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Narjess Khalifa, Hui-Hsiung Kuo, Habib Ouerdiane
and Benedykt Szozda

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Narjess Khalifa[1], Hui-Hsiung Kuo[2], Habib Ouerdiane[3] and Benedykt Szozda[4]

[1] University of Tunis El Manar, narjeskhalifa@yahoo.fr
[2]Louisiana State University, kuo@math.lsu.edu
[3]University of Tunis El Manar, habib.ouerdiane@fst.rnu.tn
[4]Aarhus University, szozda@imf.au.dk

Abstract

In this paper we use the new stochastic integral introduced by Ayed and Kuo (2008) and the results obtained by Kuo et al. (2012b) to find a solution to a drift-free linear stochastic differential equation with anticipating initial condition. Our solution is based on well-known results from classical Itô theory and anticipative Itô formula results from Kuo et al. (2012b). We also show that the solution obtained by our method is consistent with the solution obtained by the methods of Malliavin calculus, e.g. Buckdahn and Nualart (1994).

Keywords: adapted stochastic processes, anticipating stochastic differential equations, Brownian motion, Itô integral, instantly independent stochastic processes, linear stochastic differential equations, stochastic integral

AMS Subject Classification: 60H05, 60H20

1 Introduction

The aim of the present paper is to establish a solution to linear stochastic differential equation with an anticipating initial condition of a certain form, namely

$$\begin{cases} dX_t = \alpha_t X_t dB_t + \beta_t X_t dt & t \in [a,b] \\ X_a = p(B_b - B_a). \end{cases} \tag{1.1}$$

In the case with $X_a = x \in \mathbb{R}$, it is a well-known fact that the unique solution is given by

$$X_t = x \exp\left\{ \int_a^t \alpha_s dB_s + \int_a^t \left(\beta_s - \tfrac{1}{2}\alpha_s^2\right) ds \right\}. \tag{1.2}$$

For details, see for example, (Kuo, 2006, Section 11.1). The significance of our result lays in the fact that the solution X_t of Equation (1.1) is an anticipating stochastic process and it cannot be obtained by the classical tools from the Itô theory of stochastic integration. Instead, we use the integral of adapted and instantly independent processes introduced by Ayed and Kuo (2008, 2010) and further developed by Kuo et al. (2012a,b, 2013). In contrast to results obtained by Buckdahn and Nualart (1994) and Esunge (2009), our results do not rely on Malliavin calculus or white noise analysis and are anchored in basic probability theory.

The reminder of this paper is organized as follows. In Section 2 we recall all the necessary definitions and previous results used in the rest of the paper. Section 3 contains a simple example that illustrates our methods and Section 4 presents our main result, Theorem 4.1. We conclude with several examples in Section 5.

2 Preliminary Definitions

In this section we fix the notation and recall several definitions used in the remainder of this work.

We denote by $C^k(\mathbb{R})$ the space of all functions $f : \mathbb{R} \to \mathbb{R}$ that are k times continuously differentiable, and by $C^\infty(\mathbb{R})$ the space of functions whose derivatives of all orders exist and are continuous. The space of all smooth functions whose Maclaurin series converges for all $x \in \mathbb{R}$ is denoted by $\mathcal{M}^\infty$, that is

$$\mathcal{M}^\infty = \Big\{ f \in C^\infty(\mathbb{R}) \,\Big|\, f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} x^k \text{ for all } x \in \mathbb{R} \Big\},$$

where $f^{(k)}(x)$ stands for the k-th derivative of $f(x)$.

