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Abstract

The density function of the gamma distribution is used as shift kernel in Brownian semistationary processes modelling the timewise behaviour of the velocity in turbulent regimes. This report presents exact and asymptotic properties of the second order structure function under such a model, and relates these to results of von Karmann and Horwath. But first it is shown that the gamma kernel is interpretable as a Green's function.

1 Introduction

As a model for the timewise development of the velocity, measured at a fixed point in space and under homogeneous physical conditions, the stationary stochastic process

$$Y_t = \mu + \int_{-\infty}^{t} g(t-s)\sigma_s \mathrm{d}B_s + \int_{-\infty}^{t} q(t-s)\sigma_s^2 \mathrm{d}s \tag{1.1}$$

has been extensively studied probabilistically and compared to empirical and simulated data, see [BNSch07], [BNSch08], cf. also [BNSch04], [BNESch05] and [BNBlSch04]. Here B is Brownian motion, g and q are shift kernels and σ is a positive stationary process representing the volatility or intermittency of the turbulent field.

The specification of g in the gamma form

$$g(x) = \Gamma(\nu)^{-1} x^{\nu-1} e^{-\lambda x}$$

is of some particular interest, because of its simplicity and because it allows explicit analytic calculations.

Tempo-spatial extensions of this model, where the gamma kernel is again of central use, are considered in [BNSch07]. In view of the physical setting and the closeness with which this model type corresponds to reality it seems natural to enquire whether the gamma kernel has an interpretation as a Green's function. Such an interpretation is provided in Section 2, with reference to the Riemann-Liouville and Caputo concepts of fractional differentiation.

Of special interest in the context of turbulence are the cases where $\frac{1}{2} < \nu < 1$ and $1 < \nu < \frac{3}{2}$. The dynamics of the process (1.1) is significantly different in the two cases, cf. Section 3. Focussing on the main component of (1.1), i.e. taking

$$Y_t = \Gamma(\nu)^{-1} \int_{-\infty}^{t} (t-s)^{\nu-1} e^{-\lambda(t-s)} \sigma_s \mathrm{d}B_s, \tag{1.2}$$

and assuming for simplicity that σ and B are independent, we note that the restriction $\nu > \frac{1}{2}$ is needed for the stochastic integral to be well defined and that (1.2) constitutes a semimartingale only if $\nu = 1$ or $\nu > \frac{3}{2}$. These aspects and some of their consequences are discussed in [BNSch09], [BNCP11a], [BNCP11b] and [BNSch12].

The specification (1.1) is a particular null-spatial case of the concept of tempo-spatial ambit fields, which are random fields of the form

$$\begin{aligned} Y_t(x) = \mu &+ \int_{A_t(x)} g(s,\xi;t,x)\sigma_s(\xi) L(\mathrm{d}\xi, \mathrm{d}s) \\ &+ \int_{D_t(x)} q(s,\xi;t,x) a_s(\xi) \mathrm{d}\xi \mathrm{d}s. \end{aligned} \tag{1.3}$$

where $A_t(x)$, and $D_t(x)$, termed *ambit sets*, are deterministic subsets of $\mathbb{R} \times \mathbb{R}^k$, g and q are deterministic functions, $\sigma \geq 0$ is a stochastic field referred to as the *intermittency* or *volatility*, and L is a *Lévy basis*, i.e. an independently scattered, infinitely divisible and homogeneous random measure, and a is another random field, often traken as σ^2. Multivariate and stationary versions of such ambit fields are straightforward to formulate, yielding in particular a flexible modelling framework for the three-dimensional velocity vector field in homogeneous isotropoic turbulence.

The representation of the gamma kernel as a Green's function is given in Section 2, and Section 3 presents an overview of the properties of the second order structure function

$$S_2(t) = \mathrm{E}\left\{(Y_{s+t} - Y_s)^2\right\}$$

when Y is the stationary process given by (1.2). In particular, the relation to the von Karman [vKar48] specification of S_2 is discussed. Various requisite background material on the Bessel functions K_ν and I_ν is provided in Section 4.

