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METRICAL RESULTS ON SYSTEMS OF SMALL LINEAR FORMS

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M. HUSSAIN AND S. KRISTENSEN

ABSTRACT. In this paper the metric theory of Diophantine approximation associated with the small linear forms is investigated. Khintchine–Groshev theorems are established along with Hausdorff measure generalization without the monotonic assumption on the approximating function.

1. INTRODUCTION

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a function tending to 0 at infinity referred to as an *approximation* function. An $m \times n$ matrix $X = (x_{ij}) \in \mathbb{I}^{mn} := [0,1]^{mn}$ is said to be ψ–*approximable* if the system of inequalities

$$|q_1 x_{1i} + q_2 x_{2i} + \cdots + q_m x_{mi}| \leq \psi(|\mathbf{q}|) \quad \text{for} \quad (1 \leq i \leq n),$$

is satisfied for infinitely many $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$. Here and throughout, the system $q_1 x_{1i} + \cdots + q_m x_{mi}$ of n linear forms in m variables will be written more concisely as $\mathbf{q}X$, where the matrix X is regarded as a point in $\mathbb{I}^{mn}$ and $|\mathbf{q}|$ denotes the supremum norm of the integer vector $\mathbf{q}$. The set of ψ–approximable points in $\mathbb{I}^{mn}$ will be denoted by $W_0(m,n;\psi)$;

$$W_0(m,n;\psi) := \{X \in \mathbb{I}^{mn} : |\mathbf{q}X| < \psi(|\mathbf{q}|) \text{ for i.m. } \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}\},$$

where 'i.m.' means 'infinitely many'. For a monotonic approximating function, the metric theory has been established for the set $W_0(m,n;\psi)$ in [17] (the dimension of this set was obtained in [10]) and it's generalization to mixed case in [8]. The aim of this paper is to discuss the metric theory for the set $W_0(m,n;\psi)$ without the monotonicity assumption on the approximating function.

It is worth relating the above to the set of ψ–approximable matrices as is often studied in classical Diophantine approximation. In such a setting studying the metric structure of the lim sup-set

$$W(m,n;\psi) = \{X \in \mathbb{I}^{mn} : \|\mathbf{q}X\| < \psi(|\mathbf{q}|) \text{ for i.m. } \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}\},$$

where $\|x\|$ denotes the distance of x to the nearest integer vector, is a central problem and the theory is well established, see for example [1, 2, 6, 12, 22]. In the case that the approximating function is monotonic, the main result in this setting is the Khintchine-Groshev theorem which gives an elegant answer to the question of the size of the set $W(m,n;\psi)$. The result links the measure of the set to the convergence or otherwise of a series that depends only on the approximating function and is the template for many results in the field of metric number theory. The following is an

improved modern version of this fundamental result – see [2] and references within. Given a set X, $| X |_k$ denotes k-dimensional Lebesgue measure of X.

Theorem(Khintchine-Groshev). *Let ψ be an approximating function. Then*

$$|W(m,n;\psi)|_{mn} = \begin{cases} 0 & if \quad \sum_{r=1}^{\infty} r^{m-1}\psi(r)^n < \infty, \\ \\ 1 & if \quad \sum_{r=1}^{\infty} r^{m-1}\psi(r)^n = \infty \ \ and \ \psi \ is \ monotonic. \end{cases}$$

The convergence part is reasonably straightforward to establish from the Borel–Cantelli Lemma and is free from any assumption on ψ. The divergence part constitutes the main substance of the Khintchine–Groshev theorem and involves the monotonicity assumption on the approximating function. It is worth mentioning that in the original statement of the theorem [15, 19, 20] the stronger hypothesis that $q^m\psi(q)^n$ is monotonic was assumed. In the one-dimensional case ($m = n = 1$), it is well known that the monotonicity hypothesis in the Khintchine-Groshev theorem is absolutely crucial. Indeed, Duffin and Schaeffer [11] constructed a non-monotonic function for which $\sum_{q=1}^{\infty} \psi(q)$ diverges but $|W(1,1;\psi)| = 0$. In other words the Khintchine-Groshev theorem is false without the monotonicity hypothesis and the conjectures of Catlin [7] and Duffin and Schaeffer [11] provide appropriate alternative statements, see [1] for the details and generalizations of Duffin-Schaeffer and Catlin conjectures to the linear forms. Beyond the one-dimensional case the monotonicity assumption on the approximating function is completely removed. The proof is attributed to various authors for different values of m. For $m = 1$, Khintchine–Groshev theorem without the monotonicity of ψ was proved by Gallagher [14]. For $m = 2$, this was recently proved by Beresenevich and Velani in [4]. For $m \geq 3$ it can be derived from Schmidt [21, Theorem 2] or Sprindžuk's [22, §1.5, Theorem 15].

