# Collective Interference of Composite Two-Fermion Bosons

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(Received 17 September 2012; published 28 December 2012)

The composite character of two-fermion bosons manifests itself in the interference of many composites as a deviation from the ideal bosonic behavior. A state of many composite bosons can be represented as a superposition of different numbers of perfect bosons and fermions, which allows us to provide the full Hong–Ou–Mandel-like counting statistics of interfering composites. Our theory quantitatively relates the deviation from the ideal bosonic interference pattern to the entanglement of the fermions within a single composite boson.

DOI: 10.1103/PhysRevLett.109.260403 PACS numbers: 05.30.Fk, 03.65.Ud, 05.30.Jp

The quantum statistics of bosons is most apparent in correlation functions and counting statistics. Characteristic bosonic signatures are encountered for thermal states, which feature the Hanbury Brown–Twiss effect [1–4], as well as in meticulously prepared Fock-states [5–8], which exhibit Hong–Ou–Mandel-like (HOM) interference. Deviations from the ideal bosonic pattern in HOM setups are often caused by inaccuracies in the preparation of Fock-states and in the alignment of the setup, which induce partial distinguishability between the particles [5,9–11]. Another source for deviations from perfect bosonic behavior has received only little attention, limited to mixed states [12,13]: since most bosons are composites ("cobosons") made of an even number of fermions, reminiscences of underlying fermionic behavior are expected in many-coboson interference. In analogy to partially distinguishable particles [10,11], one can intuitively anticipate that the many-coboson wave function partially behaves in a fermionic way, with impact on the resulting counting statistics.

Here, we investigate such compositeness effects in HOM interferometry of cobosons. The ideal bosonic interference pattern is jeopardized by the Pauli principle that acts on the underlying fermions, an effect that becomes relevant when the constituents populate only a small set of single-fermion states. The effective number of single-fermion states can be related to the entanglement between the fermions, via the Schmidt decomposition. Not only does entanglement, thus, guarantee the irrelevance of the Pauli-principle for coboson states, but it also constitutes the very many-body coherence property that ensures that many-coboson interference matches the ideal bosonic pattern [7,8]. The many-coboson wave function can be described as a superposition of different numbers of perfect bosons and fermions, with weights that are determined by the Schmidt coefficients. Using that intuitive representation, we compute the exact counting statistics in many-coboson interference and provide direct experimental observables for compositeness. Properties of the collective wave function of the fermionic constituents can, thus, be extracted from coboson interference signals, while in the limit of truly many particles, particularly simple forms for the interference pattern emerge.

The bottom line of our discussion, the observable competition of fermions for single-particle states, is a rather general phenomenon that is not restricted to any particular physical system. To render our analysis of many-coboson interference tangible, however, we focus on an interferometric setup that can be realized with trapped ultracold atoms [14].

We consider strongly bound bi-fermion pairs that are trapped in a two-dimensional potential landscape with different horizontal and vertical coupling rates [15], as depicted in Fig. 1, which is described by the Hamiltonian

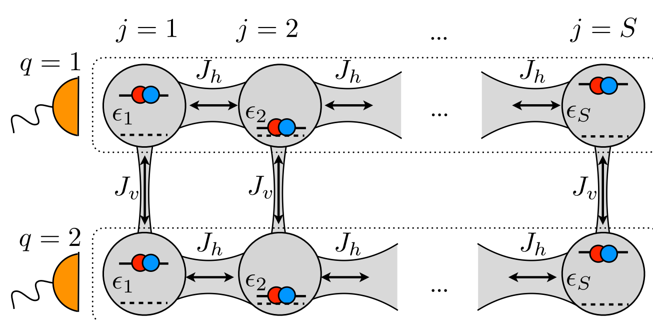


FIG. 1 (color online). Setup for the interference of engineered cobosons. $N_1$ ($N_2$) strongly bound bi-fermions are prepared in the upper (lower) lattice at $J_v \ll J_h$, such that each bi-fermion is governed by the local energies $\epsilon_j$ and the tunneling rate $J_h$. The barrier between the lattices is then ramped down, such that $J_v \gg J_h$ and vertical tunneling takes place. The total number of bi-fermions in the upper and lower lattice is then counted. The graphical representation is adapted from Ref. [15].

