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Stochastic lot-sizing problems have been addressed quite extensively, but relatively few studies also consider marketing factors, such as pricing. In this paper, we address a joint stochastic lot-sizing and pricing problem with capacity constraints and backlogging for a firm that produces a single item over a finite multi-period planning horizon. Thece-dependent demands. The stochastic demand is captured by the scenario analysis approach, and this leads to a multiple-stage stochastic programming problem. Given the complexity of the stochastic programming problem, it is hard to determine optimal prices and lot sizes simultaneously. Therefore, we decompose the joint lot-sizing and pricing problem with stochastic demands and capacity constraints into a multi-phase decision process. In each phase, we solve the associated sub-problem to optimality. The decomposed decision process corresponds to a practically viable approach to decision-making. In addition to incorporating market uncertainty and pricing decisions in the traditional production and inventory planning process, our approach also accommodates the complexity of time-varying cost and capacity constraints. Finally, our numerical results show that the multi-phase heuristic algorithm solves the example problems effectively.

Keywords: capacitated lot sizing; pricing; stochastic model; dynamic programming; decomposition

1. Introduction

In most manufacturing or service organisations, production planning and pricing of finished products are two important decision areas. Conventionally, pricing decisions are made by the marketing and sales department, while the production department must satisfy the demand that results from those pricing decisions. Both departments will be evaluated by their respective performance. However, although marketing departments often have general information about production capacity, they do not have enough detailed knowledge about how production can be scheduled. As a result, such decentralised decision-making and use of local performance measures may lead to sub-optimality of the overall operations performance. During the past two decades, there has been an increasing interest in coordinating production planning decisions with other business decisions, especially marketing or finance decisions. The recent interest in procedures for sales and operations planning (S&OP) and demand management reflect the perceived needs to further coordinate demand and supply in enterprise planning. This paper attempts to contribute to such improved coordination by suggesting an effective and practically appealing planning procedure.

Revenue management research and practices have also shown the benefits of integrating production and pricing decisions. However, most successful research and applications are found in the airline and hotel industries. In the manufacturing/retailing sector, many firms have achieved improvements on production and inventory management by pursuing advanced technologies, but they may still suffer opportunity losses because of unbalanced demand and supply. Therefore, attention is now increasingly focused on the demand side of the supply–demand equation, including reexamining pricing strategies and developing software technologies for better demand management. These developments generate needs for models that coordinate production/inventory control and pricing strategies. Such models are particularly important in industries, where price-dependent demand plays an important role and production/inventory decisions can be complemented with pricing strategies to improve the firm's bottom line (Chen and Simchi-Levi 2004a; Harrison, Lee, and John 2004).

Because production capacity is often inflexible in the short term, firms often adjust their production/inventory levels in order to maximise its profit. Price is used as a market clearing mechanism or as a tool to match demand with a limited but partially controllable supply. In environments where capacity is tightly constrained, marginal production costs depend on both the volume of output and the production planning decisions. In such environments, it is particularly important to coordinate production planning with pricing decisions. In this paper, we investigate a joint dynamic lot-sizing and pricing problem with

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stochastic demand and capacity constraints. Our study is inspired by Deng and Yano (2006) who address the joint lot-sizing and pricing problem with deterministic price-dependent demand. However, in many settings, to be realistic, demands over a planning cycle should also be treated as uncertain. Therefore, we extend the problem in their paper to consider stochastic demands. For many consumer durable products, capacity costs and demand fluctuations over economic planning cycles are significant, and manufacturers/retailers wish to avoid overbuilding inventory as well as losing revenue from carrying insufficient stock. In addition, firms face set-up costs or learning curve effects that create economies of scale. Thus, the effects of stochastic demands, capacity limitations, and economies of scale should ideally be considered simultaneously when planning production/inventory and setting prices.

In our study, we consider a single-item firm with production capacity constraints and stochastic demand, that makes production and pricing decisions in each planning period over a finite horizon. This problem can be labelled as the joint capacitated stochastic lot-sizing and pricing problem. Fixed and variable production costs are incurred in the production process. In each period, excess production (demand) is carried over to the next period and thereby incurs holding (backlogging) costs. In addition, the final product demand in each period is affected by the product price and a random factor. The deterministic lot-sizing and pricing problem is already NP-hard. The stochastic demand factors further cause the difficulty to solve the joint lot-sizing and pricing problem. We employ scenario analysis which is a commonly applied approach to model random demand.

The capacitated stochastic lot-sizing and pricing problem is first formulated using a stochastic mixed integer programming (MIP) model. Given the complexity of this model, a three-phase heuristic procedure is then developed in order to find a good pricing and production policy for the firm. In the first phase, an initial price vector is obtained by solving a deterministic counterpart of the problem. In the second phase, we use the initial price vector and the demand scenario tree to solve a profit maximising stochastic and dynamic lot-sizing problem. Finally, in the third phase, we fix the lot-sizing decisions determined in the second phase and adjust the product prices such that the expected profit is increased, if possible. We also consider reversing the order of the phases two and three as an alternative algorithm. In addition to reducing the computational complexity, the decomposed decision process corresponds to a practically appealing approach to real-world decision-making. For example, a manufacturer may need to plan procurement decisions in advance in order to sign contracts with its suppliers for component supply in future periods. Based on a forecast of the expected demands, the manufacturer could determine a production schedule at the beginning of a planning horizon, and then make pricing decisions dynamically with respect to the current state of the inventory and production system. One of the main insights provided here is that there is a significant benefit from this even if the joint lot-sizing and pricing problem is only solved heuristically.

Our paper explores the potential to improve a manufacturing firm's profitability by coordinating production and pricing decisions in a market environment characterised by demand uncertainty. The main contribution is the design of a procedure for making the joint dynamic lot-sizing and pricing decisions. The rest of the paper is organised as follows. In Section 2, related papers are reviewed. To the best of our knowledge, this is the first study to consider and solve a stochastic joint dynamic lot-sizing and pricing problem with capacity constraints and fixed production/order costs. Please refer the major features of the study to the Table 1. The general stochastic lot-sizing and pricing problem is modelled as a mixed integer stochastic programming problem in Section 3. In Section 4, in order to ease the complexity of the stochastic programming problem, we reformulate the lot-sizing and pricing problem as a multi-stage stochastic MIP model based on demand scenario-tree structures. In Section 5, a multi-phase heuristic algorithm including an initial solution search phase, a dynamic production lot-sizing phase and a dynamic pricing procedure is presented. The numerical study in Section 6 indicates that the multi-phase algorithm is effective in terms of obtaining good quality solutions. Finally, Section 7 contains our conclusions.

2. Literature review

This study is related to two main streams of literature, viz. dynamic lot-sizing and dynamic pricing. Numerous studies on dynamic lot-sizing problems have been published since the 1950's, see for example, Atamturk and Kücükyavuz (2008), van Hoesel and Wagelmans (1996), Bitran and Yanasse (1982), and the references therein. Recently, Buschkühl et al. (2010) present a review of four decades of research on dynamic lot sizing with capacity constraints. They discuss both different modeling approaches and different algorithmic solution approaches. Lot-sizing problems with capacity constraints and stochastic demands are well known to be complex and computationally challenging. The study of stochastic lot-sizing problems is significant, because demands are often difficult to predict in real life and companies have to react dynamically, as new information is successively revealed.

However, the modelling and computational complexities of stochastic problems pose challenges for applications. Therefore, much effort has been exerted on stochastic lot-sizing problems aiming at developing effective algorithms. A relatively early contribution is by Aviv and Federgruen (1997), who solve a capacitated stochastic lot-sizing problem, but with no fixed setup cost included. Guan (2011), who considers fixed setup costs, is closely related to our study. His paper treats a stochastic lot-sizing problem and proposes a dynamic programming framework to solve the problem with time-varying capacity constraints and a cost minimisation objective. The dynamic programming algorithm solves the stochastic capacitated lot-sizing problem to optimality with a computational complexity of $\mathcal{O}(n^4)$, where n is the number of nodes in a scenario tree that represents the stochastic demand. The problem structure is similar to the structure of the model obtained in the second phase of our decision problem, although we focus on profit maximising lot sizing. We adopt the basic idea from Guan (2011) and calculate the value of our objective function by a backward dynamic programming procedure.

Market 'scenarios' are sometimes used in practice by retailers in developing marketing plans for alternative contingencies (Agrawal, Smith, and Tsay 2002). Scenario analysis is comprehensive and relatively easy to implement. There is also an established precedent in the extant literature of using scenarios to model uncertainty in a variety of contexts (Genc, Reynolds, and Sen 2005). In addition, heuristic approaches have been a common and reasonable choice to solve stochastic lot sizing or similar problems. Recent examples include Beraldi et al. (2006) who develop a solution strategy for multi-item stochastic lot-sizing problems based on heuristics that use a time-partitioning policy and an enhanced approach that focuses on a representative scenario. Levi et al. (2008) develop an approximation algorithm that applies a cost-accounting scheme to decide order policies for a multi-period capacitated inventory system with stochastic demands.