We denote by $\mathcal{S}(\mathbb{R})$ the Schwartz class of rapidly decreasing functions, that is

$$\mathcal{S}(\mathbb{R}) = \Big\{ f \in C^\infty(\mathbb{R}) \,\Big|\, \sup_{x\in\mathbb{R}} |x^n f^{(m)}(x)| < \infty, \text{ for all } m, n \in \mathbb{N} \Big\}. \tag{2.1}$$

It is a well known fact that $\mathcal{S}(\mathbb{R})$ is closed under the Fourier transform, which we define as $\hat{f}(\zeta) = \int_{\mathbb{R}} f(x) e^{-2\pi i x\zeta}\, dx$, with the inverse Fourier transform given by $f(x) = \int_{\mathbb{R}} \hat{f}(\zeta) e^{2\pi i x\zeta}\, d\zeta$. In this setting, we have the following property of the Fourier transform

$$\widehat{\left(\frac{d}{dx} f(x)\right)}(\zeta) = 2\pi i \zeta \hat{f}(x). \tag{2.2}$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, B_t be a standard Brownian motion on $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{t\in[0,\infty)}$ be a right-continuous, complete filtration such that:

1. for each $t \in [0, \infty)$, the random variable B_t is $\mathcal{F}_t$-measurable;
2. for all s and t such that $0 \le s < t$, the random variable $B_t - B_s$ is independent of $\mathcal{F}_s$.

Following Ayed and Kuo (2008), we say that a stochastic process X_t is *instantly independent* with respect to $(\mathcal{F}_t)_{t\in[0,\infty)}$ if for each $t \in [0, \infty)$, the random variable X_t is independent of $\mathcal{F}_t$. Recall that if f_t is adapted and φ_t is instantly independent with respect to $(\mathcal{F}_t)$, the Itô integral of the product of f and φ is defined as the limit

$$\int_a^b f_t \varphi_t dB_t = \lim_{\|\Delta_n\| \to 0} \sum_{i=0}^{n} f_{t_{i-1}} \varphi_{t_i} (B_{t_i} - B_{t_{i-1}}), \tag{2.3}$$

whenever the limit exists in probability. Note that if $\varphi \equiv 1$, then the integral defined in Equation (2.3) reduces to the ordinary Itô integral for adapted processes. This kind of integral was introduced by Ayed and Kuo (2008, 2010) and studied further by Kuo et al. (2012a,b, 2013).

Following the notation of Kuo (2006), we denote by $L^2_{\text{ad}}(\Omega \times [a,b])$ the space of all adapted stochastic processes X_t such that $\mathbb{E}[\int_a^b X_t^2 \, dB_t] < \infty$. It is a well-known fact that the Itô integral is well-defined for processes from $L^2_{\text{ad}}(\Omega \times [a,b])$.

As in the Itô theory of stochastic integration, the key tool used in this work will be the Itô formula. We state below one of the results of Kuo et al. (2012b) where the authors provide several formulas of this type. Multidimensional version and further generalizations of Itô formulas together with an anticipative version of the Girsanov theorem can be found in Kuo et al. (2013+).

Theorem 2.1 (Kuo et al. (2012b, Corollary 6.2))**.** *Suppose that*

$$\theta(t,x,y) = \tau(t) f(x) \varphi(y),$$

where $\tau \in C^1(\mathbb{R})$*,* $f \in C^2(\mathbb{R})$*, and* $\varphi \in \mathcal{M}^\infty$*. Let*

$$X_t = \int_a^t \alpha_s \, dB_s + \int_a^t \beta_s \, ds,$$

where $\alpha, \beta \in L^2_{ad}(\Omega \times [a,b])$*. Then*

$$\begin{aligned} \theta(t, X_t, B_b - B_a) = \theta(a, X_a, B_b - B_a) &+ \int_a^t \frac{\partial \theta}{\partial x}(s, X_s, B_b - B_a) \, dX_s \\ &+ \frac{1}{2} \int_a^t \frac{\partial^2 \theta}{\partial x^2}(s, X_s, B_b - B_a) \, (dX_s)^2 \\ &+ \int_a^t \frac{\partial^2 \theta}{\partial x \partial y}(s, X_s, B_b - B_a) \, (dX_s)(dB_s) \\ &+ \int_a^t \frac{\partial \theta}{\partial t}(s, X_s, B_b - B_a) \, ds. \end{aligned} \tag{2.4}$$

