AARHUS UNIVERSITY
DEPARTMENT OF MATHEMATICS

ISSN: 1397–4076

RENORMALIZED TWO-BODY LOW-ENERGY SCATTERING

by E. Skibsted

Preprint Series No. 3

March 2012

Publication date: 2012/03/30

Published by

Department of Mathematics
Aarhus University
Ny Munkegade 118, Bldg. 1530
DK-8000 Aarhus C
Denmark

institut@imf.au.dk
http://imf.au.dk

For more preprints, please visit
http://imf.au.dk/en/research/publications/

RENORMALIZED TWO-BODY LOW-ENERGY SCATTERING

E. SKIBSTED

ABSTRACT. For a class of long-range potentials, including ultra-strong perturbations of the attractive Coulomb potential in dimension $d \geq 3$, we introduce a stationary scattering theory for Schrödinger operators which is regular at zero energy. In particular it is well defined at this energy, and we use it to establish a characterization there of the set of generalized eigenfunctions in an appropriately adapted Besov space generalizing parts of [DS1]. Principal tools include global solutions to the eikonal equation and strong radiation condition bounds.

CONTENTS

1. Introduction 2
1.1. A priori quantum bounds 3
1.2. Open problems 5
1.3. Ideas of procedure and results 5
2. Class of potentials 6
2.1. Conditions 7
3. Eikonal equation 9
3.1. Geometric properties 10
3.2. Diagonalization 11
3.3. Outgoing approximate generalized eigenfunctions 12
4. Quantum bounds 17
4.1. Microlocalization for ϵ-small perturbation 17
4.2. Preliminary considerations 18
4.3. Strong radiation condition bounds 20
5. Distorted Fourier transform 26
5.1. Asymptotics of generalized eigenfunctions 30
6. Characterization of generalized eigenfunctions 31
6.1. Concluding remarks 34
References 34

1. Introduction

For a class of long-range potentials we introduce a stationary scattering theory for Schrödinger operators $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$ which is regular at zero energy. In particular it is well defined at this energy, and we use it to establish a characterization there of the set of generalized eigenfunctions in an appropriately adapted Besov space. The analogue of this characterization at positive energies for potentials obeying $\langle x\rangle^{\mu+|\alpha|}|\partial^\alpha V(x)| \leq C_\alpha$ for some $\mu > 0$ is well known [AH, Hö, GY]. It goes as follows:

For all $\lambda > 0$ and all generalized eigenfunctions, $(H-\lambda)u_\lambda = 0$, in the Besov space $B(|x|)^*$ there exist unique $\tau, \tilde{\tau} \in L^2(S^{d-1})$ such that

$$u_\lambda(x) - C|x|^{-(d-1)/2}\left(\mathrm{e}^{\mathrm{i}S(x,\lambda)}\tau(\omega) + \mathrm{e}^{-\mathrm{i}S(x,\lambda)}\tilde{\tau}(\omega)\right) \in B(|x|)^*_0. \tag{1.1}$$

Here $S(\cdot,\lambda) = \sqrt{\lambda}|x| + o(|x|)$ is a solution to the eikonal equation, $\omega = x/|x|$ and $B(|x|)^*_0 \subset B(|x|)^*$ are specified by

$$u \in B(|x|)^* \Leftrightarrow u \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ and } \sup_{R>1} R^{-1}\|F(|x|<R)u\| < \infty,$$
$$u \in B(|x|)^*_0 \Leftrightarrow u \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ and } \lim_{R\to\infty} R^{-1}\|F(|x|<R)u\| = 0.$$

Moreover we can write $\tilde{\tau}(\omega) = (S(\lambda)^{-1}\tau)(-\omega)$ where the operator $S(\lambda)$ is a unitary operator on $L^2(S^{d-1})$ named the scattering matrix at energy λ. The family of these operators is connected to a scattering operator from time-dependent scattering theory by a Legendre transformation.

The (inverse) scattering matrix at energy λ is determined by (1.1): For all $\tau \in L^2(S^{d-1})$ there exist a unique $\tilde{\tau} \in L^2(S^{d-1})$ and a unique generalized eigenfunction $u_\lambda \in B(|x|)^*$ such that the asymptotics (1.1) is fulfilled. Whence indeed the set of generalized eigenfunctions in $B(|x|)^*$ at any positive energy λ is characterized by (1.1). The variable ω may be thought of as the observable asymptotic normalized velocity, see [DS1] for discussion.

We refer to [Me, Va] for a related approach to stationary scattering theory for a class of geometric models.

For a class of potentials, negative at infinity and to leading order spherically symmetric, the above constructions were extended down to (and including) zero energy [DS1]. We refer to [DS2, Fr] for explicit calculations of the scattering matrix at zero energy and to [Ya, SW] for related one-dimensional results on asymptotics of scattering quantities. Whence in particular the set of generalized zero energy eigenfunctions in an appropriately adapted Besov space is characterized in [DS1] for the restrictive class of potentials. Since in turn this class of potentials is close to being optimal for the existence in Classical Mechanics of asymptotic normalized velocity at zero energy the given characterization result may be viewed as "best possible". Nevertheless the purpose of this paper is to provide a similar characterization of generalized zero energy eigenfunctions for a bigger class of potentials than considered in [DS1]. Again we obtain a parametrization by $L^2(S^{d-1})$ howewer the isomorphism is different. Rather than involving functions on a sphere of asymptotic normalized velocities it will be in terms of functions on a sphere of *initial velocities*. In this sense our approach will be in the spirit of [ACH] where a distorted Fourier transform is constructed for order zero potentials at high energies in terms of a family of initially controlled geodesics. We prove low-energy radiation condition bounds of independent interest.

The class of potentials to be studied in this paper is introduced in Section 2. In the remaining part of the present section we review various background results for a somewhat bigger class. The zero energy characterization problem makes sense for this bigger class (at least to some degree), see Subsection 1.2. Whence the class considered in the bulk of the paper may not be optimal for the characterization problem although a further extension would involve difficult problems to overcome, see Subsection 1.3.

1.1. **A priori quantum bounds.** We give an account of some recent results [Sk]. These include Besov space bounds of the resolvent at low energies in any dimension for a class of potentials that are negative and obey a virial condition with these conditions imposed at infinity only. There are two boundary values of the resolvent at zero energy which are characterized by radiation conditions. These radiation conditions are zero energy versions of the well-known Sommerfeld radiation condition.

We consider the Schrödinger operator $H = -\Delta + V$ on $\mathcal{H} = L^2(\mathbb{R}^d)$, $d \geq 1$, where the potential V obeys the following condition. We use the notation $\langle x \rangle = \sqrt{x^2+1}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and for $\mu \in (0,2)$ the notation $s_0 = 1/2 + \mu/4$.

Condition 1.1. Let $V = V_1 + V_2$ be a real-valued function defined on $\mathbb{R}^d$; $d \geq 1$. There exists $\mu \in (0,2)$ such that the following conditions (1)–(5) hold.

(1) There exists $\epsilon_1 > 0$ such that $V_1(x) \leq -\epsilon_1 \langle x \rangle^{-\mu}$.
(2) $V_1 \in C^\infty(\mathbb{R}^d)$. For all $\alpha \in \mathbb{N}_0^d$ there exists $C_\alpha > 0$ such that
$$\langle x \rangle^{\mu+|\alpha|} |\partial^\alpha V_1(x)| \leq C_\alpha.$$
(3) There exists $\tilde{\epsilon}_1 > 0$ such that $-|x|^{-2}\left(x \cdot \nabla(|x|^2 V_1)\right) \geq -\tilde{\epsilon}_1 V_1$.
(4) There exists $\delta, C, R > 0$ such that
$$|V_2(x)| \leq C|x|^{-2s_0-\delta},$$
for $|x| > R$.
(5) $V_2 \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$, where $p = 2$ if $d = 1,2,3$ and $p > d/2$ if $d \geq 4$.

Due to (4) and (5) the operator $V_2(-\Delta + i)^{-1}$ is a compact operator on $L^2(\mathbb{R}^d)$, cf. [RS, Theorem X.20]. Whence H is self-adjoint. The Schrödinger operator with an attractive Coulomb potential in dimension $d \geq 3$ is a particular example.

Let $\theta \in (0,\pi)$, $\lambda_0 > 0$ and define
$$\Gamma_{\theta,\lambda_0} = \{z \in \mathbb{C} \setminus \{0\} \,\big|\, \arg z \in (0,\theta),\ |z| \leq \lambda_0\}. \tag{1.2}$$

For a Hilbert space $\mathcal{H}$ (which in our case will be $L^2(\mathbb{R}^d)$) we denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$ (a similar notation will be used for Banach spaces). A $\mathcal{B}(\mathcal{H})$-valued function $T(\cdot)$ on $\Gamma_{\theta,\lambda_0}$ is said to be uniformly Hölder continuous in $\Gamma_{\theta,\lambda_0}$ if there exist $C, \gamma > 0$ such that
$$\|T(z_1) - T(z_2)\| \leq C|z_1 - z_2|^\gamma \text{ for all } z_1, z_2 \in \Gamma_{\theta,\lambda_0}.$$

We denote the resolvent of H by $R(z) = (H - z)^{-1}$. The notation $B(|x|)$ and $B(|x|)^*$ refers to the Besov space for the operator of multiplication by $|x|$ and its dual space, respectively.

Proposition 1.2 (LAP). *Suppose Condition 1.1. For all $s > s_0$ the family of operators $T(z) = \langle x\rangle^{-s}R(z)\langle x\rangle^{-s}$ is uniformly Hölder continuous in $\Gamma_{\theta,\lambda_0}$. In particular the limits*

$$T(0+\mathrm{i}0) = \langle x\rangle^{-s}R(0+\mathrm{i}0)\langle x\rangle^{-s} = \lim_{\Gamma_{\theta,\lambda_0}\ni z\to 0} T(z),$$
$$T(0-\mathrm{i}0) = \langle x\rangle^{-s}R(0-\mathrm{i}0)\langle x\rangle^{-s} = \lim_{\Gamma_{\theta,\lambda_0}\ni z\to 0} T(\bar z)$$

exist in $\mathcal{B}(L^2(\mathbb{R}^d))$.

There exists $C > 0$ such that for all $z \in \Gamma_{\theta,\lambda_0}$

$$\|(|z| + \langle x\rangle^{-\mu})^{1/4}R(z)(|z| + \langle x\rangle^{-\mu})^{1/4}\|_{\mathcal{B}(B(|x|),B(|x|)^*)} \le C. \tag{1.3}$$

1.1.1. *Zero energy Sommerfeld radiation condition.* We shall give an outline of some microlocal estimates and characterizations of solutions to the equation $Hu = v$. In particular we estimate and characterize the "outgoing" solution whose existence is provided by Proposition 1.2. This particular solution is given as follows in terms of Besov spaces. First note that the relevant Besov space at zero energy is $B^\mu := B(\langle x\rangle^{2s_0}) = \langle x\rangle^{-\mu/4}B(|x|)$, cf. (1.3). We have the following characterization of the corresponding dual space

$$u \in (B^\mu)^* \Leftrightarrow u \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ and } \sup_{R>1} R^{-s_0}\|F(|x| < R)u\| < \infty.$$

A slightly smaller space is given by

$$u \in (B^\mu)^*_0 \Leftrightarrow u \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ and } \lim_{R\to\infty} R^{-s_0}\|F(|x| < R)u\| = 0.$$

Now suppose $v \in B^\mu$. Then due to Proposition 1.2 there exists the weak-star limit

$$u = R(0+\mathrm{i}0)v = \mathrm{w}^\star\text{–}\lim_{\Gamma_{\theta,\lambda_0}\ni z\to 0} R(z)v \in (B^\mu)^*.$$

Note that indeed this u is a (distributional) solution to the equation $Hu = v$.

To state microlocal properties of this solution we first introduce for all $\lambda \ge 0$ the function

$$f = f_\lambda(x) = (\lambda + K\langle x\rangle^{-\mu})^{1/2};\ x \in \mathbb{R}^d, \tag{1.4}$$

where $K := \epsilon_1\tilde\epsilon_1/(2-\mu)$ with the ϵ's given in Condition 1.1. In terms of f_0 we introduce symbols

$$a_0 = \frac{\xi^2}{f_0(x)^2},\quad b_0 = \frac{\xi}{f_0(x)}\cdot\frac{x}{\langle x\rangle}, \tag{1.5}$$

and we have, using here Weyl quantization,

$$\mathrm{Op}^{\mathrm{w}}(\chi_-(a_0)\tilde\chi_-(b_0))u \in (B^\mu)^*_0 \text{ for all } \chi_- \in C^\infty_{\mathrm{c}}(\mathbb{R}) \text{ and } \tilde\chi_- \in C^\infty_{\mathrm{c}}((-\infty,1)). \tag{1.6a}$$

These estimates are accompanied by "high energy estimates", stated as follows: Let us note that

$$f^{-2}_{|z|}(x)\big|V_1(x) - z\big| \le C_0' := \max(C_0/K, 1),$$

where C_0 is given in Condition 1.1 (2) (i.e. the constant with $\alpha = 0$). Consider real-valued $\chi_- \in C^\infty_{\mathrm{c}}(\mathbb{R})$ such that $\chi_-(t) = 1$ in a neighbourhood of $[0, C_0']$, and let $\chi_+ := 1 - \chi_-$. For all such functions χ_+

$$\mathrm{Op}^{\mathrm{w}}(\chi_+(a_0))u \in (B^\mu)^*_0. \tag{1.6b}$$

The support property of $\tilde\chi_-$ in (1.6a) mirrors that the particular solution studied is the outgoing one, and we refer to (1.6a) as the *outgoing Sommerfeld radiation condition.* This condition (in fact a weaker version) suffices for a characterization as expresssed in the following result. Here and henceforth $L^2_m := \langle x\rangle^{-m}L^2(\mathbb{R}^d)$.

Proposition 1.3 (Uniqueness of outgoing solution, data in B^μ)**.** *Suppose $v \in B^\mu$. Suppose u is a distributional solution to the equation $Hu = v$ belonging to the space L^2_m for some $m \in \mathbb{R}$, and suppose that there there exists $\kappa \in (0, 1]$ such that*

$$\mathrm{Op^w}(\chi_-(a_0)\tilde\chi_-(b_0))u \in (B^\mu)^*_0 \textit{ for all } \chi_- \in C^\infty_{\mathrm c}(\mathbb{R}) \textit{ and } \tilde\chi_- \in C^\infty_{\mathrm c}((-\infty, \kappa)). \tag{1.7}$$

Then $u = R(0 + \mathrm{i}0)v$. In particular (1.6a) *and* (1.6b) *hold.*

The "incoming" solution $u = R(0 - \mathrm{i}0)v$ can be characterized similarly. These results generalize [DS1, Proposition 4.10] at zero energy.

Remark 1.4. There are similar results for positive energies. For $R(\lambda+\mathrm{i}0)$ we have the same conclusion $u = R(\lambda + \mathrm{i}0)v$ for an outgoing solution to $(H - \lambda)u = v$. This means more precisely that if we replace the Besov spaces by replacing $s_0 \to s_0 = 1/2$ in the definition of these spaces and change the localization symbols a_0, b_0 in (1.5) and (1.7) by replacing $f_0 \to f_\lambda$ there, then indeed the solution u is given by $u = R(\lambda + \mathrm{i}0)v$. This result is known for larger classes of potentials, see [Hö, Theorem 30.2.10] and [GY].

1.2. **Open problems.** Define under Condition 1.1 the operator

$$\delta(0) = (2\pi\mathrm{i})^{-1}(R(0 + \mathrm{i}0) - R(0 - \mathrm{i}0)) = \pi^{-1}\,\mathrm{Im}(R(0 + \mathrm{i}0)) \in \mathcal{B}(B^\mu, (B^\mu)^*),$$

and note that its range

$$\mathrm{Ran}(\delta(0)) \subseteq \mathcal{E}_0 := \{u \in (B^\mu)^* \,|\, Hu = 0\}.$$

Under some stronger conditions it follows from [DS1, Theorem 8.2] that $\mathrm{Ran}(\delta(0)) = \mathcal{E}_0$ (proved in terms of wave matrices at zero energy). Equality and characterization of $\mathcal{E}_0$ are open problems under Condition 1.1, in fact it is only known that $\delta(0) \neq 0$, see [FS]. More specifically "scattering theory at zero energy" in the spirit of [DS1, Theorem 8.2] is an open problem under Condition 1.1. In this paper we address these problems for an intermediate class of potentials, i.e. a smaller class than the one defined by Condition 1.1 but bigger than the one studied in [DS1].

1.3. **Ideas of procedure and results.** Let us give an outline of a possible procedure for solving the problems posed in the proceeding subsection. This procedure will be implemented for the subclass of potentials to be introduced in Section 2. The corresponding (main) results are stated more precisely in Theorem 6.3. For simplicity we assume in the discussion below that V is negative.

First we need a global solution to the eikonal equation (or at least solving outside a compact set)

$$|\nabla_x S(x, \lambda)|^2 = \lambda - V(x);\ \lambda \geq 0.$$

The existence for $\lambda = 0$ is not known under Condition 1.1. Potentially we could define $S(\cdot, \lambda)$ to be the distance in the metric $g_\lambda = (\lambda - V)\mathrm{d}x^2$ to the origin in $\mathbb{R}^d$, i.e. $S(x, \lambda) = d_{g_\lambda}(x, 0)$. This is the so-called maximal solution to the eikonal equation. However under Condition 1.1 it is a problematic choice, in fact for $d \geq 2$ it might be expected that in some generic sense this $S(\cdot, \lambda) \notin C^1(\mathbb{R}^d \setminus \{0\})$.

However for the subclass of potentials to be considered the above geometric construction is manageable and we shall consider the corresponding geodesic flow

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\Phi = (\lambda - V(\Phi))^{-1}\nabla_x S(\Phi,\lambda),\ \ \Phi(0,\sigma)=0,\ \tfrac{\mathrm{d}}{\mathrm{d}s}\Phi(s,\sigma)_{|s=0} = (\lambda - V(0))^{-1/2}\sigma;$$
$$(s,\sigma)\in[0,\infty)\times S^{d-1}.$$

In particular it turns out that this flow is a diffeomorphism $\Phi:\mathbb{R}_+\times S^{d-1}\to\mathbb{R}^d\setminus\{0\}$.

Next for an appropriate Jacobian type function J, see (3.7a), we propose to introduce

$$F^+(\lambda)v = \mathcal{G}\text{–}\lim_{s\to\infty}\,(J^{1/2}\mathrm{e}^{-\mathrm{i}S(\cdot,\lambda)}R(\lambda+\mathrm{i}0)v)(\Phi(s,\cdot)), \tag{1.8}$$

where $\mathcal{G}:=L^2(S^{d-1},\mathrm{d}\sigma)$. This is for v in an appropriate dense subset of $L^2(\mathbb{R}^d)$, and using an integration by parts and Stone's formula we then derive the following formula for the orthogonal projection onto the continuous subspace of H:

$$\|P_{\mathrm{c}}v\|^2 = \int_0^\infty \|F^+(\lambda)v\|^2_{\mathcal{G}}\,\mathrm{d}\lambda.$$

This leads to the *distorted Fourier transform*

$$F^+ := \int_0^\infty \oplus F^+(\lambda)\,\mathrm{d}\lambda.$$

This map is a partial isometry diagonalizing H_{c}, i.e. $F^+H_{\mathrm{c}} = M_\lambda F^+$. We show the existence of the limit (1.8) by using some new low-energy radiation condition bounds valid under the conditions of Section 2. The reader may consult (5.10) for a somewhat cleaner definition.