Given the progress of research on lot-sizing problems, an interesting challenge is to solve the joint stochastic lot-sizing and pricing problem effectively and efficiently. The issue of coordinating pricing and production/inventory decisions has attracted significant research attention. The need to integrate inventory control and pricing strategies was recognised early by Whitin (1955). Federgruen and Heching (1999) and Petruzzi and Dada (1999) are two landmark studies on the joint pricing and inventory decision considering demand uncertainties. Eliashberg and Steinberg (1991), Yano and Gilbert (2004) and Chan et al. (2004) and provide comprehensive literature surveys on pricing and inventory coordination.

Recently, Chen and Hall (2010) consider the coordination of pricing and scheduling decisions in a make-to-order environment. Salviettia and Smith (2008) discuss the joint economic lot-sizing and pricing problem, and analyse structural properties of optimal solutions. Joint pricing and inventory management problems then remain a fruitful research topic until lately. Rezaei and Davoodi (2012) study a joint pricing, lot-sizing and supplier selection problem. They formulate the problem as a multi-objective non-linear programming model and propose a genetic algorithm to solve the problem. González-Ramírez, Smith, and Askin (2011) address a multi-product capacitated lot-sizing problem with pricing. The study presents a heuristic procedure to solve the problem. However, these papers only consider static or deterministic price-dependent demand.

Tim, Soulaymane, and Ali (2010) and Chan, Simchi-Levi, and Swann (2006) extend the deterministic pricing and lot-sizing problem and consider stochastic demands factors. Tim, Soulaymane, and Ali (2010) address the problem of a magazine publishing firm facing stochastic demand over multiple periods. A dynamic programming formulation is provided and a single-stage reduction that admits a classical news-vendor characterisation is then proposed. Production capacity constraints are not considered in their paper. Chan, Simchi-Levi, and Swann (2006) address a number of decision-making strategies including fixed pricing, delayed production and delayed pricing. They are discussed with respect to both deterministic and stochastic models. Their study also provides heuristic algorithms. It considers capacity constraints and non-stationary cost parameters, but fixed set-up costs are not taken into account.

As noted above, research in the area of yield and revenue management has demonstrated that major benefits can be derived by coordinating production and pricing decisions. Bitran and Caldentey (2003) present a comprehensive review on pricing models for revenue management. They review pricing models for a single product and for multiple products, from the perspective of deterministic as well as stochastic demands and state that the natural way to tackle a stochastic pricing problem is by using stochastic programming techniques. Elmaghraby and Keskinocak (2003) also review studies on coordination of dynamic pricing and inventory decisions. Hamister and Suresh (2008) address a production and pricing model with a price-dependent and auto-correlated demand function, and inventory holding and shortage cost based on Maccini and Zabel (1996). Their study analyses the impact of a random demand factor. Chen and Simchi-Levi (2004a,b) present approaches to coordinate inventory control and pricing strategies over finite and infinite planning horizons. Recently, Feng (2010) examines an integrated decision-making process regarding pricing for uncertain demand and sourcing from uncertain supply. Kazaz and Webster (2011) study the role of a yield-dependent trading cost structure influencing the optimal choice of the selling price and production quantity for a firm that operates under supply uncertainty in the agricultural industry. However, most of these studies are based on some restrictive assumptions, for example, capacity constraints are often not considered.

While many complex joint production planning and pricing problems remain unsolved, some practical applications of existing theories and methods suggest directions for further academic research. Metcalf (1982) assumes that production managers also have pricing authority, and explores the relationship between production/inventory and pricing decision through a case analysis. Mantrala and Rao (2001) develop a stochastic dynamic programming model-based decision-support system, specifically to help retail-store buyers of fashion goods decide on optimal merchandise order quantities and markdown prices. Swann (2004) develops a case study to illustrate the benefits of integrating production and pricing decisions.

Table 1. The main characteristics of the relevant literatures and our study.

Literatures	Capacity constraints	Setup costs	Multiple periods	Deterministic demands	Demand uncertainty	Pricing	Miscellaneous
van Hoesel and Wagelmans (1996)	X	X	X	X			
Atamturk and Kücükyavuz (2008)	X	X	X	X			
Aviv and Federgruen (1997)	X		X		X		
Guan (2011)	X	X	X		X		
Summaa and Wolsey (2008)	X	X	X		X		
Beraldi et al. (2006)		X	X		X		
Levi et al. (2008)	X		X		X		
Chen and Hall (2010)			X	X		X	
Salviettia and Smith (2008)	X	X		X		X	Multi-machines
Rezaei and Davoodi (2012)							Multi-items
González-Ramírez, Smith, and Askin (2011)	X	X	X	X		X	
Federgruen and Heching (1999)							
Petruzzi and Dada (1999)			X		X	X	
Chen and Simchi-Levi (2004a,b)		X	X		X	X	
Tim, Soulaymane, and Ali (2010),			X		X	X	
Chan, Simchi-Levi, and Swann (2006)	X		X		X	X	Discrete prices
Our study	X	X	X		X	X	Continuous prices

Li and Atkins (2002) address the coordination issues of production and marketing departments in a firm. Specifically, they propose a linear transfer pricing mechanism to align department managers' objectives with those of the firm.

Based on this review of related literature, we conclude that there is a need for further research on the capacitated stochastic lot-sizing and pricing problem. We clarify the main characteristics of the lot-sizing literatures in Table 1 below. Each row describes a class of studies with specific features indicated by X. The listed literatures are only representative studies for each class study. While there may be more analogous studies for each class, we do not intend to exhaust them for the sake of brevity. The last row of Table 1 shows the main features of our study.

3. Model formulation

We consider a single-product firm whose production plan and unit price are periodically reviewed in a finite planning horizon with T periods. At the beginning of each period, the firm makes decisions on the production quantity in each period and the unit price based on capacity, costs and available demand information in order to maximise future expected profit. For simplicity, it is assumed that the production quantity becomes available instantly without considering a lead time. A constant lead time can be incorporated in our model and the algorithms will not change significantly. However, a stochastic lead time would make the problem dramatically more complex, and therefore needs to be addressed as a new topic.

The demand for the product in each period depends on the price and a randomness component. Let $p_t \in [p^{\min}, p^{\max}]$ denote the unit selling price of the product and let $\epsilon_t \in [\epsilon^L, \epsilon^U]$ denote the random demand factor in period t, $t = 1, \ldots, T$. Thereby, the demand in period t can be specified either as $d_t(p_t) = D_t(p_t) + \epsilon_t$ in the additive form, or as $d_t(p_t) = D_t(p_t)\epsilon_t$ in the multiplicative form. One interpretation of this model is that the shape of the demand curve is deterministic, while the scaling parameter representing the size of market is random (Petruzzi and Dada 1999). $D_t(p_t)$ is a decreasing and invertible function of price p_t. The random demand component ϵ_t is assumed to be price independent with $E(\epsilon_t) = 0$ for the additive uncertainty and $E(\epsilon_t) = 1$ for the multiplicative form of uncertainty.

We apply the commonly used linear price-dependent demand function $D_t(p_t) = A_t - \beta_t p_t$, where A_t represents an expected market scaling factor of the product, and β_t represents its price sensitivity in period t. In order to assure non-negative demand for the whole range of p_t, we require that $\epsilon_t \geq -D(p^{\min})$ in the additive case. To simplify the description, we apply the additive demand function $d_t(p_t) = A_t - \beta_t p_t + \epsilon_t$ throughout the rest of this paper. However, the multiplicative demand function can easily be adopted in our approach, and the algorithmic steps and complexities will not be significantly affected. In addition, we assume that expected revenue $E[p_t d_t(p_t)]$ in period t is concave, i.e. $\beta_t \geq 0$ and that demands in consecutive periods are stochastically independent. The linear price-dependent demand function reflects the most fundamental characteristics of price-dependent demand functions. Moreover, given the highly complex capacitated stochastic lot-sizing and pricing problem, a linear demand function is a good starting point to explore the benefits of coordinated lot-sizing and pricing decisions.

As in general lot-sizing problems, we assume that *fixed* as well as *variable* production/order costs are incurred when a batch is produced. In addition, excess production will be stocked and incur linear inventory carrying costs. Unsatisfied demand will be backlogged with a linear penalty cost. We assume that all cost parameters may fluctuate in arbitrary ways over the planning horizon. They are defined as follows:

K_t = fixed setup cost for a production lot in period t, $t = 1, \ldots, T$
a_t = unit production cost for the product in period t, $t = 1, \ldots, T$
h_t = inventory carrying cost for each unit of product in period t, $t = 1, \ldots, T$
b_t = backlogging cost for each unit of product in period t, $t = 1, \ldots, T$.

The production quantity in each period is restricted by the available capacity C_t in each period. One reason why it might be relevant to consider a time-varying capacity is that a firm often actually operates with multiple items. The capacity available for an individual item may therefore vary between periods.