Equivalently, we can write the Equation (2.4) *in a differential form as*

$$\begin{aligned} d\theta(t, X_t, B_b - B_a) = \frac{\partial \theta}{\partial x}(t, X_t, B_b - B_a) \, dX_t &+ \frac{1}{2} \frac{\partial^2 \theta}{\partial x^2}(s, X_t, B_b - B_a) \, (dX_t)^2 \\ &+ \frac{\partial^2 \theta}{\partial x \partial y}(t, X_t, B_b - B_a) \, (dX_t)(dB_t) \\ &+ \frac{\partial \theta}{\partial t}(t, X_t, B_b - B_a) \, ds. \end{aligned} \tag{2.5}$$

3 A Motivational Example

In this section, we present an example that illustrates the method for obtaining a solution of Equation (1.1). We begin with the simplest possible case of Equation (1.1), that is we set $\alpha \equiv 1, \beta \equiv 0$ and $p(x) = x$, and restrict our considerations to the interval $[0, 1]$. Thus we wish to find a solution to

$$\begin{cases} dX_t = X_t\, dB_t, & t \in [0,1] \\ X_0 = B_1. \end{cases} \tag{3.1}$$

The natural guess for the solution of Equation (3.1) is obtained by putting B_1 for x in Equation (1.2) to obtain

$$X_t = B_1 \exp\left\{B_t - \tfrac{1}{2}t\right\}.$$

Using the Itô formula, it is easy to show that the process X_t is not a solution of Equation (3.1), but it is a solution of

$$dX_t = X_t\, dB_t + e^{B_t - \frac{1}{2}t} dt, \tag{3.2}$$

which is obviously different from Equation (3.1). The failure of this approach comes from the fact that we do not account for the new factor in the equation, namely B_1. To account for B_1 in Equation (3.1), we can introduce a correction term to X_t that will counteract the dt term appearing in Equation (3.2).

Now, we will use the following as an ansatz for the solution of Equation (3.1)

$$X_t = (B_1 - \xi(t)) \exp\left\{B_t - \tfrac{1}{2}t\right\}, \tag{3.3}$$

where $\xi(t)$ is a deterministic function. The reason for this particular choice is simple. We see that the difference between Equations (3.2) and (3.1) is the term $\exp\{B_t - \frac{1}{2}t\}\, dt$, and to counteract this, we need to introduce another dt-term with the opposite sign. Looking at the Itô formula in Theorem 2.1, we see that we have to introduce a correction factor that depends only on t.

We use the Itô formula from Theorem 2.1 with $\theta(t, x, y) = (y - \xi(t))e^{x - \frac{1}{2}t}$, and

$$\begin{aligned} \theta_t &= -\xi'(t)e^{x-\frac{1}{2}t} - \tfrac{1}{2}(y - \xi(t))e^{x-\frac{1}{2}t}, \\ \theta_x &= (y - \xi(t))e^{x-\frac{1}{2}t}, \\ \theta_{xx} &= (y - \xi(t))e^{x-\frac{1}{2}t}, \\ \theta_{xy} &= e^{x-\frac{1}{2}t}, \end{aligned}$$

to obtain

$$\begin{aligned} d\theta(t, B_t, B_1) =\; & (B_1 - \xi(t))\, e^{B_t - \frac{1}{2}}\, dB_t + \tfrac{1}{2}\, (B_1 - \xi(t))\, e^{B_t - \frac{1}{2}}\, dt \\ & + e^{B_t - \frac{1}{2}t}\, dt - \left(\xi'(t)e^{B_t-\frac{1}{2}t} + \tfrac{1}{2}(B_1 - \xi(t))e^{B_t - \frac{1}{2}t}\right) dt \\ =\; & (B_1 - \xi(t))\, e^{B_t - \frac{1}{2}}\, dB_t + \left(e^{B_t - \frac{1}{2}t} - \xi'(t)e^{B_t - \frac{1}{2}t}\right) dt. \end{aligned}$$

So for $X_t = \theta(t, B_t, B_1)$ to be the solution of Equation (3.1), function $\xi(t)$ has to satisfy the following ordinary differential equation

$$\begin{cases} \xi'(t) = 1, & t \in [0,1] \\ \xi(0) = 0. \end{cases} \tag{3.4}$$

Thus, with $\xi(t) = t$, process X_t given in Equation (3.3) is a solution to stochastic differential equation (3.1), that is

$$X_t = (B_1 - t) \exp\left\{B_t - \tfrac{1}{2}t\right\} \tag{3.5}$$

solves Equation (3.1).