 

$$\hat{H} = -\frac{J_h}{2}\sum_{q=1}^{2}\sum_{j=1}^{S-1}\hat{d}^\dagger_{q,j}\hat{d}_{q,j+1} - \frac{J_v}{2}\sum_{j=1}^{S}\hat{d}^\dagger_{1,j}\hat{d}_{2,j} + \text{H.c.} + \sum_{q=1}^{2}\sum_{j=1}^{S}\epsilon_j\,\hat{d}^\dagger_{q,j}\hat{d}_{q,j}, \quad (1)$$

where $\hat{d}^\dagger_{q,j} = \hat{a}^\dagger_{q,j}\hat{b}^\dagger_{q,j}$ creates a bi-fermion consisting of an $a$- and a $b$-type fermion in the $j$th site of the upper or lower lattice ($q = 1, 2$); $J_h$ ($J_v$) is the effective tunneling strength along (between) the lattices, and $\epsilon_j$ defines a local energy landscape [14]. We assume that, initially, $J_h \gg J_v$, and multicoboson states are prepared in the horizontally extended lattice $q$ by [16,17]

$$\hat{c}^\dagger_q = \sum_{j=1}^{S}\sqrt{\lambda_j}\,\hat{d}^\dagger_{q,j} = \sum_{j=1}^{S}\sqrt{\lambda_j}\,\hat{a}^\dagger_{q,j}\hat{b}^\dagger_{q,j}. \quad (2)$$

A coboson is, thus, a horizontally delocalized bi-fermion, and the $S$ coefficients $\lambda_j$ are then the *Schmidt coefficients* of the two-fermion state.

The distribution $\vec{\lambda}$ is conveniently characterized by its power sums

$$M(m) = \sum_{j=1}^{S}\lambda_j^m, \quad (3)$$

where normalization implies $M(1) = 1$ and $M(2) = P$ is the purity of either reduced single-fermion state. We consider an initial state of $N_1$ cobosons in the upper and $N_2$ cobosons in the lower lattice [14],

$$|\Psi\rangle = \frac{(\hat{c}^\dagger_1)^{N_1}}{\sqrt{\chi_{N_1}N_1!}}\frac{(\hat{c}^\dagger_2)^{N_2}}{\sqrt{\chi_{N_2}N_2!}}|0\rangle, \quad (4)$$

where we assume $N_1 \geq N_2$, and $\chi_N$ is the coboson normalization factor [16–20], a symmetric polynomial [21] given by $\chi_N = \Omega(\underbrace{\{1,\ldots,1\}}_{N})$, with

$$\Omega(\{x_1,\ldots,x_N\}) = \sum_{\substack{p_1,\ldots,p_N \\ 1\leq p_j\leq S}}^{i\neq j\Rightarrow p_i\neq p_j}\prod_{q=1}^{N}\lambda_{p_q}^{x_q}. \quad (5)$$

To assess the behavior of the cobosons, we let the bi-fermions tunnel vertically between the two lattices by setting $J_v \gg J_h$ and letting the system evolve for a time of the order $1/J_v$. Thus, beam splitter-like dynamics couples the two lattices, while tunneling processes within the lattices, induced by $J_h$, can be neglected on this time scale. The Schmidt modes $j$ are, therefore, left unchanged. Time evolution until $t$ implements a beam splitter with reflectivity $R = \cos^2(tJ_v/2)$. In principle, the counting statistics of bi-fermions in the two lattices can be obtained by integrating the dynamics induced by Eq. (1) for the initial state $|\Psi\rangle$ given in Eq. (4) and taking the expectation values of the counting operators

$$\hat{A}_{n_1,n_2} = \sum_{\substack{j_1<j_2<\cdots<j_{n_1} \\ l_1<l_2<\cdots<l_{n_2}}}^{1\leq j_k,l_m\leq S}\prod_{k=1}^{n_1}\hat{d}^\dagger_{1,j_k}\hat{d}_{1,j_k}\prod_{m=1}^{n_2}\hat{d}^\dagger_{2,l_m}\hat{d}_{2,l_m}, \quad (6)$$

which witness the probability to find exactly $n_1$ ($n_2$) bi-fermions in the first (second) lattice. This procedure, however, is computationally expensive and does not offer an intuitive physical picture. By exploiting the symmetry properties of the state [Eq. (4)], one can show [14] that the behavior of cobosons is imitated exactly by a superposition of states with a different number of perfect bosons and fermions, in analogy to partially distinguishable particles [10,11]. When the distribution of the bi-fermions along the lattices is neglected, $|\Psi\rangle$ exhibits the same total counting statistics in the two lattices as the state