Apart from the unit prices, and in alignment with the classical dynamic lot-sizing problem, the decision variables are:

x_t = amount to be produced or procured in period t, $t = 1, \ldots, T$
y_t = binary production setup variable in period t, $t = 1, \ldots, T$
s_t^+ = inventory of the product at the end of period t, $t = 1, \ldots, T$
s_t^- = backlog of the product at the end of period t, $t = 1, \ldots, T$.
$s_t = s_t^+ - s_t^-$, net inventory at the end of period t, $t = 1, \ldots, T$.

Based on the specifications above, the deterministic version of the general capacitated lot-sizing and pricing problem allowing backlogging can be specified as the following non-linear MIP problem:

$$\mathbf{P} : \pi = \max \sum_{t=1}^{T} \left[p_t D_t(p_t) - \left(a_t x_t + K_t y_t + h_t s_t^+ + b_t s_t^- \right) \right] \tag{1}$$

subject to

$$D_t(p_t) = A_t - \beta_t p_t, \quad \forall\, t = 1, \ldots, T \tag{2a}$$

$$s_{t-1}^+ - s_{t-1}^- + x_t = s_t^+ - s_t^- + D_t(p_t), \quad \forall\, t = 1, \ldots, T \tag{2b}$$

$$x_t \leq C_t y_t, \quad \forall t = 1, \ldots, T \tag{2c}$$

$$y_t = \begin{cases} 1, & \text{if} \quad x_t > 0 \\ 0, & \text{otherwise} \end{cases}; \quad \forall\, t = 1, \ldots, T \tag{2d}$$

$$s_0^+ = s_0^- = 0; \tag{2e}$$

$$s_T^+ = s_T^- = 0; \tag{2f}$$

$$p_t, D_t, x_t, s_t^+, s_t^- \geq 0, \quad \forall\, t = 1, \ldots, T. \tag{2g}$$

The objective function (1) maximises the profit over the planning horizon. Equation (2a) is the deterministic price-dependent demand function. Constraints (2b) are the inventory balance conditions. Production is restricted by the capacity constraints (2c). Constraints (2d) define the binary setup variables. Constraints (2e) and (2f) set the initial and end of season inventory and backlog to zero. All variables and demands are restricted to be non-negative by Constraints (2g).

However, if demand is specified by the stochastic price-dependent function $d_t(p_t) = D_t(p_t) + \epsilon_t$, then problem **P** becomes a stochastic non-linear MIP problem. As it is well known that even the deterministic capacitated lot-sizing problem with general cost structure is *NP-Hard* (Bitran and Yanasse 1982), it is immediately concluded that the corresponding stochastic lot-sizing and pricing problem is also a highly complex problem. In order to tackle the intractability of the non-linear stochastic programming problem analogous to problem **P**, we model the random demand by using discrete demand scenarios. Based upon a specific demand scenario structure, we then formulate a multi-stage stochastic MIP model and develop a multi-phase heuristic algorithm for solving the resulting stochastic lot-sizing and pricing problem.

4. Scenario-tree generation

We now assume that the random demand for the lot-sizing and pricing problem evolves as a discrete time stochastic process with finite probability space. The market size together with a random demand factor determines the demand in a particular period. A sequence which consists of possible realisations of random demand in each period over the entire planning horizon is called a scenario. The demand is revealed sequentially over time. All possible scenarios form a tree-like structure and are

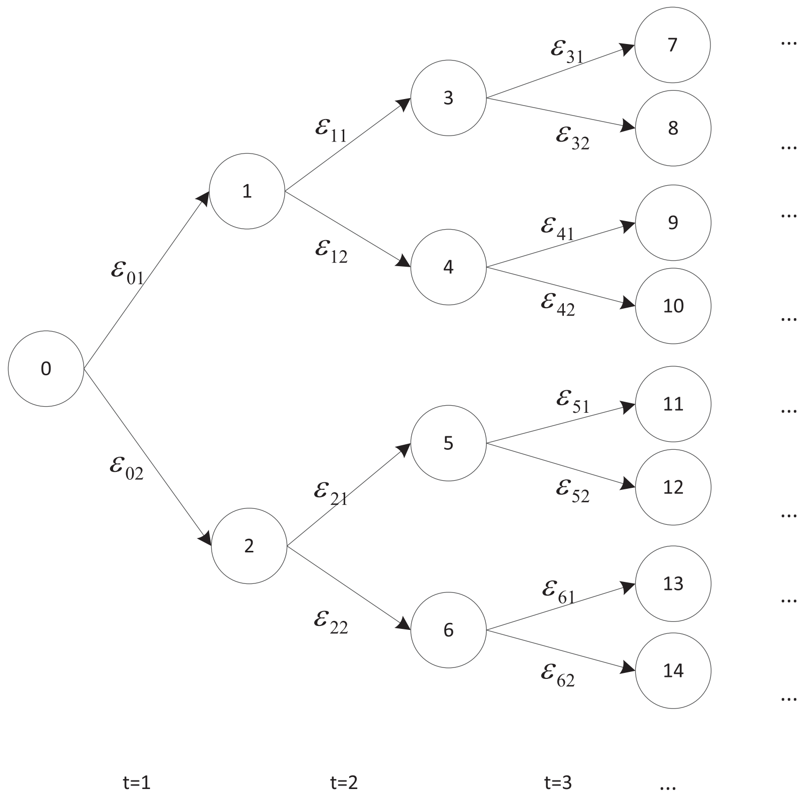

Figure 1. Stochastic demand scenario tree with $n = 2$.

therefore named scenario trees. The practical interpretation is that as the planning period moves forward one period in time, demand realised in the previous period is revealed. The new production and pricing decisions for the imminent planning period will then be made based on the current inventory and available information about future costs, capacities and demand scenarios.

The simplest method to generate demand scenarios is to describe the random demand as a discrete distribution with a finite number of outcomes for each period. As an example, we have chosen to use the properties of the N-point ($N_t = n^t$) discrete uniform distribution in each stage, where n is the number of states considered for the random demand component ϵ_t. As stated in Section 2, the price-demand relationship is governed by $d_t(p_t) = A_t - \beta_t p_t + \epsilon_t$. For $t = 0$, we assume $\epsilon_0 = 0$, and for $t = 1, 2 \ldots, T$, $\epsilon_t = \{\epsilon_{t1}, \epsilon_{t2}, \ldots, \epsilon_{tn}\}$. An example of a scenario tree with $n = 2$ is shown in Figure 1.

Figure 1 shows a scenario tree with a symmetric structure in which the number of branches is the same for all decision nodes in the same period. However, our approach is not restricted to problems with symmetric demand scenario structures. It is also applicable to asymmetric structures or exogenous state points in each period. Other scenario generation approaches exist, but it is not our intention to consider those in this study. For further discussions on scenario generation, we refer to Høyland and Wallace (2001) and Brandimarte (2006).

The scenario tree can be further specified using the following notation which is commonly applied in the literature (Beraldi et al. 2006; Summaa and Wolsey 2008). Denote the scenario tree as the graph $\mathcal{T} = (\mathcal{V}, \epsilon)$ over T periods, where node $i \in \mathcal{V}$ in period t of the tree represents the state of the system that can be distinguished by the information available up to period t. In addition, $\mathcal{V}(0)$ is used to represent the entire tree, and $\mathcal{V}(i)$ represents a subtree with root node i.

The time period for node i, $i \in \mathcal{V}$ is specified as $t(i)$. The set of nodes on the path from the root node to node i is defined as $\mathcal{P}(i) \subset \mathcal{V}$, and the root node is denoted by 0. The set of node i's direct descendants is denoted by $\mathcal{C}(i) \subseteq \mathcal{V}$. Each node i in the scenario tree, except the root node, has a unique precedent i^-, $i^- \in \mathcal{V}$. Let $\mathcal{P}(i, j)$ be the path from node i to node j, and let $\mathcal{L}$ denote the set of leaf nodes. In addition, let the probability associated with the state represented by node i be $q_i = \Sigma_{j \in \mathcal{C}(i)} q_j$, $i \notin \mathcal{L}$. A scenario is $\mathcal{P}(0, j)$, $j \in \mathcal{L}$, and the probability for each scenario equals the probability of each leaf node, i.e. q_j, $j \in \mathcal{L}$.

The production and price decisions for node i have to be made before the demand in node i is realised, but after observing the realisations of demands, production quantities and prices along the path from the root node to node i^-. Hence, in a multi-stage decision situation, the scenario tree captures the sequence in which information is revealed. Let $\Im = \{1, 2, \ldots, N_T\}$

be the scenario set, and let τ be an individual scenario. The node in scenario τ and period t can then be represented as τ_t. If $\tau_t = \tau_t^{'}$, then $\tau_{j-1} = \tau_{j-1}^{'}$, for all $j = 1, \ldots, t$. If the probability of each scenario is q_τ, then $\sum_{\tau \in \Im} q_\tau = 1$.