2 Gamma kernel as Green's function

For any $\gamma \in (0,1)$ and $n \in \mathbb{N}$ the Caputo fractional derivative $D^{n,\gamma}$ is, in its basic form, defined by

$$D^{n,\gamma} f(x) = \Gamma(1-\gamma)^{-1} \int_a^x (x-\xi)^{-\gamma} f^{(n)}(\xi) \, \mathrm{d}\xi$$

where f denotes any function on the interval $[a, \infty)$ which is n times differentiable there and such that $f^{(n)}$ is absolutely continuous on $[a, \infty)$. This concept was introduced by [Caputo67] and has since been much generalised and extensively applied in a great variety of scientific and technical areas. For a comprehensive exposition of this and other concepts of fractional differentiation, see [KilSriTru06], cf. also [LiQianChen11], [AdCre05] and [Mai10].

Now, for functions f on $\mathbb{R}$ let M_λ with $\lambda \geq 0$ be the operator $M_\lambda f(x) = e^{\lambda x} f(x)$ and, for $0 < \gamma < 1$ and $a \in \mathbb{R}$, define the operator $H_\lambda^{n,\gamma}$ by

$$H_\lambda^{n,\gamma} f(x) = M_\lambda^{-1} D D^{n,\gamma} M_\lambda f(x)$$

where D indicates ordinary differentiation and $D^{n,\gamma}$ is the Caputo fractional derivative.[1]

Now, suppose that $1 < \nu < \frac{3}{2}$ and consider the equation

$$H_\lambda^{1,\nu-1} f(x) = \phi(x) \tag{2.1}$$

where ϕ is assumed known. We seek the solution f to this equation, stipulating that $f(a)$ should be equal to 0, and it turns out to be

$$f(x) = \Gamma(\nu)^{-1} \int_a^x (x-\xi)^{\nu-1} e^{-\lambda(x-\xi)} \phi(\xi) \mathrm{d}\xi. \tag{2.2}$$

In other words,

$$g(x) = g(x; \nu, \lambda) = \Gamma(\nu)^{-1} x^{\nu-1} e^{-\lambda x} \tag{2.3}$$

is the Green's function corresponding to the operator $H_\lambda^{1,\nu-1}$ when $1 < \nu < \frac{3}{2}$.

The verification is by direct calculation. With f given by (2.2) we find

$$(M_\lambda f)(x) = \Gamma(\nu)^{-1} \int_a^x (x-\xi)^{\nu-1} e^{\lambda\xi} \phi(\xi) \mathrm{d}\xi$$

so

$$(M_\lambda f)'(x) = \Gamma(\nu-1)^{-1} \int_a^x (x-\xi)^{\nu-2} e^{\lambda\xi} \phi(\xi) \mathrm{d}\xi$$

and hence, for any $\gamma \in (0,1)$,

$$\begin{aligned}
H_\lambda^{1,\gamma} f(x) &= \Gamma(\nu-1)^{-1} \Gamma(1-\gamma)^{-1} e^{-\lambda x} D \int_a^x (x-\xi)^{-\gamma} \int_a^\xi (\xi-\eta)^{\nu-2} e^{\lambda\eta} \phi(\eta) \mathrm{d}\eta \mathrm{d}\xi \\
&= \Gamma(\nu-1)^{-1} \Gamma(1-\gamma)^{-1} e^{-\lambda x} D \int_a^x e^{\lambda\eta} \phi(\eta) \int_\eta^x (x-\xi)^{-\gamma} (\xi-\eta)^{\nu-2} \mathrm{d}\xi \mathrm{d}\eta \\
&= \Gamma(\nu-1)^{-1} \Gamma(1-\gamma)^{-1} e^{-\lambda x} D \int_a^x (x-\eta)^{-\gamma+\nu-1} e^{\lambda\eta} \phi(\eta) \mathrm{d}\eta \\
&\quad \times \int_0^1 (1-w)^{(1-\gamma)-1} w^{(\nu-1)-1} \mathrm{d}w \\
&= \Gamma(\nu-\gamma)^{-1} e^{-\lambda x} D \int_a^x (x-\eta)^{-\gamma+\nu-1} e^{\lambda\eta} \phi(\eta) \mathrm{d}\eta.
\end{aligned}$$