$$|\psi\rangle = \sum_{p=0}^{N_2}\sqrt{w_p}\,|\phi(p)\rangle, \quad \text{with} \quad (7)$$

$$|\phi(p)\rangle = \left[\prod_{q=1}^{2}\frac{(\hat{g}^\dagger_q)^{N_q-p}}{\sqrt{(N_q-p)!}}\right]\left[\prod_{j=1}^{p}\hat{f}^\dagger_{1,j}\hat{f}^\dagger_{2,j}\right]|0\rangle, \quad (8)$$

where $\hat{g}^\dagger_q$ ($\hat{f}^\dagger_{q,j}$) creates a boson ($j$-type fermion) in the lattice $q$. The weight of the component with $p$ pairs of fermionically behaving bi-fermions depends on the Schmidt coefficients and reads [14]

$$w_p = \binom{N_1}{p}\binom{N_2}{p}\frac{p!}{\chi_{N_1}\chi_{N_2}}\Omega(\{\underbrace{2,\ldots,2}_{p},\underbrace{1,\ldots,1}_{N_1+N_2-2p}\}). \quad (9)$$

Combinatorially speaking, $w_p$ is the probability that, given two groups of $N_1$ and $N_2$ objects with properties distributed according to $\vec{\lambda}$, and assuming that all objects in either group carry different properties, one finds $p$ pairs of objects with the same property when the two groups are merged. In the present context, $w_p$ denotes the population of the state components in which the Pauli principle affects $p$ pairs of bi-fermions. The term $|\phi(0)\rangle$, thus, describes perfect bosonic behavior, its weight $w_0 = \chi_{N_1+N_2}/(\chi_{N_1}\chi_{N_2})$ can be bound via the purity $P$ and the particle numbers $N_1$, $N_2$ [17,20]:

$$\frac{(L-N_1)!(L-N_2)!}{(L-N_1-N_2)!L!} \leq w_0 \leq \frac{(1-\sqrt{P})(1+\sqrt{P}(N_1+N_2-1))}{(1+\sqrt{P}(N_2-1))(1+\sqrt{P}(N_1-1))}, \quad (10)$$

where $L = \lceil\frac{1}{P}\rceil$.

We can now derive the counting statistics of cobosons after time evolution until $t = \pi/2/J_v$, which corresponds to a balanced beam splitter with $R = T = 1/2$. The probability $P_{\text{tot}}(m)$ to find $m$ cobosons in the upper lattice is the sum of the resulting probabilities from the different contributions in Eq. (7),

$$P_{\rm tot}(m) = \sum_{p=0}^{N_2} w_p P(m, p), \qquad (11)$$

where $P(m, p)$ is the probability to find $m$ particles of any species in the upper lattice, given the state $|\phi(p)\rangle$ defined in Eq. (8) and the beam splitter reflectivity $R = 1/2$ [14].

The simplest case is given by two interfering cobosons ($N_1 = N_2 = 1$), for which we find $w_0 = P$ and $w_1 = 1 - P$:

$$P_{\rm tot}(1) = P, \qquad P_{\rm tot}(0) = P_{\rm tot}(2) = \frac{1-P}{2}. \qquad (12)$$

For $P \to 1$, the Pauli principle dominates and one always finds one particle in each lattice. In contrast to the interference of unbound boson pairs that can break up dynamically [22], a perfect bosonic dip emerges here in the limit of vanishing purity, $P \to 0$.

Higher-order power sums $M(m)$ with $m \geq 3$ become relevant when more than two cobosons interfere. For example, the interference of $N_2 = 1$ with $N_1$ cobosons reflects the normalization ratio $\chi_{N+1}/\chi_N$ [16–18,23,24]:

$$P_{\rm tot}(m) = \frac{\chi_{N_1+1}}{\chi_{N_1}} P(m, 0) + \left(1 - \frac{\chi_{N_1+1}}{\chi_{N_1}}\right) P(m, 1). \qquad (13)$$

In general, the balance between all the weights $w_0, \ldots, w_{N_2}$ governs the counting statistics. Since the weights $w_p$ depend on power sums $M(m)$ up to order $N_1 + N_2$ [14], the characteristics of the distribution $\vec{\lambda}$ can be established through interference signals. For $N_1 = N_2 = 2$, we illustrate the decomposition [Eq. (7)] in Fig. 2.

