CONGRUENCE PROPERTIES OF INDUCED REPRESENTATIONS

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ABSTRACT. In this paper we study representations of the projective modular group induced from the Hecke congruence group of level 4 with Selberg's character. We show that the well known congruence properties of Selberg's character are equivalent to the congruence properties of the induced representations. Concerning this congruence property, it turns out that working with the induced representations is easier than with Selberg's character itself. We also show that the kernels of the induced representations determine an infinite sequence of noncongruence groups, whose noncongruence property can not be detected by Zograf's geometric method. They belong to the class of character groups of type I for the principal congruence subgroup $\Gamma(4)$ and have, contrary to the noncongruence groups determined by Selberg's character which all have genus $g = 0$, arbitrary genus $g \geq 0$.

CONTENTS

1. INTRODUCTION

In this paper we study some arithmetic congruence properties of unitary representations of the projective modular group $\mathrm{PSL}(2,\mathbb{Z})$ induced from special representations χ of the Hecke congruence subgroup $\Gamma_0(n)$. To explain the problem we consider in detail here one instructive example, namely the one parameter family of Selberg's one dimensional representations $\chi = \chi_\alpha$ of the group $\Gamma = \Gamma_0(4)$, $\alpha \in [0, 1]$ (see [14] and the definition below in section 2). In [14] Selberg proved the remarkable result that each point s on the critical line $\Re s = 1/2$ is a limit point of zeros of Selberg's zeta function $Z(s, \chi_\alpha) = 0$ when α tends to zero. This result reflects the fact that for each α from (0,1) the multiplicity of the continuous spectrum of

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the automorphic Laplacian $A(\Gamma, \chi_\alpha)$ is equal to one and for $\alpha = 0$ or 1 this multiplicity is equal to three (see for example [14] or [15]). From the point of view of the spectral theory of automorphic functions this family of representations χ_α hence is singular when α tends to 0 or to 1. We believe that the general characteristics of the spectrum of the automorphic Laplacian $A(\Gamma, \chi_\alpha)$ or the zeros of the corresponding Selberg zeta function are depending strongly on arithmetic properties of α being a rational or irrational number. In particular, the spectrum will depend on properties of the algebraic kernel $\ker \chi_\alpha = \{g \in \Gamma_0(4) | \chi_\alpha(g) = 1\}$ for a given α from $[0, 1]$. It is not difficult to see from the definition of χ_α, that for irrational α the kernel $\ker \chi_\alpha$ is a subgroup of infinite index in $\Gamma_0(4)$. For each rational value of $\alpha = n/d$, $d > n$ and $(n, d) = 1$, one can construct a system of generators for the kernel (see [2]). For $d = 2, 3, \dots$ it is a cofinite, normal subgroup of $\Gamma_0(4)$ of index d. Notice, that it is not a normal subgroup of $\mathrm{PSL}(2, \mathbb{Z})$. The kernel is generated by $d + 2$ parabolic elements $S_1, S_2, \dots, S_{d+2}$ with one relation

$$S_1 S_2 \dots S_{d+2} = Id_{2\times 2}. \tag{1.1}$$

It turns out that the kernel is independent of n for $\alpha = n/d$, $(n, d) = 1$. Hence for these values of α we can denote the kernel $\ker \chi_\alpha$ by Γ_d. The group Γ_d is a Fuchsian group of the first kind with signature $(g; m_1, m_2, \dots, m_k; h) = (0; 0, 0, ...0; d + 2)$ (see for example [15]), where $2g$ is the number of hyperbolic generators, k is the number of the elliptic generators, m_j is the order of elliptic generator e_j, and h is the number of parabolic generators of the group. According to the Gauss-Bonnet formula the hyperbolic area a_d of the canonical fundamental domain F_d of Γ_d is equal to

$$a_d = 2\pi(2g - 2 + \sum_{j=1}^{k}(1 - 1/m_j) + h) = 2\pi d. \tag{1.2}$$

An obvious problem then is to determine the values of d for which Γ_d is a congruence subgroup of $\mathrm{PSL}(2, \mathbb{Z})$, that is, contains a principal congruence subgroup. For the definition of the principal congruence subgroups see (2.3). The spectral theory of automorphic functions as well as the classical theory of modular forms is very well developed for these congruence subgroups compared to the noncongruence ones. For example, in the spectral theory of automorphic functions one can prove Weyl's asymptotic law for the counting function $N(\lambda)$ of discrete eigenvalues λ of the automorphic Laplacian A for a congruence subgroup,

$$N(\lambda) = \frac{a(F)}{4\pi}\lambda + o(1), \quad \lambda \to \infty \tag{1.3}$$

where $N(\lambda) = \sharp\,\{\lambda_j < \lambda\}$ and $a(F)$ is the hyperbolic area of the fundamental domain of the group.

In [9] Newman proved that for $\alpha \in (0, 1/2]$ only $\alpha = 1/8, 2/8, 3/8, 4/8$ are congruence values, that means the corresponding kernels are congruence subgroups of $\mathrm{PSL}(2, \mathbb{Z})$. Indeed, Newman considered in [9] conjugate subgroups of the principal congruence group $\Gamma(2)$ which is conjugate to the Hecke group $\Gamma_0(4)$. Also Phillips and Sarnak [11] discussed the character of $\Gamma(2)$ conjugate to χ_α. About Newman's proof see also below. In terms of the groups Γ_d Newman's result says that the only congruence subgroups are those with $d = 1, 2, 4, 8$.