Based on the scenario-tree approach, our stochastic lot-sizing and pricing problem can be written as the multi-stage stochastic MIP problem $\mathbf{P_s}$.

$$\mathbf{P_s} : \pi = \max \sum_{\tau \in \Im} q_\tau \sum_{t=1}^{T} [p_{\tau_t} d_{\tau_t}(p_{\tau_t}) - (a_t x_{\tau_t} + K_t y_{\tau_t} + h_t s^+_{\tau_t} + b_t s^-_{\tau_t})] \tag{3}$$

subject to

$$d_{\tau_t}(p_{\tau_t}) = A_{\tau_t} - \beta_{\tau_t} p_{\tau_t} + \epsilon_{\tau_t}, \quad \forall\, t = 1, 2, \ldots, T; \quad \tau \in \Im; \tag{4a}$$

$$s^+_{\tau_{(t-1)}} - s^-_{\tau_{(t-1)}} + x_{\tau_t} = d_{\tau_t}(p_{\tau_t}) + s^+_{\tau_t} - s^-_{\tau_t}; \quad \forall\, t = 1, 2, \ldots, T; \quad \tau \in \Im; \tag{4b}$$

$$x_{\tau_t} \le C_t y_{\tau_t}, \quad \forall\, t = 1, 2, \ldots, T; \quad \tau \in \Im; \tag{4c}$$

$$y_{\tau_t} = \begin{cases} 1, & \text{if} \quad x_{\tau_t} > 0 \\ 0, & \text{otherwise} \end{cases}, \quad \forall\, t = 1, 2, \ldots, T; \quad \tau \in \Im; \tag{4d}$$

$$s_0^+ = s_0^- = 0; \quad s_T^+ = s_T^- = 0; \tag{4e}$$

$$p_{\tau_t}, d_{\tau_t}, x_{\tau_t}, s^+_{\tau_t}, s^-_{\tau_t} \ge 0, \quad \forall\, t = 1, 2, \ldots, T; \quad \tau \in \Im. \tag{4f}$$

$$x_{\tau t} = x_{\tau' t}, \quad \text{if} \quad \tau_{t-1} = \tau_{t-1}^{'} \quad \forall\, t = 1, 2, \ldots, T; \quad \tau \in \Im \tag{4g}$$

$$p_{\tau t} = p_{\tau' t}, \quad \text{if} \quad \tau_{t-1} = \tau_{t-1}^{'} \quad \forall\, t = 1, 2, \ldots, T; \quad \tau \in \Im \tag{4h}$$

The objective function (3) seeks to maximise the expected total profit over the entire planning horizon. Constraints (4a) are deterministic because the demand uncertainty is resolved by using multiple scenarios. Apart from the expansion to express a scenario-tree structure, the objective function (3) and constraints (4b)–(4f) in problem $\mathbf{P_s}$ have the same meanings as those in problem **P** in Section 3. For the sake of brevity, we do not repeat them here. The constraints (4g)–(4h) are non-anticipative constraints. They guarantee consistency between decisions in different scenarios that include the same nodes, i.e. they guarantee a unique solution for each node. This assures that the solution is implementable. If $(\tau_0, \ldots, \tau_{t-1}) = (\tau_0^{'}, \ldots, \tau_{t-1}^{'})$, scenarios τ and $\tau^{'}$ share the same nodes until period $t-1$. Thus, they should have the same solution in period t.

In principal, the scenario-based problem $\mathbf{P_s}$ could be solved by using a commercial solver. However, with the increase of the number of states and time periods, the size of the problem increases exponentially. It would take excessive time to solve any reasonable size problem. The computational complexity of multi-stage stochastic MIP problems makes the use of efficient heuristic strategies one of few realistic alternatives to solve real-world applications in an acceptable amount of time.

In this study, we design a sequential heuristic algorithm based on decomposition. The details of the algorithm are addressed in the following sections.

5. Multi-phase algorithm

In this section, we specify a heuristic algorithm with multiple phases to solve problem $\mathbf{P_s}$. The lot-sizing and pricing decisions are made via a sequential procedure including the following three major phases. In each phase, the stated sub-problem is solved to optimality.

(1) Solve problem **P**, which yields a dynamic price vector and a lot-size vector as the starting point for the overall solution;
(2) Solve a scenario-based lot-sizing problem by dynamic programming. The price vector obtained in Phase 1 is used to generate the production plan. The algorithm in this phase has polynomial time complexity;
(3) Solve a delayed dynamic pricing problem based on the production plan obtained in Phase 2. This step helps to balance the production and sales. The algorithm in this phase has Pseudo-polynomial time complexity.

Below, we specify these phases in further details.

5.1 *Phase 1: Solving the deterministic lot-sizing and pricing problem*

The purpose of this phase is to determine an initial price vector $\bar{\mathbf{p}} = \{p_1, p_2, \ldots, p_T\}$ for the entire planning horizon by solving a deterministic pricing and lot-sizing problem **P**. Using $E[d_t(p_t)]$ for the period demands then seems natural as a starting point. In addition, because the revenue term $p_t E[d_t(p_t)]$ is concave, the objective function (1) is piecewise concave

(Zangwill 1966). However, any other relevant point forecast could be used. From a practical point of view, this is a strength of the suggested procedure.

For the deterministic dynamic pricing and lot-sizing problem without backlogging, Deng and Yano (2006) explore properties of optimal solutions and develop algorithms for a few special problem instances with specific cost and capacity parameter characteristics. We extend one of their results to the case with backlogging. Using the inventory decomposition property and the property of optimal solutions presented by Florian and Klein (1971), we can prove the following property of the optimal solution for the joint lot-sizing and pricing problem with backlogging:

Proposition 1 *For any instance of deterministic pricing and lot-sizing problem with backlogging, an optimal solution exists consisting of capacity constrained sequences Q_{uv}, $1 \leq u, v \leq T$, where*

$$Q_{uv} = \{x_t, t = u+1, \ldots, v | s_u = s_v = 0; s_t \neq 0 \quad u < t < v\} \tag{5}$$

Within each sequence Q_{uv}, for one period at most, the production level is positive and less than capacity ($0 < x_t < C_t$), and, for all other periods, production is then either zero or at the capacity level ($x_t = 0$ or $x_t = C_t$).

Proof Consideration of the pricing decision will not change the structure of the feasible region of the lot-sizing problem. For any feasible price vector $\mathbf{p}$, the corresponding demand stream is $\{d_t(p_t), t = 1, \ldots, T\}$, and the remaining problem has the same structure as that of Florian and Klein (1971). Thus, the result follows directly from the proof of Theorems $1'$ and 2 in Florian and Klein (1971). □

Although, according to Proposition 1, there exists an optimal solution structure for problem **P**, an exponential time algorithm is required for the general problem. For cases with constant capacity or special cost structures, polynomial time algorithms exist (Deng and Yano 2006). In such cases, any of those algorithms could be applied in our Phase 1. Heuristics algorithms with limited loss of optimality for problem **P** could also be used (see, e.g. Haugen, Olstad, and Pettersen 2007a,b).

5.2 *Phase 2: Solving the scenario-based lot-sizing problem*

Given the price vector $\bar{\mathbf{p}}$ determined in Phase 1, the profit maximising stochastic lot-sizing and pricing problem $\mathbf{P_s}$ is reduced to a cost minimisation problem. Guan (2011) studies the scenario-based stochastic lot-sizing problem with capacity constraint and backlogging to minimise the expected total cost. Thus, the property of optimal solutions explored by Guan (2011) in his Proposition (1) continues to hold for the profit maximising stochastic capacitated lot-sizing problem in Phase 2.

Lemma 1 *Given a price vector, for any instance of the profit maximising stochastic capacitated lot-sizing problem with backlogging, an optimal solution $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{s}^*)$ exists such that for each node $i \in \mathcal{V}$,*

$$if\ 0 < x_i^* < C_i, then\ x_i^* + s_{i-}^* = d_{ik} - \sum_{j \in S} C_j\ for\ some\ nodes\ k \in \mathcal{V}(i)\ and$$
$$S \subseteq \mathcal{P}(k) \setminus \mathcal{P}(i), and\ x_j^* = 0 \quad \text{or} \quad x_j^* = C_j \quad for\ j \in \mathcal{P}(k) \setminus \mathcal{P}(i).$$

where $d_{ik} = \sum_{j \in \mathcal{P}(k) \setminus \mathcal{P}(i)} d_j$, and $s_i = s_i^+ - s_i^-$ is the net inventory in each node.

In other words, an optimal solution satisfies the condition that if the firm produces at node i, then together with some nodes in the path $\{i, k\}$ producing at full capacity, it produces exactly enough to satisfy the accumulated demand d_{ik}.

There are the three possible situations of no production, production at full capacity, and production at less than full capacity. Considering s as the state transition variable, the profit in each case can be calculated based on the following value functions, respectively.