[1] The differentiation term $DD^{n,\gamma}$ may be viewed as a special case of the more general definition

$$D^{m,n,\gamma} = D^m D^{n,\gamma}$$

where m, like n, is a nonnegative integer and $0 < \gamma < 1$. Then $D^{m,0,\gamma}$ equals the Riemann-Liouville fractional derivative while $D^{0,n,\gamma}$ is the Caputo fractional derivative.

Consequently, for $\gamma = \nu - 1$ we have

$$H_\lambda^{1,\nu-1} f(x) = e^{-\lambda x} D \int_a^x e^{\lambda\eta}\phi(\eta)\mathrm{d}\eta = \phi(x).$$

On the other hand, in case $\nu \in \left(\frac{1}{2}, 1\right)$ the relevant equation is

$$H_\lambda^{0,\nu} f(x) = \phi(x),$$

and the solution is again of the form (2.2). In fact,

$$\begin{aligned}
H_\lambda^{0,\gamma} f(x) &= \Gamma(\nu)^{-1}\Gamma(1-\gamma)^{-1} e^{-\lambda x} D \int_a^x (x-\xi)^{-\gamma} \int_a^\xi (\xi-\eta)^{\nu-1} e^{\lambda\eta}\phi(\eta)\mathrm{d}\eta\mathrm{d}\xi \\
&= \Gamma(\nu)^{-1}\Gamma(1-\gamma)^{-1} e^{-\lambda x} D \int_a^x e^{\lambda\eta}\phi(\eta)\mathrm{d}\eta \int_\eta^x (x-\xi)^{-\gamma}(\xi-\eta)^{\nu-1}\mathrm{d}\xi \\
&= \Gamma(\nu)^{-1}\Gamma(1-\gamma)^{-1} e^{-\lambda x} D \int_a^x (x-\eta)^{-\gamma+\nu} e^{\lambda\eta}\phi(\eta)\mathrm{d}\eta \\
&\quad\times \int_0^1 (1-w)^{-\gamma} w^{\nu-1}\mathrm{d}\xi \\
&= \Gamma(\nu)^{-1}\Gamma(1-\gamma)^{-1} e^{-\lambda x} D \int_a^x (x-\eta)^{-\gamma+\nu} e^{\lambda\eta}\phi(\eta)\mathrm{d}\eta \\
&\quad\times \int_0^1 (1-w)^{(1-\gamma)-1} w^{\nu-1}\mathrm{d}\xi \\
&= \Gamma(1-\gamma+\nu)^{-1} e^{-\lambda x} D \int_a^x (x-\eta)^{-\gamma+\nu} e^{\lambda\eta}\phi(\eta)\mathrm{d}\eta
\end{aligned}$$

and with $\gamma = \nu$ we have

$$H_\lambda^{0,\nu} f(x) = \phi\,(x)\,.$$

Thus, in both cases, $\frac{1}{2} < \nu < 1$ and $1 < \nu < \frac{3}{2}$, the gamma kernel (2.3) occurs as the Green's function. In the former case the differential operator is of Riemann-Liouville type and in the latter of Caputo type.

Note. Introducing the operator I_λ^ν by

$$I_\lambda^\nu \phi = M_\lambda^{-1} D^{0,\nu-1} M_\lambda \phi,$$

we may reexpress formula (2.2) as

$$f = I_\lambda^\nu \phi$$

and the calculation above shows that

$$H_\lambda^{1,\nu-1} I_\lambda^\nu = I \tag{2.4}$$

where I denotes the identity operator. Thus $H_\lambda^{1,\nu-1}$ is the left inverse of I_λ^ν.