The fact that Γ_d for large d can not be a congruence subgroup follows already from a remarkable geometric result of Zograf (see [19], [20]), based on previous results of Yang and Yau [17] respectively Hersh [4], together with Selberg's famous theorem on small eigenvalues (see [13]). We recall here these two Theorems of Zograf [18] and Selberg [13]:

Theorem 1.1 (Zograf)**.** *Let* Γ *be a discrete cofinite subgroup of* $\mathrm{PSL}(2,\mathbb{R})$ *of signature* $(g; m_1, m_2, \ldots, m_k; h)$ *and* $a(F)$ *be the hyperbolic area of its fundamental domain* F*. Assume* $a(F) \geq 32\pi(g(\Gamma)+1)$*,* $g = g(\Gamma)$*. Then the set of eigenvalues of the automorphic Laplacian* $A(\Gamma)$ *in* $(0, 1/4)$ *is not empty and*

$$\lambda_1 < \frac{8\pi(g(\Gamma)+1)}{a(F)} \tag{1.4}$$

where λ_1*,* $0 < \lambda_1 \leq \lambda_2, \ldots$ *is the first non zero eigenvalue of* $A(\Gamma)$*.*

Theorem 1.2 (Selberg)**.** *Let* Γ *be a congruence subgroup of* $\mathrm{PSL}(2,\mathbb{Z})$*. Then*

$$(3/16) \leq \lambda_1 \tag{1.5}$$

where the notations are the same as in Theorem 1.1.

Selberg's eigenvalue conjecture for congruence subgroups is indeed $\lambda_1 \geq 1/4$ (see [13]). Notice that the interval $[0, 1/4)$ is free from the continuous spectrum of the automorphic Laplacian $A(\Gamma)$ which is real and given by $[1/4, \infty)$. If we now combine these two theorems, we get

$$3/16 < \frac{8\pi(g(\Gamma)+1)}{a(F)} \tag{1.6}$$

If we assume that for a given d the group Γ_d, which has vanishing genus g, is a congruence subgroup, we get from (1.6) that $3/16 < 8\pi/2\pi d$ or $d < 64/3$ and hence there are only finitely many d with Γ_d a congruence subgroup.

The groups Γ_d are typical examples of so called character groups of congruence subgroups G of the modular group $\mathrm{PSL}(2,\mathbb{Z})$. Quite generally one calls a subgroup Γ of an arithmetic subgroup $G \subset \mathrm{PSL}(2,\mathbb{Z})$ a character group of G if $\Gamma = \ker\phi$ for some group homomorphism $\phi : G \to A$ onto a finite abelian group A. The character group Γ is of type I if $\phi(g) \neq id$ for some parabolic element $g \in G$, otherwise it is of type II. The groups Γ_d are obviously character groups of type I of the congruence subgroup $\Gamma_0(4) \subseteq \mathrm{PSL}(2,\mathbb{Z})$ since $\Gamma_d = \ker\chi_\alpha$, $\alpha = \frac{n}{d}$, $n < d$, $(n,d) = 1$.

Many of the examples of noncongruence subgroups are such character groups. It is known for instance [7], that most type II character groups in any genus g are noncongruence. That there are infinitely many such noncongruence groups in any genus $g \geq 0$ has been shown by G. Jones [5]. In the transfer operator approach to Selberg's zeta function for the group $\Gamma_0(4)$ with Selberg's character χ_α [3] this character is replaced by the monomial representation U_α of $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathbb{C}^6$ induced from χ_α. Selberg's zeta function $Z(\Gamma_0(4);\chi_\alpha;s)$ thereby gets expressed as $Z(\Gamma_0(4);\chi_\alpha;s) = \det(1 - \mathcal{L}_{s,U_\alpha})$, where the transfer operator $\mathcal{L}_{s,U_\alpha}$ contains the induced representation U_α. The zeros of $Z(\Gamma_0(4);\chi_\alpha;s)$ are determined by the eigenvalues and resonances of the automorphic Laplacian $A(\Gamma_0(4),\chi_\alpha)$ and hence depend on the induced representation U_α being congruence or not, that means $\ker U_\alpha$ being a congruence subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ or not. As we will show $\ker U_\alpha$ is congruence iff $\ker\chi_\alpha$ is congruence, that means $\alpha = 0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}$ for $0 \leq \alpha \leq \frac{1}{2}$. The groups $\Gamma_d = \ker\chi_\alpha$, $\alpha = \frac{n}{d}$, $n < d$, $(n,d) = 1$ all have vanishing genus and hence Zograf's Theorem gives an infinite number of noncongruence groups among them. Surprisingly the groups $\Lambda_\alpha := \ker U_\alpha$ can have arbitrary large genus $g(\Lambda_\alpha)$

such that $32\pi(g(\Lambda_\alpha)+1) > a(F_{\Lambda_\alpha})$, the area of fundamental domain F_{Λ_α} of the group Λ_α.

Consider namely the group homomorphism $\phi_\alpha : \Gamma(4) \to A_\alpha$ with $\phi_\alpha(g) = U_\alpha(g)$. Then A_α for α rational is a finite abelian group. It turns out, that $\ker \phi_\alpha = \{g \in \Gamma(4) | U_\alpha(g) = id\}$ is congruence iff $\ker \chi_\alpha$ is congruence, that means $\alpha = 0, 1/8, 2/8, 3/8, 4/8$ for $0 \leq \alpha \leq 1/2$. Contrary to the groups Γ_d which all have genus $g(\Gamma_d) = 0$ the groups $\Lambda_\alpha = \ker \phi_\alpha = \ker U_\alpha$ can have arbitrary genus: indeed, if $\alpha = \alpha_N$ such that the generators of the group A_{α_N} have order N then we get $g(\ker \phi_{\alpha_N}) = 1 + 2N^3 - 3N^2$. The surface $a(F_{\alpha_N})$ of its fundamental domain F_{α_N} on the other hand turns out to be $a(F_{\alpha_N}) = \frac{\pi}{3} 24 N^3$. Hence Zograf's Theorem together with Selberg's result on the lower bound $\lambda_1 > \frac{3}{16}$ for the lowest nontrivial eigenvalue of the Laplacian for any congruence subgroup, which show the noncongruence property of Γ_d for infinitely many d's, can not be applied to the groups $\Lambda_\alpha = \ker \phi_\alpha = \ker U_\alpha$. To prove the congruence properties of these groups we have to apply therefore some different algebraic approach.