We point out that the solution in Equation (3.5) coincides with the one that can be obtained by methods of Buckdahn and Nualart (1994), where in Proposition 3.2 authors state that the unique solution of Equation (3.1) has the form

$$X_t = g(t,x)\Big|_{x=B_1} \exp\left\{B_t - \tfrac{1}{2}t\right\},$$

where g solves the following partial differential equation

$$\begin{cases} g_t(t,x) = -g_x(t,x), & t \in (0,1] \\ g(0,x) = x. \end{cases}$$

In our case, $g(t,x) = x - t$.

4 General Case

Theorem 4.1 gives the solution to Equation (1.1) for a certain class of coefficients α_t and β_t, and initial conditions $p(x)$ with $x = B_b - B_a$. The proof of this theorem uses the idea of a correction term introduced in the previous section.

Theorem 4.1. *Suppose that $\alpha \in L^2([a,b])$ and $\beta \in L^2_{ad}(\Omega \times [a,b])$. Suppose also that $p \in \mathcal{M}^\infty \cap \mathcal{S}(\mathbb{R})$. Then the stochastic differential equation*

$$\begin{cases} dX_t = \alpha_t X_t \, dB_t + \beta_t X_t \, dt, & t \in [a,b] \\ X_a = p(B_b - B_a), \end{cases} \tag{4.1}$$

has a unique solution given by

$$X_t = \left[p(B_b - B_a) - \xi(t, B_b - B_a)\right] Z_t, \tag{4.2}$$

where

$$\xi(t,y) = \int_a^t \alpha_s p'\left(y - \int_s^t \alpha_u \, du\right) ds, \tag{4.3}$$

and

$$Z_t = \exp\left\{\int_a^t \alpha_s \, dB_s + \int_a^t \left(\beta_s - \tfrac{1}{2}\alpha_s^2\right) ds\right\}.$$

Remark 4.2. Before we proceed with proof of Theorem 4.1, let us remark that if $a = 0$, $\alpha_t \equiv \alpha$ and $\beta_t \equiv \beta$, that is the coefficients are constant and evolution starts at 0, we can again apply the results of Proposition 3.2 of Buckdahn and Nualart (1994). In our notation, the above mentioned proposition states that the solution to Equation (4.1) has the form

$$X_t = g(t, B_1) \exp\left\{\alpha B_t + \left(\beta - \tfrac{1}{2}\alpha^2\right) t\right\}, \tag{4.4}$$

where $g(t,x)$ is the solution of the following partial differential equation

$$\begin{cases} g_t(t,x) = -\alpha g_x(t,x) & t \in (0,b) \\ g(0,x) = p(x). \end{cases} \tag{4.5}$$

Hence in order to show that our solution and the one given by Equation (4.4) coincide, it is enough to show that $g(t,x) = p(x) - \xi(t,x)$ solves Equation (4.5). Note that in the case of constant coefficients, $g(t,x) = p(x - \alpha t)$. Now it is a matter of simple computation to check that g solves Equation (4.5).