Now we can address the problems of Subsection 1.2 (under these conditions). Indeed

$$\mathrm{Ran}(\delta(0)) = \mathcal{E}_0 = \{u\in(B^\mu)^*|\,Hu=0\}$$

follows from the following properties:

$$\begin{aligned}&F^+(0):B^\mu\to\mathcal{G}\text{ is onto,}\\&F^+(0)^*:\mathcal{G}\to\mathcal{E}_0\text{ is a bi-continuous isomorphism,}\\&\delta(0)=F^+(0)^*F^+(0).\end{aligned}\tag{1.9}$$

Furthermore note that (1.9) constitutes a parametrization of $\mathcal{E}_0$. The isomorphism $F^+(0)^*$, named the *wave matrix at zero energy*, is given more explicitly as follows: For any $u=2\pi\mathrm{i}F^+(0)^*\tau\in\mathcal{E}_0$

$$u(x) - J^{-1/2}(x)(\mathrm{e}^{\mathrm{i}S(x,0)}\tau(\sigma) - \mathrm{e}^{-\mathrm{i}S(x,0)}\tilde\tau(\sigma))\in(B^\mu)^*_0;\ x=\Phi(t,\sigma). \tag{1.10}$$

The function $\tilde\tau\in\mathcal{G}$ in (1.10) is uniquely determined from u and it is of the form $\tilde\tau = S(0)^{-1}\tau$ where $S(0)$ is a unitary operator on $\mathcal{G}$. This operator is called the *scattering matrix at zero energy.* Combined with similar constructions for $\lambda>0$ the scattering matrix $S(\lambda)$ is strongly continuous in $\lambda\geq 0$. Whence this *renormalized* stationary scattering theory is *regular at zero energy.*

2. Class of potentials

We introduce the class of potentials to be studied in this paper. The zero energy dynamics for this class of potentials is generically qualitatively very different (unless $d=1$) from the one for potentials in the smaller class of [DS1]. We give an example to that effect.

2.1. **Conditions.**

Condition 2.1 (Unperturbed potential)**.** Let $V = V_1 + V_2$ be a real-valued function defined on $\mathbb{R}^d$; $d \geq 1$. There exists $\mu \in (0, 2)$ such that the following conditions (1)–(4) hold.

(1) There exists $\epsilon_1 > 0$ such that $V_1(x) \leq -\epsilon_1 \langle x \rangle^{-\mu}$.
(2) $V_1 \in C^\infty(\mathbb{R}^d)$. For all $\alpha \in \mathbb{N}_0^d$ there exists $C_\alpha > 0$ such that
$$\langle x \rangle^{\mu+|\alpha|} |\partial^\alpha V_1(x)| \leq C_\alpha.$$
(3) $V_1(x) = V_{\mathrm{rad}}(|x|)$ is spherically symmetric, and there exists $\tilde{\epsilon}_1 > 0$ such that
$$-2V_{\mathrm{rad}}(r) - rV'_{\mathrm{rad}}(r) \geq -\tilde{\epsilon}_1 V_{\mathrm{rad}}(r).$$
(4) V_2 is compactly supported, and $V_2 \in L^p(\mathbb{R}^d)$, where $p = 2$ if $d = 1, 2, 3$ and $p > d/2$ if $d \geq 4$.

Given Condition 2.1 we consider the class $\mathcal{W}$ of real-valued smooth functions W on $\mathbb{R}^d$ obeying that for all $\alpha \in \mathbb{N}_0^d$
$$\sup_{x \in \mathbb{R}^d} \langle x \rangle^{\mu+|\alpha|} |\partial^\alpha W(x)| < \infty. \tag{2.1a}$$
Given $l \in \mathbb{N}$ we say that $W_\epsilon \in \mathcal{W}$ is *ϵ-small* if for some $\epsilon > 0$
$$\max_{|\alpha| \leq l} \sup_{x \in \mathbb{R}^d} \langle x \rangle^{\mu+|\alpha|} |\partial^\alpha W_\epsilon(x)| \leq \epsilon. \tag{2.1b}$$
Clearly this quantity depends on the given l, however we prefer for the above teminology of ϵ-smallness to suppress this dependence. If in a given context l is not specified, it is tacitly understood that $l = 2$ (although for example $l = 1$ suffices for Proposition 4.1). We use $l = 4$ in Lemma 3.2 stated below. Similarly we need $l \geq 4$ in Lemma 3.3 and Proposition 4.2 (a sufficient choice $l = l(\mu, d)$ can be calculated, however we shall not bother). Consequently our main result Theorem 6.3 will depend on some fixed $l \geq 4$ in the definition (2.1b) of ϵ-small perturbations.

We shall study potentials of the form $V_\epsilon = V + W_\epsilon$ where $V = V_{\mathrm{rad}} + V_2$ agrees with Condition 2.1 and $W_\epsilon \in \mathcal{W}$ is an ϵ-small perturbation. The class of such potentials V_ϵ, say $\mathcal{V}_\epsilon$, is a particular subclass of the one defined by Condition 1.1; here we need ϵ small. In fact at various other points of the paper we need to take $\epsilon > 0$ small, however this will be expressible in terms of V_{rad} only, which henceforth is considered as fixed. For convenience we assume throughout the paper that
$$V_{\mathrm{rad}}(r) = V_{\mathrm{rad}}(0) \text{ for } r \leq R := (-V_{\mathrm{rad}}(0))^{-1/2}, \tag{2.2a}$$
and similarly for perturbations that
$$W_\epsilon(x) = 0 \text{ for } |x| \leq R. \tag{2.2b}$$
We can freely assume (2.2a) and (2.2b). As for V_{rad} the property (2.2a) can be assumed possibly upon making ϵ_1 smaller (but not changing $\tilde{\epsilon}_1$) and changing V_2. Although this is an elementary fact it is not completely obvious. Let us give a proof: Decompose $1 = \chi_+ + \chi_-$ where $\chi_+, \chi_- \in C^\infty(\mathbb{R}_+)$ are monotone, $\chi_+(t) = 1$ for $t \geq 2$ and $\chi_+(t) = 0$ for $t \leq 1$. Introduce
$$V_n(r) = V_{\mathrm{rad}}(r)\chi_+(r/n) - n^{-2}\chi_-(r/n);\ n \in \mathbb{N}. \tag{2.3}$$
We claim that for any n big enough such that $\epsilon_1 \langle 2n \rangle^{-\mu} \geq n^{-2}$ indeed Conditions 2.1(1)–(3) and (2.2a) hold with ϵ_1 replaced by n^{-2}, new constants C_α, the

same constant $\tilde\epsilon_1$ and with $R = n$, respectively. To see that indeed the same $\tilde\epsilon_1$ works we consider the estimates

$$\begin{aligned} -rV_n'(r) &\geq (2-\tilde\epsilon_1)V_n(r) + (2-\tilde\epsilon_1)n^{-2}\chi_-(r/n) - \tfrac{r}{n}\chi_-'(r/n)(-V_{\mathrm{rad}}(r) - n^{-2}) \\ &\geq (2-\tilde\epsilon_1)V_n(r) - \tfrac{r}{n}\chi_-'(r/n)(\epsilon_1\langle 2n\rangle^{-\mu} - n^{-2}) \\ &\geq (2-\tilde\epsilon_1)V_n(r); \end{aligned}$$

we assumed that $\tilde\epsilon_1 \leq 2$. The other statements are obvious. Similarly (2.2b) can be assumed by changing V_2 and possibly by taking ϵ smaller.

2.1.1. *Example.* Let $g \in C^\infty(\mathbb{R})$ be 2π-periodic with $\max g' \geq 1 - \mu/2$. Let $\chi \in C^\infty(\mathbb{R}_+)$ obey $\chi(r) = 0$ for $r < 1$ and $\chi(r) = 1$ for $r > 2$. Similarly introduce for $\mu \in (0,2)$ and (large) $n \in \mathbb{N}$ a function $h = h_n \in C^\infty(\mathbb{R}_+)$ obeying

$$\begin{cases} h(r) = r/n & \text{for } r \leq n \\ h(r) = (1-\mu/2)^{-1}r^{1-\mu/2} + C_n & \text{for } r \geq 2n \\ h'(r) > \max(0, -rh''(r)) & \text{for } r > 0 \end{cases}.$$

Note that the construction (2.3) with $V_{\mathrm{rad}}(r) = -r^{-\mu}$ leads to the particular example $h_n(r) = \int_0^r \sqrt{-V_n(t)}\mathrm{d}t$. We construct in dimension $d = 2$ a potential in terms of a parameter $\epsilon \geq 0$ and polar coordinates (r,θ) (i.e. $x = (r\cos\theta, r\sin\theta)$):

$$\begin{aligned} S_\epsilon(x,\lambda=0) &= h_n(r)\exp\{\epsilon g(\theta - \epsilon\ln r)\chi(r/n)\}, \\ V_\epsilon(x) &= -|\nabla S_\epsilon(x,\lambda=0)|^2. \end{aligned}$$

Clearly $V_{\epsilon=0}(x) = V_{\mathrm{rad}}(r)$ obeys Condition 2.1 and (2.2a) (the latter with $R = n$). Moreover clearly $W_\epsilon(x) := V_\epsilon(x) - V_{\mathrm{rad}}(r)$ satisfies (2.1a) and (2.2b). Morever for any $l \in \mathbb{N}$ there exists $C > 0$, sufficiently large and possibly depending on n, such that the potential W_ϵ is $(C\epsilon)$-small. So up to a linear reparametrization also (2.1b) is satisfied.

This example does not fit into the framework of [DS1]. In fact for the class studied in [DS1] classical zero energy scattering orbits have asymptotic normalized velocities. This would for the above example mean that there exist $\lim_{t\to\pm\infty}\theta(t)$. However this cannot be as the following arguments show: Consider the flow (in polar coordinates)

$$\begin{cases} \dot r(= \tfrac{\mathrm{d}}{\mathrm{d}s}r(s)) &= (-V_\epsilon(x))^{-1}\partial_r S_\epsilon(x,\lambda=0) \\ \dot\theta(= \tfrac{\mathrm{d}}{\mathrm{d}s}\theta(s)) &= (-V_\epsilon(x)r^2)^{-1}\partial_\theta S_\epsilon(x,\lambda=0) \\ (r,\theta)(s=1) &= (n,\sigma). \end{cases} \tag{2.4a}$$

Noticing that for $\epsilon > 0$ small $\partial_r S_\epsilon > 0$ we can consider θ as a function of r determined by the single equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = F(r,\theta) := \frac{\epsilon}{r}\frac{g'\chi}{rh'/h + \epsilon\frac{r}{n}g\chi' - \epsilon^2 g'\chi}. \tag{2.4b}$$

Here of course g and χ are functions of $\psi := \theta - \epsilon\ln r$ and r/n, respectively. For $r \geq 2n$ (2.4b) reduces to

$$\frac{\mathrm{d}\psi}{\mathrm{d}r} = \frac{\epsilon}{r}\Big(\frac{g'}{1-\frac{\mu}{2}-\epsilon^2 g'} + O(r^{\mu/2-1}) - 1\Big) = \frac{\epsilon}{r}\Big(\frac{(1+\epsilon^2)g' - 1 + \frac{\mu}{2}}{1-\frac{\mu}{2}-\epsilon^2 g'} + O(r^{\mu/2-1})\Big).$$

Introducing a new time, $\mathrm{d}\tau/\mathrm{d}r = \epsilon r^{-1}$, we obtain

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = \frac{(1+\epsilon^2)g'(\psi) - 1 + \frac{\mu}{2}}{1 - \frac{\mu}{2} - \epsilon^2 g'(\psi)} + O(\mathrm{e}^{(\mu/2-1)\tau/\epsilon}). \tag{2.4c}$$

Note that to leading order (2.4c) is autonomous. Any solution ψ to (2.4c) converges to a root of the corresponding fixed point equation $g'(\psi) = (1-\mu/2)(1+\epsilon^2)^{-1}$, say $\psi \to \psi_0$. In particular going back to the time s of (2.4a) we conclude that $\theta - \epsilon \ln r \to \psi_0$ as $s \to \infty$, and since $\ln r \to \infty$ indeed also $\theta \to \infty$. So the asymptotic normalized velocity does not exist for the flow (2.4a). Noticing that (2.4a) defines a class of zero energy scattering orbits in a reparametrized time we conclude that indeed these orbits do not have asymptotic normalized velocity.

In Subsection 3.1 we study a flow of the type (2.4a) for general ϵ-small perturbations in any dimension (extended as well to any non-negative energy).

3. Eikonal equation

One reason for considering $\mathcal{V}_\epsilon$ with ϵ small only is that Classical Mechanics is particularly nice for this class. Whence (cf. [CS]) there exists a global solution to the eikonal equation

$$|\nabla S_\epsilon|^2 = K_\epsilon; \tag{3.1}$$
$$K_\epsilon(x) = K_\epsilon(x,\lambda) := \lambda - V_{\mathrm{rad}}(|x|) - W_\epsilon(x),\ \lambda \geq 0.$$

We also introduce

$$K_0(r) = \lambda - V_{\mathrm{rad}}(r),$$
$$f(r,\lambda) = \sqrt{K_0(r)}$$
$$S_0(x) = S_0(|x|) = \int_0^{|x|} f(r,\lambda)\mathrm{d}r.$$

As used in [CS] we have uniformly in $r, \lambda \geq 0$

$$crf(r,\lambda) \leq S_0(r) \leq Crf(r,\lambda).$$

Notice also that S_0 is a solution to (3.1) if $W_\epsilon = 0$.

Due to [CS] we have

Proposition 3.1. *Let V_{rad} be given as in Condition 2.1 (assuming also* (2.2a)*) and let $l \geq 2$. There exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0,\epsilon_0]$ and all ϵ-small perturbations W_ϵ (assuming* (2.2b)*) there exists a family of real-valued smooth functions $\{S_\epsilon \in C^\infty(\mathbb{R}^d \setminus \{0\}) \,|\, \lambda \geq 0\}$ with the following properties:*

1. *$|\nabla S_\epsilon(x)|^2 = K_\epsilon(x)$ for $x \in \mathbb{R}^d \setminus \{0\}$.*
2. *$S_\epsilon(x) = S_0(x) = f(0,\lambda)|x|$ for $r = |x| \leq R = (-V_{\mathrm{rad}}(0))^{-1/2}$.*
3. *For all $r_0 > 0$, uniformly in W_ϵ with $\epsilon \in (0,\epsilon_0]$*

$$\max_{|\alpha|\leq l} \sup_{\lambda \geq 0} \sup_{|x|\geq r_0} \langle x\rangle^{|\alpha|} \left|S_0(x)^{-1}\partial_x^\alpha S_\epsilon(x)\right| < \infty.$$

(4) *Uniformly in* W_ϵ, $\lambda \geq 0$ *and* $x \in \mathbb{R}^d \setminus \{0\}$

$$\begin{aligned} S_\epsilon(x) &= S_0(r)(1 + O(\epsilon)),\\ \nabla S_\epsilon(x) &= f(r,\lambda)\;(\langle\hat{x}| + O(\epsilon^{3/4}));\ \hat{x} := x/r,\\ \nabla^2 S_\epsilon(x) &= \tfrac{f(r,\lambda)}{r}\;(P_\perp + \tfrac{rf'(r,\lambda)}{f(r,\lambda)}P + O(\epsilon^{1/2}));\\ &\qquad P = P(\hat{x}) := |\hat{x}\rangle\langle\hat{x}|,\ P_\perp := I - P. \end{aligned}$$

(5) *For all* $\alpha \in \mathbb{N}_0^d$

$$\partial_x^\alpha S_\epsilon \in C(\mathbb{R}^d \setminus \{0\} \times [0,\infty));\ S_\epsilon = S_\epsilon(x,\lambda).$$

We remark that for $l = 2$ the bounds (3) follow from (4). Having $l > 2$ influences only on (3) and requires, according to the proposition, an $\epsilon_0 > 0$ possibly depending on l. It is tempting to conjecture that one could take $l = 2$ in the proposition and replace the constraint of (3), $|\alpha| \leq l$, by $|\alpha| \leq k$ where k is arbitrarily given. The new bounds would be uniform in perturbations from any bounded family (bounded in terms of the seminorms (2.1a)). However this is an open problem, and in fact it is not known whether $\epsilon_0 > 0$ can be chosen independently of l, although there are weaker estimates than (3) indeed independent of l, cf. [CS, Proposition 1.2]. The latter deficiency gives rise to a slight complication when dealing with S_ϵ in the context of pseudodifferential operators, see (4.8c).

3.1. **Geometric properties.** The construction of the function S_ϵ of Proposition 3.1 is given by a geometric procedure: We consider the metric $g_\epsilon = K_\epsilon \mathrm{d}x^2$ on the manifold $M = \mathbb{R}^d$ and the origin $o = 0 \in M$. Then for all $x \in M$ the number $S_\epsilon(x)$ is the distance in this metric to o, i.e. $S_\epsilon(x) = d_{g_\epsilon}(x, o)$. The function S_ϵ is called the *maximal* solution to the eikonal equation.

3.1.1. *Flow.* In the metric g_ϵ the unit-sphere in the tangent space TM_o at the origin $o = 0$ is given by $f(0,\lambda)^{-1}S^{d-1}$ where S^{d-1} is the standard unit-sphere in $\mathbb{R}^d$. We shall use the notation σ for generic points of S^{d-1} and we let $\mathrm{d}\sigma$ denote the standard Euclidean surface measure on S^{d-1}. The exponential mapping at the origin for the metric g_ϵ defines a diffeomorphism $\Phi : \mathbb{R}_+ \times S^{d-1} \to \mathbb{R}^d \setminus \{0\}$

$$\Phi(s,\sigma) = \exp_o(sf(0,\lambda)^{-1}\sigma),$$

and we have the flow property

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\Phi = (K_\epsilon^{-1}\nabla S_\epsilon)(\Phi);\ s > 0, \sigma \in S^{d-1}. \tag{3.2a}$$

Since by assumption, cf. (2.2a) and (2.2b), the conformal factor K_ϵ is constant for $r = |x| \leq R = (-V_{\mathrm{rad}}(0))^{-1/2}$ we have explicitly

$$\Phi(s,\sigma) = sf(0,\lambda)^{-1}\sigma \text{ for } s \leq 1.$$

Whence we can supplement (3.2a) by the "initial condition"

$$\Phi(1,\sigma) = f(0,\lambda)^{-1}\sigma. \tag{3.2b}$$

The assertion above that Φ is a diffeomorphism can be proved taking (3.2a) and (3.2b) as a definition of the map. Notice the consequences of (3.1), (3.2a) and (3.2b) that the distance $d_{g_\epsilon}(x,o) = S_\epsilon(x) = s$. This point of view is taken in the proof of an analogous statement [ACH, Proposition 2.2]. However the mapping property can also be viewed as an independent part of the proof of Proposition 3.1

given in [CS]. The flow Φ constitutes a family of reparametrized Hamiltonian orbits for the Hamiltonian

$$h_\epsilon = \xi^2 + V_{\rm rad}(|x|) + W_\epsilon(x) \tag{3.3}$$

at energy λ. It is continuous in λ, i.e. $\Phi \in C(\mathbb{R}_+ \times S^{d-1} \times [0,\infty))$; $\Phi = \Phi(s,\sigma,\lambda)$.