(i) No production

$$\pi_{NP}(i, s) = p_{t(i)} d_i - \max\left\{h_{t(i)}(s - d_i), -b_{t(i)}(s - d_i)\right\} + \sum_{\ell \in \mathcal{C}(i)} q_\ell \pi(\ell, s - d_i) \tag{6}$$

(ii) Production at full capacity

$$\begin{aligned}\pi_{CP}(i, s) = {} & p_{t(i)} d_i - K_{t(i)} - a_{t(i)} C_{t(i)} - \max\left\{h_{t(i)}(C_{t(i)} + s - d_i), -b_{t(i)}(C_{t(i)} + s - d_i)\right\} \\ & + \sum_{\ell \in \mathcal{C}(i)} q_\ell \pi(\ell, C_{t(i)} + s - d_i)\end{aligned} \tag{7}$$

(iii) Production at less than full capacity

$$\pi_{FP}(i,s) = \max_{\mathcal{S},k\in\mathcal{V}(i):d_{ik}-\sum_{j\in\mathcal{S}}C_{t(j)}-s<d_{ik}-\sum_{j\in\mathcal{S}}C_{t(j)}} \pi_P^{k,\mathcal{S}}(i,s) \tag{8}$$

where,

$$\begin{aligned}\pi_P^{k,\mathcal{S}}(i,s) &= p_{t(i)}d_i - K_{t(i)} - a_{t(i)}\left(d_{ik} - \sum_{j\in\mathcal{S}} C_{t(j)} - s\right)\\ &\quad - \max\left\{h_{t(i)}\left(d_{ik} - \sum_{j\in\mathcal{S}} C_{t(j)} - d_i\right), -b_{t(i)}\left(d_{ik} - \sum_{j\in\mathcal{S}} C_{t(j)} - d_i\right)\right\}\\ &\quad + \sum_{\ell\in\mathcal{C}(i)} q_\ell \pi\left(\ell, \left(d_{ik} - \sum_{j\in\mathcal{S}} C_{t(j)} - d_i\right)\right)\end{aligned} \tag{9}$$

Furthermore, for any $s \leq \max_{j\in\mathcal{V}(i)} d_{ij}$, the backward equations can be obtained from

$$\pi(i,s) = \max\{\pi_{NP}(i,s), \pi_{CP}(i,s), \pi_{FP}(i,s)\} \tag{10}$$

For each node, the three cases of value functions are all piecewise linear and right-continuous. Because the maximum of linear functions is still linear, and the sum of linear functions is still linear, the value function $\pi(i,s)$ is also piecewise linear and right-continuous. Therefore, the value functions can be calculated by means of stored break points, values at break points and their right slopes. Based on these value functions, we can apply the dynamic programming algorithm of Guan (2011) with a polynomial time complexity in Phase 2 of our algorithm. We name it the dynamic lot-sizing procedure and describe it briefly below.

Dynamic lot-sizing procedure:

Step 1 For all leaf nodes $i \in \mathcal{L}$, calculate the value functions $\pi(i,s)$; Store all breakpoints, evaluations at the breakpoints and their right slopes; The value functions are piecewise linear in net inventory s. When $C_{t(i)}(b_{t(i)} - a_{t(i)}) > K_{t(i)}$, the value function for node $i \in \mathcal{L}$ is shown in Equation (11).

$$\pi(i,s) = \begin{cases} p_{t(i)}d_i - K_{t(i)} - a_{t(i)}C_{t(i)} + b_{t(i)}(s + C_{t(i)} - d_i), & s < C_{t(i)} - d_i\\ p_{t(i)}d_i - K_{t(i)} - a_{t(i)}(d_i - s), & C_{t(i)} - d_i \leq s < \underline{d}_i\\ p_{t(i)}d_i - b_{t(i)}(d_i - s), & \underline{d}_i \leq s < d_i\\ p_{t(i)}d_i - h_{t(i)}(s - d_i), & s \geq d_i\end{cases} \tag{11}$$

where $\underline{d}_i = d_i - \frac{K_{t(i)}}{b_{t(i)} - a_{t(i)}}$. When $C_{t(i)}(b_{t(i)} - a_{t(i)}) \leq K_{t(i)}$, backlogging is less costly than production, and the value function is further simplified. We illustrate these properties for a leaf node i in Figure 2.

Step 2 Choose a node $i \in \mathcal{V}$, and calculate the value functions $\pi(\ell, s)$ of all its direct descendent nodes $\ell \in \mathcal{C}(i)$.

Step 3 Calculate the total value function for all nodes in $\mathcal{C}(i)$, i.e. $\sum_{\ell\in\mathcal{C}(i)} q_\ell \pi(\ell, s)$.

Step 4 Calculate the value functions, $\pi_{NP}(i,s)$, $\pi_{CP}(i,s)$ and $\pi_{FP}(i,s)$, respectively, and store $\pi(i,s)$, using Equation (10).

Step 5 If the root node $i \in \mathcal{V}$ has not yet been chosen, go to step 2. Otherwise, go to step 6.

Step 6 Retrieve the solution including the production quantities and the expected profit, and obtain values for consequential solution variables such as inventory quantities and setup policy. Stop.

By the dynamic lot-sizing procedure, the maximum expected profit for the entire planning horizon and the production decision for the root node $i = 0$ are found. If the initial stock is zero, the optimal production quantities x_i^*, $i \in \mathcal{V}$ will be determined by $\pi(0,0)$. The breakpoints are generated according to the piecewise linear curve of the corresponding value function. We refer to Guan (2011) for further details of the dynamic programming procedure.

Thus, from Phase 2, we obtain production quantities when considering demand uncertainty. However, the lot-sizing plan may not be globally optimal, because of the predetermined dynamic prices. As it is a common practice in revenue management, price adjustments could then be used to improve the potential profit with the evolution of demands under a fixed supply. Therefore, with the fixed production schedule determined by the dynamic lot-sizing procedure in Phase 2, we suggest in Phase 3 to adjust prices over both scenarios and time periods during the entire planning horizon.

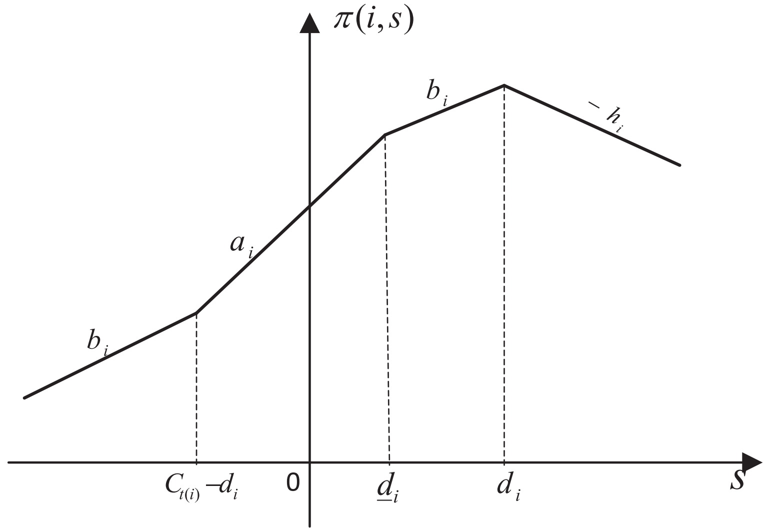

Figure 2. Graphical illustration of value function in a leaf node.

5.3 *Phase 3: Solving the delayed dynamic pricing problem*

The price vector $\bar{\mathbf{p}}$ and production policy $\mathbf{x}^* = \{x_i, i \in \mathcal{V}\}$ are obtained by the algorithms in the first two phases. However, the price vector is determined by only considering average demand or some other point estimate. With the given production policy, the profit might be increased if prices are adjusted based on a given inventory position and realised demand information. The stochastic programming problem $\mathbf{P}_s$ can then be reformulated as a dynamic pricing problem, and dynamic prices on all nodes can thus be determined. The objective is to maximise the profit-to-go considering the given inventory position in each node.