The paper is organized as follows. In chapter 2 we recall Selberg's character χ_α for the group $\Gamma_0(4)$ and the induced 6-dim. monomial representation of $\mathrm{PSL}(2,\mathbb{Z})$, which for the trivial character χ_0 is a 6-dim. permutation representation. In chapter 3 we discuss the groups $G_\alpha = U_\alpha(\mathrm{PSL}(2,\mathbb{Z}))$, $0 \leq \alpha \leq 1/2$, and their relations to the group $U_0(\mathrm{PSL}(2,\mathbb{Z}))$. We determine the kernel $\ker U_0$ of the representation U_0 and introduce the group $A_\alpha = \{U_\alpha(g) | g \in \ker U_0\}$. The group A_α is a finitely generated abelian normal subgroup of G_α and is of finite order iff α is rational. The factor group G_α / A_α turns out to be isomorphic to the group G_0. In chapter 4 we show that $\ker U_\alpha$ is congruence iff the character χ_α is congruence. To determine independently those α-values for which $\ker U_\alpha$ is congruence we use Wohlfahrt's concept of the level of a finite index subgroup Γ of $\mathrm{PSL}(2,\mathbb{Z})$ and a formula of M. Newman for the genus $g(\ker U_\alpha)$. We derive a necessary condition for $\ker U_\alpha$ to be congruence and determine these congruence subgroups.

2. The induced representation U_α of $\mathrm{PSL}(2,\mathbb{Z})$

The projective modular group $\mathrm{PSL}(2,\mathbb{Z})$ is defined by

$$\mathrm{PSL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}, ad - bc = 1 \right\} / \{\pm 1\}. \tag{2.1}$$

This group is generated by the elements

$$T = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathrm{S} = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.2}$$

with the relations $S^2 = (ST)^3 = Id$.

The principal congruence subgroup $\Gamma(n)$ of level n is defined to be

$$\Gamma(n) := \left\{ \gamma \in \mathrm{PSL}(2,\mathbb{Z}) | \gamma \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod n \right\}. \tag{2.3}$$

The index μ of $\Gamma(n)$ in $\mathrm{PSL}(2,\mathbb{Z})$ is given by (see for example [12])

$$\mu = [\mathrm{PSL}(2,\mathbb{Z}) : \Gamma(n)] = \frac{1}{2} n^3 \prod_{p|n} (1 - \frac{1}{p^2}). \tag{2.4}$$

where p runs over all primes dividing n. The Hecke congruence subgroup of level 4 of $\mathrm{PSL}(2,\mathbb{Z})$ is defined by

$$\Gamma_0(4)=\{\gamma\in\mathrm{PSL}(2,\mathbb{Z})\,|\,\gamma_{21}=0\mod 4\}\,. \tag{2.5}$$

This group is freely generated by

$$T=\pm\begin{pmatrix}1&1\\0&1\end{pmatrix}\qquad\text{and}\qquad ST^4S=\pm\begin{pmatrix}-1&0\\4&-1\end{pmatrix}. \tag{2.6}$$

A set of right cosets of $\Gamma_0(4)$ in $\mathrm{PSL}(2,\mathbb{Z})$ is given by

$$\begin{aligned}&\Gamma_0(4)\backslash\mathrm{PSL}(2,\mathbb{Z})=\\&\left\{\Gamma_0(4),\Gamma_0(4)S,\Gamma_0(4)ST,\Gamma_0(4)ST^2,\Gamma_0(4)ST^3,\Gamma_0(4)ST^2S\right\}.\end{aligned} \tag{2.7}$$

Selberg's character χ_α, $0\le\alpha\le1$,

$$\chi_\alpha:\Gamma_0(4)\to\mathrm{Aut}\mathbb{C}, \tag{2.8}$$

on the group $\Gamma_0(4)$ is defined by the following assignments:

$$\chi_\alpha(T)=\exp(2\pi i\alpha),\quad \chi_\alpha(ST^4S)=1. \tag{2.9}$$

Next we recall the representation U_α of $\mathrm{PSL}(2,\mathbb{Z})$ induced from Selberg's character χ_α for $\Gamma_0(4)$. Let R be a fixed ordered set of representatives of $\Gamma_0(4)\backslash\mathrm{PSL}(2,\mathbb{Z})$. The representation $U_\alpha:\mathrm{PSL}(2,\mathbb{Z})\to\mathrm{End}\mathbb{C}^6$ is then given by

$$[U_\alpha(g)]_{i,j}=\delta_{\Gamma_0(4),\chi_\alpha}(r_igr_j^{-1}),\quad r_i\in R,\quad 1\le i,j\le 6 \tag{2.10}$$

where

$$\delta_{\Gamma_0(4),\chi_\alpha}(\gamma)=\begin{cases}\chi_\alpha(\gamma) & \gamma\in\Gamma_0(4),\\ 0 & \gamma\notin\Gamma_0(4).\end{cases} \tag{2.11}$$

Obviously $U_\alpha(g)$ is a 6 dimensional monomial matrix. For $\alpha=0$ U_0 is the representation of $\mathrm{PSL}(2,\mathbb{Z})$ induced from the trivial one dimensional representation of $\Gamma_0(4)$. Hence $U_0(g)$ is a permutation matrix with entries given by

$$[U_0(g)]_{ij}=\delta_{\Gamma_0(4)}(r_igr_j^{-1}),\quad r_i\in R,\quad 1\le i,j\le 6 \tag{2.12}$$

where

$$\delta_{\Gamma_0(4)}(\gamma)=\begin{cases}1 & \gamma\in\Gamma_0(4),\\ 0 & \gamma\notin\Gamma_0(4).\end{cases} \tag{2.13}$$