It is readily verified that $W_0(1,n;\psi) = \{\mathbf{0}\}$ as any $x = (x_1, x_2, \ldots, x_n) \in W_0(1,n;\psi)$ must satisfy the inequality $|qx_j| < \psi(q)$ infinitely often. As $\psi(q) \to 0$ as $q \to \infty$ this is only possible if $x_j = 0$ for all $j = 1, 2, \ldots, n$. Thus when $m = 1$ the set $W_0(1,n;\psi)$ is a singleton and must have both zero measure and dimension. We will therefore assume that $m \geq 2$.

Notation. To simplify notation the Vinogradov symbols $\ll$ and $\gg$ will be used to indicate an inequality with an unspecified positive multiplicative constant depending only on m and n. If $a \ll b$ and $a \gg b$ we write $a \asymp b$, and say that the quantities a and b are comparable. A *dimension function* is an increasing continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(r) \to 0$ as $r \to 0$. Throughout the paper, $\mathcal{H}^f$ denotes the f–dimensional Hausdorff measure which will be fully defined in section 3.1. Finally, for convenience, for a given approximating function ψ, define the function

$$\Psi(r) := \frac{\psi(r)}{r}.$$

2. STATEMENT OF THE RESULTS

The main results below depend critically on assumptions on m and n. In order to get beyond the Duffin–Schaeffer counterexample (see below), we will always assume that $m + n > 3$. However, an additional phenomenon occurs when the number of forms is greater than or equal to to the number of variables ($m \leq n$), and we

will have to treat each case separately. Our first result concerns the case when the number of variables exceeds the number of forms.

Theorem 2.1. *Let $m > n$, $m + n > 3$ and ψ be an approximating function. Let f, $r^{-n^2} f(r)$ and $r^{-(m-n-1)n} f(r)$ be dimension functions such that $r^{-mn} f(r)$ is monotonic. Then*

$$\mathcal{H}^f\left(W_0\left(m,n;\psi\right)\right) = \begin{cases} 0 & \text{if} \quad \sum\limits_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-(m-1)n} r^{m-1} < \infty, \\[4ex] \mathcal{H}^f(\mathbb{I}^{mn}) & \text{if} \quad \sum\limits_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-(m-1)n} r^{m-1} = \infty. \end{cases}$$

As in most of the statements the convergence part is reasonably straightforward to establish and is free from any assumptions on m, n and the approximating function. This fact was already established in [17, Theorem 4]. It is the divergence statement which constitutes the main substance and this is where conditions come into play.

The requirement that $r^{-mn} f(r)$ be monotonic is a natural and not particularly restrictive condition. Note that if the dimension function f is such that $r^{-mn} f(r) \to \infty$ as $r \to 0$ then $\mathcal{H}^f(\mathbb{I}^{mn}) = \infty$ and Theorem 2.1 is the analogue of the classical result of Jarník (see [9, 18]). In the case when $f(r) := r^{mn}$ the Hausdorff measure $\mathcal{H}^f$ is proportional to the standard mn–Lebesgue measure supported on $\mathbb{I}^{mn}$ and the result is the natural analogue of the Khintchine–Groshev theorem for $W_0\left(m, n; \psi\right)$.