Dynamic pricing procedure:
Denote the profit-to-go function in node i as $G(i, s)$, where s is the net inventory of node i at the beginning of period $t(i)$. In order to keep the decision space finite, we first consider integer inventory levels s. The procedure includes the following steps:

Step 0 Initialisation;

Step 0.1 Specify the profit-to-go function in period $T + 1$ as $G(i, \cdot) = 0$;
Step 0.2 Calculate the minimum and maximum prices for each node as $p_i^{\min} = a_{t(i)}$ and $p_i^{\max} = \min\{A_i/\beta_i, p^U\}$, where p^U is a fixed upper bound on price if it exists;

Step 1 Determine $G(i, s)$ for all leaf nodes;
For node $i \in \mathcal{L}$ (in period $t = T$), find the price p_i that maximises the profit-to-go function

$$G^*(i, s) = \max_{p^{\min} \le p_i \le p^{\max}} G(i, s) \tag{12}$$

where

$$G(i, s) = p_i d_i(p_i) - \max\left\{h_{t(i)}\left(s + x_i - d_i(p_i)\right), -b_{t(i)}\left(s + x_i - d_i(p_i)\right)\right\} \tag{13}$$

Because the revenue term $p_i d_i(p_i)$ is concave, $G(i, s)$ is piecewise concave in price for any given value of s. Thus, it is straightforward to determine the optimal price $p_i^*(s) = \arg\max_{p_i} G(i, s)$. From Equation (13), we find optimal prices for a given s by using the following steps:

Step 1.1 Let $p_1^*(i, s)$ be the solution to the problem

$$G_1^*(i, s) = \max_{p_i^{\min} \le p_i \le p_i^{\max}} \left\{p_i d_i(p_i) + b_{t(i)}(s + x_i - d_i(p_i))\right\} \tag{14}$$

Let $p_i^u(s) = \frac{A_{t(i)} + b_t \beta_{t(i)} + \epsilon_i}{2\beta_{t(i)}}$. If $s + x_i \le d_i(p_i^u(s))$, then $p_1^*(i, s) = \min\{p_i^u(s), p_i^{\max}\}$. Otherwise, let $d_i(p_i) = s + x_i$, and $p_1^*(i, s) = (A_{t(i)} + \epsilon_i - s - x_i)/\beta_{t(i)}$. Given $p_1^*(i, s)$, calculate the profit-to-go function and store its value as $G_1^*(i, s)$.

Step 1.2 Let $p_2^*(i,s)$ be the solution to the problem

$$G_2^*(i,s) = \max_{p_i^{\min} \le p_i \le p_i^{\max}} \left\{ p_i d_i(p_i) - h_{t(i)}(s + x_i - d_i(p_i)) \right\} \tag{15}$$

Let $p_i^l(s) = \frac{A_{t(i)} - h_{t(i)}\beta_{t(i)} + \epsilon_i}{2\beta_{t(i)}}$. If $s + x_i \ge d_i(p_i^l(s))$, then $p_2^*(i,s) = \max\{p_i^l(s), p_i^{\min}\}$. Otherwise, let $d_i(p_i) = s + x_i$, and denote $p_2^*(i,s) = (A_{t(i)} + \epsilon_i - s - x_i)/\beta_{t(i)}$. Given $p_2^*(i,s)$, calculate the profit-to-go function and store its value as $G_2^*(i,s)$.

Step 1.3 Compare $G_1^*(i,s)$ and $G_2^*(i,s)$, and store the corresponding best price as the optimal price when the inventory position is $s + x_i$;

Step 1.4 Repeat steps 1.1–1.3, for all $s \in [-x_i, \sum_{i \in \mathcal{P}(i)} x_i]$. When s falls outside this range, the price remains at its minimum or maximum level, and the value of $G(i,s)$ will decrease at the linear rate of h_T or b_T. Thereby, it can be computed from existing function values or simply be neglected because no improved profit can be obtained.

Step 2 Recursive procedure for the non-leaf nodes;
For all nodes in period $t = T-1, \ldots, 1$, we can calculate the profit-to-go function recursively for each possible net inventory level s from

$$G^*(i,s) = \max_{p_i^{\min} \le p_i \le p_i^{\max}} G(i,s) \tag{16}$$

where

$$\begin{aligned} G(i,s) = {} & p_i d_i(p_i) - \max\left\{h_{t(i)}(s + x_i - d_i(p_i)), -b_{t(i)}(s + x_i - d_i(p_i))\right\} \\ & + \sum_{\ell \in \mathcal{C}(i)} q_\ell G(\ell, s + x_i - d_i(p_i)), \quad i \in \mathcal{V} \end{aligned} \tag{17}$$

and $s \in [-\max_k\{x_{ik}\}, x_{0i}]$, $x_{ik} = \sum_{j \in \mathcal{P}(k) \setminus \mathcal{P}(i)} x_j$, and $x_{0i} = \sum_{j \in \mathcal{P}(i)} x_j$, $k \in \mathcal{L} \cap \mathcal{C}(i)$. Similarly to the case for the leaf nodes, when s falls outside this range, the value of $G(i,s)$ will decrease at a rate which is greater than $h_{t(i)}$ or $b_{t(i)}$, and a price change will not extract positive profit.
To obtain $G^*(i,s)$, for a given net inventory level of s, we calculate $G(i,s)$ for the demand range $[d_i(p_i^{\max}), d_i(p_i^{\min})]$, and determine the optimal demand value d_i for node i. Since the price-dependent demand function is invertible, the corresponding optimal price can then be determined. The optimal price is $p_i^*(s) = \arg\max_{p_i} G(i,s)$. Specifically, the following steps are taken:

Step 2.1 Let $d_i = d_i^0 = d_i(p_i^{\max})$, calculate the profit-to-go function, and denote the resulting value by $G_0(i,s)$;

Step 2.2 Let $d_i = d_i + 1$. If $d_i \le d_i(p_i^{\min})$, calculate the profit-to-go function, denote its value by $G(i,s)$, and go to Step 2.3; otherwise, go to Step 3.

Step 2.3 If $G(i,s) > G_0(i,s)$, let $G_0(i,s) = G(i,s)$, and go to step 2.2; and otherwise, go to step 3.

Step 3 Retrieve solutions, stop.

Starting from the root node, the prices and the inventory states can be determined from the inventory balance relationship $s_i = s_{i^-} + x_i - d_i(p_i)$, and from $p_i^*(s_{i^-}) = \arg\max_{p_i} G(i, s_{i^-})$. The total profit is obtained from $G(0,0)$ if the initial inventory is zero.

In Step 2 of the dynamic pricing procedure above, we use Proposition 2 below.

PROPOSITION 2 *If the inventory s is continuous, the profit-to-go function $G(i,s)$ is concave in s for all nodes $i \in \mathcal{V}$.*

Proof (1) According to step 1 above, for a leaf node i, and for any net inventory level s, price p_i is non-increasing in s, because

$$\frac{\partial p_i}{\partial s} = \begin{cases} 0, & s < d_i(\max\{p_i^u(s), p_i^{\max}\}) \\ 0, & s > d_i(\min\{p_i^l(s), p_i^{\min}\}) \\ \frac{-1}{\beta_{t(i)}}, & \text{otherwise} \end{cases} \tag{18}$$

Therefore, the profit-to-go function $G(i,s)$ consists of three segments, and their respective slopes are

$$\frac{\partial G(i,s)}{\partial s} = \begin{cases} b_t, & s < d_i(\max\{p_i^u(s), p_i^{\max}\}) \\ -h_t, & s > d_i(\min\{p_i^l(s), p_i^{\min}\}) \\ \frac{\partial p_i}{\partial s} d_i(p_i(s)), & \text{otherwise} \end{cases} \tag{19}$$

Hence, if $\frac{\partial p_i}{\partial s} = 0$, then $\frac{\partial^2 G(i,s)}{\partial s^2} = 0$, and if $\frac{\partial p_i}{\partial s} = \frac{1}{-\beta_{t(i)}}$, we have $\frac{\partial^2 G(i,s)}{\partial s^2} = \frac{-1}{\beta_{t(i)}} < 0$.

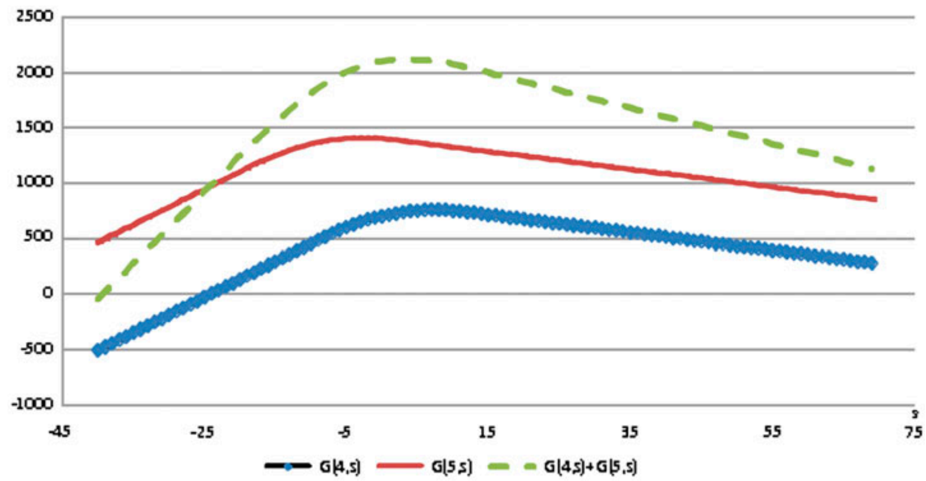

Figure 3. Graphical illustration of the profit-to-go functions in leaf nodes (for numerical data see Section 6).