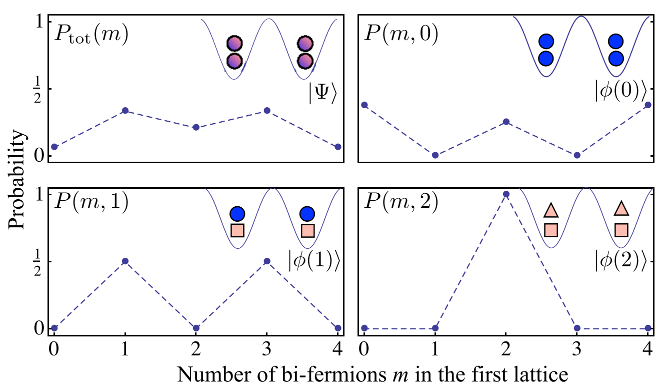


FIG. 2 (color online). Counting statistics for the coboson-state $|\Psi\rangle$ with $N_1 = N_2 = 2$, and of its components with different numbers of bosons and fermions $|\phi(p)\rangle$, $p = 0, 1, 2$. Dark blue circles represent bosonically behaving bi-fermions, light orange squares and triangles stand for fermionically behaving bi-fermions. The total counting statistics $P_{\rm tot}(m)$ is the weighted sum [Eq. (11)] over the different components of the wave function. While $|\phi(0)\rangle$ exhibits perfect bosonic behavior, $|\phi(p \geq 1)\rangle$ are partially fermionic, which leaves a signature in the counting statistics. Here, $R = 1/2$ and $\lambda_1 = \cdots = \lambda_4 = 1/4$, such that $w_0 = w_2 = 1/6$, $w_1 = 2/3$.

The ideal boson interference pattern $P(m, 0)$ is jeopardized by the finite purity $P = 1/4$, the contributions of the single fermion-pair and double fermion-pair part in the wave function lead to the altered signal $P_{\rm tot}(m)$.

Distributions with the same purity $P$ may have different higher-order power sums $M(m)$, with consequently distinct counting statistics. Keeping $P$ constant, the counting statistics is extremized by two particular distributions: the upper bound in Eq. (10) is saturated by peaked distribution $\vec{\lambda}^{(p)}$ with $\lambda_1^{(p)} > \lambda_2^{(p)} = \cdots = \lambda_S^{(p)}$, in the limit $S \to \infty$; the lower bound is saturated by the uniform distribution $\vec{\lambda}^{(u)}$ with $\lambda_1^{(u)} \leq \lambda_2^{(u)} = \cdots = \lambda_{L \equiv \lceil 1/P \rceil}^{(u)}$, for fractional purities $P = 1/L$ [20]. The counting statistics for $N_1 = N_2 = 6$ is shown in Fig. 3. The weights $w_k^{u(p)}$ of the uniform (peaked) distributions differ considerably (see lower panel), which is reflected by the counting statistics [upper panel, note that $P(m) = P(12 - m)$ due to symmetry]. Only one Schmidt coefficient in the peaked distribution is finite in the limit $S \to \infty$; thus, only the weights $w_0^{(p)}$ and $w_1^{(p)}$ are nonvanishing: the interference patterns of 12 and of 10 bosons take turns. Instead, all weights $w_{0 \leq j \leq 6}^{(u)}$ alternate for the uniform distribution. Kinks emerge at fractional values of $P$, when a new nonvanishing Schmidt coefficient emerges. For $P \to 1/6$, fully fermionic behavior is attained, and one always finds six cobosons in each lattice.