Corollary 2.2. *Let $m > n$ and $m + n > 3$. Let ψ be an approximating function, then*

$$\left|W_0\left(m,n;\psi\right)\right|_{mn} = \begin{cases} 0 & \text{if} \quad \sum\limits_{r=1}^{\infty} \psi(r)^n r^{m-n-1} < \infty, \\[4ex] 1 & \text{if} \quad \sum\limits_{r=1}^{\infty} \psi(r)^n r^{m-n-1} = \infty. \end{cases}$$

In the results above, the condition $m + n > 3$ is absolutely necessary. For $m = 1$ the set $W_0(1, n; \psi)$ is singleton as already remarked. For $m = 2, n = 1$, the Duffin–Schaeffer counter example can be exploited to show that there exists a function ψ such that

$$\sum_{r=1}^{\infty} \psi(r) = \infty \quad \text{but} \quad |W_0(2, 1; \psi)|_2 = 0.$$

Indeed, the Duffin–Schaeffer counter example provides us with a function ψ, such that the set

$$\mathcal{DS} = \{y \in \mathbb{R} : |qy - p| < \psi(q) \text{ for infinitely many } p, q \in \mathbb{Z}\}$$

is Lebesgue null, while the sum $\sum \psi(r) = \infty$. Using this function as a ψ in the definition of $W_0(2, 1; \psi)$ and assuming the measure of the latter set to be positive, using the ideas below in the proof of Theorem 2.1, this will imply that $\mathcal{DS}$ has positive Lebesgue measure.

For $m \leq n$ the conditions on the dimension function in Theorem 2.1 change. This change is due to the fact that if $X \in W_0(m, n; \psi)$ and $m \leq n$ then a linear system of equations given by X is over-determined and the set of solutions lies in a subset of strictly lower dimension than mn. Hence, the corresponding set of ψ-approximable systems of forms will concentrate on a lower dimensional surface. This is proved in [17] where it is shown that for $m \leq n$ the set $W_0\left(m, n; \psi\right)$ lie on

a $(m-1)(n+1)$–dimensional hypersurface Γ. Therefore, naturally, we expect the analogue of Theorem 2.1 holds on Γ.

Theorem 2.3. *Let* $2 < m \leq n$ *and* ψ *be an approximating function. Let* f, $r^{-n^2} f(r)$, $r^{-(m-n-1)n} f(r)$ *and* $r^{-(n-m+1)(m-1)} f(r)$ *be dimension functions such that* $r^{-(m-1)(n+1)} f(r)$ *is monotonic. Then*

$$\mathcal{H}^f(W_0(m,n;\psi)) = 0 \qquad if \quad \sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-(m-1)n} r^{m-1} < \infty.$$

On the other hand, if

$$\sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-(m-1)n} r^{m-1} = \infty,$$

then

$$\mathcal{H}^f(W_0(m,n;\psi)) = \begin{cases} \infty & if \quad r^{-(m-1)(n+1)} f(r) \to \infty \ as \ r \to 0, \\ \\ K & if \quad r^{-(m-1)(n+1)} f(r) \to C \ as \ r \to 0, \end{cases}$$

for some fixed constant $0 \leq C < \infty$, *where* $0 < K < \infty$.

Note that for a dimension function f which satisfies $r^{-(m-1)(n+1)} f(r) \to C > 0$ as $r \to 0$ the measure $\mathcal{H}^f$ is comparable to standard $(m-1)(n+1)$-dimensional Lebesgue measure and in the case when $f(r) = r^{(m-1)(n+1)}$, we obtain the following analogue of the Khintchine-Groshev theorem.

Corollary 2.4. *Suppose* $2 < m \leq n$ *and assume that the conditions of Theorem 2.3 hold for the dimension function* $f(r) := r^{(m-1)(n+1)}$. *Then*

$$|W_0(m,n;\psi)|_{(m-1)(n+1)} = \begin{cases} 0 & if \quad \sum_{r=1}^{\infty} \psi(r)^{m-1} < \infty, \\ \\ K & if \quad \sum_{r=1}^{\infty} \psi(r)^{m-1} = \infty, \end{cases}$$

where $0 < K < \infty$.

3. MACHINERY

The machinery required for the proofs of both the theorems is the Mass Transference Principle along with 'slicing' technique. We merely state the results and refer the reader to [3] for further details.