3.1.2. *Surface measure.* The mapping $\Phi(s,\cdot) : S^{d-1} \to \mathcal{S}_\epsilon(s) := \{x \in \mathbb{R}^d \mid S_\epsilon(x) = s\}$ induces a measure on S^{d-1} by pullback $\mathrm{d}\omega = \Phi(s,\cdot)^*\mathrm{d}A(x)$ where $\mathrm{d}A(x)$ refers to the Euclidean surface measure on $\mathcal{S}_\epsilon(s)$. A computation using (3.2a) and (3.2b) shows that explicitly

$$\begin{aligned} &\mathrm{d}\omega = K_\epsilon^{1/2}(x) m_\epsilon(x)\,\mathrm{d}\sigma;\\ &m_\epsilon(x) = f(0,\lambda)^{2-d} K_\epsilon^{-1}(x) \exp\Big(\int_1^s (K_\epsilon^{-1}\triangle S_\epsilon)(\Phi(t,\sigma))\,\mathrm{d}t\Big),\ x = \Phi(s,\sigma). \end{aligned} \tag{3.4}$$

Indeed, take local coordinates $\theta_1,\dots,\theta_{d-1}$ on S^{d-1}, write (3.2a) as $\dot\eta = F(\eta)$ and let A be the $d\times(d-1)-$matrix with entries $a_{ki} = \partial_{\theta_i}\eta^k$. The pullback $\mathrm{d}\omega$ is computed from the metric $g_{ij} = (A^TA)_{ij}$ noting that the determinant $|g|$ obeys

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\ln|g| = \mathrm{tr}((A^TA)^{-1}\tfrac{\mathrm{d}}{\mathrm{d}s}(A^TA)) = \mathrm{tr}((B^T+B)P) = 2K_\epsilon^{-1}\triangle S_\epsilon - \tfrac{\mathrm{d}}{\mathrm{d}s}\ln K_\epsilon(\Phi),$$

where $B = F'$ (the Jacobian matrix) and $P_{kl} = \delta_{kl} - (\partial_k S_\epsilon)(\partial_l S_\epsilon)K_\epsilon^{-1}$. We integrate and obtain

$$\mathrm{d}\omega = |g|^{1/2}\mathrm{d}\theta = f(0,\lambda)K_\epsilon^{-1/2}(x)\exp\Big(\int_1^s (K_\epsilon^{-1}\triangle S_\epsilon)(\Phi(t,\sigma))\mathrm{d}t\Big) f(0,\lambda)^{1-d}\,\mathrm{d}\sigma,$$

showing (3.4).

3.1.3. *Volume measure.* In combination with (3.4) the co-area formula, cf. [Ev, Theorem C.5], yields for (reasonable) functions ϕ on $\mathbb{R}^d$

$$\begin{aligned} \int \phi(x)\,\mathrm{d}x &= \int_0^\infty \mathrm{d}s \int_{\mathcal{S}_\epsilon(s)} \phi K_\epsilon^{-1/2}\,\mathrm{d}A(x)\\ &= \int_0^\infty \mathrm{d}s \int_{S^{d-1}} (\phi m_\epsilon)(\Phi(s,\cdot))\,\mathrm{d}\sigma. \end{aligned} \tag{3.5}$$

Let $\mathcal{B}_\epsilon(s) := \{x \in \mathbb{R}^d | S_\epsilon(x) \le s\}$ for $s > 0$. Clearly $\partial\mathcal{B}_\epsilon(s) = \mathcal{S}_\epsilon(s)$ and whence the Gauss integration theorem, cf. [Ev, Theorem C.1], yields for $j = 1,\dots,d$

$$\begin{aligned} \int_{\mathcal{B}_\epsilon(s)} (\partial_j\phi)(x)dx &= \int_{\mathcal{S}_\epsilon(s)} \phi(\partial_j S_\epsilon)K_\epsilon^{-1/2}\,\mathrm{d}A(x)\\ &= \int_{S^{d-1}} (\phi(\partial_j S_\epsilon)m_\epsilon)(\Phi(s,\cdot))\,\mathrm{d}\sigma. \end{aligned} \tag{3.6}$$

3.2. **Diagonalization.** Under the conditions of Section 2 we consider the Hamiltonian $H = -\Delta + V_\epsilon$ on $\mathcal{H} = L^2(\mathbb{R}^d)$. Denoting the corresponding continuous part by $H_{\rm c}$ we aim at constructing a diagonalizing transform taking $H_{\rm c} \to M_\lambda$ where M_λ is multiplication by λ in $\widetilde{\mathcal{H}} := L^2(\mathbb{R}_+, \mathrm{d}\lambda; \mathcal{G})$ with $\mathcal{G} := L^2(S^{d-1}, \mathrm{d}\sigma)$. Here we explain our procedure leaving the details of implementation to Section 5. It goes as follows, assuming below $v \in L^2_3$ (recall $L^2_m := \langle x\rangle^{-m}L^2(\mathbb{R}^d)$): By Stone's formula, cf. [RS],

$$\|P_{\rm c}v\|^2 = \pi^{-1}\lim_{\lambda_0\to\infty}\int_0^{\lambda_0} \langle v, (\mathrm{Im}\,R(\lambda+\mathrm{i}0))v\rangle\,\mathrm{d}\lambda = \pi^{-1}\int_0^\infty \langle v, (\mathrm{Im}\,R(\lambda+\mathrm{i}0))v\rangle\,\mathrm{d}\lambda.$$

Whence writing $u = R(\lambda + \mathrm{i}0)v$, $p_j = -\mathrm{i}\partial_j$ and using (3.6)

$$\|P_{\mathrm{c}}v\|^2 = \pi^{-1}\int_0^\infty \operatorname{Im}\langle (H-\lambda)u, u\rangle\,\mathrm{d}\lambda$$
$$= \pi^{-1}\int_0^\infty \lim_{s\to\infty} \operatorname{Re}\sum_{j=1}^d \int_{S^{d-1}} (\overline{(p_j u)}u(\partial_j S_\epsilon)m_\epsilon)(\Phi(s,\cdot))\,\mathrm{d}\sigma\mathrm{d}\lambda.$$

Next we substitute $p_j u = (p_j - \partial_j S_\epsilon)u + (\partial_j S_\epsilon)u$. The contribution from the first term will be shown to vanish in the limit $s \to \infty$. Whence we have

$$\|P_{\mathrm{c}}v\|^2 = \pi^{-1}\int_0^\infty \lim_{s\to\infty}\int_{S^{d-1}} (|u|^2 K_\epsilon m_\epsilon)(\Phi(s,\cdot))\,\mathrm{d}\sigma\mathrm{d}\lambda.$$

We are lead to define

$$F^+(\lambda)v = \mathcal{G}\text{–}\lim_{s\to\infty} \pi^{-1/2}(\mathrm{e}^{-\mathrm{i}S_\epsilon}K_\epsilon^{1/2}m_\epsilon^{1/2}R(\lambda+\mathrm{i}0)v)(\Phi(s,\cdot)), \tag{3.7a}$$

yielding

$$\|P_{\mathrm{c}}v\|^2 = \int_0^\infty \|F^+(\lambda)v\|^2_{\mathcal{G}}\,\mathrm{d}\lambda.$$

Finally the "distorted Fourier transform"

$$F^+ := \int_0^\infty \oplus F^+(\lambda)\,\mathrm{d}\lambda$$

diagonalizes H_{c}, i.e. $F^+H_{\mathrm{c}} = M_\lambda F^+$.

Similarly we can define the "distorted Fourier transform"

$$F^- := \int_0^\infty \oplus F^-(\lambda)\,\mathrm{d}\lambda,$$

where

$$F^-(\lambda)v = \mathcal{G}\text{–}\lim_{s\to\infty} \pi^{-1/2}(\mathrm{e}^{\mathrm{i}S_\epsilon}K_\epsilon^{1/2}m_\epsilon^{1/2}R(\lambda-\mathrm{i}0)v)(\Phi(s,\cdot)). \tag{3.7b}$$

3.3. **Outgoing approximate generalized eigenfunctions.** We conclude this section by stating and proving a technical result motivated by the formulas (3.7a) and (3.7b). This enable us to construct outgoing and sufficiently well approximate generalized eigenfunctions which in turn are used to construct exact generalized eigenfunctions.

Let $\tau \in C^\infty(S^{d-1})$ and $\lambda \geq 0$ be given. Define a function $\tilde{u}$ by

$$\tilde{u} = \tilde{u}(x) = \pi^{1/2}(\chi \mathrm{e}^{\mathrm{i}S_\epsilon}K_\epsilon^{-1/2}m_\epsilon^{-1/2})(x)\tau(\sigma);\ x = \Phi(s,\sigma),\ \chi(x) = \chi(|x|). \tag{3.8}$$

Here $\chi(r) = \chi(r > 2)$ is a cutoff function; see Subsection 4.1 for the precise definition. A short computation (using for example (4.10) stated below) shows that

$$(H-\lambda)\tilde{u} = -\pi^{1/2}\chi\mathrm{e}^{\mathrm{i}S_\epsilon(x)}\triangle_x((K_\epsilon^{-1/2}m_\epsilon^{-1/2})(x)\tau(\sigma)) + \text{ compactly supported term.}$$

It will be important for us that also the first term to the right is small at infinity.

Lemma 3.2. *Let $\varepsilon > 0$ be given and suppose $l = 4$. Then for $\epsilon_0 > 0$ sufficiently small, for all $\tau \in C^\infty(S^{d-1})$ and all $\lambda_0 > 0$ there exists $C > 0$ such that uniformly in W_ϵ with $\epsilon \in (0,\epsilon_0]$ and in $\lambda \in [0,\lambda_0]$:*

$$\forall|\alpha| \leq 2\,\forall|x| \geq 1 : \left|K_\epsilon^{1/2}m_\epsilon^{1/2}\partial_x^\alpha((K_\epsilon^{-1/2}m_\epsilon^{-1/2})(x)\tau(\sigma))\right| \leq C\langle x\rangle^{\varepsilon-|\alpha|}. \tag{3.9a}$$

In particular the function $\tilde{u}$ *of* (3.8) *obeys*

$$K_\epsilon^{1/2} m_\epsilon^{1/2}(H-\lambda)\tilde{u} = O(\langle x\rangle^{\varepsilon-2}). \tag{3.9b}$$

Proof. Let

$$T_1(x) = \int_1^s (K_\epsilon^{-1}\triangle S_\epsilon)(\Phi(t,\sigma))\mathrm{d}t,$$
$$T_2(x) = \tau(\sigma);\ x = \Phi(s,\sigma).$$

We need to show that

$$\forall|\alpha| \le 2\,\forall|x|\ge 1: \left|\partial_x^\alpha T_j(x)\right| \le C\langle x\rangle^{\varepsilon-|\alpha|};\ j=1,2. \tag{3.10}$$

For that we shall use the diffeomorphism $\psi : \mathbb{R}^d \to \mathbb{R}^d$ given by

$$y = \Psi(x) = S_0(x)\hat{x} = \int_0^{|x|} f(r,\lambda)\mathrm{d}r\,|x|^{-1}x,$$

and invoke results of [CS] for the model metric

$$\tilde{g}_\epsilon = (\Psi^*)^{-1}g_\epsilon;\ g_\epsilon = K_\epsilon \mathrm{d}x^2.$$

This idea of changing framework is actually behind Proposition 3.1 too. Here we shall use the bounds

$$\forall|\alpha|\le 2: |\partial_x^\alpha \Psi(x)| \le C\langle x\rangle^{-|\alpha|}\langle\Psi(x)\rangle, \tag{3.11a}$$
$$\forall|\beta|\le 2: |\partial_y^\beta \Psi^{-1}(y)| \le C\langle y\rangle^{-|\beta|}\langle\Psi^{-1}(y)\rangle, \tag{3.11b}$$

which are uniform in $\lambda\in[0,\lambda_0]$.

Step I. We note the representation

$$\Phi(t,\sigma) = \Psi^{-1}(\tilde{\gamma}_{\Psi(x)}(t/S_\epsilon(x)));\ x = \Phi(s,\sigma) = \Phi(S_\epsilon(x),\sigma), \tag{3.12}$$

where, using notation of [CS], $\tilde{\gamma}_y(t) = ty + \kappa_y(t)$ is the unique geodesic in the metric $\tilde{g}_\epsilon$ emanating from $0\in\mathbb{R}^d$ with value y at time one. Whence we can rewrite $T_j(x)$ as

$$T_1(x) = \int_1^{S_\epsilon(x)} \phi(\tilde{\gamma}_{\Psi(x)}(t/S_\epsilon(x)))\mathrm{d}t;\ \phi = (K_\epsilon^{-1}\triangle S_\epsilon)\circ\Psi^{-1},$$
$$T_2(x) = \tau(f(0,\lambda)\Psi^{-1}(\tilde{\gamma}_{\Psi(x)}(1/S_\epsilon(x)))).$$

Due to Proposition 3.1(3) and (3.11b) we have the bounds (since we have assumed that $l=4$)

$$\forall|\beta|\le 2: |\partial_y^\beta\phi| \le C\langle y\rangle^{-1-|\beta|}. \tag{3.13}$$

Step II. We prove Sobolev bounds of model geodesics. As in [CS, Section 6] introduce the Sobolev spaces $\mathcal{H}^p := W_0^{1,p}(0,1)^d$, $1<p<\infty$, consisting of absolutely continuous functions $h:[0,1]\to\mathbb{R}^d$ vanishing at the endpoints and having $\dot{h}\in L^p(0,1)^d = L^p(]0,1[,\mathbb{R}^d)$ (we use the notation L^p for this vector-valued L^p space). The space $\mathcal{H}^p$ is equipped with the norm

$$\|h\|_{\mathcal{H}^p} = \|\dot{h}\|_p = \Big(\int_0^1 |\dot{h}(t)|^p dt\Big)^{1/p}.$$

Due to [CS, Proposition 6.8], with reference to the model geodesic $\tilde\gamma_y(t) = ty + \kappa_y(t)$, we have $\kappa_y \in \mathcal{H}^p$ for any prescribed $p \in [2, \infty)$, and for all sufficiently small $\epsilon > 0$

$$\forall|\beta| \le 2 : \|\partial_y^\beta \kappa\|_{\mathcal{H}^p} \le C_p\langle y\rangle^{1-|\beta|}. \tag{3.14a}$$

We claim that any such fixed p the following generalization holds. For all $k \in \{0, 1, 2\}$:

$$\forall|\beta| \le 2 : \|t^{k-1}\partial_y^\beta \tilde\gamma_y^{(k)}(t)\|_p \le C_p\langle y\rangle^{1-|\beta|}. \tag{3.14b}$$

Here $\tilde\gamma_y^{(k)}$ refers to the k'th time-derivative of $\gamma = \tilde\gamma_y$. Due to (3.14a) and the Hardy inequality [CS, Lemma 6.1] only the case $k = 2$ needs to be proved. But since γ is a geodesic for the metric $\tilde g_\epsilon$ the second derivative $\gamma^{(2)}$ is a sum of expressions $\phi_{jk}(\gamma_y)(\dot\gamma_y)^j(\dot\gamma_y)^k$ where

$$\forall|\beta| \le 2 : |\partial_z^\beta \phi_{jk}| \le C\langle z\rangle^{-1-|\beta|}. \tag{3.15}$$

We use the product and chain rules to calculate derivatives ∂_y^β, $|\beta| \le 2$, of any such expression. Then we can obtain the desired bound for any term in the resulting expansion by combining (3.14b) for $k = 0, 1$ (and some bigger values of p), (3.15), the a priori bounds

$$ct|y| \le |\gamma_y(t)| \le Ct|y|, \tag{3.16}$$

cf. [CS, Lemma 2.1], and the generalized Hölder estimate. We omit the details. The reader may consult [CS, Section 6] for similar arguments.

Step III. We can treat $T_1(x)$ by combining Proposition 3.1(3), (3.13), (3.14b) and the generalized Hölder estimate. The smaller $\varepsilon > 0$ is given the bigger $p \ge 2$ in (3.14b) is needed. The estimations are straightforward. Let us for completeness do it in details for $|\alpha| = 1$:

$$\partial^\alpha T_1 = (\partial^\alpha S_\epsilon)K_\epsilon^{-1}\triangle S_\epsilon + \int_1^{S_\epsilon} \nabla\phi \cdot \big((\partial_y\gamma)_\Psi(t/S_\epsilon)\cdot\partial^\alpha\Psi - \dot\gamma_\Psi(t/S_\epsilon)\tfrac{t}{S_\epsilon^2}(\partial^\alpha S_\epsilon)\big)\,\mathrm{d}t; \tag{3.17}$$

The first term is $O(\langle x\rangle^{-1})$. For the second term we estimate for $\delta = \min(\varepsilon, 1)$

$$|(\nabla\phi)(\gamma_\Psi)| \le C\big|\tfrac{t}{S_\epsilon}\Psi\big|^{\delta-2}$$

and substitute $t \to tS_\epsilon$ leading to the upper bound of the integral

$$Cf\big|\Psi\big|^{\delta-2}\int_0^1 t^{\delta-1}(S_\epsilon t^{-1}|\partial_y\gamma)_\Psi(t)| + |\dot\gamma_\Psi(t)|)\,\mathrm{d}t.$$

We choose $p \ge 2$ so big that $(\delta-1)/(1-1/p) > -1$ yielding in turn, using (3.14b), the upper bounds

$$C_1(\delta)fS_\epsilon^{\delta-1} \le C_2(\delta)S_\epsilon^\delta\langle x\rangle^{-1} = O(\langle x\rangle^{\varepsilon-1}).$$

The case $|\alpha| = 2$ is treated similarly differentiating (3.17) except that now there is one term involving $\dot\gamma_\Psi(1)$. For this term we use the formula

$$\dot\gamma_y(t) = 2\int_{1/2}^1 \Big(\gamma_y^{(1)}(s) + \int_s^t \gamma_y^{(2)}(t')\mathrm{d}t'\Big)\mathrm{d}s \tag{3.18a}$$

with $t = 1$ and invoke again (3.14b).