The operator I_λ^ν also has a right inverse (where, again, $1 < \nu < \frac{3}{2}$). To determine that, let

$$J_\lambda^{1,\gamma} = M_\lambda^{-1} D^{1,\gamma} M_\lambda$$

(where $\gamma \in (0,1)$). Then

$$\begin{aligned}
I_\lambda^\nu J_\lambda^{1,\gamma} f(x) &= M_\lambda^{-1} D^{0,\nu-1} M_\lambda \Gamma(\gamma)^{-1} e^{-\lambda x} \int_a^x (x-\xi)^{-\gamma} \left(e^{\lambda\xi} f(\xi)\right)' \mathrm{d}\xi \\
&= \Gamma(\gamma)^{-1} e^{-\lambda x} D^{0,\nu-1} \int_a^x (x-\xi)^{-\gamma} \left(e^{\lambda\xi} f(\xi)\right)' \mathrm{d}\xi \\
&= \frac{1}{\Gamma(\gamma)\Gamma(\nu-1)} e^{-\lambda x} \int_a^x (x-\xi)^{1-\nu} \int_a^\xi (\xi-\eta)^{-\gamma} \left(e^{\lambda\eta} f(\eta)\right)' \mathrm{d}\eta \mathrm{d}\xi \\
&= \frac{1}{\Gamma(\gamma)\Gamma(\nu-1)} e^{-\lambda x} \int_a^x \left(e^{\lambda\eta} f(\eta)\right)' \mathrm{d}\eta \int_\eta^x (x-\xi)^{1-\nu} (\xi-\eta)^{-\gamma} \mathrm{d}\xi \\
&= \frac{1}{\Gamma(\gamma)\Gamma(\nu-1)} e^{-\lambda x} \int_a^x (x-\eta)^{2-\nu-\gamma} \left(e^{\lambda\eta} f(\eta)\right)' \mathrm{d}\eta \\
&\quad \times \int_0^1 (1-w)^{2-\nu-1} w^{1-\gamma-1} \mathrm{d}w \\
&= \frac{B(2-\nu, 1-\gamma)}{\Gamma(\gamma)\Gamma(1-\nu)} e^{-\lambda x} \int_a^x (x-\eta)^{2-\nu-\gamma} \left(e^{\lambda\eta} f(\eta)\right)' \mathrm{d}\eta.
\end{aligned}$$

So, for $\gamma = 2-\nu$ we have

$$I_\lambda^\nu J_\lambda^{1,\gamma} f(x) = f(x),$$

i.e. $J_\lambda^{1,2-\nu}$ is the right inverse of I_λ^ν. □

Remark. In view of the link to fractional differentiation established in the present Section it is pertinent briefly to refer to the broad range of studies of the relevance of (multi)fractional calculus to turbulence modelling existing in the literature. Some links between that literature and the ambit modelling approach considered in the present note are given in [BNESch05].

Another related line of study is that of space-time fractional diffusion equations and the possibility of interpreting the associated Green's functions as probability densities, see [MaiLuPag01] and [MaiPagGo02].

3 Second order structure function

The null-spatial case of the general stationary ambit field specification has the form

$$Y_t = \mu + \int_{-\infty}^t g(t-s)\sigma_s \mathrm{d}B_s + \int_{-\infty}^t q(t-s) a_s \mathrm{d}s. \tag{3.1}$$

The variance function S_2 of the increments of Y, i.e.

$$S_2(u) = \mathrm{E}\left\{(Y_{t+u} - Y_t)^2\right\},$$

is in the turbulence literature called the second order structure function.

Considering, for simplicity, only the main stochastic term of (3.1), that is letting

$$Y_t = \int_{-\infty}^t g(t-s)\sigma_s \mathrm{d}B_s, \tag{3.2}$$

we have that

$$S_2(u) = 2\mathrm{E}\left\{\sigma_0^2\right\} ||g||^2 \bar{r}(u)$$

where

$$||g||^2 = \int_0^\infty g^2(s)\,\mathrm{d}s$$

and where

$$\bar{r}(u) = 1 - r(u)$$

with $r(u)$ denoting the autocorrelation function of the process (3.2).