Furthermore, for any break point of the net inventory level s, if $s = d_i(\max\{p_i^u(s), p_i^{\max}\})$, $p_i^*(i,s) = \max\{p_i^u(s), p_i^{\max}\}$, and then $\frac{\partial G(i,s)}{\partial s} \geq b_t$. Similarly, if $s = d_i(\min\{p_i^l(s), p_i^{\min}\})$, then $\frac{\partial G(i,s)}{\partial s} \geq -h_t$. In addition, a new breakpoint of s is always calculated based on the previous breakpoint. The value of profit-to-go function in a smaller break point is always the initial solution of the next larger break point. Hence, the function $G(i,s)$ is continuous and $\frac{\partial^2 G(i,s)}{\partial s^2} \leq 0$. Thus, $G(i,s)$ in a leaf node is concave in s. A graphical illustration is given in Figure 3.

(2) For non-leaf nodes, the pricing decision in node i is not restricted by the price decisions in its children nodes, and thus price p_i is non-increasing in s. Hence, the concavity of the function $G(i,s)$ is determined by the immediate profit generated in node i:

$$\tilde{G}(i,s) = p_i d_i(p_i) - \max\{h_t(s + x_i - d_i(p_i)), -b_t(s + x_i - d_i(p_i))\} \tag{20}$$

If node i is a direct parent node of some leaf nodes, $\tilde{G}(i,s)$ has the same structure as the profit-to-go function of its children node obtained by shifting the inventory variable to the right by $d_i(p_i)$ units. Hence, $\tilde{G}(i,s)$ is also concave. Because the sum of concave functions is concave, $\sum_{\ell \in \mathcal{C}(i)} G(\ell, s)$ is concave, and the function $G(i,s) = \tilde{G}(i,s) + \sum_{\ell \in \mathcal{C}(i)} q_\ell G(\ell, s)\}$ is concave in s. Finally, by the induction, the profit-to-go function of nodes in other earlier periods are also concave. □

According to Proposition 2, in principle, our algorithm can also deal with the case of continuous inventory levels, and solve the sub-problem to optimality. However, because the optimal prices in non-leaf nodes cannot be described in closed form, the solution can only be obtained by searching on discrete values of s.

5.4 *Overview of algorithm*

First, the proposed algorithm is validated because the multi-phase algorithm specified above is guaranteed to generate a feasible solution. In addition, Phase 1 provides a deterministic approximation solution of the original stochastic problem. Due to the increasing decision flexibility, each phase of the algorithm is guaranteed never to decrease (and generally will increase) system performance. The deterministic solution can be regarded as a lower bounder (the worst case) solution of our algorithm.

Furthermore, the dynamic lot-sizing procedure (Phase 2) and the delayed pricing procedure (Phase 3) are actually independent of each other. Firms can choose the sequence which is the most suitable to their business practices. For example, the companies with high pricing flexibility can choose alternative 1, and the companies with higher production flexibility may choose alternative 2. Therefore, by reversing the order of these two phases, an alternative solution algorithm is generated. It can also be terminated after completion of any of the phases. The alternative algorithm includes the following three phases:

Phase 1 Solve problem **P**. The optimal lot-size vector is the production plan to be used as input for a delayed pricing procedure in Phase 2'.

Phase 2' Based on the production plan obtained from Phase 1, use the dynamic pricing procedure to determine node-dependent prices.

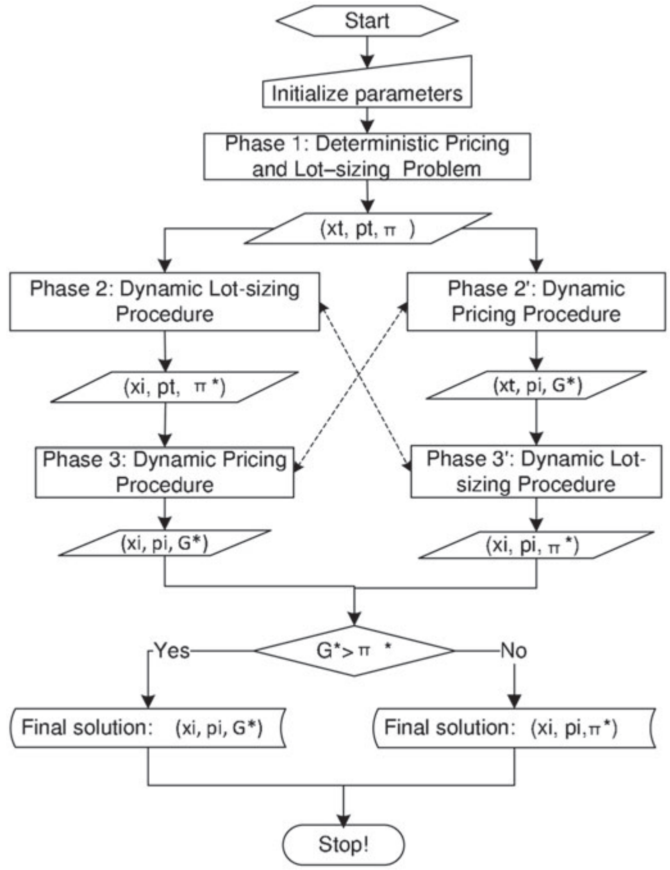

Figure 4. Flowchart illustration of the two alternative multi-phase algorithms.

Phase 3' Given the node-dependent prices and demands from Phase 2', the dynamic lot-sizing procedure is applied to fine-tune the time-varying production plan into node-dependent lot-size decisions.

An overview of the two alternative three-phase algorithms is shown by a flow chart in Figure 4. For use in the figure, we specify the following notation:

$x_t =$ the solution for the production quantity in period $t, t = 1, \ldots, T$ from Phase 1 of the multi-phase algorithms, i.e. from the deterministic pricing and lot-sizing model.
$p_t =$ the solution for the price in period $t, t = 1, \ldots, T$ from Phase 1 of the multi-phase algorithms, i.e. from the deterministic pricing and lot-sizing model.
$\pi =$ the value of the objective function from Phase 1 of the algorithms.
$x_i =$ the solution for the production quantity in node $i, i \in \mathcal{V}$ from the dynamic lot-sizing procedure with profit maximising objective.
$p_i =$ the optimal solution for the price in node $i, i \in \mathcal{V}$ from the dynamic pricing procedure.
$\boldsymbol{\pi}^* =$ the value of the objective function of the dynamic lot-sizing procedure with profit maximising objective.
$G^* =$ the optimal value of the objective function of the dynamic stochastic pricing procedure.

6. Numerical study

In this section, we first use numerical instances to illustrate the performance of the algorithms developed in the previous sections. We construct multiple hypothetical instances with different planning horizons, cost structures and stochastic demand factors.

Table 2. Parameters for the small example problem.

Notation		Parameters				
T	4	t	1	2	3	4
n	2	h_t	7	4	10	4
N	15	b_t	11	15	16	17
		a_t	18	12	9	18
ε_i	$\pm 0.3 * A_t$	C_t	51	60	56	55
β_t	1	K_t	112	130	120	98
		A_t	56	60	57	61

Table 3. Solutions for the example using two alternative multi-phase algorithms.

Phase 1: Lot-sizing and Pricing solution from deterministic model															
t	1	2	3	4											
p_t	39.5	36	33	40											
x_t	0	40.5	45	0											
Expected total profit					6603										
*Algorithm*1															
Phase 2: Dynamic lot-sizing procedure															
i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
d_i	17	15	33	15	33	15	33	12	30	12	30	12	30	12	30
x_i	17	42	60	0	18	0	18	0	18	0	18	0	18	0	18
Expected total profit					7417										
Phase 3: Dynamic pricing procedure															
i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
p_i	39	38	36	33	35	34	37	38	38	36	36	39	39	36	36
d_i	17	13	33	15	31	14	29	14	32	16	34	13	31	16	34
Expected total profit					7570										
*Algorithm*2															
Phase 2': Dynamic pricing procedure															
i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
p_i	40	37	46	33	39	33	36	11	29	23	43	20	38	35	53
d_i	1	16	21	21	25	18	23	26	26	22	26	24	26	19	26
Expected total profit					7389										
Phase 3': Dynamic lot-sizing procedure															
i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
x_i	25	60	60	0	0	0	0	0	0	0	0	0	0	0	0
Expected total profit					8253										
Optimal solution															
node	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
x_i	0	36	45	41	50	41	50	0	0	0	0	0	0	0	0
p_i	40	31	41	28	38	29	38	31	49	31	49	31	49	31	49
Expected optimal profit					8841										

Because the problem structure is complex, and the algorithms includes multiple phases, we first provide a replicable example to illustrate the procedures before turning to the full numerical study. The costs and other parameters of the example problem are shown in Table 2, and the solutions are presented in Table 3.

We test both alternative algorithms described in Figure 4 using exactly the same instances. Note that the solutions obtained with the two algorithms are quite different. In this example, algorithm 1 (the left-hand flow in Figure 4) yields the highest profit often each of the last two phases, but this is not always the case for other instances. However, with both algorithms,

Table 4. Average numerical results and comparisons of test problem instances using multi-phase algorithms.