Step IV. We need to treat $T_2(x)$. In addition to (3.18a) we shall use

$$\gamma_y(t) = \int_0^t \gamma_y^{(1)}(s)\mathrm{d}s. \tag{3.18b}$$

The case $|\alpha| = 0$ is trivial. We treat the case $|\alpha| = 1$ in details leaving the remaining case $|\alpha| = 2$ to the reader (it is very similar apart from an application of (3.15) for one term arising after yet another differentiation):

$$\partial^\alpha T_2 = \nabla(\tau \circ (f(0,\lambda)\Psi^{-1})) \cdot \partial^\alpha \gamma_\Psi(1/S_\epsilon). \tag{3.19}$$

Here the first factor is evaluated in $\gamma_\Psi(1/S_\epsilon) \in S^{d-1}$ and whence, cf. (3.11b), it is bounded (uniformly in λ). For the second factor of (3.19) we compute

$$\partial^\alpha \gamma_\Psi(1/S_\epsilon) = (\partial_y\gamma)_\Psi(1/S_\epsilon) \cdot \partial^\alpha \Psi - \dot\gamma_\Psi(1/S_\epsilon)S_\epsilon^{-2}\partial^\alpha S_\epsilon. \tag{3.20}$$

We look at the first term. Using the Hölder estimate, (3.14b) and (3.18b) we estimate

$$|(\partial_y\gamma)_\Psi(t)| \le C_p t^{1/p'};\, 1/p' + 1/p = 1,$$

which is used with $t = 1/S_\epsilon$. Moreover due to (3.11a) we have $|\partial^\alpha\Psi| \le C\langle x\rangle^{-1}\langle\Psi\rangle$, so altogether

$$|(\partial_y\gamma)_\Psi(1/S_\epsilon) \cdot \partial^\alpha\Psi| \le C_p S_\epsilon^{1-1/p'}\langle x\rangle^{-1};\, |x| \ge 1.$$

If $p \ge 2$ is chosen big enough then $1 - 1/p' = 1/p \le \varepsilon$, so the first term of (3.20) conforms with (3.10) with $j = 2$ and $|\alpha| = 1$.

We look at the second term. Using the Hölder estimate, (3.14b) and (3.18a) we estimate

$$|\dot\gamma_\Psi(t)| \le C_p t^{-1/p}\langle\Psi\rangle,$$

which again is used with $t = 1/S_\epsilon$, yielding

$$|\dot\gamma_\Psi(1/S_\epsilon)| \le C_p S_\epsilon^{1+1/p};\, |x| \ge 1.$$

Moreover $|S_\epsilon^{-2}\partial^\alpha S_\epsilon| \le CS_\epsilon^{-1}\langle x\rangle^{-1}$, so altogether

$$|\dot\gamma_\Psi(1/S_\epsilon)S_\epsilon^{-2}\partial^\alpha S_\epsilon| \le C_p S_\epsilon^{1/p}\langle x\rangle^{-1},$$

which again conforms with (3.10) with $j = 2$ and $|\alpha| = 1$ provided $p \ge 2$ is chosen as above. □

Remark. The similar result [ACH, Proposition 2.5] also contains a loss of decay (in Lemma 3.2 expressed by the power $\langle x\rangle^\varepsilon$). Such loss can in general not be avoided. This can seen using the example in Subsection 2.1.1.

3.3.1. *Generalized eigenfunctions.* We learn from (3.5) and (3.9b) that

$$(H-\lambda)\tilde u \in f^{1/2}L^2_\delta; \quad \delta < \tfrac{3}{2} - \tfrac{\mu}{2} - \varepsilon. \tag{3.21}$$

In particular we can choose $\delta > \frac{1}{2}$ in (3.21) provided $\varepsilon > 0$ is small enough. With such δ we can define the generalized eigenfunctions

$$u^\pm = u^\pm(\cdot,\lambda) = \tilde u - R(\lambda \pm \mathrm{i}0)(H-\lambda)\tilde u. \tag{3.22}$$

Since intuitively u^+ is a purely outgoing exact eigenfunction it should be zero. This is the content of the following result.

Lemma 3.3. *There exist $l \geq 4$ and $\epsilon_0 > 0$ such that for all ϵ-small perturbations W_ϵ with $\epsilon \in (0, \epsilon_0]$ the generalized eigenfunction u^+ of* (3.22) *vanishes for any $\tau \in C^\infty(S^{d-1})$ and any $\lambda \geq 0$.*

Proof. By Proposition 1.2 the second term of (3.22) is in $f^{-1/2}B(|x|)^*$. The first term is also in this space due to an explicit calculation using (3.5) and the Besov space bound (5.13c) (stated below), see (5.17) for a more general statement. So we conclude that $f^{1/2}u^\pm \in B(|x|)^*$.

Let us argue for $\lambda = 0$ only. The case $\lambda > 0$ can be treated similarly using Remark 1.4. To conclude that indeed $u^+ = 0$ for $\lambda = 0$ it suffices due to Proposition 1.3 to show, with reference to the notation (1.5), that for some small positive κ

$$\mathrm{Op}^{\mathrm{w}}(\chi_-(a_0)\tilde{\chi}_-(b_0))u^+ \in (B^\mu)^*_0 \text{ for all } \chi_- \in C_c^\infty(\mathbb{R}) \text{ and } \tilde{\chi}_- \in C_c^\infty((-\infty, \kappa)). \tag{3.23}$$

The contribution from the second term of (3.22), $R(\lambda + \mathrm{i}0)(H - \lambda)\tilde{u}$, is treated by (1.6a) (here we may have $\kappa = 1$).

As for the contribution from the first term, $\tilde{u}$, a computation using (3.9a) shows that $f^{-1}(p - \nabla S_\epsilon)\tilde{u} \in (B^\mu)^*_0$. On the other hand due to Proposition 3.1(4) for small $\kappa, \epsilon > 0$ the symbol $f^{-1}(\xi - \nabla S_\epsilon)$ is elliptic on the support of any symbol $\chi_-(a_0)\tilde{\chi}_-(b_0)$ as in (3.23) which intuitively yields the desired bound. However at this point some care must be taken in that $\partial_j S_\epsilon$ is singular at zero and the good bounds of Proposition 3.1(3) are only valid for $|\alpha| \leq l$ (which consequently must be chosen sufficiently large). A similar deficiency will arise in Section 4, see (4.8c) and the discussion there. Let us give an elaboration: First it is more convenient to use S_0 (modified by a cutoff near infinity) rather than S_ϵ. Then we have good bounds of all derivatives well suited for the calculus of pseudodifferential operators (see Section 4 for some details). Whence by this calculus we can use the ellipticity property (with the above replacement and for small $\kappa > 0$) to write, abbreviating $T = \mathrm{Op}^{\mathrm{w}}(\chi_-(a_0)\tilde{\chi}_-(b_0))$,

$$T = T\sum_{j=1}^{d} T_j(f(r,0)^{-1}p_j - \langle x\rangle^{-1}x_j) + \widetilde{T}\langle x\rangle^{\mu/2-1},$$

where T_j and $\widetilde{T}$ are bounded pseudodifferential operators. We apply this identity to $\tilde{u}$. The last term contributes by a term in $(B^\mu)^*_0$. As for the first term we use (3.9a) (with $\varepsilon < 1 - \mu/2$) and get a similar contribution and in addition the term

$$T\sum_{j=1}^{d} T_j\phi_j\tilde{u};\ \phi_j = f(r,0)^{-1}\chi(r > 1)\partial_j S_\epsilon - \langle x\rangle^{-1}x_j.$$

Next using a statement like (4.8c) (see the discussion there) we can write

$$T\sum_{j=1}^{d} T_j\phi_j = \sum_{j=1}^{d}\phi_j T_j T + \widetilde{T}\langle x\rangle^{\mu/2-1}$$

for some $\widetilde{T} \in \mathcal{B}((B^\mu)^*_0)$. Note at this point that we need Proposition 3.1(3) for an appropriate $l = l(\mu, d)$. By Proposition 3.1(4) we then conclude that

$$T\tilde{u} - O(\epsilon^{3/4})T\tilde{u} \in (B^\mu)^*_0.$$

Consequently (for small $\epsilon > 0$) also $T\tilde{u} \in (B^\mu)^*_0$. □

4. Quantum bounds

In this section we collect various microlocal resolvent bounds that will be useful in Section 5 for proving the existence of the limit (3.7a) for $v \in L^2_3$ as well as for proving some continuity properties. Our main result Proposition 4.2 has some independent interest, in particular it is new even for spherically symmetric potentials.

4.1. Microlocalization for ϵ-small perturbation.

Let $\tilde r$ denote a smooth convex function of $r \geq 0$ equal to $1/2$ for $r \leq 1/4$ and equal to r for $r \geq 1$. We introduce for $\lambda \geq 0$ the symbols

$$a_\lambda = a_\lambda(x,\xi) = \frac{\xi^2}{f_\lambda(r)^2}, \quad b_\lambda = b_\lambda(x,\xi) = \frac{\xi}{f_\lambda(r)} \cdot \frac{x}{\tilde r}, \tag{4.1}$$

given in terms of the function $\tilde r$ of $r = |x|$ and the function $f = f_\lambda = f_\lambda(x) = f_\lambda(r) = f(r,\lambda)$ of Section 3. Note that $b_\lambda^2 \leq a_\lambda$. We shall state microlocal properties in terms of these observables. For zero energy the resolvent bounds of this subsection are stronger than similar estimates obtainable using the observables (1.5), see [Sk, Proposition 3.5 ii)]. They are in the spirit of [DS1, Proposition 4.1] and [Sk, Lemmas 3.2 and 3.3]. We shall use Weyl quantization of symbols in a uniform symbol class $S_{\mathrm{unif}}(m_{|z|}, g_{|z|})$,

$$g = g_\lambda = \langle x\rangle^{-2}\mathrm{d}x^2 + f_\lambda(x)^{-2}\mathrm{d}\xi^2.$$

The word *uniform* refers to the requirement that bounds of derivatives are uniform in z in the closure of $\Gamma_{\theta,\lambda_0} \subset \mathbb{C}$, say $z \in \Gamma^{\mathrm{clos}}_{\theta,\lambda_0}$. Precisely a symbol $c = c_z \in S_{\mathrm{unif}}(m_{|z|}, g_{|z|})$ with z in this set, if and only if c obeys the bounds

$$|\partial_x^\gamma \partial_\xi^\beta c_z(x,\xi)| \leq C_{\gamma,\beta} m_{|z|}(x,\xi)\langle x\rangle^{-|\gamma|} f_{|z|}^{-|\beta|}(x). \tag{4.2}$$

For example $a_{|z|}$ is defined for $z \in \Gamma^{\mathrm{clos}}_{\theta,\lambda_0}$ (obviously) and $a_{|z|} \in S_{\mathrm{unif}}(a_{|z|}+1, g_{|z|})$, and similarly for the symbol h_ϵ defined in (3.3) $h_\epsilon, h_\epsilon - z \in S_{\mathrm{unif}}(f^2_{|z|}(a_{|z|}+1), g_{|z|})$. For the corresponding calculus the quantity $\langle x\rangle^{\mu/2-1}$ plays the role of a "uniform Planck constant". We refer to [Sk] for a more elaborate discussion. The corresponding class of Weyl quantized operators is denoted by $\Psi_{\mathrm{unif}}(m_{|z|}, g_{|z|})$.

Consider real-valued $\chi_- \in C^\infty_{\mathrm{c}}(\mathbb{R})$ such that $\chi_-(t) = 1$ in a neighbourhood of $[0,1]$ and such that $\chi_-'(t) \leq 0$ for $t > 0$. Let correspondingly $\chi_+ = 1 - \chi_-$. Consider $\tilde\chi_- \in C^\infty(\mathbb{R})$ with $\tilde\chi_-' \in C^\infty_{\mathrm{c}}((-1,1))$, $\tilde\chi_-(-1) = 1$ and $\tilde\chi_-(1) = 0$. Let $\tilde\chi_+ = 1 - \tilde\chi_-$. Clearly the bound (4.3) below (involving the function χ_+) is an energy bound. The bound (4.4) (involving the functions χ_- and $\tilde\chi_-$) is a microlocal bound whose classical analogue is partly explained after Proposition 4.1.

Proposition 4.1. *Let functions χ_-, χ_+ and $\tilde\chi_-$ be given as above. There exists $\epsilon_0 > 0$ such the following three properties hold: For all $\theta \in (0,\pi)$, $\lambda_0 > 0$, $\delta > 1/2$, $t \geq 0$ and ϵ-small perturbations W_ϵ with $\epsilon \in (0,\epsilon_0]$ there exists $C > 0$ such that*

(i) with $T_+(z) := \langle x\rangle^{t-\delta} f_{|z|}^{1/2}\,\mathrm{Op}^{\mathrm{w}}(a_{|z|}\chi_+(a_{|z|}))R(z) f_{|z|}^{1/2}\langle x\rangle^{-t-\delta}$ for $z \in \Gamma_{\theta,\lambda_0}$

$$\|T_+(z)\| \leq C, \tag{4.3}$$

(ii) with $T_-(z) := \langle x\rangle^{t-\delta} f_{|z|}^{1/2}\,\mathrm{Op}^{\mathrm{w}}(\chi_-(a_{|z|})\tilde\chi_-(b_{|z|}))R(z) f_{|z|}^{1/2}\langle x\rangle^{-t-\delta}$ for $z \in \Gamma_{\theta,\lambda_0}$

$$\|T_-(z)\| \leq C, \tag{4.4}$$

(iii) uniformly in $\lambda \in [0, \lambda_0]$ *there exist*

$$T_\pm(\lambda + \mathrm{i}0) := \lim_{\Gamma_{\theta,\lambda_0} \ni z \to \lambda} T_\pm(z) \text{ in } \mathcal{B}(L^2). \tag{4.5}$$

There are analogous properties for $\bar{z} \in \Gamma_{\theta,\lambda_0}$. By the calculus (or the same proof) we can replace the symbol $a_{|z|}\chi_+(a_{|z|})$ in (i) by $\chi_+(a_{|z|}$. In particular the combination of (i) and (ii) yields an effective microlocalization $R(z)v \approx \mathrm{Op^w}(\chi_-(a_{|z|})\tilde{\chi}_+(b_{|z|}))R(z)v$. Let us for later applications choose the localization more concretely: Let $\chi(\cdot < 1)$ be a decreasing smooth function on $\mathbb{R}$ with $\chi(t < 1) = 1$ for $t \leq 1/2$ and $\chi(t < 1) = 0$ for $t \geq 1$. Introduce for $\kappa > 0$ (small) the functions $\chi(t < \kappa) := \chi(t/\kappa < 1)$ and $\chi(t > \kappa) := 1 - \chi(t < \kappa)$. Choose $\chi_- = \chi_\kappa^- = \chi(\cdot - 1 < \kappa)$ and $\tilde{\chi}_+ = \tilde{\chi}_\kappa^+ = \chi(1 - \cdot < \kappa)$. This leads to the introduction of the symbols

$$\chi_\kappa = \chi_{\kappa,|z|} = \chi_\kappa^-(a_{|z|})\tilde{\chi}_\kappa^+(b_{|z|}) \in S_{\mathrm{unif}}(1, g_{|z|});\ \kappa > 0. \tag{4.6}$$

The proof of Proposition 4.1 (not to be given in details here) is similar to the ones of [Sk, Lemmas 3.2 and 3.3] using instead of [Sk, (3.12)] the following computation, cf. [DS1, (4.30)]: Let $h_{\mathrm{rad}} = \xi^2 + V_{\mathrm{rad}}(r)$. The Poisson bracket with $b = b_\lambda$ (i.e. the derivative of b along the flow generated by h_{rad}) is given by

$$\{h_{\mathrm{rad}}, b\} = \tfrac{2f}{\tilde{r}}\big(1 - \tfrac{rV'_{\mathrm{rad}}}{2f^2}\big)(1 - b^2) + \tfrac{2}{f\tilde{r}}(h_{\mathrm{rad}} - \lambda). \tag{4.7a}$$

Due to Condition 2.1(3) the factor $1 - \frac{rV'_{\mathrm{rad}}}{2f^2} \geq \tilde{\epsilon}_1/2$. For the Hamiltonian h_ϵ of (3.3) we have uniformly $x, \xi \in \mathbb{R}^d$ and $\lambda \geq 0$

$$\{h_\epsilon, b\} = \tfrac{2f}{\tilde{r}}\big(1 - \tfrac{rV'_{\mathrm{rad}}}{2f^2}\big)(1 - b^2) + \tfrac{2}{f\tilde{r}}(h_\epsilon - \lambda) + O(\epsilon)\tfrac{f}{\tilde{r}}. \tag{4.7b}$$

Whence uniformly in a set of the form $\{b^2 \leq 1 - \delta\}$, $\delta > 0$, and $\lambda \geq 0$

$$\begin{aligned}\{h_\epsilon, b\} &\geq \tilde{\epsilon}_1 \tfrac{f}{\tilde{r}}(1 - b^2) + \tfrac{2}{f\tilde{r}}(h_\epsilon - \lambda) - \epsilon C \tfrac{f}{\tilde{r}} \\ &\geq \tfrac{f}{\tilde{r}}(\tilde{\epsilon}_1\delta - C\epsilon) + \tfrac{2}{f\tilde{r}}(h_\epsilon - \lambda).\end{aligned} \tag{4.7c}$$

We learn from (4.7c) that provided ϵ is taken small the observable b grows along the flow generated by h_ϵ on any set $\{b^2 \leq 1 - \delta,\ h_\epsilon = \lambda\}$. This is part of the classical analogue of (4.4). For κ-depending symbols used as input in Proposition 4.1 the bounds (4.7c) indicate an optimal choice, $\epsilon_0 \approx \kappa$. In fact, and more precisely, the proof of Proposition 4.1 shows that we can choose $\epsilon_0 = \kappa/C$ for some $C > 0$ in the regime $\kappa > 0$ small, whence allowing us to write $R(z)v \approx \mathrm{Op^w}(\chi_{\kappa,|z|})R(z)v$ for all (κ/C)-small perturbations. This will be one reason for considering perturbations of a spherically symmetric potential only. Another reason originates in the construction of S_ϵ, i.e. Proposition 3.1.

4.2. **Preliminary considerations.**

We assume in this subsection that $V_2 = 0$. Whence we consider here the quantization of (3.3), $H = H_\epsilon$.