From now on we choose R to be the following fixed ordered set

$$R=\left\{Id,S,ST,ST^2,ST^3,ST^2S\right\}. \tag{2.14}$$

With this choice of R one finds for the generators S and T of $\mathrm{PSL}(2,\mathbb{Z})$:

$$U_\alpha(S)=\begin{pmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&\exp(-2\pi i\alpha)&0\\0&0&0&0&0&1\\0&0&\exp(2\pi i\alpha)&0&0&0\\0&0&0&1&0&0\end{pmatrix} \tag{2.15}$$

and

$$U_\alpha(T) = \begin{pmatrix} \exp(2\pi i\alpha) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \exp(-2\pi i\alpha) \end{pmatrix}. \tag{2.16}$$

3. The group $U_\alpha(\mathrm{PSL}(2,\mathbb{Z}))$

In this section we study the group $G_\alpha = U_\alpha(\mathrm{PSL}(2,\mathbb{Z}))$, that is,

$$G_\alpha := \{U_\alpha(g) | g \in \mathrm{PSL}(2,\mathbb{Z})\}\,. \tag{3.1}$$

Since $\mathrm{PSL}(2,\mathbb{Z})$ is generated by S and T, the group G_α is generated by $U_\alpha(S)$ and $U_\alpha(T)$. For the following we need to introduce some definitions and auxiliary results.

The general linear group $\mathrm{GL}(\mathrm{n},\mathbb{C})$ is the group of all invertible $n \times n$ matrices with entries in $\mathbb{C}$. A matrix is called monomial if each row and column has exactly one nonzero element ([1], page 48). We denote by $M(6,\mathbb{C})$ the group of all monomial matrices in $\mathrm{GL}(6,\mathbb{C})$. We also denote by $\Delta(6,\mathbb{C})$ the group of all diagonal matrices in $\mathrm{GL}(6,\mathbb{C})$. The group $M(6,\mathbb{C})$ is the normalizer of $\Delta(6,\mathbb{C})$ in $\mathrm{GL}(6,\mathbb{C})$ (see [1], page 48, Exercise 7). Hence, $\Delta(6,\mathbb{C})$ is normal in $M(6,\mathbb{C})$ (see for example [1], page 34, third paragraph). A permutation matrix is a monomial matrix in which all nonzero elements are equal to one. We denote by W the set of all 6 dimensional permutation matrices in $\mathrm{GL}(6,\mathbb{C})$. W is a subgroup of $\mathrm{GL}(6,\mathbb{C})$ which is called the Weyl group (see [1], page 42, Proposition 4). The group W is isomorphic to S_6, the symmetric group of degree 6 (see [1], page 42, Proposition 5). The group $M(6,\mathbb{C})$ has the following semidirect product structure (see [1], page 48, Exercise 7)

$$M(6,\mathbb{C}) = \Delta(6,\mathbb{C}) \rtimes W. \tag{3.2}$$

Hence, each element m in $\in M(6,\mathbb{C})$ has a unique expression as $m = \delta w$ where $\delta \in \Delta(6,\mathbb{C})$ and $w \in W$ (see [1], page 21, second item).

The generators of G_α, namely $U_\alpha(S)$ and $U_\alpha(T)$ belong to $M(6,\mathbb{C})$. Hence, G_α is a subgroup of $M(6,\mathbb{C})$:

$$G_\alpha \leq M(6,\mathbb{C}). \tag{3.3}$$

Lemma 3.1. *Let U_0 be the representation of $\mathrm{PSL}(2,\mathbb{Z})$ induced from the trivial one dimensional representation of $\Gamma_0(4)$. Then each element $U_\alpha(g) \in G_\alpha$ has a unique representation as*

$$U_\alpha(g) = D_\alpha(g) U_0(g), \tag{3.4}$$

where $D_\alpha(g) \in \Delta(6,\mathbb{C})$.

Proof. To $g \in \mathrm{PSL}(2,\mathbb{Z})$ we assign the diagonal matrix $D_\alpha(g) \in \Delta(6,\mathbb{C})$ with entries

$$[D_\alpha(g)]_{ik} = \delta_{ik}\chi_\alpha(r_i g r(i)^{-1}), \quad 1 \leq i,k \leq 6. \tag{3.5}$$

Here, r_i and $r(i)$ are elements of the set R of representatives of $\Gamma_0(4)\backslash\mathrm{PSL}(2,\mathbb{Z})$ with $r(i)$ uniquely determined by the condition $r_i g r(i)^{-1} \in \Gamma_0(4)$. Then we have

$$[D_\alpha(g)U_0(g)]_{ij} = \sum_{k=1}^{6} [D_\alpha(g)]_{ik} [U_0(g)]_{kj} \,. \tag{3.6}$$

Inserting (3.5) into this identity we get

$$[D_\alpha(g)U_0(g)]_{ij} = \chi_\alpha(r_i g r(i)^{-1}) [U_0(g)]_{ij} \,. \tag{3.7}$$

But according to the definition of U_0 in (2.12), we get

$$[D_\alpha(g)U_0(g)]_{ij} = \chi_\alpha(r_i g r(i)^{-1})\delta_{\Gamma_0(4)}(r_i g r_j^{-1}). \tag{3.8}$$

Hence

$$[D_\alpha(g)U_0(g)]_{ij} = \begin{cases} \chi_\alpha(r_i g r_j^{-1}) & \text{if } r_i g r_j^{-1} \in \Gamma_0(4), \\ 0 & \text{if } r_i g r_j^{-1} \notin \Gamma_0(4) \end{cases} \tag{3.9}$$

or

$$[D_\alpha(g)U_0(g)]_{ij} = \delta_{\Gamma_0(4),\chi_\alpha}(r_i g r_j^{-1}). \tag{3.10}$$