	Demand states	$n=2$						$n=3$			
	Planning horizon (T)	3	4	5	6	7	8	3	4	5	6
Capacity	Number of Leaves (nodes)	4(7)	8(15)	16(31)	32(63)	64(127)	128(255)	9(13)	27(40)	81(121)	243(364)
Low	Optimal profit (*)	4189	17,685					5763			
	Phase 1 profit (1)	2732	13,352	14,748	16,225	21,227	103,648	2947	12,841	44,611	107,764
	Phase 2 profit (2)	4029	15,544	20,385	26,077	34,249	190,996	4171	19,020	73,922	187,661
	Phase 3 profit (3)	4052	16,205	23,814	30,112	40,349	202,532	4931	24,124	78,868	231,404
	Computational Time 1	13.6	36.4	86.0	231.0	1529.6	3750.0	10.1	68.4	1604.0	2637.0
	Relative improvement from phase 1 = ((2)−(1))/(1)	47.5%	16.4%	38.2%	60.7%	61.3%	84.3%	41.5%	48.1%	65.7%	74.1%
	Relative improvement from phase 2 = ((3)−(2))/(2)	0.6%	4.3%	16.8%	15.5%	17.8%	6.0%	18.2%	26.8%	6.7%	23.3%
	Relative difference to optimal = ((*)−(3))/(3)	3.4%	9.1%					16%			
	Phase 2' profit (4)	3859	15,702	22,376	28,466	40,057	169,545	4256	19,051	69,433	161,281
	Phase 3' profit (5)	3958	16,118	24,522	31,553	47,721	211,457	5149	24,866	82,159	215,580
	Computational Time 2	15	20	82	248	1369	3394	8	74	1601	1974
	Relative improvement from phase 1 = ((4)−(1))/(1)	41.3%	17.6%	51.7%	75.4%	88.7%	63.6%	44.4%	48.4%	55.6%	49.7%
	Relative improvement from phase 2' = ((5)−(4))/(4)	2.5%	2.6%	9.6%	10.8%	19.1%	24.7%	21.0%	30.5%	18.3%	33.7%
	Relative difference to optimal = ((*)–(5))/(5)	5.8%	9.7%					11.9%			
High	Optimal profit (**)	1943	9804					13936			
	Phase 1 profit (6)	1171	6599	12,171	53,239	27,246	50,219	9067	32,885	74,380	64,526
	Phase 2 profit (7)	1400	8751	18,347	64,441	37,246	65,775	11,952	38,332	106,603	104,524
	Phase 3 profit (8)	1539	9321	21,504	67,681	40,250	82,703	12,014	40,150	113,877	143,178
	Computational Time 3	11.0	38.7	172.4	492.5	1082.8	4886.3	20.6	76.3	2717.3	3342.0
	Relative improvement from phase 1 = ((7)−(6))/(6)	19.6%	32.6%	50.7%	21.0%	36.7%	31.0%	31.8%	16.6%	43.3%	62.0%
	Relative improvement from phase 2 = ((8)−(7))/(7)	9.9%	6.5%	17.2%	5.0%	8.1%	25.7%	0.5%	4.7%	6.8%	37.0%
	Relative difference to optimal = ((**)−(8))/(8)	26.3%	5.2%					16.0%			
	Phase 2' profit (9)	1760	8921	15,581	63,514	38,595	87,171	10,209	38,093	96,364	108,904
	Phase 3' profit (10)	1858	9326	21,097	67,384	43,441	88,301	12,528	39,539	114,256	148,414
	Computational Time 4	10.0	30.6	80.4	456.4	1030.8	4406.6	20.1	32.7	2708.6	3318.0
	Relative improvement from phase 1 = ((9)−(6))/(6)	50.2%	35.2%	28.0%	19.3%	41.7%	73.6%	12.6%	15.8%	29.6%	68.8%
	Relative improvement from phase 2' = ((10)−(9))/(9)	5.6%	4.5%	35.4%	6.1%	12.6%	1.3%	22.7%	3.8%	18.6%	36.3%
	Relative difference to optimal = ((**)−(10))/(10)	4.6%	5.1%					11.2%			

profit improvements are achieved in the last two phases. The main improvement is obtained in the first of the two phases. This pattern is typical also in the larger numerical study reported next. It also needs to mention that even for such a small example, the mixed integer quadratic model as Ps is very difficult to be extracted and solved by commercial solver like CPLEX.

We further test the algorithms using multiple randomly generated problem instances. The values of the parameters used in the instances are generated randomly. The cost data are taken from Summaa and Wolsey (2008), where holding costs h_t are random values in the interval [1, 11] and backlogging costs b_t are random values in the interval [3, 33]. Variable production costs a_t are random values in [0, 20] and fixed setup costs K_t are random values in $25 \times [0, 80]$. Two capacity levels C_t are considered which show the effects of low capacity in the range [30, 50] and high capacity in the range [75, 100]. We use time-varying capacity levels in the test, but of course the algorithms work for constant capacity as well. The probabilities for the scenarios are all equal. Data for market size A_t and price sensitivity β_t are drawn from uniform distributions in the range [50, 100] and [0, 2] respectively. The random demand factor ϵ_t is also drawn from a uniform distribution in the range $[-0.3A_t, 0.3A_t]$. Finally, the minimum price equals the variable production cost, $p^{\min} = a_t$, and the maximum price equals the ratio between market size and price sensitivity, $p^{\max} = A_t/\beta_t$.

The test problem instances have varying number of nodes ranging from small instances with 3 periods and 2 states in each node to larger cases with 6 periods and 3 states per node. The model and algorithm is coded in Matlab 7.5 and CPLEX11.0, and was run on a Dell PC with 2.5 GHz Intel processor and 4GB memory. Our interest is mainly in solving the stochastic problem. Therefore, for simplicity, we applied CPLEX 11.0 in the numerical study to solve problem **P** in Phase 1 to optimality without any further restrictions on cost and capacity parameters. The computation results are the average over 5 replications of each instance. For the smallest instance, the optimal profit solution is obtained by solving problem $\mathbf{P_s}$ using the demand scenario tree for each particular replication. The Phase 1 profits are calculated by substituting the production and pricing solutions from Phase 1 into the realised scenario tree for each instance and evaluating. The summary of results obtained are presented in Table 4.

The numerical results indicate that the multi-phase algorithms can solve the stochastic lot-sizing and pricing problem effectively. First, the relative average profit differences to the optimal solutions of the small instances are quite small. Second, in all instances, the overall profit of the stochastic problem is improved consistently from the deterministic approximation in Phase 1, in some cases dramatically. Moreover, the main improvement based on Phase 1 is obtained in the second phase of either algorithm. In other words, relative improvement from phase 1 (phase 1') is greater than relative improvement from phase 2 (phase 2'). On the other hand, improvements are also obtained in the last phase because the values of relative improvement from phase 2 (phase 2') are positive in all instances. In addition, no algorithm dominates the other in terms of final average profit. Hence, the results in Table 4 show that coordination of lot-sizing and pricing decisions outperforms the sole lot-sizing or sole pricing in both algorithms. Therefore, as a hybrid approach, we suggest to use both algorithms as indicated in Figure 4 for a particular problem and then choose the solution that brings the higher total profit. The users can also choose to stop the algorithm at any phase because the phases are independent from each other.

7. Conclusions

We have considered a capacitated dynamic lot-sizing problem combined with pricing decisions, in which demand is stochastic and can be backlogged. The resulting problem was first formulated as a multi-stage stochastic mixed integer programming model. In order to tackle the combined complexity of stochastic programming and the non-linear nature of the pricing problem, we have developed two alternative three-phase algorithms including delayed lot-sizing and delayed pricing decisions. Although the overall procedure is heuristic, the algorithms provide optimal solutions for each individual decision stage. Using the multi-stage algorithm, we have managed to obtain solutions to the complex joint lot-sizing and pricing problem with stochastic demand. Our numerical study indicates that the algorithms are effective for small- and medium-sized problems.

In addition, the structure of the multi-phase decision process provides a practically reasonable approach to integrated decision-making. It is appealing from a practical point of view, because its phases correspond to what appears like a natural sequence of decisions in practical settings including production/inventory and pricing decisions. We would like to investigate the real-world examples to implement the approach and further extend the study. However, the multi-phase character of the algorithm generally causes loss of optimality. In general, the scenario-based multi-phase algorithms are not efficient for very large instances. The properties of the integrated optimal solutions are not generalised easily because the limitation of the considered scenarios. Hence, it is necessary to further look into the problem so that the general properties of joint capacitated stochastic lot-sizing and pricing decisions are explored which may lead to develop more efficient approaches. Furthermore, we only addressed the single item problem, in the future research, we will also investigate the stochastic lot-sizing and pricing problem with multiple items. Since the lead time often plays important role in production and pricing decisions, it is also interesting to consider stochastic lead times in future.

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