4.2.1. *Calculus considerations.* By the calculus the family of symbols (4.6) has the properties that for all $n \in \mathbb{N}$ and all $c = c_z \in S_{\mathrm{unif}}(a_{|z|} + 1, g_{|z|})$

$$(I - \mathrm{Op^w}(\chi_{2\kappa,|z|}))\,\mathrm{Op^w}(\chi_{\kappa,|z|}) \in \Psi_{\mathrm{unif}}(\langle x\rangle^{-n}\langle\xi\rangle^{-2}, g_{|z|}), \tag{4.8a}$$

$$[\mathrm{Op^w}(c_z), \mathrm{Op^w}(\chi_{\kappa,|z|})]\,\mathrm{Op^w}(\chi_{\kappa/2,|z|}) \in \Psi_{\mathrm{unif}}(\langle x\rangle^{-n}\langle\xi\rangle^{-2}, g_{|z|}). \tag{4.8b}$$

Note that in particular (4.8b) applies to $c_z = h - z$ and any function $c_z = c_z(x, \xi) = \phi_z(x) \in S_{\mathrm{unif}}(1, g_{|z|})$. We shall need the following modification of the latter statement. Consider $\phi_z \in C^N(\mathbb{R}^d)$, with $z \in \Gamma^{\mathrm{clos}}_{\theta,\lambda_0}$, obeying uniform bounds

$$|\partial_x^\alpha \phi_z(x)| \le C_\alpha f_{|z|}(x)\langle x\rangle^{-|\alpha|};\ |\alpha| \le N.$$

Now for any given $n \in \mathbb{N}$ we can find $N = N(n, \mu, d)$ such that for all $\phi_z \in C^N(\mathbb{R}^d)$ obeying these bounds we have (uniformly in $z \in \Gamma^{\mathrm{clos}}_{\theta,\lambda_0}$)

$$[\phi_z(x), \mathrm{Op^w}(\chi_{2\kappa,|z|})]\,\mathrm{Op^w}(\chi_{\kappa,|z|}) = \langle x\rangle^{-n} B \langle x\rangle^{-n};\ \|B\| \le C. \tag{4.8c}$$

This statement can be proved by the symbolic calculus and an explicit estimation of an associated oscillatory integral. Note that the constant C in (4.8c) can be chosen proportional to a natural norm of ϕ_z, and whence the bound is an example of a familiar continuity property of the calculus of pseudodifferential operators. We shall apply it to $\phi_z = \chi(r > 2)\partial_j S_\epsilon(x, |z|) = \chi(r > 2)\partial_j S_\epsilon(x, |z|)$, $j = 1, \ldots, d$. Note that we here need $l = N + 1$ in Proposition 3.1(3). If n in (4.8c) is taken large possibly (in fact likely so) $\epsilon_0 > 0$ in Proposition 3.1 must then be small (since in practise $N = l - 1$ large is needed for (4.8c) for given large n).

The last preliminary property we will discuss is an application of the Fefferman-Phong inequality [Hö, Theorem 18.6.8] (uniform version), concretely bounds for the symbol $b_{|z|}$ and $\chi_{2\kappa,|z|}$: For all $\kappa > 0$ there exists $C = C_\kappa > 0$ so that for all $z \in \Gamma^{\mathrm{clos}}_{\theta,\lambda_0}$

$$\mathrm{Op^w}(\chi_{2\kappa,|z|})\,\mathrm{Op^w}(b_{|z|} - 1 + 2\kappa)\,\mathrm{Op^w}(\chi_{2\kappa,|z|}) \ge -C(\langle x\rangle f_{|z|})^{-2}, \tag{4.9a}$$

$$\mathrm{Op^w}(\chi_{2\kappa,|z|})\,\mathrm{Op^w}(b_{|z|} - 1 - 2\kappa)\,\mathrm{Op^w}(\chi_{2\kappa,|z|}) \le C(\langle x\rangle f_{|z|})^{-2}, \tag{4.9b}$$

$$\mathrm{Op^w}(\chi_{2\kappa,|z|})\,\mathrm{Op^w}((b_{|z|} - 1)^2 - (2\kappa)^2)\,\mathrm{Op^w}(\chi_{2\kappa,|z|}) \le C(\langle x\rangle f_{|z|})^{-2}. \tag{4.9c}$$

4.2.2. *Radiation operators.* We shall combine Propositions 3.1 and 4.1 to obtain radiation condition bounds similar to some of [HS1, HS2] for positive energies (see also [Sa1, Sa2]). Our method is different in that it is purely stationary whereas [HS1, HS2] rely on propagation estimates. Whence we introduce for $\lambda \ge 0$ and any given ϵ-small perturbation W_ϵ *radiation operators* defined in terms of the function $S_\epsilon = S_\epsilon(x, \lambda)$ from Proposition 3.1 as

$$\gamma = p - \nabla S_\epsilon,\ \gamma_j = p_j - \partial_j S_\epsilon,\ j = 1, \ldots, d,\ \text{and}\ \gamma_\| = \mathrm{Re}(\nabla S_\epsilon \cdot \gamma).$$

Using (3.1) we obtain, cf. [HS1, HS2],

$$2\gamma_\| = (H - \lambda) - \gamma^2. \tag{4.10}$$

Next we compute the Heisenberg derivative, say denoted by $\mathrm{D} = \mathrm{i}[H, \cdot]$, of γ. The involved operators are local and we shall only need the computation for $r \ge 1$.

$$\begin{aligned}
\mathrm{D}\gamma &= -2\nabla^2 S_\epsilon \gamma + \mathrm{i}2\nabla\triangle S_\epsilon\\
&= -\tfrac{2f}{r}\big(\gamma - (1 + \tfrac{rV'_{\mathrm{rad}}}{2f^2}) f^{-1}|\hat{x}\rangle \nabla S_\epsilon \cdot \gamma + O(\epsilon^{1/2})\gamma\big) + \mathrm{i}2\nabla\triangle S_\epsilon\\
&= -\tfrac{2f}{r}\big(\gamma + F f^{-1}(2\gamma_\| + \mathrm{i}\triangle S_\epsilon) + O(\epsilon^{1/2})\gamma\big) + \mathrm{i}2\nabla\triangle S_\epsilon\\
&= -\tfrac{2f}{r}\big(\gamma + F f^{-1}((H - \lambda) - \gamma^2) + O(\epsilon^{1/2})\gamma\big) - 2\mathrm{i}(\tfrac{F}{r}\triangle S_\epsilon - \nabla\triangle S_\epsilon);\\
&\qquad F := \tfrac{\tilde{f}}{\tilde{r}} x,\ \tilde{f} = -\tfrac{1}{2}(1 + \tfrac{rV'_{\mathrm{rad}}}{2f^2}).
\end{aligned} \tag{4.11}$$

Here we used Proposition 3.1(4) and (4.10). The meaning of $O(\epsilon^{1/2})$ is the same as in the proposition, i.e. it is a uniform bound. We can simplify the right hand side using Proposition 3.1(3) (assuming $l \geq 3$) and conclude that

$$\mathrm{D}\gamma = -\tfrac{2f}{r}\Big((I + O(\epsilon^{1/2}))\gamma + Ff^{-1}(H-\lambda) - Ff^{-1}\gamma^2 - \mathrm{i}r^{-1}O(\epsilon^0)\Big), \tag{4.12}$$

where as above the estimates are uniform in W_ϵ, $\lambda \geq 0$ and x with $r = |x| \geq 1$.

Next we compute

$$\mathrm{Re}(f^{-1}F \cdot \gamma) = \mathrm{Re}(\tilde{f}(\mathrm{Op}^{\mathrm{w}}(b) - 1 + O(\epsilon^{3/4}))). \tag{4.13}$$

Effectively the right hand side will be "small"; we will use it to treat the third term in (4.12). Here it is also useful to note that

$$[\gamma_i, \gamma_j] = 0 \leq i, j \leq d. \tag{4.14}$$

4.3. **Strong radiation condition bounds.** We introduce for $k \in \mathbb{N}$

$$X = \langle x \rangle = (1 + r^2)^{1/2} \text{ and } X_k = X(1 + r^2/k)^{-1/2},$$

and "propagation observables"

$$P_1 = \sum_i Q_i^* Q_i; \qquad Q_i = X_k^{1-\epsilon'}\chi(r)\gamma_i \,\mathrm{Op}^{\mathrm{w}}(\chi_{\kappa,|z|}), \tag{4.15a}$$

$$P_2 = \sum_{i,j} Q_{ij}^* Q_{ij}; \qquad Q_{ij} = X_k^{2(1-\epsilon')}\chi(r)\gamma_i\gamma_j \,\mathrm{Op}^{\mathrm{w}}(\chi_{\kappa,|z|}), \tag{4.15b}$$

where $\gamma = \gamma(\lambda = |z|)$, $\chi(r) = \chi(r > 2)$ and $\epsilon' \in (0, 1]$ needs to be specified. Note that the powers of X_k are bounded factors and that pointwise $X_k \uparrow X$ for $k \to \infty$.

We compute the Poisson brackets

$$\{h, \chi(r)\} = 2\chi'(r)\hat{x} \cdot \xi = 2f\chi'(r)\, b, \tag{4.16a}$$

$$\{h, X\} = 2X^{-1}x \cdot \xi = \tfrac{2f}{r}\phi\, bX; \qquad \phi = \tfrac{r\tilde{r}}{X^2}, \tag{4.16b}$$

$$\{h, X_k\} = \tfrac{2f}{r}(\phi - \phi_k)\, bX_k; \qquad \phi_k = \tfrac{r\tilde{r}}{k+r^2}. \tag{4.16c}$$

Note for (4.16c) that

$$0 \leq \phi - \phi_k = \tfrac{(k-1)r\tilde{r}}{(k+r^2)X^2} \leq 1. \tag{4.17}$$

Yet another property we will use (tacitly) are the uniform bounds

$$|\partial_x^\alpha f_\lambda^s| \leq C_{\alpha,s} f_\lambda^s \langle x \rangle^{-|\alpha|};\ \lambda \geq 0,\ x \in \mathbb{R}^d,$$
$$|\partial_x^\alpha X_k^s| \leq C_{\alpha,s} X_k^s \langle x \rangle^{-|\alpha|};\ k \in \mathbb{N},\ x \in \mathbb{R}^d.$$

The main result of this section is

Proposition 4.2. *There exist $l = l(\mu, d) \in \mathbb{N}$ ($l \geq 4$ is used explicitly) and $\epsilon_0, C_0 > 0$ with $\sqrt{C}_0\epsilon_0 \leq 1$, such that for all ϵ-small perturbations W_ϵ with $\epsilon \in (0, \epsilon_0]$ and with $\epsilon' = C_0\sqrt{\epsilon}$ the following bounds* (4.18a) *and* (4.18b) *hold uniformly in λ in intervals of the form $I = [0, \lambda_0]$. We consider in these bounds components γ_i, γ_j, $1 \leq i, j \leq d$, of $\gamma = \gamma_\epsilon(\lambda) = p - \nabla S_\epsilon(x, \lambda)$.*

$$\|X^{1-\epsilon'}(\tfrac{f_\lambda}{r})^{1/2}\chi(r)\gamma_i R(\lambda + \mathrm{i}0) f_\lambda^{1/2} X^{-3/2}\| \leq C, \tag{4.18a}$$

$$\|X^{2(1-\epsilon')}(\tfrac{f_\lambda}{r})^{1/2}\chi(r)\gamma_i\gamma_j R(\lambda + \mathrm{i}0) f_\lambda^{1/2} X^{-5/2}\| \leq C. \tag{4.18b}$$

Proof. Due to resolvent equations we can assume that $V_2 = 0$. Throughout the proof the notations H and $R(z)$ refer to this case. Fix $\theta \in (\pi/2, \pi)$ and $\lambda_0 > 0$. We shall prove microlocal bounds of states $u = R(z)v$ in terms of quantities related to P_1 of (4.15a) and P_2 of (4.15b), respectively, where $z \in \Gamma_{\theta,\lambda_0}$. In particular we consider below $\gamma_i = p_i - \partial_i S_\epsilon(x, |z|)$ with $z \in \Gamma_{\theta,\lambda_0}$. We could choose to take $\kappa > 0$ in the definition of the factors $\mathrm{Op^w}(\chi_{\kappa,|z|})$ to be proportional to ϵ with a sufficiently large constant of proportionality, cf. a discussion at the end of Subsection 4.1. However the larger choice $\kappa = \epsilon^{3/4}$ suffices and will be used below. In any case for the corresponding lozalization operators $B = \mathrm{Op^w}(\chi_{\kappa,|z|})$ and $B = \mathrm{Op^w}(\chi_{\kappa/2,|z|})$ we can use the bounds of Proposition 4.1 for ϵ-small perturbations (more precisely we have such bounds upon replacing the pseudodifferential operators there by $I - B$). This is done in (4.20b) and (4.23b) below (for (4.18a)). We choose $\epsilon_0 > 0$ in agreement with any such application as well as being in agreement with Proposition 3.1 with an l (the one to be used in the proposition) choosen sufficiently large. How large l must be depends for (4.18a) partly on applications below of Proposition 3.1 and the symbolic calculus property (4.8c), used in (4.26) and (4.29). See (4.35) for the case of (4.18b) (used in (4.34) and (4.37)). Of course it is legitimate to take ϵ_0 smaller if needed. The choice $\epsilon' = C_0\sqrt{\epsilon}$ for some (large) constant C_0 (rather than ϵ' being proportional to ϵ) is needed (and best possible) in our treatment of the contribution from the term $O(\epsilon^{1/2})\gamma$ in (4.12) in the computation and estimation of a commutator, see (4.24) below. We fix an applicable C_0 for (4.18a) at the end of Step I (this constant will also work for (4.18b), see the end of Step II).

Step I. We show (4.18a) by first establishing the bound

$$\begin{aligned} &\langle P_1'\rangle_u \leq C_1\|f_{|z|}^{-1/2}X^{3/2}v\|^2 + C_2\tfrac{|(z-|z|)|^2}{\operatorname{Im} z}\|X^{-1}X_k^{1-\epsilon'}u\|^2; \\ &P_1' = \sum_i Q_i^*\tfrac{f_{|z|}}{r}Q_i. \end{aligned} \tag{4.19}$$

Here we have suppressed the dependence of $|z|$ in Q_i (as above). The constants are independent of $z \in \Gamma_{\theta,\lambda_0}$ and $k \in \mathbb{N}$ (however dependent on ϵ and possibly also W_ϵ). Whence we conclude by first letting $\operatorname{Im} z \to 0$ (for fixed $\lambda = \operatorname{Re} z \geq 0$) and then letting $k \to \infty$, that at all energies $\lambda \in [0, \lambda_0]$

$$\sum_i \langle Q_i^*\tfrac{f_\lambda}{r}Q_i\rangle_u \leq C_1\|f_\lambda^{-1/2}X^{3/2}v\|^2,$$

and whence

$$\|X^{1-\epsilon'}(\tfrac{f_\lambda}{r})^{1/2}\chi(r)\gamma_i\,\mathrm{Op^w}(\chi_{\kappa,\lambda})R(\lambda + \mathrm{i}0)v\| \leq \sqrt{C_1}\|f_\lambda^{-1/2}X^{3/2}v\|. \tag{4.20a}$$

On the other hand we have, cf. Proposition 4.1,

$$\|X^{1-\epsilon'}(\tfrac{f_\lambda}{r})^{1/2}\chi(r)\gamma_i(I - \mathrm{Op^w}(\chi_{\kappa,\lambda}))R(\lambda + \mathrm{i}0)v\| \leq C_3\|f_\lambda^{-1/2}X^{3/2}v\|. \tag{4.20b}$$

Clearly (4.18a) follows from (4.20a) and (4.20b).

To show (4.19) we calculate the expectation

$$\langle \mathrm{i}[H,P_1]\rangle_u = 2\operatorname{Im} z\,\langle P_1\rangle_u + 2\operatorname{Im}\langle P_1 u, v\rangle, \tag{4.21a}$$

$$\langle \mathrm{i}[H,P_1]\rangle_u = 2\sum_i \operatorname{Re}\langle Q_i u, \mathrm{i}[H,Q_i]u\rangle, \tag{4.21b}$$

$$\begin{aligned} \mathrm{i}[H,Q_i] &= T_i^1 + T_i^2; \\ T_i^1 &= \mathrm{i}[H, X_k^{1-\epsilon'}\chi(r)\gamma_i]\operatorname{Op^w}(\chi_{\kappa,|z|}), \\ T_i^2 &= X_k^{1-\epsilon'}\chi(r)\gamma_i \mathrm{i}[H,\operatorname{Op^w}(\chi_{\kappa,|z|})]. \end{aligned} \tag{4.21c}$$

The idea of the proof is to show that (4.21a) "tends" to be non-negative while (4.21b) "tends" to be non-positive. To keep the notation at a minimum we abbreviate $f_{|z|} = f$ in the remaining part of the proof of the proposition.

Clearly indeed the first term to the right in (4.21a) is non-negative (to be used in (4.25) stated below).

The second term to the right in (4.21a) can be estimated as

$$\begin{aligned} |2\operatorname{Im}\langle P_1 u, v\rangle| &\le \delta\langle P_1'\rangle_u + \delta^{-1}\sum_i \langle r/f\rangle_{Q_i v} \\ &\le \delta\langle P_1'\rangle_u + \delta^{-1}C_1\|X^{3/2-\epsilon'}f^{1/2}v\|^2 \\ &\le \delta\langle P_1'\rangle_u + \delta^{-1}C_2\|f^{-1/2}X^m v\|^2;\ m \ge 3/2-\epsilon'. \end{aligned} \tag{4.22}$$

We choose $\delta > 0$ suitably small later.

As for (4.21b) we substitute (4.21c). The contribution from the terms T_i^2 is estimated similarly using (4.8b) (for suitable n) and (1.3)

$$\begin{aligned} &2\sum_i |\langle Q_i u, T_i^2 \operatorname{Op^w}(\chi_{\kappa/2,|z|})u\rangle| \\ &\quad\le \tfrac{\delta}{2}\langle P_1'\rangle_u + \delta^{-1}C_1\|f^{-1/2}X^m v\|^2;\ m > 1/2, \end{aligned} \tag{4.23a}$$

and by using Proposition 4.1

$$\begin{aligned} &2\sum_i |\langle Q_i u, T_i^2(I - \operatorname{Op^w}(\chi_{\kappa/2,|z|}))u\rangle| \\ &\quad\le \tfrac{\delta}{2}\langle P_1'\rangle_u + \delta^{-1}C_2\|f^{-1/2}X^m v\|^2;\ m > 3/2-\epsilon'. \end{aligned} \tag{4.23b}$$

It remains to consider the contribution from T_i^1. We split

$$T_i^1 = S_i^1 + S_i^2 + S_i^3;$$

where

$$\begin{aligned} S_i^1 &= X_k^{1-\epsilon'}\chi(r)(\mathrm{D}\gamma_i)\operatorname{Op^w}(\chi_{\kappa,|z|}), \\ S_i^2 &= X_k^{1-\epsilon'}(\mathrm{D}\chi(r))\gamma_i\operatorname{Op^w}(\chi_{\kappa,|z|}), \\ S_i^3 &= (\mathrm{D}X_k^{1-\epsilon'})\chi(r)\gamma_i\operatorname{Op^w}(\chi_{\kappa,|z|}), \end{aligned}$$

and intend to use (4.12), (4.16a) and (4.16c) to treat the contribution to (4.21b) from the three terms, respectively.