Hence from (2.10) we see

$$D_\alpha(g)U_0(g) = U_\alpha(g). \tag{3.11}$$

Since $U_0(g)$ is a permutation matrix in W and G_α is a subgroup of $M(6,\mathbb{C})$, this decomposition according to (3.2) is unique. □

As mentioned already, the group $\Delta(6,\mathbb{C})$ is normal in the group of monomial matrices $M(6,\mathbb{C})$ and G_α is a subgroup of $M(6,\mathbb{C})$. Hence $A_\alpha := G_\alpha \cap \Delta(6,\mathbb{C})$ is normal in G_α. By definition A_α is the group of all diagonal matrices in G_α. Hence, according to lemma 3.1, A_α is the image of the kernel of the representation U_0 under the map U_α, that means

$$A_\alpha = \{U_\alpha(\gamma) | \gamma \in \ker U_0\} \,. \tag{3.12}$$

Lemma 3.2. *Let U_0 be the representation of $\mathrm{PSL}(2,\mathbb{Z})$ induced from the trivial one dimensional representation of $\Gamma_0(4)$. Then the kernel of U_0,*

$$\ker U_0 = \{g \in \mathrm{PSL}(2,\mathbb{Z}) | U_0(g) = Id_{6\times 6}\} \,, \tag{3.13}$$

coincides with the principal congruence subgroup of level 4, that is,

$$\ker U_0 = \Gamma(4). \tag{3.14}$$

Proof. Since $\Gamma(4) \lhd \mathrm{PSL}(2,\mathbb{Z})$, for each $\gamma \in \Gamma(4)$ and $r \in R$ there exists a $\gamma' \in \Gamma(4)$ such that $r\gamma r^{-1} = \gamma'$. Thus, according to (2.12) and (2.13), $\Gamma(4) \subset \ker U_0$. Next we prove $\ker U_0 \subset \Gamma(4)$. Let $\gamma \in \ker U_0$. Then, according to (2.12) and (2.13), for each $r \in R$ one has $r\gamma r^{-1} \in \Gamma_0(4)$. Since $Id \in R$, necessarily $\gamma \in \Gamma_0(4)$. For $\gamma \in \Gamma_0(4)$ one finds $S\gamma S^{-1} \in \Gamma^0(4)$, but $S \in R$ and therefore $S\gamma S^{-1} \in \Gamma_0(4)$. Hence γ itself must belong to $\Gamma_0(4) \cap \Gamma^0(4)$. Next consider the conjugation of the element $\gamma = \begin{pmatrix} a & 4b \\ 4c & d \end{pmatrix} \in \Gamma_0(4) \cap \Gamma^0(4)$ by $ST \in R$. A simple calculation shows that $ST\gamma(ST)^{-1} \in \Gamma_0(4)$ iff $a \equiv d \mod 4$. This and the fact that $\det\gamma = 1$ yields $\gamma \in \Gamma(4)$. Hence $\ker U_0 \subset \Gamma(4)$. This completes the proof. □

Corollary 3.1. *The normal subgroup A_α of G_α in* (3.12) *is given by*

$$A_\alpha = \{U_\alpha(\gamma) | \gamma \in \Gamma(4)\} . \tag{3.15}$$

According to this corollary generators of A_α can be calculated explicitly from generators of $\Gamma(4)$. A set of generators of $\Gamma(4)$ are given for instance by (see for example [6], page 104)

$$\begin{aligned} g_1 = T^4 &= \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \\ g_2 = ST^{-4}S &= \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \\ g_3 = T^{-1}ST^4ST &= \begin{pmatrix} -5 & -4 \\ 4 & 3 \end{pmatrix}, \\ g_4 = T^{-2}ST^{-4}ST^{-2} &= \begin{pmatrix} 7 & -12 \\ -4 & 7 \end{pmatrix}, \\ g_5 = TST^{-4}ST^{-1} &= \begin{pmatrix} -5 & 4 \\ -4 & 3 \end{pmatrix}. \end{aligned} \tag{3.16}$$

The corresponding generators of A_α are obtained by calculating the induced representation U_α for each generator of $\Gamma(4)$:

$$A_1(\alpha) := U_\alpha(g_1) = \mathrm{diag}(\exp(8\pi i\alpha), 1, 1, 1, 1, \exp(-8\pi i\alpha)), \tag{3.17}$$

$$A_2(\alpha) := U_\alpha(g_2) = \mathrm{diag}(1, \exp(-8\pi i\alpha), 1, \exp(8\pi i\alpha), 1, 1), \tag{3.18}$$

$$A_3(\alpha) := U_\alpha(g_3) = \mathrm{diag}(1, 1, \exp(8\pi i\alpha), 1, \exp(-8\pi i\alpha), 1) \tag{3.19}$$

where $\mathrm{diag}(a_1, \dots, a_6)$ denotes a 6 dimensional diagonal matrix with entries a_i. The generators g_4 and g_5 do not lead to new generators for A_α. In fact, we have

$$U_\alpha(g_5) = U_\alpha(g_3), \quad U_\alpha(g_4) = [U_\alpha(g_1)U_\alpha(g_2)]^{-1} . \tag{3.20}$$

Thus, the group A_α is generated by three elements,

$$A_\alpha = \langle A_1(\alpha), A_2(\alpha), A_3(\alpha) \rangle . \tag{3.21}$$

Next we consider the factor group G_α / A_α.