Proof. The uniqueness of a solution follows from linearity of Equation (4.1) and standard arguments. To prove the existence of a solution, first observe that Z_t is a solution of a stochastic differential equation given by

$$\begin{cases} dZ_t = \alpha_t Z_t \, dB_t + \beta_t Z_t \, dt, & t \in [a,b] \\ Z_a = 1. \end{cases}$$

Consider

$$\begin{aligned} dX_t &= d\big[\big(p(B_b - B_a) - \xi(t, B_b - B_a)\big) Z_t\big] \\ &= d\big[p(B_b - B_a) Z_t\big] - d\big[\xi(t, B_b - B_a) Z_t\big], \end{aligned}$$

where

$$\xi(t,y) = \sum_{n=0}^{\infty} \xi_n(t) y^n, \qquad \text{for all } t \geq 0,\ y \in \mathbb{R}. \tag{4.6}$$

Note that since the function $z\xi_n(t)y^n$ satisfies the assumptions of the Theorem 2.1, we can write

$$\begin{aligned} d(Z_t \xi(t, B_b - B_a)) &= d\Big(Z_t \sum_{n=0}^{\infty} \xi_n(t)(B_b - B_a)^n\Big) \\ &= \sum_{n=0}^{\infty} d\left(Z_t \xi_n(t)(B_b - B_a)^n\right) \\ &= \sum_{n=0}^{\infty} \Big[\xi_n(t)(B_b - B_a)^n \, dZ_t + Z_t \xi'(t)(B_b - B_a)^n \, dt \\ &\qquad + Z_t \xi_n(t) n (B_b - B_a)^{n-1} (dZ_t)(dB_t)\Big] \\ &= \xi(t, B_b - B_a) \, dZ_t + Z_t \frac{\partial \xi}{\partial t}(t, B_b - B_a) \, dt \\ &\quad + \frac{\partial \xi}{\partial y}(t, B_b - B_a)(dZ_t)(dB_t). \end{aligned} \tag{4.7}$$

Using Theorem 2.1 and Equation (4.7) we obtain

$$\begin{aligned}
dX_t &= p(B_b - B_a)dZ_t + p'(B_b - B_a)(dZ_t)(dB_t) \\
&\quad - \left[\frac{\partial \xi}{\partial t}(t, B_b - B_a)Z_t\, dt + \xi(t, B_b - B_a)\, dZ_t + \frac{\partial \xi}{\partial y}(t, B_b - B_a)(dZ_t)(dB_t)\right] \\
&= \big[p(B_b - B_a) - \xi(t, B_b - B_a)\big]\, dZ_t \\
&\quad + \Big[p'(B_b - B_a)(dZ_t)(dB_t) - \frac{\partial \xi}{\partial t}(t, B_b - B_a)Z_t\, dt \\
&\qquad - \frac{\partial \xi}{\partial y}(t, B_b - B_a)(dZ_t)(dB_t)\Big].
\end{aligned}$$

So for X_t to be a solution of Equation (4.1), we need

$$p'(B_b - B_a)(dZ_t)(dB_t) - \frac{\partial \xi}{\partial t}(t, B_b - B_a)Z_t\, dt - \frac{\partial \xi}{\partial y}(t, B_b - B_a)(dZ_t)(dB_t) = 0 \tag{4.8}$$

for all $t \in [a, b]$. Note that

$$\begin{aligned}
(dZ_t)(dB_t) &= \big(\alpha_t Z_t\, dB_t + \beta_t Z_t\, dt\big)(dB_t) \\
&= \alpha_t Z_t\, dt.
\end{aligned} \tag{4.9}$$

Putting together Equations (4.8) and (4.9) yields

$$p'(B_b - B_a)\alpha_t Z_t\, dt - \frac{\partial \xi}{\partial t}(t, B_b - B_a)Z_t\, dt - \frac{\partial \xi}{\partial y}(t, B_b - B_a)\alpha_t Z_t\, dt = 0,$$

or equivalently,

$$\Big[p'(B_b - B_a)\alpha_t - \frac{\partial \xi}{\partial t}(t, B_b - B_a) - \frac{\partial \xi}{\partial y}(t, B_b - B_a)\alpha_t\Big] X_t\, dt = 0.$$