We shall discuss the structure of r and $\bar{r}$ under the gamma assumption (2.3) with $\nu > \frac{1}{2}$.

3.1 Bessel function representation of S_2

The autocorrelation function r of the process (3.2) is expressible in terms of the type K Bessel functions (and hence the same is the case for the second order structure function corresponding to the process (3.1)).

For the gamma choice (2.3) of g we find

$$\int_0^\infty g^2(s)\,\mathrm{d}s = \Gamma(2\nu-1)\,2^{-2\nu+1}\lambda^{-2\nu+1} \tag{3.3}$$

and, with $u > 0$,

$$\int_0^\infty g(u+s)\,g(s)\,\mathrm{d}s = \Phi_\nu(u;\lambda)$$

where we have introduced the notation

$$\Phi_\nu(u;\lambda) = e^{-\lambda u}\int_0^\infty (u+s)^{\nu-1}\,s^{\nu-1}e^{-2\lambda s}\mathrm{d}s. \tag{3.4}$$

By [3.383.3][2], $\Phi_\nu(u;\lambda)$ may, for any $\nu > 0$, be reexpressed as

$$\Phi_\nu(u;\lambda) = \sqrt{\frac{2}{\pi}}\Gamma(\nu)\,2^{-\nu}\lambda^{-\nu+\frac{1}{2}}u^{\nu-\frac{1}{2}}K_{\nu-\frac{1}{2}}(\lambda u) \tag{3.5}$$

where, as usual, K_ν is the notation for a modified Bessel function of the third kind. Further, defining

$$\bar{K}_\nu(x) = x^\nu K_\nu(x) \tag{3.6}$$

we can write $\Phi_\nu(u;\lambda)$ as

$$\Phi_\nu(u;\lambda) = \frac{\Gamma(\nu)}{\Gamma\left(\frac{1}{2}\right)}2^{-\nu+\frac{1}{2}}\lambda^{-2\nu+1}\bar{K}_{\nu-\frac{1}{2}}(\lambda u).$$

It follows, using (3.3), that

$$r(u) = \frac{\Gamma(\nu)}{\Gamma\left(\frac{1}{2}\right)\Gamma(2\nu-1)}2^{-\nu+\frac{1}{2}}\bar{K}_{\nu-\frac{1}{2}}(\lambda u)$$

[2] Here and in the following code numbers in square brackets refer to formulae given in [GrRy96].

whence, by the use of the doubling formula [8.335.1]

$$\Gamma(2\nu - 1) = 2^{2\nu-2} \frac{\Gamma(\nu)\,\Gamma\left(\nu - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}, \tag{3.7}$$

we find that

$$r(u) = 2^{-\nu+\frac{3}{2}} \Gamma\left(\nu - \frac{1}{2}\right)^{-1} \bar{K}_{\nu-\frac{1}{2}}(\lambda u). \tag{3.8}$$

3.2 Relation to von Karman specification

In a paper from 1948 [vKar48] von Karmann discussed the behaviour of the double correlation functions in three dimensional homogeneous and isotropic turbulence. These functions are defined by

$$\phi(r) = \frac{\overline{u(x_1, x_2, x_3)\, u(x_1 + r, x_2, x_3)}}{\overline{u^2}} \tag{3.9}$$

and

$$\psi(r) = \frac{\overline{u(x_1, x_2, x_3)\, u(x_1, x_2 + r, x_3)}}{\overline{u^2}} \tag{3.10}$$

where u denotes the main component of the three-dimensional velocity vector (i.e. the component in the mean wind direction). Due to the continuity equation for incompressible fluids the functions f and g are related by

$$\psi(r) = \phi(r) + \tfrac{r}{2}\phi'(r), \tag{3.11}$$

see [vKarHorw38], cf. also Section 6.2.1 of [Frisch95].