The seeked negativity comes from the terms S_i^1 more precisely from the contribution from the first term to the right in (4.12). Thus, by the Cauchy Schwarz inequality,

$$\begin{aligned} &2\sum_i \operatorname{Re}\langle Q_i u, X_k^{1-\epsilon'}\chi(r)\tfrac{-2f}{r}(\gamma_i + (O(\epsilon^{1/2})\gamma)_i)\operatorname{Op^w}(\chi_{\kappa,|z|})u\rangle \\ &\qquad \le (-4+\tilde C\sqrt{\epsilon})\langle P_1'\rangle_u. \end{aligned} \tag{4.24}$$

To bound the contribution from the second term to the right in (4.12) we use that F is (uniformly) bounded and estimate

$$\begin{aligned} &2\sum_i \operatorname{Re}\langle Q_i u, X_k^{1-\epsilon'}\chi(r)\tfrac{-2F^i}{r}(H-|z|)\operatorname{Op^w}(\chi_{\kappa,|z|})u\rangle \\ &\qquad \le\ -4(z-|z|)\sum_i \operatorname{Re}\langle Q_i u, X_k^{1-\epsilon'}\chi(r)\tfrac{F^i}{r}\operatorname{Op^w}(\chi_{\kappa,|z|})u\rangle \\ &\qquad\quad + \delta\langle P_1'\rangle_u + \delta^{-1}C_1\|f^{-1/2}X^m v\|^2 \\ &\qquad \le 2\operatorname{Im} z\,\langle P_1\rangle_u + \tilde C_2\tfrac{|(z-|z|)|^2}{\operatorname{Im} z}\|X^{-1}X_k^{1-\epsilon'}u\|^2 \\ &\qquad\quad + \delta\langle P_1'\rangle_u + \delta^{-1}C_1\|f^{-1/2}X^m v\|^2;\ m>1/2. \end{aligned} \tag{4.25}$$

To bound the contribution from the third term to the right in (4.12) we "redistribute" the factors of components in γ^2 and use (1.3), (4.8a), (4.8c), (4.9c), (4.13) and (4.14) estimating with $u_j := \operatorname{Op^w}(\chi_{2\kappa,|z|})(2f/r)^{1/2}Q_j u$

$$\begin{aligned} &2\sum_i \operatorname{Re}\langle Q_i u, X_k^{1-\epsilon'}\chi(r)\tfrac{2F^i}{r}\gamma^2\operatorname{Op^w}(\chi_{\kappa,|z|})u\rangle \\ &\qquad \le 2\sum_j \operatorname{Re}\langle \tfrac{2}{r}F\cdot\gamma Q_j u, Q_j u\rangle + C_1\|f^{-1/2}X^m v\|^2 \\ &\qquad = 2\sum_j \operatorname{Re}\langle \tilde f(\operatorname{Op^w}(b) - 1 + O(\epsilon^{3/4}))\rangle_{u_j} + C_2\|f^{-1/2}X^m v\|^2 \\ &\qquad \le (2\kappa(\sup|\tilde f|^2+1) + C_1\epsilon^{3/4})\langle 2P_1'\rangle_u + C_3\|f^{-1/2}X^m v\|^2 \\ &\qquad = C_4\epsilon^{3/4}\langle P_1'\rangle_u + C_3\|f^{-1/2}X^m v\|^2;\ m>1/2. \end{aligned} \tag{4.26}$$

To bound the contribution from the fourth term to the right in (4.12) it is convenient (although not necessary) to symmetrize (assuming then $l\ge 4$). This gives with $\tilde u := X_k^{1-\epsilon'}\operatorname{Op^w}(\chi_{\kappa,|z|})u$

$$\begin{aligned} &2\sum_i \operatorname{Re}\langle Q_i u, X_k^{1-\epsilon'}\chi(r)\mathrm{i}\tfrac{2f}{r^2}O_i(\epsilon^0)\operatorname{Op^w}(\chi_{\kappa,|z|})u\rangle \\ &\qquad \le C_1\langle \tfrac{2f}{\tilde r^3}\rangle_{\tilde u} \\ &\qquad \le C_2\|f^{-1/2}X^m v\|^2;\ m>1/2. \end{aligned} \tag{4.27}$$

Clearly the contribution from from the terms S_i^2 are bounded similarly, cf. (4.16a),

$$2\sum_i \operatorname{Re}\langle Q_i u, S_i^2 u\rangle \le C\|f^{-1/2}X^m v\|^2;\ m>1/2. \tag{4.28}$$

It remains to examine the contribution from from the terms S_i^3. We shall use the following statements (4.8a), (4.8c), (4.9b), (4.16c) and (4.17) estimating with $u_i := \mathrm{Op^w}(\chi_{2\kappa,|z|})((\phi-\phi_k)2f/r)^{1/2}Q_iu$,

$$\begin{aligned}2\sum_i &\mathrm{Re}\langle Q_iu, S_i^3\,\mathrm{Op^w}(\chi_{\kappa,|z|})u\rangle\\ &\le 2(1-\epsilon')\sum_i\langle \mathrm{Op^w}(b)\rangle_{u_i} + C_1\|f^{-1/2}X^m v\|^2\\ &\le 2(1-\epsilon')(1+2\kappa)\sum_i \|u_i\|^2 + C_2\|f^{-1/2}X^m v\|^2\\ &\le 4(1-\epsilon')(1+2\epsilon^{3/4})\langle P_1'\rangle_u + C_3\|f^{-1/2}X^m v\|^2;\ m>1/2.\end{aligned}\tag{4.29}$$

Now by combining (4.22)–(4.29) with (4.21a)–(4.21c) we obtain

$$\begin{aligned}(4-\tilde C\sqrt{\epsilon}-3\delta-C_4\epsilon^{3/4}-4(1-\epsilon')(1+2\epsilon^{3/4}))\langle P_1'\rangle_u&\\ \le C_\delta\|f^{-1/2}X^m v\|^2+\tilde C_2\tfrac{|(z-|z|)|^2}{\mathrm{Im}\,z}\|X^{-1}X_k^{1-\epsilon'}u\|^2;\ m>3/2-\epsilon'.&\end{aligned}\tag{4.30}$$

We choose $\delta=\sqrt{\epsilon}$ and fix $C_0=\frac14(\tilde C+5)$. Then (possibly by taking $\epsilon_0>0$ smaller) we conclude the bound

$$\sqrt{\epsilon}\langle P_1'\rangle_u \le C_\delta\|f^{-1/2}X^m v\|^2+\tilde C_2\tfrac{|(z-|z|)|^2}{\mathrm{Im}\,z}\|X^{-1}X_k^{1-\epsilon'}u\|^2;\ m=3/2,$$

whence (4.19) follows.

Step II. We show (4.18b) by establishing the bound

$$\begin{gathered}\langle P_2'\rangle_u \le C_1\|f^{-1/2}X^{5/2}v\|^2 + C_2\tfrac{|(z-|z|)|^2}{\mathrm{Im}\,z}(\|X^{-2}X_k^{2(1-\epsilon')}u\|^2+k^2\langle P_1'\rangle_u)+C_3\langle P_1'\rangle_u;\\ P_2'=\sum_i Q_{ij}^*\tfrac{f}{r}Q_{ij}.\end{gathered}\tag{4.31}$$

Here P_1' is given as in (4.19). Due to (4.19) we can proceed as above letting first $\mathrm{Im}\,z\to 0$ (for fixed $\lambda=\mathrm{Re}\,z\ge 0$) and then $k\to\infty$. Then again we invoke Proposition 4.1. Whence it suffices for (4.18b) to show (4.31).

For (4.31) we proceed similarly as in Step I giving now less details. We replace P_1 by P_2 in (4.21a)–(4.21c) and need to show "essential positivity" and "essential negativity" of the expression to the right of the analogous (4.21a) and (4.21b), respectively. The most interesting contribution to the analogous commutator (4.21b) is the one from an expression like S_i^1. More precisely this term is now replaced by

$$S_{ij}^1 = X_k^{2(1-\epsilon')}\chi(r)(\mathrm{D}(\gamma_i\gamma_j))\,\mathrm{Op^w}(\chi_{\kappa,|z|}).$$

We write

$$\mathrm{D}(\gamma_i\gamma_j)=(\mathrm{D}\gamma_i)\gamma_j+\gamma_i(\mathrm{D}\gamma_j),$$

and invoke again (4.12) which contains four terms.

As for the first term the analogous of (4.24) reads (using the constant $\tilde C$ from (4.24))

$$\begin{aligned} 2\sum_{i,j} &\operatorname{Re}\langle Q_{ij}u, X_k^{2(1-\epsilon')}\chi(r) \\ &\Big(\tfrac{-2f}{r}(\gamma_i + (O(\epsilon^{1/2})\gamma)_i)\gamma_j + \gamma_i\tfrac{-2f}{r}(\gamma_j + (O(\epsilon^{1/2})\gamma)_j)\Big)\operatorname{Op^w}(\chi_{\kappa,|z|})u\rangle \\ &\qquad \le (-8 + 2\tilde C\sqrt{\epsilon} + \delta)\langle P_2'\rangle_u + \delta^{-1}C_1\langle P_1'\rangle_u. \end{aligned} \tag{4.32}$$

As for the analogous of (4.25) we have, using the bound $X_k^2 \le k$ and the first identity of (4.11), and by arguing as in (4.23a)–(4.23b),

$$\begin{aligned} 2\sum_{i,j} &\operatorname{Re}\langle Q_{ij}u, X_k^{2(1-\epsilon')}\chi(r)\Big(\tfrac{-2F^i}{r}(H-|z|)\gamma_j + \gamma_i\tfrac{-2F^j}{r}(H-|z|)\Big)\operatorname{Op^w}(\chi_{\kappa,|z|})u\rangle \\ &\le 2\operatorname{Im} z\,\langle P_2\rangle_u + \tilde C_1\tfrac{|(z-|z|)|^2}{\operatorname{Im} z}(\|X^{-2}X_k^{2(1-\epsilon')}u\|^2 + k^2\langle P_1'\rangle_u) \\ &\quad + \delta\langle P_2'\rangle_u + \delta^{-1}C_2(\|f^{-1/2}X^{3/2}v\|^2 + \langle P_1'\rangle_u). \end{aligned} \tag{4.33}$$

As for the analogous of (4.26) we obtain by redistributing components of γ^2, using the notation $u_{mn} = \operatorname{Op^w}(\chi_{2\kappa,|z|})(4f/r)^{1/2}Q_{mn}$,

$$\begin{aligned} 2\sum_{i,j} &\operatorname{Re}\langle Q_{ij}u, X_k^{2(1-\epsilon')}\chi(r)\Big(\tfrac{2F^i}{r}\gamma^2\gamma_j + \gamma_i\tfrac{2F^j}{r}\gamma^2\Big)\operatorname{Op^w}(\chi_{\kappa,|z|})u\rangle \\ &\le 2\sum_{m,n}\operatorname{Re}\langle \tilde f(\operatorname{Op^w}(b) - 1 + O(\epsilon^{3/4}))\rangle_{u_{mn}} + \tfrac{\delta}{2}\langle P_2'\rangle_u + C_{1,\delta}\|f^{-1/2}Xv\|^2 \\ &\le (2\kappa(\sup|\tilde f|^2 + 1) + C_1\epsilon^{3/4})\langle 4P_2'\rangle_u + \delta\langle P_2'\rangle_u + C_{2,\delta}\|f^{-1/2}Xv\|^2 \\ &= \tilde C_3\epsilon^{3/4}\langle P_2'\rangle_u + \delta\langle P_2'\rangle_u + C_{2,\delta}\|f^{-1/2}Xv\|^2. \end{aligned} \tag{4.34}$$

Here we used twice that $(rf)^{-1} = O(r^{\mu/2-1})$ (whence up to a compactly supported term this function is bounded by δ/C), and we used a uniform bound similar to (4.8c), for example

$$[(4f/r)^{1/2}X_k^{2(1-\epsilon')}\chi(r)\gamma_m\gamma_n, \operatorname{Op^w}(\chi_{2\kappa,|z|})]\operatorname{Op^w}(\chi_{\kappa,|z|}) = X^{-2}BX^{-1}f^{1/2};\ \|B\| \le C. \tag{4.35}$$

As for the analogous of (4.27) we have (using $l \ge 4$)

$$\begin{aligned} 2\sum_{i,j} &\operatorname{Re}\langle Q_{ij}u, X_k^{2(1-\epsilon')}\chi(r)\Big(\mathrm{i}\tfrac{2f}{r^2}O_i(\epsilon^0)\gamma_j + \gamma_i\mathrm{i}\tfrac{2f}{r^2}O_j(\epsilon^0)\Big)\operatorname{Op^w}(\chi_{\kappa,|z|})u\rangle \\ &= 4\sum_{i,j}\operatorname{Re}\langle Q_{ij}u, X_k^{2(1-\epsilon')}\chi(r)\Big(\mathrm{i}\tfrac{2f}{r^2}O_i(\epsilon^0)\gamma_j + \tfrac{f}{r^3}O_{ij}(\epsilon^0)\Big)\operatorname{Op^w}(\chi_{\kappa,|z|})u\rangle \\ &\le \delta\langle P_2'\rangle_u + \delta^{-1}C_1(\|f^{-1/2}Xv\|^2 + \langle P_1'\rangle_u). \end{aligned} \tag{4.36}$$

The analogue of (4.28) is obvious.

The analogue of (4.29) reads with $u_{ij} := \operatorname{Op^w}(\chi_{2\kappa,|z|})((\phi - \phi_k)2f/r)^{1/2}Q_{ij}u$

$$\begin{aligned} 2\sum_{i,j} \operatorname{Re}\langle Q_{ij}u, (\mathrm{D}X_k^{2(1-\epsilon')})\chi(r)\gamma_i\gamma_j \operatorname{Op^w}(\chi_{\kappa,|z|})u\rangle \\ &\le 4(1-\epsilon')\sum_{i,j}\langle \operatorname{Op^w}(b)\rangle_{u_{ij}} + C_1\|f^{-1/2}Xv\|^2 \\ &\le 4(1-\epsilon')(1+2\kappa)\sum_{i,j}\|u_{ij}\|^2 + \delta\langle P_2'\rangle_u + C_2\|f^{-1/2}Xv\|^2 \\ &\le (8(1-\epsilon')(1+2\epsilon^{3/4}) + \delta)\langle P_2'\rangle_u + C_3\|f^{-1/2}Xv\|^2. \end{aligned} \tag{4.37}$$

Here we used the bound (4.35) trivially modified by an insertion of a factor $(\phi - \phi_k)$.

Collecting bounds we get (similar to (4.30))

$$\begin{aligned} &(8 - 2\tilde{C}\sqrt{\epsilon} - 7\delta - \tilde{C}_3\epsilon^{3/4} - 8(1-\epsilon')(1+2\epsilon^{3/4}))\langle P_2'\rangle_u \\ &\quad \le C_1(\delta)\|f^{-1/2}X^{5/2}v\|^2 + \tilde{C}_1\tfrac{|(z-|z|)|^2}{\operatorname{Im} z}(\|X^{-2}X_k^{2(1-\epsilon')}u\|^2 + k^2\langle P_1'\rangle_u) + C_2(\delta)\langle P_1'\rangle_u. \end{aligned} \tag{4.38}$$

Picking $\delta = \sqrt{\epsilon}$ and $C_0 = \frac{1}{4}(\tilde{C} + 5)$ in (4.38) (as in Step I) the left hand side bounds $\sqrt{\epsilon}\langle P_2'\rangle_u$ from above, and whence (4.31) follows. □

5. Distorted Fourier transform

We prove the existence of the limit (3.7a) for all $\lambda \ge 0$ and all $v \in L_3^2$. For that we first compute the derivative $\frac{\mathrm{d}}{\mathrm{d}s}$ along the flow $\Phi(s,\cdot)$

$$\tfrac{\mathrm{d}}{\mathrm{d}s}(\mathrm{e}^{-\mathrm{i}S_\epsilon}K_\epsilon^{1/2}m_\epsilon^{1/2}R(\lambda + \mathrm{i}0)v) = \mathrm{i}\mathrm{e}^{-\mathrm{i}S_\epsilon}K_\epsilon^{-1/2}m_\epsilon^{1/2}\gamma_\parallel(\lambda)R(\lambda+\mathrm{i}0)v. \tag{5.1}$$

It suffices to show that the right hand side is integrable as a $\mathcal{G}$-valued function. For the latter purpose we use (3.5), the identity $S_\epsilon(\Phi(s,\cdot)) = s$ and the Cauchy Schwarz inequality to conclude that in turn it suffices to find $\delta > 0$ such that

$$\|S_\epsilon^{1/2+\delta}K_\epsilon^{-1/2}\gamma_\parallel(\lambda)R(\lambda+\mathrm{i}0)v\| \le C < \infty. \tag{5.2}$$

We plug in (4.10). Since

$$crf(r,\lambda) \le S_\epsilon(x) \le Crf(r,\lambda) \tag{5.3}$$

the contribution from the first term $(H - V_2 - \lambda)$ is in $\mathcal{H}$ for any $\delta \le 2$ (then we have $(fr)^{1/2+\delta}K_\epsilon^{-1/2} \le CX^3$ and by assumption $X^3v \in \mathcal{H}$). For the contribution from the second term $-\gamma^2$ we use (4.18b) with $i = j$. Again since $X^3v \in \mathcal{H}$ we only need to examine the weight $X^{2(1-\epsilon')}f^{1/2}r^{-1/2}$ to the left in (4.18b), in particular we need to specify applicable δ and ϵ': More precisely we need to specify these parameters such that the function

$$S_\epsilon^{1/2+\delta}K_\epsilon^{-1/2}X^{-2(1-\epsilon')}f^{-1/2}r^{1/2} \text{ is bounded.}$$

Using (5.3) this property will follow from boundedness of

$$X^{2\epsilon'+\delta-1}f^{\delta-1},$$

which in turn for $\delta \le 1$ will follow from boundedness of

$$X^{2\epsilon'+(\delta-1)(1-\mu/2)}.$$

The latter boundedness is achieved for any $\delta \in (0,1)$ (henceforth taken fixed) and for all sufficient small $\epsilon, \epsilon' = C_0\sqrt{\epsilon} > 0$. We have shown (5.2) for all $\lambda \ge 0$ and all $v \in L^2_3$. Since the bound (5.2) is uniform in $\lambda \in [0, \lambda_0]$ for any $\lambda_0 > 0$ and the function $[0, \lambda_0] \ni \lambda \to (\mathrm{e}^{-\mathrm{i}S_\epsilon} K_\epsilon^{1/2} m_\epsilon^{1/2} R(\lambda + \mathrm{i}0)v)(\Phi(s, \cdot)) \in \mathcal{G}$ is continuous for any (large) fixed $s > 1$, we conclude that also the function

$$[0, \infty) \ni \lambda \to F^+(\lambda)v \in \mathcal{G} \text{ is continuous.} \tag{5.4}$$

Clearly (by time-reversal invariance) we conclude the existence of (3.7b) for all $v \in L^2_3$ also. Similarly $F^-(\lambda)v$ is continuous in $\lambda \geq 0$.