Lemma 3.3. *The factor group G_α/A_α is isomorphic to the modularity group $G(4) = \mathrm{PSL}(2, \mathbb{Z})/\Gamma(4)$.*

Proof. First, we consider the group homomorphisms

$$\begin{aligned} h_1 &: \mathrm{PSL}(2, \mathbb{Z}) \to G_\alpha \\ h_1(g) &= U_\alpha(g) \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} h_2 &: \Gamma(4) \to A_\alpha \\ h_2(g) &= U_\alpha(g). \end{aligned} \tag{3.23}$$

According to lemma 3.1, $\ker h_1 \le \ker U_0$ whereas lemma 3.2 shows $\ker U_0 = \Gamma(4)$. Thus $\ker h_1 \le \Gamma(4)$. Moreover, h_2 is the restriction of h_1 to $\Gamma(4)$ and hence $\ker h_1 = \ker h_2 \le \Gamma(4)$. From the definitions of h_1 and h_2 it is also clear that

$$\ker U_\alpha = \ker h_1 = \ker h_2. \tag{3.24}$$

According to the so called "fundamental theorem on homomorphisms" (see for example [1], page 10), the following isomorphisms hold

$$\mathrm{PSL}(2,\mathbb{Z})/\ker U_\alpha \cong G_\alpha = U_\alpha(\mathrm{PSL}(2,\mathbb{Z})) \tag{3.25}$$

and

$$\Gamma(4)/\ker U_\alpha \cong A_\alpha. \tag{3.26}$$

Then the "second isomorphism theorem" (see [1], page 12) yields the desired result,

$$G_\alpha/A_\alpha \cong \mathrm{PSL}(2,\mathbb{Z})/\Gamma(4) = \mathrm{G}(4). \tag{3.27}$$

□

4. Congruence characters and representations

According to definition of U_α in (2.10) an element $g \in \mathrm{PSL}(2,\mathbb{Z})$ is in the kernel of the representation U_α iff for all elements r in the set R, the set of representatives of $\Gamma_0(4)\backslash\mathrm{PSL}(2,\mathbb{Z})$ given in (2.14), one has $\delta_{\Gamma_0(4),\chi_\alpha}(rgr^{-1}) = 1$. But according to (2.11) this is equivalent to $rgr^{-1} \in \ker\chi_\alpha$ for all $r \in R$. Therefore we have

$$\ker U_\alpha = \left\{ g \in \mathrm{PSL}(2,\mathbb{Z}) \,|\, rgr^{-1} \in \ker\chi_\alpha, \forall r \in R \right\}. \tag{4.1}$$

Then we get

Theorem 4.1. $\ker U_\alpha$ *is congruence if and only if* $\ker\chi_\alpha$ *is congruence.*

Proof. Since $Id \in R$, according to (4.1) $\ker U_\alpha$ is a subgroup of $\ker\chi_\alpha$

$$\ker U_\alpha \leq \ker\chi_\alpha. \tag{4.2}$$

Thus, if $\ker U_\alpha$ is a congruence subgroup then also $\ker\chi_\alpha$ is a congruence subgroup. To prove the converse, consider the kernel $\ker U_\alpha$ in (4.1), which is given by the following intersection of sets

$$\ker U_\alpha = r_1 \ker\chi_\alpha r_1^{-1} \cap r_2 \ker\chi_\alpha r_2^{-1} \cap \ldots \cap r_6 \ker\chi_\alpha r_6^{-1}, \quad r_i \in R. \tag{4.3}$$

If $\ker\chi_\alpha$ is congruence, then $\Gamma(n) \leq \ker\chi_\alpha$ for some $n \in \mathbb{N}$. Since $\Gamma(n)$ is normal in $\mathrm{PSL}(2,\mathbb{Z})$ one gets $\Gamma(n) \subset r\ker\chi_\alpha r^{-1}$ for all $r \in R$. Therefore, according to (4.3), $\Gamma(n) \leq \ker U_\alpha$. Hence $\ker U_\alpha$ is also a congruence subgroup. □

Definition 4.1. *Selberg's character* χ_α *is called a congruence character if* $\ker\chi_\alpha$ *is a congruence subgroup.*

Definition 4.2. *We call* U_α *a congruence representation if* $\ker U_\alpha$ *is a congruence subgroup.*

As a corollary of theorem 4.1 we then have

Corollary 4.1. *The representation* U_α *is congruence iff Selberg's character* χ_α *is congruence.*

Next we are going to determine the α-values for which $\ker U_\alpha$ is congruence without refering to Selberg's character χ_α, and also the corresponding principal congruence sbgroups. This will lead also to an infinite family of noncongruence character groups of type I with arbitrary large genus. For this denote by $N = N(\alpha)$ the order of a generator of the group A_α defined in (3.21). Obviously all generators of this group have the same order, which for α irrational for example is given by $N = \infty$. We recall that (see (3.26))

$$\Gamma(4)/\ker U_\alpha \cong A_\alpha. \tag{4.4}$$

Hence, the index $\mu(\alpha) = [\mathrm{PSL}(2,\mathbb{Z}) : \ker U_\alpha]$ of $\ker U_\alpha$ in $\mathrm{PSL}(2,\mathbb{Z})$ is equal to the number of elements of A_α times $[\mathrm{PSL}(2,\mathbb{Z}) : \Gamma(4)] = 24$, the index of $\Gamma(4)$ in $\mathrm{PSL}(2,\mathbb{Z})$. Thus we have

$$\mu(\alpha) = 24N^3 = 24N(\alpha)^3. \tag{4.5}$$

From this formula it is clear that for irrational α the subgroup $\ker U_\alpha$ is of infinite index in $\mathrm{PSL}(2,\mathbb{Z})$; that means it can not be congruence. In the following let α be a rational number with $N(\alpha) = N$, $N \in \mathbb{N}$.