Von Karmann set up a series of physically based assumptions concerning this type of turbulence and supplementing these assumptions with some speculative reasoning he arrived at the following proposal for the functional form of ϕ

$$\phi(\kappa r) = \frac{2^{2/3}}{\Gamma(1/3)} (\kappa r)^{1/3} K_{1/3}(\kappa r) \tag{3.12}$$

where κ is a length scale parameter. A main point in von Karmann's argument was that the spectral density corresponding to this functional form interpolates smoothly between behaving as a fourth power near the origin and decaying at exponential rate $-5/3$ for large frequencies. (Both of these traits correspond to well documented empirical behaviour, and the $5/3$ rate is the spectral counterpart to Kolmogorov's $2/3$ law.) In the same paper von Karmann compared this form, or rather that of ψ, to data obtained at California Institute of Technology and found a fair agreement between the observations and ψ, as determined from (3.11).

Formula (3.12) is a special case, obtained for $\nu = \frac{5}{6}$ and $\lambda = \kappa$, of the general form of autocorrelation function (3.8)[3], which of course has been derived from a completely different set of assumptions; in particular, von Karmann's derivation is not based on any specified probability structure or model. (The general form (3.8),

[3] This connection to von Karmann's approach has been noted independently by Emil Hedevang.

obtained by a Fourier inversion, was proposed as correlation function by [Tatar61] (Russian Edition 1959). That form is also known as the Whittle-Matérn correlation function, see [GuGnei05].) Note that the von Karmann-Tatarski specification refers to spatial correlations whereas that of (3.8) concerns timewise correlation. However, the Taylor Frozen Field hypothesis provides a direct physical link between the two results.

In the paper [HedSch12] the authors, in extension of von Karmann's approach, show that a convolution of two gamma kernels allow to fit observed spectra not only over the dissipation and inertial ranges, as in the case (3.12), but also over the energy domain.

If ϕ has the form (3.8) then ψ as determined from (3.11) is given by

$$\psi\left(r\right) = 2^{-\nu+\frac{3}{2}}\Gamma\left(\nu-\tfrac{1}{2}\right)^{-1}\left(\bar{K}_{\nu-\frac{1}{2}}\left(r\right)+\tfrac{1}{2}r\bar{K}'_{\nu-\frac{1}{2}}(r)\right).$$

Now it follow from elementary properties of the Bessel functions K (formulae (4.1), (4.2), (4.3) and (4.4) of the Appendix) that we have the simple relation

$$\bar{K}'_{\nu}(x) = -x\bar{K}_{\nu-1}(x) \tag{3.13}$$

and hence that

$$\begin{aligned}\psi\left(r\right) &= 2^{-\nu+\frac{3}{2}}\Gamma\left(\nu-\tfrac{1}{2}\right)^{-1}\left(\bar{K}_{\nu-\frac{1}{2}}\left(r\right)-\tfrac{1}{2}r^{2}\bar{K}_{\nu-\frac{3}{2}}(r)\right)\\ &= 2^{-\nu+\frac{3}{2}}\Gamma\left(\nu-\tfrac{1}{2}\right)^{-1}\left(\bar{K}_{\nu-\frac{1}{2}}\left(r\right)-\tfrac{1}{2}r^{\nu+\frac{1}{2}}K_{\nu-\frac{3}{2}}(r)\right)\\ &= 2^{-\nu+\frac{3}{2}}\Gamma\left(\nu-\tfrac{1}{2}\right)^{-1}\left(\bar{K}_{\nu-\frac{1}{2}}\left(r\right)-\tfrac{1}{2}r^{\nu+\frac{1}{2}}K_{\frac{3}{2}-\nu}(r)\right)\end{aligned}$$

i.e.

$$\psi(r) = 2^{-\nu+\frac{3}{2}}\Gamma\left(\nu-\tfrac{1}{2}\right)^{-1}\left(\bar{K}_{\nu-\frac{1}{2}}\left(r\right)-\tfrac{1}{2}r^{2\nu-1}\bar{K}_{\frac{3}{2}-\nu}(r)\right). \tag{3.14}$$

3.3 Asymptotic behaviour

In determining the asymptotic behaviour of the autocorrelation function (3.8) we may, without loss of generality assume that $\lambda = 1$.