3.1. **Hausdorff Measure and Dimension.** Below is a brief introduction to Hausdorff f–measure and dimension. For further details see [5, 12]. Let $F \subset \mathbb{R}^n$. For any $\rho > 0$ a countable collection $\{B_i\}$ of balls in $\mathbb{R}^n$ with diameters $\operatorname{diam}(B_i) \leq \rho$ such that $F \subset \bigcup_i B_i$ is called a ρ–cover of F. Define

$$\mathcal{H}_\rho^f(F) = \inf \sum_i f(\operatorname{diam}(B_i)),$$

where the infimum is taken over all possible ρ–covers of F. The Hausdorff f–measure of F is

$$\mathcal{H}^f(F) = \lim_{\rho \to 0} \mathcal{H}_\rho^f(F).$$

In the particular case when $f(r) = r^s$ with $s > 0$, we write $\mathcal{H}^s$ for $\mathcal{H}^f$ and the measure is referred to as s–dimensional Hausdorff measure. The Hausdorff dimension of F is denoted by $\dim F$ and is defined as

$$\dim F := \inf\{s \in \mathbb{R}^+ : \mathcal{H}^s(F) = 0\}.$$

3.2. **Slicing.** We now state a result which is the key ingredient in the proof of Theorems 2.1 and 2.3. The result was used in [3] to prove the Hausdorff measure version of the W. M. Schmidt's inhomogeneous linear forms theorem in metric number theory. The authors refer to the technique as 'slicing'. Before we state the result it is necessary to introduce a little notation.

Suppose that V is a linear subspace of $\mathbb{R}^k$, $V^\perp$ will be used to denote the linear subspace of $\mathbb{R}^k$ orthogonal to V. Further $V + a := \{v + a : v \in V\}$ for $a \in V^\perp$.

Lemma 3.1. *Let $l, k \in \mathbb{N}$ be such that $l \leq k$ and let f and $g : r \to r^{-l}f(r)$ be dimension functions. Let $B \subset \mathbb{I}^k$ be a Borel set and let V be a $(k-l)$–dimensional linear subspace of $\mathbb{R}^k$. If for a subset S of $V^\perp$ of positive $\mathcal{H}^l$ measure*

$$\mathcal{H}^g\left(B \cap (V + b)\right) = \infty \quad \forall\, b \in S,$$

then $\mathcal{H}^f(B) = \infty$.

3.3. **A Hausdorff measure version of Khintchine–Groshev theorem.** As an application of the mass transference principle for system of linear forms developed in [3] the Hausdorff measure version of the Khintchine–Groshev theorem is established without the monotonic assumption on the approximating function in [1, Theorem 15]. The additional assumption that ψ is monotonic was assumed in [1] for the case $m = 2$, but subsequently removed in [4].

Theorem 3.2. *Let ψ be an approximating function and $m + n > 2$. Let f and $r^{-(m-1)n}f(r)$ be dimension functions such that $r^{-mn}f(r)$ is monotonic. Then,*

$$\mathcal{H}^f\left(W\left(m, n; \psi\right)\right) = \begin{cases} 0 & \text{if} \quad \sum\limits_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-(m-1)n}r^{m+n-1} < \infty \\[4mm] \mathcal{H}^f(\mathbb{I}^{mn}) & \text{if} \quad \sum\limits_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-(m-1)n}r^{m+n-1} = \infty\,. \end{cases}$$

Theorem 3.2 along with Lemma 3.1 will be used to prove the infinite measure case of Theorem 2.1.

4. Proof of Theorem 2.1

As stated earlier the condition $r^{-mn}f(r)$ is not a restrictive condition. The statement of the Theorem essentially reduces to two cases, finite measure case, *i.e.*, when $r^{-mn}f(r) \to C > 0$ as $r \to 0$ and to the infinite measure case which corresponds to $r^{-mn}f(r) \to \infty$ as $r \to 0$. Therefore, we split the proof of the Theorem 2.1 into two parts, the finite measure case and the infinite measure case.

Before proceeding, we will need the following key lemma, which will make our proofs work.