Based on the Gauss-Bonnet formula, we can determine the number of generators of $\ker U_\alpha$. As can be seen from (4.4) $\ker U_\alpha \leq \Gamma(4)$. Thus $\ker U_\alpha$ has no elliptic elements. The Gauss-Bonnet formula for a group Γ without elliptic elements shows (see for example [15], page 15), that

$$|F| = 2\pi(2g - 2 + h), \tag{4.6}$$

where $|F|$ is the area of the fundamental domain, g is the genus, and h is the number of cusps of Γ. It is also known that the number of generators of Γ is $2g + h$ (see [15], page 14). For the group $\ker U_\alpha$ we have

$$|F| = \mu(\alpha)\frac{\pi}{3}, \tag{4.7}$$

where $\pi/3$ is the area of the fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})$ and $\mu(\alpha)$ is the index of $\ker U_\alpha$ in $\mathrm{PSL}(2,\mathbb{Z})$ as given in (4.5). Hence, the number of generators of $\ker U_\alpha$ is given by

$$2g + h = 4N^3 + 2. \tag{4.8}$$

The number of free generators on the other hand is given by (see [15], page 14)

$$\mathcal{N}(\alpha) = 2g + h - 1 = 4N^3 + 1. \tag{4.9}$$

Before continuing further we recall the concept of the width of a cusp (see [16], page 529).

Definition 4.3. *For $x \in \mathbb{Q} \cup \{\infty\}$ a cusp of the group $\Gamma \leq \mathrm{PSL}(2,\mathbb{Z})$ and $\sigma \in \mathrm{PSL}(2,\mathbb{Z})$ with $\sigma\infty = x$, let $P \in \Gamma$ be a primitive parabolic element with $Px = x$. If*

$$\sigma P \sigma^{-1} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}). \tag{4.10}$$

then $|m|$ is called the width of the cusp x of Γ.

Next we need Wohlfahrt's definition of the level of a group (see [16], page 530)

Definition 4.4. *Let $\Gamma \leq \mathrm{PSL}(2,\mathbb{Z})$ and $C(\Gamma) \subset \mathbb{N}$ be the set of widths of the cusps of Γ. If $C(\Gamma)$ is nonempty and bounded in $\mathbb{N}$, the level $n(\Gamma)$ of Γ is defined to be the least common multiple of the elements of $C(\Gamma)$. Otherwise the level is defined to be zero.*

For congruence subgroups Γ F. Klein on the other hand defined the level as follows (see [16] and the references there)

Definition 4.5. *The level of a congruence subgroup is defined to be the least integer n such that $\Gamma(n) \subset \Gamma$.*

The following theorem, which is crucial for our purpose, shows that for congruence groups Wohlfahrt's and F. Klein's definition of the level are equivalent (see [16] and the references there).

Theorem 4.2. *If* Γ *is a congruence subgroup of level* n *in the sense of Wohlfahrt then* $\Gamma(n) \leq \Gamma$.

Next we determine the level of $\ker U_\alpha$ in the sense of Wohlfahrt. To this end, we note that $\ker U_\alpha$ is normal in $\mathrm{PSL}(2,\mathbb{Z})$. Hence, all cusps of $\ker U_\alpha$ have the same width (see [15], page 160 third paragraph). Thus it is enough to find the width just for one cusp. According to (3.17), for α with $N(\alpha) = N \in \mathbb{N}$, we have $U_\alpha(g_1)^N = Id_{6\times 6}$ where

$$g_1 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \tag{4.11}$$

as given in (3.16). Hence

$$g_1^N = \begin{pmatrix} 1 & 4N \\ 0 & 1 \end{pmatrix} \tag{4.12}$$

belongs to $\ker U_\alpha$ and it is obviously primitive. Thus Wohlfahrt's level $n(\alpha)$ of $\ker U_\alpha$ is given for α with $N(\alpha) = N$ by

$$n(\alpha) = 4N. \tag{4.13}$$

Next, based on a formula due to Morris Newman [10], we derive a formula for the genus of $\ker U_\alpha$. Let Γ be a normal subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ with index μ, genus g, and number of cusps h, respectively Wohlfahrt's level n. Put $t := \dfrac{\mu}{n}$. Then according to Newman (see [10], page 268 and also [15] page 160) one has

$$g = 1 + \frac{\mu}{12} - \frac{t}{2}. \tag{4.14}$$

For the group $\ker U_\alpha$ according to (4.5) and (4.13) we get $t = 6N^2$. Inserting this and (4.5) into (4.14) we get for the genus of $\ker U_\alpha$

$$g(\alpha) = 1 + 2N^3 - 3N^2. \tag{4.15}$$

From this formula and (4.8) we obtain the number $h(\alpha)$ of cusps of $\ker U_\alpha$ as follows

$$h(\alpha) = 6N^2. \tag{4.16}$$

Let us summarize all the information about $\ker U_\alpha$ in the following theorem

Theorem 4.3. *For* α *with* $N(\alpha) = N \in \mathbb{N}$ *the order of the generators of* A_α*, let* $\mu(\alpha)$ *be the index of the group* $\ker U_\alpha$ *in* $\mathrm{PSL}(2,\mathbb{Z})$*,* $g(\alpha)$ *its genus,* $h(\alpha)$ *the number of its cusps,* $n(\alpha)$ *its Wohlfahrt's level, and* $\mathcal{N}(\alpha)$ *the number of its free generators. Then we have*

- $\mu(\alpha) = 24N^3$
- $g(\alpha) = 1 + 2N^3 - 3N^2$
- $h(\alpha) = 6N^2$
- $n(\alpha) = 4N$
- $\mathcal{N}(\alpha) = 4N^3 + 1$

Since $n = 4N$ is Wohlfahrt's level of $\ker U_\alpha$, according to theorem 4.2 the following must hold if $\ker U_\alpha$ is congruence

$$\Gamma(4N) \leq \ker U_\alpha. \tag{4.17}$$

This now provides us in the following way with a criterion for $\ker U_\alpha$ to be congruent. The index of $\Gamma(4N)$ in $\mathrm{PSL}(2,\mathbb{Z})$ is given by

$$[\mathrm{PSL}(2,\mathbb{Z}) : \Gamma(4N)] = \frac{1}{2}(4N)^3 \prod_{p|4N}(1 - \frac{1}{p^2}). \tag{4.18}$$