For $u \to \infty$, $r\left(u\right)$ behaves as

$$r\left(u\right) \sim \frac{\sqrt{\pi}}{\Gamma\left(\nu-\frac{1}{2}\right)}2^{-(\nu-1)}u^{\nu-1}e^{-u}$$

which follows from the well known formula [8.451.6] $K_{\nu}\left(x\right) \sim \sqrt{\pi}2^{-1}x^{-1}e^{-x}$ as $x \to \infty$.

On the other hand, as the lag time u tends to 0 we have $r\left(u\right) \to 1$, while for the complementary autocorrelation function

$$\bar{r}(u) = 1 - r(u) = 1 - 2^{-\nu+\frac{3}{2}}\Gamma\left(\nu-\tfrac{1}{2}\right)^{-1}\bar{K}_{\nu-\frac{1}{2}}\left(u\right)$$

we find[4]

[4] The results below complement and correct some calculations referred to in Section 5: "Examples" of [BNCP11a].

Theorem. *For $u \to 0$ the complementary autocorrelation function $\bar{r}$ behaves as*

$$\bar{r}(u) \sim \begin{cases} 2^{-2\nu}\frac{\Gamma\left(\frac{3}{2}-\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)}u^{2\nu-1} + O(u^2) & \text{for } \frac{1}{2} < \nu < \frac{3}{2} \\ \frac{1}{2}u|\ln\frac{u}{2}| + O(u^3|\ln(u)|) & \text{for } \nu = \frac{3}{2} \\ \frac{1}{2}\frac{\Gamma(\nu-\frac{5}{2})}{\Gamma\left(\nu-\frac{1}{2}\right)}u^2 + O(u^3|\ln(u)|) & \text{for } \frac{3}{2} < \nu. \end{cases}$$

Proof. In proving the conclusions in the Theorem we need to distinguish between the three cases $\frac{1}{2} < \nu < \frac{3}{2}$, $\nu = \frac{3}{2}$ and $\nu > \frac{3}{2}$.

To verify the Theorem in the former case we recall the well known expansion for I_ν [cf. [Wat66], formula (2), p. 77]

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{n!\Gamma(n+\nu+1)}. \tag{3.15}$$

From this and the recursion formula (4.8) we obtain

$$\begin{aligned} I_{-\nu}(x) &= (1-\nu)\,2^\nu x^{-\nu}\sum_{n=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2n}}{n!\Gamma(n-\nu+2)} + \left(\frac{x}{2}\right)^{2-\nu}\sum_{n=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2n}}{n!\Gamma(n-\nu+3)} \\ &= x^{-\nu}\left\{(1-\nu)\,2^\nu\sum_{n=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2n}}{n!\Gamma(n-\nu+2)} + 2^\nu\sum_{n=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2n+2}}{n!\Gamma(n-\nu+3)}\right\} \\ &= x^{-\nu}\left\{\frac{2^\nu}{\Gamma(1-\nu)} + 2^\nu\sum_{n=0}^{\infty}\frac{n+2}{(n+1)!!\Gamma(n-\nu+3)}\left(\frac{x}{2}\right)^{2n+2}\right\}. \end{aligned}$$

Together with (4.7) and (3.15) this shows that for $0 < \nu < 1$

$$\bar{K}_\nu(x) \sim 2^{\nu-1}\Gamma(\nu)\left\{1 - 2^{-2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}x^{2\nu} + O(x^2)\right\} \qquad \text{as } x \downarrow 0 \tag{3.16}$$

or, equivalently,

$$2^{-\nu+1}\Gamma(\nu)^{-1}\,\bar{K}_\nu(x) \sim 1 - 2^{-2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}x^{2\nu} + O(x^2) \qquad \text{as } x \downarrow 0. \tag{3.17}$$

The first the conclusion of the Theorem now follows immediately from this and formula (3.8).