Lemma 4.1. *Let $S \subseteq \mathrm{Mat}_{(m-n) \times n}(\mathbb{R})$ of full Lebesgue measure. Let $A \subseteq \mathrm{GL}_{n \times n}(\mathbb{R})$ be a set of positive Lebesgue measure. Then, the set*

$$\Lambda = \left\{ \begin{pmatrix} X \\ XY \end{pmatrix} \in \mathrm{Mat}_{m \times n}(\mathbb{R}) : X \in A, Y \in S \right\}$$

has full Lebesgue measure inside $A \times S$.

Proof. Without loss of generality, we will assume that $|A|_{n^2} < \infty$. If this is not the case, we will take a subset of A. Suppose now for a contradiction that $|(A \times S) \setminus \Lambda| > 0$ and let Z be a point of metric density for this set. We will show that the existence of such a point violates the condition that S is full.

Fix an $\epsilon > 0$. There is a $\delta > 0$ such that

$$\frac{|\Lambda \cap B(Z, \delta)|}{|B(Z, \delta)|} < \frac{\epsilon}{2^{mn+1}},$$

where $B(Z, \delta)$ denotes the ball centred at Z of radius δ. By definition of the Lebesgue measure, we may take a cover $\mathcal{C}$ of $\Lambda \cap B(Z, \delta)$ by hypercubes in $\mathbb{R}^{mn}$ such that

$$\sum_{C \in \mathcal{C}} \text{diam}(C)^{mn} < \frac{\epsilon}{2^{mn}} |B(Z, \delta)| = \epsilon \delta^{mn}.$$

The latter equality follows as we are working in the supremum norm, so that a ball of radius δ is in fact a hypercube of side length 2δ. We let $A_0 \subseteq A$ be the set of those $X \in A$ for which there is a $Y \in S$ such that $\begin{pmatrix} X \\ XY \end{pmatrix} \in B(Z, \delta)$. Note that by Fubini's Theorem A_0 has positive Lebesgue measure. In fact, the measure is equal to $2^{n^2} \delta^{n^2}$.

For any $X \in A_0$ we define the set

$$B(X) = \left\{ \begin{pmatrix} X \\ XY \end{pmatrix} \in B(Z, \delta) : Y \in S \right\}.$$

Note that

$$\mathcal{C}(X) = \left\{ \left(\begin{pmatrix} X \\ \text{Mat}_{(m-n) \times n}(\mathbb{R}) \end{pmatrix} \cap C \right) \in \text{Mat}_{m \times n}(\mathbb{R}) : C \in \mathcal{C} \right\}.$$

is a cover of $B(X)$ by $(m-n)n$-dimensional hypercubes.

As in [16], we define for each $C \in \mathcal{C}$ a function,

$$\lambda_C(X) = \begin{cases} 1 & \text{if } \left(\begin{pmatrix} X \\ \text{Mat}_{(m-n) \times n}(\mathbb{R}) \end{pmatrix} \cap C \right) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that

$$\int_{A_0} \lambda_C(X) dX \leq \text{diam}(C)^{n^2},$$

where the integral is with respect to the $n \times n$-dimensional Lebesgue measure. Also,

$$\sum_{C \in \mathcal{C}(X)} \text{diam}(C)^{(m-n)n} = \sum_{C \in \mathcal{C}} \lambda_C(X) \text{diam}(C)^{(m-n)n}.$$

We integrate the latter expression with respect to X to obtain

$$\int_{A_0} \sum_{C \in \mathcal{C}(X)} \text{diam}(C)^{(m-n)n} dX = \sum_{C \in \mathcal{C}} \int_{A_0} \lambda_C(X) dX \, \text{diam}(C)^{(m-n)n}$$

$$\leq \sum_{C \in \mathcal{C}} \text{diam}(C)^{mn} < \epsilon |B(Z, \delta)|.$$

Since the right hand side is an integral of a non-negative function over a set of positive measure, there must be an $X_0 \in A_0$ with

$$\sum_{C \in \mathcal{C}(X_0)} \operatorname{diam}(C)^{(m-n)n} < \frac{\epsilon |B(Z, \delta)|}{\mu(A_0)} = \frac{\epsilon \delta^{mn}}{2^{n^2} \delta^{n^2}} = \frac{\epsilon}{2^{n^2}} \delta^{n(m-n)}. \qquad (1)$$