According to (4.17) this must be bigger or equal to the index $\mu(\alpha)$ of $\ker U_\alpha$ in $\mathrm{PSL}(2,\mathbb{Z})$, that is

$$\frac{1}{2}(4N)^3 \prod_{p|4N}(1 - \frac{1}{p^2}) \geq 24N^3 \tag{4.19}$$

or

$$\frac{4}{3}\prod_{p|4N}(1 - \frac{1}{p^2}) \geq 1. \tag{4.20}$$

But this inequality holds if and only if $N = 1$ or $N = 2$. This means that the only possibilities for α are those for which $N(\alpha) \in \{1, 2\}$.

Let α_1 and α_2 denote the α-values for which $N(\alpha_1) = 1$ and $N(\alpha_2) = 2$, respectively. We are going to prove U_{α_1} and U_{α_2} are congruence. To this end recall that $\Gamma(4)/\ker U_\alpha \cong A_\alpha$. Since A_{α_1} is the trivial group, one has $\ker U_{\alpha_1} = \Gamma(4)$ and therefore $\ker U_{\alpha_1}$ is congruence. It remains to prove the congruence property of $\ker U_{\alpha_2}$. To this end we note that $A_{\alpha_2} \cong C_2 \times C_2 \times C_2$ where C_2 is the cyclic group of order 2. On the other hand, according to Mc Quillan (see [8], page 286), $\Gamma(4)/\Gamma(8)$ is also isomorphic to $C_2 \times C_2 \times C_2$. Hence, we have $A_{\alpha_2} \cong \Gamma(4)/\ker U_{\alpha_2} \cong \Gamma(4)/\Gamma(8)$. In the following lemma we give an explicit expression for this isomorphism,

Lemma 4.4. *The map*

$$\imath : \Gamma(4)/\ker U_{\alpha_2} \to \Gamma(4)/\Gamma(8) \tag{4.21}$$

defined by

$$\imath(\gamma \ker U_{\alpha_2}) = \gamma\Gamma(8) \tag{4.22}$$

is a group isomorphism.

Proof. An isomorphism $\Gamma(4)/\ker U_{\alpha_2} \cong A_{\alpha_2}$ can be defined by the map

$$\imath_1 : \Gamma(4)/\ker U_{\alpha_2} \to A_{\alpha_2} \tag{4.23}$$

given by

$$\imath_1(g \ker U_{\alpha_2}) = U_{\alpha_2}(g). \tag{4.24}$$

On the other hand, according to (3.21), $A_{\alpha_2} = U_{\alpha_2}(\Gamma(4))$ is generated by $U_{\alpha_2}(g_1)$, $U_{\alpha_2}(g_2)$ and $U_{\alpha_2}(g_3)$ with $g_i \in \Gamma(4)$, $i = 1, 2, 3$ as given in (3.16). Hence the elements g_1, g_2, and g_3 modulo $\ker U_{\alpha_2}$ generate the group $\Gamma(4)/\ker U_{\alpha_2}$. Next, a simple calculation shows that g_1, g_2, and g_3 modulo $\Gamma(8)$ are distinct elements of order 2. However $g_1, \ldots, g_5$ generate $\Gamma(4)$ and $g_3 = g_4 = g_5$ modulo $\Gamma(8)$, which shows that $g_1\Gamma(8)$, $g_2\Gamma(8)$, and $g_3\Gamma(8)$ generate $\Gamma(4)/\Gamma(8)$. This completes the proof. □

Since $\imath : \Gamma(4)/\ker U_{\alpha_2} \to \Gamma(4)/\Gamma(8)$ in (4.22) is a group isomorphism, one has $\ker \imath = \ker U_{\alpha_2}$ and hence $\gamma \in \Gamma(8)$ iff $\gamma \in \ker U_{\alpha_2}$. Thus as a corollary we get,

Corollary 4.2. *The kernel* $\ker U_{\alpha_2}$ *is given by* $\Gamma(8)$ *and hence* $\ker U_{\alpha_2}$ *is congruence.*

From the definition of the generators of A_α in (3.17), (3.18), and (3.19) it is clear that $N(\alpha_1) = 1$ iff $8\pi i \alpha_1 = 2\pi i k$ iff $\alpha_1 = (1/4)k$ with $k \in \mathbb{Z}$. Moreover, $N(\alpha_2) = 2$ iff $8\pi i \alpha_2 = \pi i k$ iff $\alpha_2 = (1/8)k$ with $k \in \mathbb{Z}$ and $(k, 2) = 1$.

We summarize our last result in the following theorem

Theorem 4.5. *The representation* U_α*, defined in* (2.10) *with* $0 \le \alpha \le 1/2$ *is congruence only for* α*-values* $0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}$. *Moreover we have*

$$\ker U_0 = \ker U_{\frac{2}{8}} = \ker U_{\frac{4}{8}} = \Gamma(4), \tag{4.25}$$

respectively

$$\ker U_{\frac{1}{8}} = \ker U_{\frac{3}{8}} = \Gamma(8). \tag{4.26}$$

This gives another proof of an analogous result of Sarnak and Phillips for Selberg's character χ_α in [11] respectively of a result of M. Newman in [9]. The noncongruence subgroups $\ker U_\alpha$ of $\Gamma(4)$, $\alpha \ne 0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}$, $0 \le \alpha \le 1/2$ are character groups of type I with arbitrary large genus $g(\alpha) = 1 + 2N^3 - 3N^2$, where $N = N(\alpha)$ denotes the order of the generators of the group $A_\alpha = \Gamma(4)/\ker U_\alpha$.

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