The second conclusion, which refers to the case $\nu = \frac{3}{2}$, relies on [8.446] according to which

$$K_1(x) \sim x^{-1} + \tfrac{x}{2}\ln\tfrac{x}{2} + O(x^3|\ln x|) \qquad \text{as } x \downarrow 0.$$

More generally, for any integer $n = 2, 3, ...$ the same formula [8.446] implies that

$$\bar{K}_n(x) \sim 2^{n-1}\Gamma(n-1) - 2^{n-3}\Gamma(n-2)x^2 + O(x^3|\ln x|) \qquad \text{as } x \downarrow 0$$

that is, for $\nu = n + \frac{1}{2}$ and $u \downarrow 0$

$$r(u) = \frac{1}{2}\frac{\Gamma(\nu-\frac{5}{2})}{\Gamma\left(\nu-\frac{1}{2}\right)}x^2 + O(x^3|\ln x|),$$

in agreement with the Table. Finally, for $\nu > \frac{3}{2}$ and $\nu \neq n + \frac{1}{2}$ the result can be established along the lines of the derivation in the case $\frac{1}{2} < \nu < \frac{3}{2}$. □

With the notation form Section 3.2 we have that $\bar{r}(u) = \bar{\phi}(u)$ where $\bar{\phi}(u) = 1-\phi(u)$; thus the asymptotic behaviour of $\bar{\phi}(u)$ as $u \to 0$ is determined by the above Theorem. As regards the asymptotic properties of the complementary autocorrelation function $\bar{\psi}(u) = 1 - \psi(u)$ of the transversal velocities it follows immediately from the Theorem and formula (3.16) that, for $\frac{1}{2} < \nu < \frac{3}{2}$ and $u \to 0$, the leading terms of the expansions of $\bar{\phi}(u)$ and $\bar{\psi}(u)$ are both of order $u^{2\nu-1}$.

4 Appendix: Bessel functions K_ν and I_ν

This Section collects a number of, mostly well known, properties of the Bessel functions K_ν and I_ν. A main reference for those properties is [GrRy96] and, as mentioned earlier, number codes given in square brackets refer to that work.

Thoughout the following ν denotes a positive real number.

From the elementary exact properties of the Bessel functions K_ν [8.486.16; 8.486.10; 8.486.13]:

$$K_\nu(x) = K_{-\nu}(x) \tag{4.1}$$

$$K_{\nu+1}(x) = 2\nu x^{-1} K_\nu(x) + K_{\nu-1}(x) \tag{4.2}$$

$$K'_\nu(x) = -K_{\nu-1}(x) - \nu x^{-1} K_\nu(x) \tag{4.3}$$

it follows that, with $\bar{K}$ as given by (3.6),

$$\bar{K}_{\nu+1}(x) = 2\nu \bar{K}_\nu(x) + x^2 \bar{K}_{\nu-1}(x). \tag{4.4}$$

The Bessel functions K_ν and I_ν are, provided that ν is not an integer, connected by [8.485]:

$$K_\nu(x) = \frac{1}{2}\frac{\pi}{\sin(\pi\nu)}\left(I_{-\nu}(x) - I_\nu(x)\right) \tag{4.5}$$

which, due to the formula [8.334.3]:

$$\Gamma(1-\nu)\Gamma(\nu) = \frac{\pi}{\sin(\pi\nu)} \tag{4.6}$$

may be rewritten as

$$K_\nu(x) = \tfrac{1}{2}\Gamma(1-\nu)\Gamma(\nu)(I_{-\nu}(x) - I_\nu(x)). \tag{4.7}$$

Further, the Bessel functions I_ν satisfy the recursion relation [8.486.1]

$$xI_{\nu-1}(x) - xI_{\nu+1}(x) = 2\nu I_\nu(x) \tag{4.8}$$

and have the power series expansion, cf. [Wat66], formula (2), p. 77,

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{n!\Gamma(n+\nu+1)}. \tag{4.9}$$

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