Hence it is enough to find $\xi(t, y)$ such that

$$\begin{cases} p'(y)\alpha_t - \frac{\partial \xi}{\partial t}(t, y) - \frac{\partial \xi}{\partial y}(t, y)\alpha_t = 0, & t \in [a, b] \\ \xi(0, y) = 0. \end{cases} \tag{4.10}$$

Thus the problem of finding a solution to the stochastic differential equation (4.1) has been reduced to that of finding a solution to the deterministic partial differential equation (4.10). In order to solve Equation (4.10), we apply the Fourier transform to both sides of Equation (4.10), to obtain

$$\widehat{p'}(\zeta)\alpha_t - \frac{\partial}{\partial t}\widehat{\xi}(t, \zeta) - 2\pi i \zeta \widehat{\xi}(t, \zeta)\alpha_t = 0. \tag{4.11}$$

Note that Equation (4.11) is an ordinary differential equation in t, with an integrating factor

$$\exp\Big\{2\pi i \zeta \int_a^t \alpha_s\, ds\Big\}.$$

Hence Equation (4.11) is equivalent to

$$\frac{\partial}{\partial t}\left(\widehat{\xi}(t,\zeta)\exp\Big\{2\pi i\zeta\int_a^t \alpha_s\,ds\Big\}\right) = \widehat{p'}(\zeta)\alpha_t\exp\Big\{2\pi i\zeta\int_a^t \alpha_s\,ds\Big\}. \tag{4.12}$$

Integration with respect to t of both sides of Equation (4.12) yields

$$\widehat{\xi}(t,\zeta)\exp\Big\{2\pi i\zeta\int_a^t \alpha_s\,ds\Big\} = \widehat{p'}(\zeta)\int_a^t \alpha_s\exp\Big\{2\pi i\zeta\int_a^s \alpha_u\,du\Big\}\,ds + \widehat{C}(\zeta), \tag{4.13}$$

for some function $\widehat{C}(\zeta) \in \mathcal{S}(\mathbb{R})$. Thus, the Fourier transform of function $\xi(t,y)$, that is a solution of Equation (4.10), is given by

$$\begin{aligned}\widehat{\xi}(t,\zeta) = \widehat{p'}(\zeta)&\int_a^t \alpha_s\exp\Big\{-2\pi i\zeta\int_s^t \alpha_u\,du\Big\}\,ds\\ &+ \widehat{C}(\zeta)\exp\Big\{-2\pi i\zeta\int_a^t \alpha_s\,ds\Big\}.\end{aligned} \tag{4.14}$$

Now, we apply the inverse Fourier transform to get

$$\begin{aligned}\xi(t,y) = &\int_{\mathbb{R}}\widehat{p'}(\zeta)\int_a^t \alpha_s\exp\Big\{-2\pi i\zeta\int_s^t \alpha_u\,du\Big\}\,ds\exp\Big\{-2\pi iy\zeta\Big\}\,d\zeta\\ &+\int_{\mathbb{R}}\widehat{C}(\zeta)\exp\Big\{-2\pi i\zeta\int_a^t \alpha_s\,ds\Big\}\exp\Big\{2\pi iy\zeta\Big\}\,d\zeta\\ = &\int_a^t \alpha_s\int_{\mathbb{R}}\widehat{p'}(\zeta)\exp\Big\{a\pi i\zeta\Big(y-\int_s^t \alpha_u\,du\Big)\Big\}\,d\zeta\,ds\\ &+\int_{\mathbb{R}}\widehat{C}(\zeta)\exp\Big\{a\pi i\zeta\Big(y-\int_s^t \alpha_u\,du\Big)\Big\}\,d\zeta\\ = &\int_a^t \alpha_s p'\Big(y-\int_s^t \alpha_u\,du\Big)\,ds + C\Big(y-\int_a^t \alpha_s\,ds\Big).\end{aligned}$$

Using the initial condition from Equation (4.10), we see that $C(y) \equiv 0$. Hence X_t as in Equation (4.2) is a solution of Equation (4.1). □

Remark 4.3. Although very tedious, it is straightforward to check that function $\xi(t,y)$ in Equation (4.3) can be expressed in the form of Equation (4.6).