There are other assertions in Subsection 3.2. As for the formula

$$\|P_{\mathrm{c}} v\|^2 = \lim_{\lambda_0 \to \infty} \int_0^{\lambda_0} \|F^+(\lambda)v\|^2_{\mathcal{G}} \,\mathrm{d}\lambda,$$

it suffices to show that for all $\lambda \geq 0$

$$\pi^{-1} \langle v, (\operatorname{Im} R(\lambda + \mathrm{i}0))v \rangle = \|F^+(\lambda)v\|^2_{\mathcal{G}}. \tag{5.5}$$

We first estimate (recall $u := R(\lambda + \mathrm{i}0)v$)

$$\begin{aligned} &\Big| \lim_{s \to \infty} \operatorname{Re} \sum_{j=1}^d \int_{S^{d-1}} (\overline{(\gamma_j(\lambda)u)} u (\partial_j S_\epsilon) m_\epsilon)(\Phi(s, \cdot)) \,\mathrm{d}\sigma \Big| \\ &\quad \leq \liminf_{s \to \infty} \sum_{j=1}^d \int_{S^{d-1}} |(\overline{(\gamma_j(\lambda)u)} u (\partial_j S_\epsilon) m_\epsilon)(\Phi(s, \cdot))| \,\mathrm{d}\sigma. \end{aligned} \tag{5.6}$$

By (3.5) and the Cauchy Schwarz inequality, for any big enough $s_0 > 0$

$$\begin{aligned} &\int_{s_0}^\infty \mathrm{d}s\, s^{-1} \sum_{j=1}^d \int_{S^{d-1}} |(\overline{(\gamma_j(\lambda)u)} u (\partial_j S_\epsilon) m_\epsilon)(\Phi(s, \cdot))| \,\mathrm{d}\sigma \\ &\quad = \sum_{j=1}^d \int_{s_0}^\infty \mathrm{d}s \int_{S^{d-1}} \big| m_\epsilon^{1/2} X^{1-\epsilon'} (\tfrac{f_\lambda}{r})^{1/2} \gamma_j(\lambda) u \big| \, \big| m_\epsilon^{1/2} X^{\epsilon'-1} (\tfrac{f_\lambda}{r})^{-1/2} (\partial_j \ln S_\epsilon) u \big| \,\mathrm{d}\sigma \\ &\quad \leq \sum_{j=1}^d \|X^{1-\epsilon'} (\tfrac{f_\lambda}{r})^{1/2} \chi(r) \gamma_j(\lambda) u\| \, \|X^{\epsilon'-1} (\tfrac{f_\lambda}{r})^{-1/2} \chi(r) (\partial_j \ln S_\epsilon) u\|. \end{aligned}$$

Since

$$|X^{\epsilon'-1} (\tfrac{f_\lambda}{r})^{-1/2} \chi(r) (\partial_j \ln S_\epsilon)| \leq C X^{\epsilon' + \mu/2 - 1} (\tfrac{f_\lambda}{r})^{1/2} \chi(r),$$

cf. (5.3), we conclude using (1.3) and (4.18a) that for all $\epsilon' = C_0\sqrt{\epsilon} \in (0, 1 - \mu/2)$ the latter integral is finite. Whence for all small $\epsilon > 0$ indeed the right hand side of (5.6) is zero. We have shown (5.5).

Throughout the rest of the paper we abbreviate $B = B(|x|)$ and $B^* = B(|x|)^*$. Note that due to (1.3) and (5.5)

$$\forall \lambda \geq 0 : \; F^+(\lambda) f^{1/2} \in \mathcal{B}(B, \mathcal{G}). \tag{5.7}$$

Next introduce

$$F^+ = \int_0^\infty \oplus F^+(\lambda) \,\mathrm{d}\lambda,$$

which due to (5.5) obeys $(F^+)^* F^+ = P_{\mathrm{c}}$. Notice that we here consider $F^+ \in \mathcal{B}(\mathcal{H}, \widetilde{\mathcal{H}})$. A short argument shows that for all $v \in (H - \lambda) C_{\mathrm{c}}^\infty(\mathbb{R}^d)$ the function $F^+(\lambda)v = 0$. Whence $F^+ H_{\mathrm{c}} \subset M_\lambda F^+$. We claim that F^+ diagonalizes H_{c}. This stronger statement is part of the following

Proposition 5.1. *The map* $F^+ : \operatorname{Ran} P_{\mathrm{c}} \to \widetilde{\mathcal{H}}$ *is a unitary diagonalizing transform, in particular*

$$\operatorname{Ran} F^+ = \widetilde{\mathcal{H}} \text{ and } F^+ H_{\mathrm{c}} = M_\lambda F^+. \tag{5.8}$$

Proof. It suffices to show the first identity of (5.8), since then indeed the restricted map $F^+ : \mathcal{H}_{\mathrm{c}}(H) = \operatorname{Ran} P_{\mathrm{c}} \to \widetilde{\mathcal{H}}$ is unitary and the second identity of (5.8) holds.

Step I. Let $\tau \in C^\infty(S^{d-1})$ and consider the function $\tilde{u}$ of (3.8). We claim that

$$\tau = F^+(\lambda)(H-\lambda)\tilde{u}. \tag{5.9}$$

Note that due to Lemma 3.3 this is formally true, however since we dont know that $(H-\lambda)\tilde{u} \in L^2_3$ a continuity argument is required. This motivates the claim that for all $v \in B$

$$F^+(\lambda) f^{1/2} v = \mathcal{G}\text{–}\lim_{S\to\infty} S^{-1} \int_0^S \pi^{-1/2} (\mathrm{e}^{-\mathrm{i}S_\epsilon} K_\epsilon^{1/2} m_\epsilon^{1/2} R(\lambda+\mathrm{i}0) f^{1/2} v)(\Phi(s,\cdot))\,\mathrm{d}s. \tag{5.10}$$

Clearly this is consistent with (5.7) if $v \in f^{-1/2} L^2_3$. To show that indeed the right hand side of (5.10) makes sense for $v \in B$ we need to show the Cauchy property. Approximating $C_{\mathrm{c}}(\mathbb{R}^d) \ni v_n \to v \in B$ it suffices to show the bound

$$\begin{aligned}\sup_{S>1} \|S^{-1} \int_0^S \pi^{-1/2} (\mathrm{e}^{-\mathrm{i}S_\epsilon} K_\epsilon^{1/2} m_\epsilon^{1/2} R(\lambda+\mathrm{i}0) f^{1/2}(v-v_n))(\Phi(s,\cdot))\,\mathrm{d}s\|_{\mathcal{G}} \\ \leq C\|v-v_n\|_B.\end{aligned} \tag{5.11}$$

We proceed a little more general and show for all $w \in B(|x|)^*$

$$\sup_{S>1} \|S^{-1} \int_0^S \pi^{-1/2} (\mathrm{e}^{-\mathrm{i}S_\epsilon} K_\epsilon^{1/2} m_\epsilon^{1/2} f^{-1/2} w)(\Phi(s,\cdot))\,\mathrm{d}s\|_{\mathcal{G}} \leq C\|w\|_{B^*}. \tag{5.12}$$

In fact given (5.12) the bound (5.11) follows using (1.3), and whence the formula (5.10) and then in turn (5.9) are justified.

To show (5.12) we first recall the Besov space bound

$$\sup_{\rho>1} \rho^{-1} \int_{|x|\leq\rho} |w|^2\,\mathrm{d}x \leq C\|w\|^2_{B^*}, \tag{5.13a}$$

and its proof: Let $R_0 = 0$ and $R_j = 2^{j-1}$ for $j \in \mathbb{N}$. Then

$$\begin{aligned}\rho^{-1} \int_{|x|\leq\rho} |w|^2\,\mathrm{d}x &\leq \sum_{j\leq J;\ R_{J-1}\leq\rho<R_J} (\rho^{-1}R_j) R_j^{-1} \int_{R_{j-1}\leq|x|<R_j} |w|^2\,\mathrm{d}x \\ &\leq \sum_{j\leq J;\ R_{J-1}\leq\rho<R_J} (\rho^{-1}R_j)\|w\|^2_{B^*} \\ &\leq 4\|w\|^2_{B^*}.\end{aligned}$$

Now for (5.12) we estimate using the Cauchy Schwarz inequality, notation from Subsection 3.1, (3.5) and (5.13b) (stated below)

$$\begin{aligned}
&\Big\|S^{-1}\int_0^S \pi^{-1/2}(\mathrm{e}^{-\mathrm{i}S_\epsilon}K_\epsilon^{1/2}m_\epsilon^{1/2}f^{-1/2}w)(\Phi(s,\cdot))\,\mathrm{d}s\Big\|_{\mathcal{G}}\\
&\le C_1S^{-1}\int_0^S \|(K_\epsilon^{1/2}m_\epsilon^{1/2}f^{-1/2}w)(\Phi(s,\cdot))\|_{\mathcal{G}}\,\mathrm{d}s\\
&\le C_1S^{-1/2}\Big(\int_0^S \|(K_\epsilon^{1/2}m_\epsilon^{1/2}f^{-1/2}w)(\Phi(s,\cdot))\|_{\mathcal{G}}^2\,\mathrm{d}s\Big)^{1/2}\\
&= C_1S^{-1/2}\Big(\int_{\mathcal{B}_\epsilon(S)} |K_\epsilon^{1/2}f^{-1/2}w|^2\,\mathrm{d}x\Big)^{1/2}\\
&\le C_2S^{-1/2}\Big(\int_{\mathcal{B}_\epsilon(S)} |f^{1/2}w|^2\,\mathrm{d}x\Big)^{1/2}\\
&\le C_3\|w\|_{B^*}.
\end{aligned}$$

In the last step we used the following analogue of (5.13a):

$$\sup_{S>1} S^{-1}\int_{\mathcal{B}_\epsilon(S)} |f^{1/2}w|^2\,\mathrm{d}x \le C(\lambda)\|w\|_{B^*}^2. \tag{5.13b}$$

Note that for $\lambda>0$ we can bound $S_\epsilon(x)\ge c|x|$ and $f^{1/2}(x)\le C$ yielding (5.13b) in this case due to (5.13a). For $\lambda=0$ we can bound $S_\epsilon(x)\ge c|x|^{1-\mu/2}$ and $f^{1/2}(x)\le C|x|^{-\mu/4}$ for $|x|\ge 1$ yielding (5.13b) in that case also. This can be seen by arguing as in the above proof of (5.13a). Consequently we have (5.13b) for all $\lambda\ge 0$, and (5.12) is proven.

For a later application let us note the following inverse of (5.13b) (proved similarly):

$$\|w\|_{B^*}^2 \le C(\lambda)\sup_{S>1} S^{-1}\int_{\mathcal{B}_\epsilon(S)} |f^{1/2}w|^2\,\mathrm{d}x. \tag{5.13c}$$

For an abstract version of (5.13b) and (5.13c) see [Sk, Lemma 2.4].

Step II. We can mimic the proof of [ACH, Theorem 1.1] using (5.9). Notice that we only need (5.9) for $\lambda>0$ (which is the analogue of [ACH, Theorem 3.3 iv)]). Details are omitted.

□

Corollary 5.2. *For all* $\tau\in C^\infty(S^{d-1})\subset\mathcal{G}$ *the generalized eigenfunction* $u^-=u^-(\lambda)$, $\lambda\ge 0$, *defined by* (3.8) *and* (3.22) *is also given by*

$$u^-(\lambda)=2\pi\mathrm{i}F^+(\lambda)^*\tau. \tag{5.14}$$

Proof. By Lemma 3.3, (5.5) and (5.9)

$$\begin{aligned}
u^- &= (R(\lambda+\mathrm{i}0)-R(\lambda-\mathrm{i}0))(H-\lambda)\tilde u\\
&= 2\pi\mathrm{i}F^+(\lambda)^*F^+(\lambda)(H-\lambda)\tilde u = 2\pi\mathrm{i}F^+(\lambda)^*\tau.
\end{aligned}$$ □

Definition 5.3. For any $\lambda\ge 0$ we define the scattering matrix $S(\lambda)\in\mathcal{B}(\mathcal{G})$ by the identity

$$F^+(\lambda)v=S(\lambda)F^-(\lambda)v;\ v\in f_\lambda^{1/2}B(|x|). \tag{5.15}$$

Proposition 5.4. *The operator $S(\lambda)$ is a well-defined unitary operator on $\mathcal{G}$. It is strongly continuous as a function of $\lambda \geq 0$. In particular the scattering matrix at zero energy $S(0)$ is uniquely determined by the diagonalizing transforms $F^{\pm}$.*

Proof. We apply (5.5), (5.7), (5.9) and their analogues for change of superscript $+ \to -$. This yields the well-definedness and the unitarity. For all $v \in L^2_3$ the functions $\{F^{\pm}(\lambda)v|\lambda \geq 0\} \in \widetilde{\mathcal{H}}$ are continuous in λ, cf. (5.4). Since moreover $F^-(\lambda)L^2_3$ is dense in $\mathcal{G}$ for any fixed λ the continuity property follows by a density argument. □

5.1. **Asymptotics of generalized eigenfunctions.** We complete this section by a discussion of the asymptotics of the generalized eigenfunctions

$$u^-_\tau(\cdot,\lambda) := 2\pi \mathrm{i} F^+(\lambda)^*\tau;\ \tau \in \mathcal{G}. \tag{5.16}$$

Notice that Corollary 5.2 provides a representation for $\tau \in C^\infty(S^{d-1})$.

Let $B^*_0 \subset B^*$ be the closure of $C_{\mathrm{c}}(\mathbb{R}^d)$ in B^*. For all $\lambda \geq 0$ and $\tau \in \mathcal{G}$ the function

$$w(x) = (K_\epsilon^{-1/2} m_\epsilon^{-1/2} f^{1/2})(x)\tau(\sigma);\ x = \Phi(t,\sigma),$$

belongs to B^* with

$$\|w\|_{B^*} \leq C\|\tau\|_{\mathcal{G}}. \tag{5.17}$$

This is due to (3.5) and (5.13c).

Next, using (5.9) and (5.15) we decompose for all $\tau \in C^\infty(S^{d-1})$

$$\begin{aligned} w^-_\tau(x) : &= \pi^{1/2}(K_\epsilon^{-1/2} m_\epsilon^{-1/2} f^{1/2})(x)(S(\lambda)^{-1}\tau)(\sigma)\\ &= \pi^{1/2}(K_\epsilon^{-1/2} m_\epsilon^{-1/2} f^{1/2})(x)(F^-(\lambda)(H-\lambda)\tilde{u})(\sigma)\\ &= w^-_1(x) + w^-_2(x);\\ &\qquad w^-_1 := \mathrm{e}^{\mathrm{i}S_\epsilon} f^{1/2} R(\lambda - \mathrm{i}0)(H-\lambda)\tilde{u}. \end{aligned}$$

While $w^-_\tau, w^-_1 \in B^*$ we have the stronger assertion for the second term,

$$w^-_2 \in B^*_0. \tag{5.18}$$

To prove (5.18) we introduce the quantity

$$w_n = \pi^{1/2}(K_\epsilon^{-1/2} m_\epsilon^{-1/2} f^{1/2})(x)(F^-(\lambda)((H-\lambda)\tilde{u} - f^{1/2}v_n))(\sigma),$$

where $C_{\mathrm{c}}(\mathbb{R}^d) \ni v_n \to v := f^{-1/2}(H-\lambda)\tilde{u} \in B$. We have $\|w_n\|_{B^*} \leq C\|v_n - v\|_B$, cf. (5.17), showing that $\|w_n\|_{B^*} \to 0$ for $n \to \infty$. Similarly

$$\mathrm{e}^{\mathrm{i}S_\epsilon} f^{1/2} R(\lambda - \mathrm{i}0) f^{1/2} v_n \to w^-_1 \text{ in } B^*.$$

We are lead to consider the quantity (for fixed $n \in \mathbb{N}$)

$$\begin{aligned} \widetilde{w}_n(x) = \pi^{1/2}(K_\epsilon^{-1/2} m_\epsilon^{-1/2} f^{1/2})(x)\big(&(F^-(\lambda) f^{1/2} v_n)(\sigma)\\ &- \pi^{-1/2}(\mathrm{e}^{\mathrm{i}S_\epsilon} K_\epsilon^{1/2} m_\epsilon^{1/2} R(\lambda - \mathrm{i}0) f^{1/2} v_n)(x)\big), \end{aligned}$$

It follows from (3.5), (5.1), (5.2) and (5.13c) by yet another approximation that $\widetilde{w}_n \in B^*_0$. Note that indeed

$$\|\widetilde{w}_n - 1_{\mathcal{B}_\epsilon(S)}\widetilde{w}_n\|_{B^*} \to 0 \text{ for } S \to \infty,$$

while obviously we have $1_{\mathcal{B}_\epsilon(S)}\widetilde{w}_n \in B^*_0$ for all $S > 1$. Whence (5.18) is proven.

Now, combining (3.22) and (5.18) we conclude that for all $\tau \in C^\infty(S^{d-1})$

$$u^-_\tau(\cdot,\lambda) - (\tilde{u} - \mathrm{e}^{-\mathrm{i}S_\epsilon} f^{-1/2} w^-_\tau) \in f^{-1/2} B^*_0.$$

This formula extends to $\mathcal{G}$ and implies an injectivety property.

Corollary 5.5. *Let $\lambda \geq 0$ and $\tau \in \mathcal{G}$ be given. Introducing the following function of $x = \Phi(t,\sigma)$,*

$$u^-_{0,\tau}(x,\lambda) = \pi^{1/2}(K_\epsilon^{-1/2} m_\epsilon^{-1/2})(x)(\mathrm{e}^{\mathrm{i}S_\epsilon(x)}\tau(\sigma) - \mathrm{e}^{-\mathrm{i}S_\epsilon(x)}(S(\lambda)^{-1}\tau)(\sigma)),$$

we have

$$u^-_\tau(\cdot,\lambda) - u^-_{0,\tau}(\cdot,\lambda) \in f_\lambda^{-1/2} B_0^*. \tag{5.19a}$$

In particular

$$2\pi\|\tau\|^2_{\mathcal{G}} = \lim_{S\to\infty} S^{-1}\int_{\mathcal{B}_\epsilon(S)} |K_\epsilon^{1/2} u^-_\tau(x,\lambda)|^2\,\mathrm{d}x. \tag{5.19b}$$

Proof. Since we know (5.19a) for $\tau \in C^\infty(S^{d-1})$ the statement for $\tau \in \mathcal{G}$ follows by approximation, cf. (5.17). As for (5.19b) we can replace u^-_τ to the right by $u^-_{0,\tau}$ due to (5.19a). Then we compute using (3.5) and the unitarity of the scattering matrix. Note that cross terms do not contribute to the limit due to oscillatory behaviour. □

6. Characterization of generalized eigenfunctions

We introduce the following class of generalized eigenfunctions.