Indeed, otherwise

$$\int_A \sum_{C \in \mathcal{C}(X)} \operatorname{diam}(C)^{(m-n)n} dX \geq \int_A \frac{\epsilon |B(Z, \delta)|}{\mu(A)} dX = \epsilon.$$

We may now estimate the $(m-n)n$-dimensional measure of $B(X_0)$ from above by this sum. This gives an upper estimate on the measure of $B(I_n)$, as X_0 is invertible. Furthermore, this estimate can be made arbitrarily small. But $B(I_n)$ is a cylinder set over S, so this is a clear contradiction since S was assumed to be full. $\qquad \square$

In applications, we will apply Lemma 4.1 with the set S being $W(m-n, n, ; \psi)$. This set is however a subset of $\mathbb{I}^{(m-n)n}$, and so not full within $\operatorname{Mat}_{(m-n) \times n}(\mathbb{R})$. It is however invariant under tranlation by integer vectors, so this causes no loss of generality.

4.1. **Finite measure.** In order to proceed, we will make some restrictions. Let $\epsilon > 0$ and $N > 0$ be fixed but arbitrary. It is to be understood that ϵ will be small eventually and N large. We will define a set $A_{\epsilon,N}$ of $m \times n$-matrices which is smaller than the whole, but which tends to the whole set as $\epsilon \to 0$. As ϵ and N are arbitrary, if we can prove that the divergence assumption implies that $W_0(m, n; \psi)$ is full inside $A_{\epsilon,N}$, this will give the full result.

For an $m \times n$-matrix X, let $\tilde{X}$ denote the $n \times n$-matrix formed by the first n rows. We will be considering a set for which $\tilde{X}$ is invertible. Evidently, the exceptional set is of measure zero within $\mathbb{R}^{mn}$. However, to make things work, we will need to work with the set

$$A_{\epsilon,N} = \left\{ X \in \operatorname{Mat}_{m \times n}(\mathbb{R}) : \epsilon < \det(\tilde{X}) < \epsilon^{-1}, \quad \max_{1 \leq i,j \leq n} |x_{ij}| \leq N \right\}$$

The set is of positive measure for ϵ small enough and N large enough, and as ϵ decreases and N increases, the set fills up $\operatorname{Mat}_{m \times n}(\mathbb{R})$ with the exception of the null-set of matrices X such that $\tilde{X}$ is singular.

We will translate the statement about small linear forms to one about usual Diophantine approximation. This will allow us to conclude from a Khintchine–Groshev theorem. We may rewrite the X as

$$X = \begin{pmatrix} \tilde{X} \\ X' \end{pmatrix} = \begin{pmatrix} I_n \\ \hat{X} \end{pmatrix} \tilde{X},$$

where X' denotes the matrix consisting of the last $m - n$ rows of the original matrix and $\hat{X}$ denotes the matrix $X'\tilde{X}^{-1}$.

Consider the set of n linear forms in $m - n$ variables defined by the matrix $\hat{X}$. Suppose furthermore that these linear forms satisfy the inequalities

$$\|\mathbf{r}\hat{X}\|_i \leq \frac{\psi(|\mathbf{r}|)}{nN}, \quad 1 \leq i \leq n, \qquad (2)$$

for infinitely many $\mathbf{r} \in \mathbb{Z}^{m-n} \setminus \{\mathbf{0}\}$, where $\|\mathbf{x}\|_i$ denotes the distance from the i'th coordinate of $\mathbf{x}$ to the nearest integer. A special case of Khintchine–Groshev states,

that the divergence condition of our theorem implies that the set of such linear forms $\hat{X}$ is full inside the set of $(m - n) \times n$-matrices, and hence in particular also in the image of $A_{\epsilon,N}$ under the map sending X to $\hat{X}$.