5 Examples

Below we give several examples of stochastic differential equations with either deterministic or anticipating initial conditions. It is interesting to compare the solutions to see how anticipating initial conditions affect the solutions.

Example 5.1 (Adapted)**.** Equation

$$\begin{cases} dX_t = X_t dB_t + X_t dt\\ X_0 = x\end{cases}$$

has solution given by

$$X_t = x\exp\left\{B_t + \tfrac{1}{2}t\right\}.$$

Example 5.2 (Anticipating, compare with Example 5.1)**.** Equation

$$\begin{cases} dX_t = X_t dB_t + X_t dt \\ X_0 = B_1 \end{cases}$$

has solution given by

$$X_t = (B_1 - t) \exp \left\{ B_t + \tfrac{1}{2} t \right\}.$$

Example 5.3 (Anticipating, compare with Example 5.1)**.** Equation

$$\begin{cases} dX_t = X_t dB_t + X_t dt \\ X_0 = e^{B_1} \end{cases}$$

has a solution given by

$$X_t = e^{B_1 - t} \exp \left\{ B_t - \tfrac{1}{2} t \right\}$$

Example 5.4 (Adapted)**.** Equation

$$\begin{cases} dX_t = \alpha_t X_t dB_t + \beta_t X_t dt \\ X_0 = x \end{cases}$$

has solution given by

$$X_t = x \exp \left\{ \int_0^t \alpha_s dB_s + \int_0^t \left(\beta_s - \tfrac{1}{2} \alpha_s^2 \right) ds \right\}.$$

Example 5.5 (Anticipating, compare with Example 5.4)**.** Equation

$$\begin{cases} dX_t = \alpha_t X_t dB_t + \beta_t X_t dt \\ X_0 = B_1 \end{cases}$$

has solution given by

$$X_t = \left(B_1 - \int_0^t \alpha_s ds \right) \exp \left\{ \int_0^t \alpha_s dB_s + \int_0^t \left(\beta_s - \tfrac{1}{2} \alpha_s^2 \right) ds \right\}.$$

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References

Wided Ayed and Hui-Hsiung Kuo. An extension of the Itô integral. *Commun. Stoch. Anal.*, 2(3):323–333, 2008.

Wided Ayed and Hui-Hsiung Kuo. An extension of the Itô integral: toward a general theory of stochastic integration. *Theory Stoch. Process.*, 16(1):17–28, 2010.

R. Buckdahn and D. Nualart. Linear stochastic differential equations and wick products. *Probability Theory and Related Fields*, 99(4):501–526, 1994.

Julius Esunge. A class of anticipating linear stochastic differential equations. *Commun. Stoch. Anal.*, 3(1):155–164, 2009.

Hui-Hsiung Kuo. *Introduction to stochastic integration.* Universitext. Springer, New York, 2006.

Hui-Hsiung Kuo, Anuwat Sae-Tang, and Benedykt Szozda. A stochastic integral for adapted and instantly independent stochastic processes. In *Stochastic Processes, Finance and Control*, Advances in Statistics, Probability and Actuarial Science, chapter 3, pages 53–71. World Scientific, 2012a.

Hui-Hsiung Kuo, Anuwat Sae-Tang, and Benedykt Szozda. The Itô formula for a new stochastic integral. *Communications on Stochastic Analysis*, 6(4):604–614, 2012b.

Hui-Hsiung Kuo, Anuwat Sae-Tang, and Benedykt Szozda. An isometry formula for a new stochastic integral. In *Proceedings of International Conference on Quantum Probability and Related Topics, May 29–June 4 2011, Levico, Italy*, volume 29 of *QP-PQ: Quantum Probability and White Noise Analysis*, pages 222–232, 2013.

Hui-Hsiung Kuo, Yun Peng, and Benedykt Szozda. Itô formula and Girsanov theorem for the new stochastic integral, 2013+. in preparation.