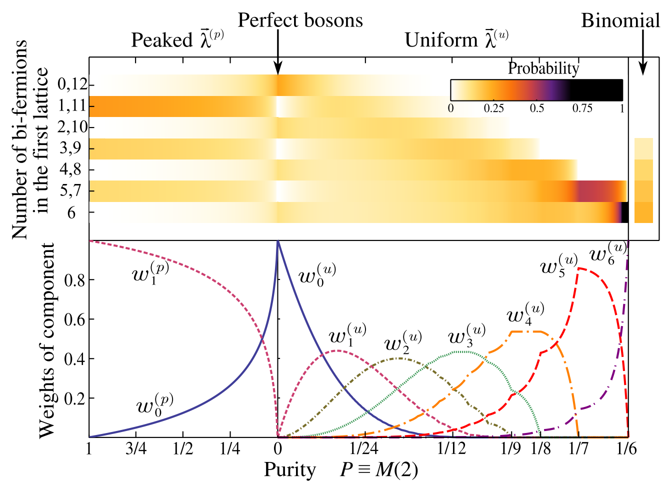


FIG. 3 (color online). Upper panel: counting statistics $P_{\rm tot}(m)$ as a function of the purity, for the uniform ($u$) (right-hand part) and peaked ($p$) (left-hand part) distributions $\vec{\lambda}^{(u/p)}$. Lower panel: corresponding weights $w^{(p/u)}$ of the coboson wave function given in Eq. (7). We set $N_1 = N_2 = 6$, $R = 1/2$. The counting statistics is perfectly bosonic for vanishing purity, $P \to 0$, while cobosons behave as fermions for the uniform distribution and $P = 1/6$. The number of nonvanishing Schmidt-coefficients in the uniform distribution is $L = \lceil 1/P \rceil$, hence the weights $w_l^{(u)}$ with $l < N - L - 1$ vanish: there are at least $N_1 - L - 1$ pairs of fermions, which results in the kinks in the weights. The binomial distribution corresponds to the statistics of distinguishable particles.

The dependence of $P_{tot}(k)$ on the power sums $M(m)$ can be used to infer the latter from measured counting statistics for different $N_1$, $N_2$. The purity $P$ follows immediately for $N_1 = N_2 = 1$ via Eq. (12); in general, $M(m)$ is inferred by the counting statistics of a total of $N_1 + N_2 = m$ cobosons. Since higher-order power sums are constrained by Jensen's and Hölder inequalities [25],

$$M(m-1)^{(m-1)/(m-2)} \leq M(m) \leq M(m-1)^{m/(m-1)}, \tag{14}$$

bounds for higher-order $M(m)$ become tighter with increasing knowledge of $M(m)$, as depicted in Fig. 4.

When the exact counting statistics cannot be retrieved and many ($N \gtrsim 1000$) cobosons are brought to interference, such as in the interference of Bose-Einstein condensates [26], the granular structure of the interference pattern becomes secondary. The impact of imperfect bosonic behavior can then be incorporated into a macroscopic wave function approach [7]; i.e., the number of particles is treated as the amplitude of a single-particle wave function. Fock-states are modeled by a random phase between the different components of the wave function. When the fractions $I_j = N_j/(N_1 + N_2)$ of ideal bosons are prepared in the two lattices, the particle fraction $I$ in the upper lattice after beam splitter dynamics obeys the probability distribution [7]

$$\mathcal{P}_{\text{MWF}}(I; I_1, I_2) = \frac{1}{\pi\sqrt{4RTI_1I_2 - (I - RI_1 - TI_2)^2}},$$

for $4RTI_1I_2 > (I - RI_1 - TI_2)^2$, while it vanishes otherwise. For cobosons, a finite fraction of fermions needs to be accounted for in each lattice. The probability distribution for the particle fraction $I$ then becomes

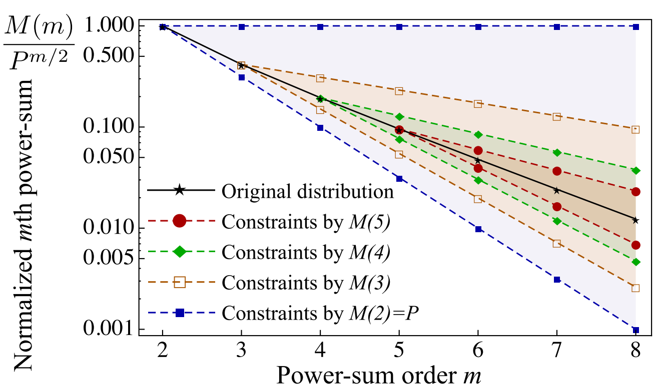


FIG. 4 (color online). Normalized power sums and constraints. The normalization to $P^{m/2}$ is chosen such that the upper bound is constant. A randomly chosen distribution $\vec{\lambda}$ leads to a certain hierarchy of power sums (black stars). The measurement of interference signals with $N_1$ and $N_2$ cobosons reveals the power sums up to order $N_1 + N_2$, which leads to the indicated constraints on higher-order $M(m)$ with $m \geq N_1 + N_2 + 1$ (blue filled squares, orange open squares, green filled diamonds, and red filled circles), according to Eq. (14).