Now, suppose that $X \in A_{\epsilon,N}$ is such that $\hat{X}$ is in the set defined by (2). We claim that X is in $W_0(m, n : \psi)$. Indeed, let $\mathbf{r}_k$ be an infinite sequence such that the inequalities (2) are satisfied for each k, and let $\mathbf{p}_k$ be the nearest integer vector to $\mathbf{r}_k\hat{X}$. Now define $\mathbf{q}_k = (\mathbf{p}_k, \mathbf{r}_k)$. The inequalities defining $W_0(m, n, \psi)$ will be satisfied for these values of $\mathbf{q}_k$, since

$$|\mathbf{q}_k X| = \left|\mathbf{q}_k \begin{pmatrix} I_n \\ \hat{X} \end{pmatrix} \tilde{X}\right| = |(\pm\|r_k\hat{X}\|_1, \dots, \pm\|r_k\hat{X}\|_n)\tilde{X}|.$$

The i'th coordinate of the first vector is at most $\psi(|q|)/nN$, so carrying out the matrix multiplication, using the triangle inequality and the fact that $|x_{ij}| \leq N$ for $1 \leq i, j \leq n$ shows that

$$|\mathbf{q}_k X|_i < \psi(|\mathbf{q}_k|).$$

Applying Lemma 4.1, the divergence part of Theorem 2.1 follows in the case of Lebesgue measure.

4.2. **Infinite measure.** The infinite measure case of the Theorem 2.1 can be easily deduced from the following lemma.

Lemma 4.2. *Let ψ be an approximating function and let f and $g : r \to r^{-n^2}f(r)$ be dimension functions with $r^{-mn}f(r) \to \infty$ as $r \to 0$. Further, let $r^{-(m-n-1)n}g(r)$ be a dimension function and $r^{-(m-n)n}g(r)$ be monotonic. If*

$$\sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-(m-1)n}r^{m-1} = \infty,$$

then

$$\mathcal{H}^f(W_0(m, n; \psi)) = \infty.$$

Proof. We define a Lipschitz map to transform our problem to a classical one. As in the finite measure case, we fix $\epsilon > 0$, $N \geq 1$ and let

$$A_{\epsilon,N} = \left\{X \in \mathrm{Mat}_{m \times n}(\mathbb{R}) : \epsilon < \det(\tilde{X}) < \epsilon^{-1}, \quad \max_{1 \leq i,j \leq n} |x_{ij}| \leq N\right\},$$

where $\tilde{X}$ denotes the $n \times n$-matrix formed by the first n rows. We also define the set

$$\tilde{A}_{\epsilon,N} = \left\{\tilde{X} \in \mathrm{GL}_n(\mathbb{R}) : \epsilon < \det(\tilde{X}) < \epsilon^{-1}, \quad \max_{1 \leq i,j \leq n} |x_{ij}| \leq N\right\}.$$

For an appropriately chosen constant $c > 0$ depending only on m, n, ϵ and N, we find that the map

$$\eta : W(m - n, n, c\psi) \times \tilde{A}_{\epsilon,N} \to W_0(m, n, \psi), \quad (Y, X) \mapsto \begin{pmatrix} X \\ YX \end{pmatrix}, \qquad (3)$$

is a Lipschitz embedding. Indeed, it is evidently injective as X is invertible for all elements of the domain. The Lipschitz condition follows as we have restricted the determinant to being positive. That the image is in $W_0(m, n, \psi)$ follows by considering the system of inequalities as above and choosing $c > 0$ accordingly as

above in the finite measure case. Consequently, the map η is bi-Lipschitz onto its image. We have,

$$\mathcal{H}^f(W_0(m, n; \psi)) \geq \mathcal{H}^f\left(\eta\left(W(m - n, n; \psi) \times \tilde{A}_{\epsilon,N}\right)\right)$$
$$\asymp \mathcal{H}^f\left(W(m - n, n; \psi) \times \tilde{A}_{\epsilon,N}\right).$$

The main idea of the proof is now to apply slicing, refer to Lemma 3.1, to a tranlate of the Borel set $B := W(m - n, n; \psi) \times \tilde{A}_{\epsilon,N} \subseteq \mathbb{I}^{mn}$. Initially, we fix an arbitrary point $X_0 \in \tilde{A}_{e,N}$. Let $\sigma : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ be the translation map sending X_0 to the origin, *i.e.*, $\sigma(X) = X - X_0$. Let $\tilde{\sigma} : \mathbb{R}^{mn} \to \mathbb{R}^{mn}$ be the map which leaves the upper $(m - n) \times n$ matrix untouched but applies σ to the lower $n \times n$ matrix. We apply these maps to all sets above. This leaves Hausdorff measure invariant, so by abuse of notation we will denote the translated sets by the same letters as the original ones.

