbb{Z}}(R/I, \mathbb{Z}) \simeq R/I.$$

Pick a unit $e \in R/I$. With the identification $\mathrm{Ext}^2_R(R/I, R) \simeq \mathrm{Ext}^1_R(I, R)$ this element correspond to a nonsplit short exact sequence

$$0 \to R \to M \to I \to 0. \tag{3}$$

The goal is to show that $M \simeq R^2$ for then the image of two generators under the surjection in (3) will generate I.

We prove first that $\mathrm{Ext}^i_R(M, R) = 0$ for all $i \geq 1$. Consider the long exact sequence associated to (3)

$$\mathrm{Hom}_R(R, R) \xrightarrow{p} \mathrm{Ext}^1_R(I, R) \to \mathrm{Ext}^1_R(M, R) \to \mathrm{Ext}^1_R(R, R) = 0.$$

By construction we have $p(\mathbb{1}) = e \in \mathrm{Ext}^1_R(I, R) \simeq R/I$. Choose an $f \in R$ with $e^{-1} = f + I$. Since p is an R-homomorphism $p(f) = fe = (f + I)e = 1$ so p is surjective and $\mathrm{Ext}^1_R(M, R) = 0$. Let now $i \geq 2$. Since R/I is Gorenstein it is Cohen-Macaulay and localizing at a prime ideal $\mathfrak{p} \in R$ containing I the Auslander-Buchsbaum formula gives us proj $\dim_R(R/I) = 2$. Therefore $0 = \mathrm{Ext}^{i+1}_R(R/I, R) \simeq \mathrm{Ext}^i_R(I, R) \simeq \mathrm{Ext}^i_R(M, R)$.

Then induction on the length of a projective resolution on a given module N gives $\mathrm{Ext}^i_R(M, N) = 0$ for all $i \geq 1$, i.e. M is projective.

Now (Quillen, 1976, Theorem 4) says that all projective modules over $\mathbb{Z}[X,Y]$ are actually free so $M \simeq R^k$. Choose any prime ideal $\mathfrak{p} \subseteq R$ not containing I. Localizing (3) at $\mathfrak{p}$ we get

$$0 \to R_{\mathfrak{p}} \to M_{\mathfrak{p}} \to R_{\mathfrak{p}} \to 0$$

showing that $k = 2$. Here we used that $I \cap (R \setminus \mathfrak{p}) \neq \emptyset$ so $I_{\mathfrak{p}}$ contains a unit. □

Remark. As mentioned in the introduction the proof ensures that the fusion ideal for a rank 2 Lie algebra can be generated by two elements. For the cases A_2 and C_2 this was previously known, in fact in both cases the ideal I_k is generated by the elements $[(k+1)\omega_1]$ and $[k\omega_1 + \omega_2]$. For the remaining type

G_2 Boysal and Kumar (2009) conjecturally described the fusion ideal as the radical of an ideal generated by three specific elements and later Douglas (2013) found an actual generating set consisting of the three elements

$$\begin{cases} [\frac{k+2}{2}\omega_2] + [\frac{k}{2}\omega_2],\ [\omega_1 + \frac{k}{2}\omega_2] \text{ and } [3\omega_1 + \frac{k-2}{2}\omega_2] & \text{if } k \text{ even} \\ [\frac{k+1}{2}\omega_2],\ [2\omega_1 + \frac{k-1}{2}\omega_2] \text{ and } [3\omega_1 + \frac{k-1}{2}\omega_2] + [3\omega_1 + \frac{k-3}{2}\omega_2] & \text{if } k \text{ odd} \end{cases}$$

Though we are garanteed that this generating set can somehow be reduced to two elements we have no method to do so.

Much of the theory here generalizes to higher ranks. When we consider a general ideal $I \subseteq R = \mathbb{Z}[X_1, \dots, X_r]$ such that $\operatorname{Hom}_{\mathbb{Z}}(R/I, \mathbb{Z}) \simeq R/I$ then Lemma 1 applied r times still gives us

$$R/I \simeq \operatorname{Hom}_{\mathbb{Z}}(R/I, \mathbb{Z}) \simeq \operatorname{Ext}^1_{\mathbb{Z}[X_1]}(R/I, \mathbb{Z}[X_1]) \simeq \\ \cdots \simeq \operatorname{Ext}^r_{\mathbb{Z}[X_1,\dots,X_r]}(R/I, \mathbb{Z}[X_1, \dots, X_r]).$$

Locally the Auslander-Buchsbaum formula still gives us $\operatorname{proj}\dim_R(R/I) = r$ so $\operatorname{Ext}^i_R(R/I, R) = 0$ for $i > r$. We also have our tool to give us global information from local data: By (Quillen, 1976, Theorem 4) any projective $\mathbb{Z}[X_1, \dots, X_r]$ is free. However only for $r = 2$ is a non-trivial extension of I by R in $\operatorname{Ext}^{r-1}_R(I, R) \simeq R/I$ equivalent to a non-split short exact sequence which is the beginning of our construction in the proof.

In Serre (1963) it was proved that a quotient of a regular local ring of codimension 2 is a complete intersection ring if and only if it is Gorenstein. After more than 50 years this result has not been generalized to higher codimensions suggesting that a generalization of Theorem 2 to higher ranks is not possible without further assumptions on the ideal.

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