$$\mathcal{P}(I) = \int_0^{I_2} dI_f \mathcal{W}(I_f)\mathcal{P}_{\text{MWF}}(I - I_f; I_1 - I_f, I_2 - I_f),$$

where $\mathcal{W}(I_f)$ is the probability distribution for the fraction of fermions $I_f$ in each lattice. For the uniform state $\vec{\lambda}^{(u)}$ with $S$ Schmidt coefficients ($P = 1/S$),

$$w_p^{(u)} = \frac{N_1!N_2!(S-N_1)!(S-N_2)!}{S!(S+p-N_1-N_2)!(N_1-p)!(N_2-p)!p!}. \tag{15}$$

The continuous limit $\mathcal{W}^{(u)}(I_f)$ is obtained for $N_1 + N_2 =: N \to \infty$, when $N_1$, $N_2$, $p$, and $S$ are scaled linearly with $N$:

$$\mathcal{W}^{(u)}(I_f) = \lim_{N\to\infty}(Nw^{(u)}_{(p=I_f\cdot N)}) = \delta(I_f - \rho I_1 I_2), \tag{16}$$

and the total number of bi-fermions per Schmidt mode is constant, $\rho = N/S$. Since the number of bi-fermions in either lattice is limited by $S$, it holds $0 < \rho \leq 1/I_1 \leq 2$. The fraction of perfect fermions is, thus, exactly the fraction of expected pairs of bi-fermions in the same Schmidt-mode, $\rho I_1 I_2$, which gives

$$\mathcal{P}^{(u)}(I) = \mathcal{P}_{\text{MWF}}(I - \rho I_1 I_2; I_1(1 - \rho I_2), I_2(1 - \rho I_1)).$$

The width $W$ of this distribution is closely related to the fraction of fermions,

$$W = 4\sqrt{RTI_1I_2(1 - \rho I_1)(1 - \rho I_2)}, \tag{17}$$

and becomes narrower with increasing number of bi-fermions per Schmidt mode, $\rho$. In principle, this may jeopardize Fock-state-interferometry with nonelementary particles such as neutral atoms, since the width of the intensity distribution is used to infer a small phase (which translates here to a reflectivity $R$).

Trapped ultracold atoms typically feature very small electron-state purities of the order of $10^{-13}$ [17,27], such that atom interferometers are not sensitive to the compositeness of the atoms. With attractively interacting fermionic atoms in tunable external potentials [28,29], the transition between fully bosonic ($P \to 0$) and fully fermionic ($P \to 1$) behavior may be implemented experimentally by varying the size of the available single-fermion space and observing the resulting interference pattern when bi-fermions are brought to interference [14].

In conclusion, even though two fermions may be arbitrarily strongly bound to a coboson with no apparent substructure, deviations from ideal bosonic behavior can be observable in many-coboson interference. Not the binding energy, but the entanglement between the fermions is observable on the level of the cobosons. The superposition [Eq. (7)] allows us to understand the partially fermionic behavior of cobosons, and ultimately, leads to simple expressions for the interference of BECs [Eq. (17)]. The methods that we have exposed can be extended immediately to larger numbers of sublattices and to more complex interference scenarios [8].

Cobosons always constitute *indistinguishable* particles; two cobosons in the two lattices share the same distribution of Schmidt coefficients $\vec{\lambda}$. The impact of partial distinguishability and the effects of compositeness can actually be discriminated in the experiment: while partially distinguishable particles can be described as a superposition of perfect bosons and distinguishable particles [10,11], cobosons exhibit the behavior of a superposition of bosons and fermions, which naturally leads to differing interference patterns in the two cases (see also the binomial distribution in Fig. 3, which is attained for distinguishable particles).

The role of entanglement for bosonic behavior is twofold: it circumvents the Pauli principle for composite bosons [16–18,20,24], and it maintains many-particle coherence. Quantum correlations between the fermions are necessary for the bosonic exchange symmetry in the relevant parts of the wave function that allows the representation in Eq. (7). If mixed states of bi-fermions are prepared instead of entangled states, the exchange symmetry and the encountered bosonic behavior break down—even though the combinatorial argument that relates to the number of accessible states remains valid. The visibility of correlation signals of, e.g., large molecules, is thus not only affected by the mixedness of the molecules at finite temperatures, but also by the consequent loss of many-particle coherence.

This work was partially supported by the Project FQM-2445 of the Junta de Andalucía and the Grant No. FIS2011-24540 of the Ministerio de Innovación y Ciencia, Spain. M. C. T. gratefully acknowledges support by the Alexander von Humboldt-Foundation through a Feodor Lynen Fellowship.

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