Let V be the space

$$\mathbb{I}^{(m-n)n} \times \{0\}^{n^2}.$$

Let

$$S = V^{\perp} := \{0\}^{(m-n)n} \times \mathbb{I}^{n^2}$$

and further it has positive $\mathcal{H}^{n^2}$-measure. Now for each $b \in S$

$$\mathcal{H}^g(B \cap (V + b)) = \mathcal{H}^g\left(\left(W(m - n, n; \psi) \times \tilde{A}_{\epsilon,N}\right) \cap (V + b)\right)$$
$$= \mathcal{H}^g\left(\left(W(m - n, n; \psi) \times \{0\}^{n^2}\right) + b\right)$$
$$\asymp \mathcal{H}^g(W(m - n, n; \psi))$$
$$= \infty \text{ if } \sum_{r=1}^{\infty} g(\Psi(r))\Psi(r)^{-(m-n-1)n}r^{m-1} = \infty.$$

The slicing lemma yields that

$$\mathcal{H}^f\left(W(m - n, n; \psi) \times \tilde{A}_{\epsilon,N}\right) = \infty \text{ if } \sum_{r=1}^{\infty} g(\Psi(r))\Psi(r)^{-(m-n-1)n}r^{m-1} = \infty.$$

Since, $g : r \to r^{-n^2}f(r)$, we have

$$\mathcal{H}^f(W_0(m, n; \psi)) = \infty \text{ if } \sum_{r=1}^{\infty} f(\Psi(r))\Psi(r)^{-(m-1)n}r^{m-1} = \infty.$$

$\square$

5. Proof of Theorem 2.3

The method of proof of Theorem 2.3 is similar to Theorem 2 of [17] which relies mainly on Theorem 2.1 and the slicing technique. To be brief, one first shows that for $m \leq n$, the set $W_0(m, n; \psi)$ must be contained in a hypersurface of dimension at most $(m - 1)(n + 1) < mn$. This follows by proving that if $X \in W_0(m, n; \psi)$, then the columns of X must be linearly dependent.

This required linear dependence can be removed from the problem by introducing another bi-Lipschitz map to a non-degenerate setting. The above methods applied for the case $m > n$ can then be applied to the non-degenerate setting in both the case of finite and infinite measure. The details are essentially the same as those in [17], and are left to the interested reader.

6. Concluding remarks

In this paper we have made no effort to remove monotonic assumption on the approximating function to prove the analogue of Khintchine–Groshev theorem for the absolute value setup. However there are still some open territories not investigated in this paper. To conclude the paper we discuss them here.

Theorem 2.1 provides a beautiful 'zero–full' criterion under certain divergent sum conditions but on the other hand Theorem 2.3 provides 'zero–positive' criterion. The later theorem relies on taking the linear combinations of the independent vectors from the former but the combinations does not span the full space. It is natural to conjecture that a 'zero–full' law for Theorem 2.3 does indeed hold.

In the current paper settings the approximating function ψ is dependent on the supremum norm of the integer vector $\mathbf{q}$. Clearly, a natural generalization is to consider multivariable approximating function, $\Psi : \mathbb{Z}^m \to \mathbb{R}^+$ and their associated set $W_0(m, n; \Psi)$.

Another natural generalisation is the case of different rates of approximation for each coordinate, *i.e.*, when we consider inequalities

$$|q_1 x_{1i} + q_2 x_{2i} + \cdots + q_m x_{mi}| \leq \psi_i(|\mathbf{q}|) \text{ for } (1 \leq i \leq n),$$

where the ψ_i are potentially different error functions. In this case, it is shown in [13] that the analogue Corollary 2.2 holds with $\psi(r)^n$ in the series replaced by $\psi_1(r) \cdots \psi_n(r)$.

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