Definition 6.1. For $\lambda \geq 0$ let

$$\mathcal{E}_\lambda = \{u \in f_\lambda^{-1/2} B^* | (H-\lambda)u = 0\}.$$

Notice that it follows from (5.7) and the definition (5.16) that for all $\tau \in \mathcal{G}$

$$u^-_\tau(\cdot,\lambda) \in \mathcal{E}_\lambda.$$

In fact it follows from (5.7), (5.13b) and (5.19b) that the map

$$\mathcal{G} \ni \tau \to u^-_\tau(\cdot,\lambda) \in \mathcal{E}_\lambda$$

is a bi-continuous linear isomorphism onto its range. The latter is identified as

Proposition 6.2. *For all $\lambda \geq 0$ the set*

$$\mathcal{E}_\lambda = \{u^-_\tau(\cdot,\lambda) | \tau \in \mathcal{G}\}. \tag{6.1}$$

Proof. Let $u_\lambda \in \mathcal{E}_\lambda$ be arbitrarily given. We need to show that it must have the form $u_\lambda = u^-_\tau(\cdot,\lambda)$ for some $\tau \in \mathcal{G}$. For that we partly mimic [DS1, Section 8]. In particular, with reference to symbols (4.1) and the corresponding localization symbols as appearing in Proposition 4.1 let us introduce

$$\chi^\pm = \chi_-(a_\lambda)\tilde{\chi}_\pm(b_\lambda) + \tfrac{1}{2}\chi_+(a_\lambda). \tag{6.2}$$

We consider in the following these functions as fixed and consider ϵ-small perturbations $W_\epsilon \in \mathcal{W}$ with $\epsilon > 0$ small exactly as in Proposition 4.1. Note the properties

$$\mathrm{Op}^{\mathrm{w}}(a_\lambda \chi_+(a_\lambda))u_\lambda,\ \mathrm{Op}^{\mathrm{w}}(\chi_+(a_\lambda))u_\lambda \in f_\lambda^{-1/2} B_0^*, \tag{6.3}$$

cf. [Sk, Lemma 3.1], in fact these functions are in any weighted L^2-space L^2_m. Whence the quantization of the second term of (6.2) contributes by a small term when applied to u_λ. The quantization of the first term localizes to an outgoing (incoming) region of phase space. A priori we only have

$$\mathrm{Op}^{\mathrm{w}}(\chi^\pm)u_\lambda,\ \mathrm{Op}^{\mathrm{w}}(\chi_-(a_\lambda)\tilde{\chi}_\pm(b_\lambda))u_\lambda \in f_\lambda^{-1/2} B^*.$$

Step I. We construct a candidate τ. Pick a non-negative $g \in C_c^\infty(\mathbb{R}_+)$ with $\int_0^\infty g(t)\mathrm{d}t = 1$, and let $G_n(s) = 1 - \int_0^{s/n} g(t)\mathrm{d}t$, $n \in \mathbb{N}$. Define

$$\tau_n = F^+(\lambda)G_n(S_\epsilon)(H-\lambda)\operatorname{Op^w}(\chi^+)u_\lambda;\ n \in \mathbb{N}.$$

We note that this family $\{\tau_n\}$ is a bounded subset of $\mathcal{G}$. In fact we have

$$\begin{aligned}\tau_n &= \mathrm{i}F^+(\lambda)\mathrm{i}[H, G_n(S_\epsilon)]\operatorname{Op^w}(\chi^+)u_\lambda\\ &= -\mathrm{i}F^+(\lambda)(\operatorname{Re}(p\cdot\nabla S_\epsilon)\tfrac{2}{n}g(S_\epsilon/n) + \mathrm{i}|\nabla S_\epsilon|^2 n^{-2}g'(S_\epsilon/n))\operatorname{Op^w}(\chi^+)u_\lambda\\ &= \tau_n^1 + \tau_n^2.\end{aligned}$$

For any $\tilde\tau \in \mathcal{G}$ we have

$$\|f_\lambda^{1/2}u_{\tilde\tau}^-(\cdot,\lambda)\|_{B^*} \le C\|\tilde\tau\|_{\mathcal{G}}, \tag{6.4a}$$

$$\|f_\lambda^{-3/2}\chi(|x|>2)\operatorname{Re}(p\cdot\nabla S_\epsilon)u_{\tilde\tau}^-(\cdot,\lambda)\|_{B^*} \le C\|\tilde\tau\|_{\mathcal{G}}, \tag{6.4b}$$

cf. (6.3). We aim at showing the uniform bounds

$$|\langle\tilde\tau, \tau_n^j\rangle| \le C\|\tilde\tau\|_{\mathcal{G}};\ j = 1,2, \tag{6.5}$$

which suffices for the boundedness.

For $j=2$ we write

$$\begin{gathered}-2\pi\mathrm{i}\langle\tilde\tau,\tau_n^2\rangle = \langle f_\lambda^{1/2}u_{\tilde\tau}^-(\cdot,\lambda), h_n^2 w\rangle;\\ h_n^2 = f_\lambda^{-1/2}|\nabla S_\epsilon|^2 n^{-2}g'(S_\epsilon/n)f_\lambda^{-1/2},\ w = f_\lambda^{1/2}\operatorname{Op^w}(\chi^+)u_\lambda.\end{gathered}$$

For $\lambda > 0$ we have $|h_n^2(x)| \le C_\lambda\langle x\rangle^{-2}$ while for $\lambda = 0$ there is the bound $|h_n^2(x)| \le C\langle x\rangle^{-2+\mu/2}$. Since B^* is continuously imbedded in $L^2_{-\delta}$ for any $\delta > 1/2$ we conclude the bound (6.5) for $j=2$ using (6.4a).

Decomposing similarly for $j=1$,

$$\begin{aligned}&\langle\tilde\tau,\tau_n^1\rangle = \langle\tilde w, h_n^1 w\rangle;\\ &\qquad h_n^1 = f_\lambda^{3/2}n^{-1}g(S_\epsilon/n)f_\lambda^{-1/2},\ w = f_\lambda^{1/2}\operatorname{Op^w}(\chi^+)u_\lambda,\end{aligned}$$

and using (6.4b) we need to show the bound

$$|\langle\tilde w, h_n^1 w\rangle| \le C\|\tilde w\|_{B^*}\,\|w\|_{B^*}.$$

For that it suffices for any $\lambda \ge 0$ to find $C > 1$ and a bounded interval I such that for all $n \in \mathbb{N}$ there exists $R \ge 1$ such that $|h_n^1(x)| \le CR^{-1}1_I(|x|/R)$ for all $x \in \mathbb{R}^d$. We recall the bounds $crf_\lambda \le S_\epsilon \le Crf_\lambda$. In particular for $\lambda > 0$ the assertion is immediate with $R = n$. For $\lambda = 0$ we choose $R = n^{1/(1-\mu/2)}$ and obtain the same conclusion. So indeed (6.5) holds, and the sequence $\{\tau_n\} \subset \mathcal{G}$ is bounded.

Take $\tau \in \mathcal{G}$ as the weak limit of some subsequence of $\{\tau_n\}$, cf. [Yo, Theorem 1 p. 126]. Upon changing the notation we can assume that

$$\tau = \operatorname*{w-\mathcal{G}-lim}_{n\to\infty} F^+(\lambda)G_n(S_\epsilon)[H, \operatorname{Op^w}(\chi^+)]u_\lambda. \tag{6.6}$$

Step II. We show that this τ works. We compute using (5.5) in the third step, and Propositions 3.1 and 4.1 in the last step, and taking $m=-3$,

$$\begin{aligned} f^{1/2}u^-_\tau(\cdot,\lambda) &= 2\pi\mathrm{i} f^{1/2}F^+(\lambda)^*\tau \\ &= 2\pi\mathrm{i}\,\mathrm{w}^\star\text{–}\mathrm{B}^*\text{–}\lim_{n\to\infty} f^{1/2}F^+(\lambda)^*F^+(\lambda)G_n(S_\epsilon)[H,\mathrm{Op^w}(\chi^+)]u_\lambda \\ &= \mathrm{w}^\star\text{–}\mathrm{B}^*\text{–}\lim_{n\to\infty} f^{1/2}(R(\lambda-\mathrm{i}0)-R(\lambda+\mathrm{i}0))G_n(S_\epsilon)[\mathrm{Op^w}(\chi^+),H-\lambda]u_\lambda \\ &= \mathrm{w}^\star\text{–}L^2_m\text{–}\lim_{n\to\infty} f^{1/2}R(\lambda-\mathrm{i}0)G_n(S_\epsilon)[H-\lambda,\mathrm{Op^w}(\chi^-)]u_\lambda \\ &\quad + \mathrm{w}^\star\text{–}L^2_m\text{–}\lim_{n\to\infty} f^{1/2}R(\lambda+\mathrm{i}0)G_n(S_\epsilon)[H-\lambda,\mathrm{Op^w}(\chi^+)]u_\lambda \\ &= w^-+w^+;\ w^\mp = f^{1/2}R(\lambda\mp\mathrm{i}0)(H-\lambda)\,\mathrm{Op^w}(\chi^\mp)u_\lambda. \end{aligned}$$

Using again Proposition 4.1 we compute

$$\begin{aligned} w^\mp &= \mathrm{w}^\star\text{–}L^2_m\text{–}\lim_{\epsilon\searrow 0} f^{1/2}R(\lambda\mp\mathrm{i}\epsilon)(H-\lambda)\,\mathrm{Op^w}(\chi^\mp)u_\lambda \\ &= f^{1/2}\,\mathrm{Op^w}(\chi^\mp)u_\lambda \mp \mathrm{w}^\star\text{–}L^2_m\text{–}\lim_{\epsilon\searrow 0}\mathrm{i}\epsilon f^{1/2}R(\lambda\mp\mathrm{i}\epsilon)\,\mathrm{Op^w}(\chi^\mp)u_\lambda \\ &= f^{1/2}\,\mathrm{Op^w}(\chi^\mp)u_\lambda. \end{aligned}$$

Whence

$$u^-_\tau(\cdot,\lambda) = f^{-1/2}(w^-+w^+) = \Big(\mathrm{Op^w}(\chi^-)+\mathrm{Op^w}(\chi^+)\Big)u_\lambda = u_\lambda.$$

□

We summarize

Theorem 6.3. *Suppose Condition 2.1 (and* (2.2a)*). There exist* $\epsilon_0>0$ *and* $l=l(\mu,d)\in\mathbb{N}$ *such that for all* ϵ*-small perturbations* W_ϵ *with* $\epsilon\in(0,\epsilon_0]$ *(assuming also* (2.2b)*) the following statements hold for all* $\lambda\geq 0$*:*

For all $\tau\in\mathcal{G}$ *there exist unique* $\tilde\tau\in\mathcal{G}$ *and* $u_\lambda\in\mathcal{E}_\lambda$ *such that (with* $x=\Phi(t,\sigma)$*)*

$$u_\lambda(x)-\pi^{1/2}(K_\epsilon^{-1/2}m_\epsilon^{-1/2})(x)(\mathrm{e}^{\mathrm{i}S_\epsilon(x)}\tau(\sigma)-\mathrm{e}^{-\mathrm{i}S_\epsilon(x)}\tilde\tau(\sigma))\in f_\lambda^{-1/2}B_0^*. \tag{6.7a}$$

Moreover for all $u_\lambda\in\mathcal{E}_\lambda$ *there exist unique* $\tau,\tilde\tau\in\mathcal{G}$ *such that* (6.7a) *holds. In particular the map* $\mathcal{G}\ni\tau\to u_\lambda\in\mathcal{E}_\lambda$ *is a linear isomorphism. It is bi-continuous, in fact*

$$2\pi\|\tau\|^2_{\mathcal{G}} = \lim_{S\to\infty} S^{-1}\int_{\mathcal{B}_\epsilon(S)}|K_\epsilon^{1/2}u_\lambda|^2\,\mathrm{d}x. \tag{6.7b}$$

There are formulas

$$u_\lambda = u^-_\tau(\cdot,\lambda) = 2\pi\mathrm{i}F^+(\lambda)^*\tau \text{ and } \tilde\tau = S(\lambda)^{-1}\tau. \tag{6.7c}$$

In particular the wave matrix $F^+(\lambda)^*:\mathcal{G}\to\mathcal{E}_\lambda$ *is a bi-continuous linear isomorphism. The maps* $F^+(\lambda)f_\lambda^{1/2}:B\to\mathcal{G}$ *and* $\delta(\lambda)=\pi^{-1}\operatorname{Im}(R(\lambda+\mathrm{i}0)):f_\lambda^{1/2}B\to\mathcal{E}_\lambda$ *are onto.*

Proof. The uniqueness of $\tilde\tau$ and u_λ in (6.7a) follows from the proof of Lemma 3.3, and the existence part (in agreement with (6.7c)) follows from (5.19a). The mapping properties mentioned in the last sentence of the theorem are consequences of Banach's closed range theorem [Yo, Theorem p. 205] and previous statements. The remaining parts of the latter are consequences of (5.19b) and Proposition 6.2. □

6.1. **Concluding remarks.** With some more effort one should be able to show that the operator $F^+(\lambda)^*$ has a somewhat regular kernel, formally given by $(F^+(\lambda)^*\delta_\sigma)(x)$. More precisely one should have

$$(F^+(\lambda)^*\tau)(x) = \int_{\mathcal{G}} \phi^+(x,\sigma,\lambda)\tau(\sigma)\,\mathrm{d}\sigma,$$

where the plane wave type eigenfunction ϕ^+ has a degree of regularity. In particular it should be continuous in all variables for $x \notin \operatorname{supp} V_2$ provided the perturbation $W_\epsilon \in \mathcal{W}$ is ϵ-small with $\epsilon > 0$ taken small enough. More regularity in $\sigma \in \mathcal{G}$ would require ϵ taken smaller. These assertions depend on possible generalizations of Proposition 4.2, cf. [HS2]. We shall not elaborate further on this issue. Note also that smoothness in the angular variable of analogous plane wave type eigenfunctions was indeed obtained in [DS1].

Another remark concerns the relationship between the scattering theory developed here and [DS1] in case of overlapping conditions (which means under the conditions of [DS1]). In the case of a spherically symmetric potential we have for all $\lambda \geq 0$ that

$$\sigma = \eta(\sigma) := \lim_{s\to\infty} \Phi(s,\sigma)/|\Phi(s,\sigma)|,$$

and the two involved solutions to the eikonal equation are identical up to a trivial explicit term. In particular the two families of S-matrices are explicitly connected as follows: For all $\lambda \geq 0$ the operator $S(\lambda)$ of this paper and the scattering matrix of [DS1], say $S_{\mathrm{DS}}(\lambda)$, are up to an explicit phase factor related as $S(\lambda) = S_{\mathrm{DS}}(\lambda)R$ where $(R\tau)(\omega) = \tau(-\omega)$, cf. the discussion at the beginning of Section 1.

More generally under the conditions of [DS1] the asymptotic normalized velocity $\eta(\sigma)$ exists for all $\lambda \geq 0$ and as a map it is a diffeomorphism on S^{d-1}. This property, Theorem 6.3 and [DS1, Theorem 8.2] yields the connection formula

$$S_{\mathrm{DS}}(\lambda)^{-1} = R\mathrm{e}^{-\mathrm{i}\phi(\cdot,\lambda)} D_\eta S(\lambda)^{-1} D_{\eta^{-1}} \mathrm{e}^{-\mathrm{i}\phi(\cdot,\lambda)}, \tag{6.8}$$

where $\phi(\omega,\lambda)$ is real and for any diffeomorphism ψ on S^{d-1} the operator D_ψ is the unitary map on $L^2(S^{d-1})$ implemented by the classical map $S^{d-1} \ni \omega \to \psi(\omega) \in S^{d-1}$, viz. $(D_\psi\tau)(\omega) = J^{1/2}(\omega)\tau(\psi^{-1}(\omega))$.

Although we shall not elaborate the formula (6.8) suggests a criterion for regularity at zero energy of a family of (inverse) scattering matrices under the conditions of Section 2: It suffices that the families of diffeomorphisms $\eta = \eta_\lambda$ and η_λ^{-1} on S^{d-1} as well as the family of phases $\phi(\cdot,\lambda)$ are regular at zero energy. Indeed in this case the right hand side of (6.8) has a limit as $\lambda \to 0$ due to Proposition 5.4. This criterion is of course not applicable for the example in Subsection 2.1. Note that under the conditions of Section 2 the form of the right hand side of (6.8) makes sense for positive energies and the expression coincides with the (inverse) scattering matrix discussed in the beginning of Section 1.

References

[ACH] S. Agmon, J. Cruz, I. Herbst: *Generalized Fourier transform for Schrödinger operators with potentials of order zero*, J. Funct. Anal. **167** (1999), 345–369.

[AH] S. Agmon, L. Hörmander: *Asymptotic properties of solutions of differential equations with simple characteristics*, J. d'Analyse Math. **30** (1976), 1–38.

[CS] J. Cruz, E. Skibsted, *Global solutions to the eikonal equation*, Preprint 2011.

[DS1] J. Dereziński, E. Skibsted, *Quantum scattering at low energies*, J. Funct. Anal., **257** (2009), 1828–1920.

[DS2] J. Dereziński, E. Skibsted, *Scattering at zero energy for attractive homogeneous potentials*, Ann. H. Poincaré **10** (2009), 549–571.

[Ev] L.C. Evans, *Partial differential equations*, Graduate Studies in Mathematics **19**, Providence, AMS 1998.

[Fr] R. Frank, *A note on low energy scattering for homogeneous long range potentials*, Ann. H. Poincaré **10** (2009), 573–575.

[FS] S. Fournais, E. Skibsted, *Zero energy asymptotics of the resolvent for a class of slowly decaying potentials*, Math. Z. **248** (2004), 593–633.

[GY] Y. Gatel, D. Yafaev, *On the solutions of the Schrödinger equation with radiation conditions at infinity: the long-range case*, Ann. Inst. Fourier, Grenoble **49** no. 5 (1999), 1581–1602.

[HS1] I. Herbst, E. Skibsted: *Time-dependent approach to radiation conditions*, Duke Math. J. **64** no. 1 (1991), 119–147.

[HS2] I. Herbst, E. Skibsted: *Free channel Fourier transform in the long-range N-body problem*, J. D'Anal. Math. **65** (1995), 297–332.

[Hö] L. Hörmander, *The analysis of linear partial differential operators. II-IV*, Berlin, Springer 1983–85.

[Me] R. Melrose, *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces*, Spectral and scattering theory (Sanda, 1992) (M. Ikawa, ed.), Marcel Dekker (1994), 85–130.

[RS] M. Reed and B. Simon, *Methods of modern mathematical physics I-IV*, New York, Academic Press 1972-78.

[Sa1] Y. Saitō, *Spectral representations for Schrödinger operators with a long-range potentials*, Lecture Notes in Matematics **727**, Berlin, Springer 1979.

[Sa2] Y. Saitō, *Schrödinger operators with a nonspherical radiation condition*, Pacific J. Math. **126** no. 2 (1987), 331–359.

[Sk] E. Skibsted: *Sommerfeld radiation condition at threshold*, Aarhus Preprint Series No. 2 June 2011.

[SW] E. Skibsted, X. P. Wang: *Two-body threshold spectral analysis, the critical case*, J. Funct. Anal. **260** (2011), 1766-1794.

[Va] A. Vasy, *Propagation of singularities in three-body scattering*, Astérique, **262** (2000).

[Ya] D. Yafaev, *The low energy scattering for slowly decreasing potentials*, Comm. Math. Phys. **85** no. 2 (1982), 177–196.

[Yo] K. Yosida, *Functional Analysis*, Springer, Berlin, 1965.

(E. Skibsted) Department of Mathematics, Aarhus Universitet, Ny Munkegade 8000 Aarhus C, Denmark

E-mail address: skibsted@imf.au.dk