

# **Information and Heterogeneous Beliefs: Cost of Capital, Trading Volume, and Investor Welfare**

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# Information and Heterogeneous Beliefs: Cost of Capital, Trading Volume, and Investor Welfare

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## Abstract

In an incomplete market setting with heterogeneous prior beliefs, we show that public information can have a substantial impact on the *ex ante* cost of capital, trading volume, and investor welfare. In a model with exponential utility investors and an asset with a normally distributed dividend, the Pareto efficient public information system is the system which enjoys the maximum *ex ante* cost of capital, and the maximum expected abnormal trading volume. The public information system facilitates improved dynamic trading opportunities based on heterogeneously updated posterior beliefs in order to take advantage of the disagreements and the differences in confidence among investors. This leads to a higher growth in the investors' certainty equivalents and, thus,

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a higher equilibrium interest rate, whereas the *ex ante* risk premium on the risky asset is unaffected by the informativeness of the public information system. In an effectively complete market setting, in which investors do not need to trade dynamically in order to take full advantage of their differences in beliefs, the *ex ante* cost of capital and the investor welfare are both higher than in the incomplete market setting, but they are independent of the informativeness of the public information system, and there is no information-contingent trade.

**Keywords:** Heterogeneous Beliefs; Public Information Quality; Dynamic Trading; Cost of Capital; Investor Welfare

**JEL Classification:** G1, G11, G12

*One of the things that microeconomics teaches you is that individuals are not alike. There is heterogeneity, and probably the most important heterogeneity here is heterogeneity of expectations. If we didn't have heterogeneity, there would be no trade. But developing an analytic model with heterogeneous agents is difficult.*

– Ken Arrow, in D. Colander, R.P.F. Holt and J. Barkley Rosser (eds.), *The Changing Face of Economics. Conversations with Cutting Edge Economists*. The University of Michigan Press, Ann Arbor, 2004, p. 301.

## 1 Introduction

Financial markets are not complete, and investors in financial markets are not alike—both in terms of preferences, wealth and beliefs. Acknowledging these facts, we develop a simple analytical model with exponential utility investors, who have heterogeneous beliefs over normally distributed dividends, which shows that the public information system plays a key role for the investors' welfare, the asset prices, and for the trading volume in the financial market. We show that the Pareto efficient public information system is the system, which enjoys the *maximum ex ante* cost of capital, and the *maximum* expected abnormal trading volume. In an incomplete market, *imperfect* public information facilitates dynamic trading opportunities based on heterogeneously updated posterior beliefs, which allow the investors to better take advantage of their disagreements and their differences in confidence.

The vast majority of prior studies in the accounting and finance literature of the impact of public information system choices, such as financial reporting regulation, on equilibrium asset prices, trading volume, and investor welfare, recognize differences in preferences and/or wealth, but assume that the investors' prior beliefs are identical, although their posterior beliefs may vary due to differences in the information they have received (see, e.g., Harsanyi 1968). In complete markets, this assumption typically leads to so-called no-trade theorems (see, e.g., Milgrom and Stokey 1982), implying that the theory cannot explain the significant trading volume in actual financial markets, for example, around earnings announcements as first documented by Beaver (1968), unless some unmodeled noise trading is injected into the price system (see, e.g., Grossman and Stiglitz 1980, Hellwig 1980, and Kyle 1985).

But why should all investors have been born equal? Some investors may be more optimistic or more confident in their estimates than others, for example, due to different *DNA* profiles or past experiences which are completely unrelated to the uncertainty and information in financial markets (see, e.g., Morris 1995, for a critical discussion of the common prior assumption in economic theory). Moreover, despite significant financial innovations over the last four decades, financial markets are probably still incomplete even if we allow for dynamic

trading strategies, for example, due to individual idiosyncratic risks (see, e.g., Krueger and Lustig 2010, and Christensen et al. 2011) or heterogeneous prior beliefs. In this paper, we develop a simple equilibrium model with heterogeneous prior beliefs and incomplete markets allowing us to study (in closed-form) the impact of public information system choices on both equilibrium asset prices, trading volume, and investor welfare.

We compare the equilibrium in the incomplete market setting to the equilibrium in an otherwise identical effectively complete market setting in which there exists a derivative security specifically targeted towards the investors' incentive to take speculative positions based on their heterogeneity in beliefs. In that economy, the investors do not need to trade dynamically in order to take full advantage of their differences in beliefs. The *ex ante* cost of capital and the investors' welfare are both higher than in the incomplete market setting, but there is no trade, and the public information system plays no role. More generally, this result suggests that the existence of derivative markets and the public information system have complementary roles in facilitating improved investor welfare in financial markets.

A large literature in accounting and finance studies the impact of information on firms' cost of equity capital both theoretically and empirically.<sup>1</sup> The general theme in this literature seems to be that more public disclosure of information will reduce firms' cost of equity capital which, in an exchange economy, is equivalent to higher stock prices. The intuition is simple. A firm's cost of equity capital is the riskless interest rate plus a risk premium. Releasing more informative public signals reduce the uncertainty about the size and the timing of future cash flows and, therefore, also the risk premium.

This intuition, however, pertains only to the cost of capital when measured after the release of information, i.e., the *ex post* cost of capital.<sup>2</sup> Christensen et al. (2010) show that if the cost of capital is measured before any signals from the information system are realized, i.e., the *ex ante* cost of capital, then the public information system has no impact on the *ex ante* cost of capital and, thus, no impact on the *ex ante* stock prices, in competitive exchange economies with *homogeneous prior beliefs* and both public and private investor information. The public information system only serves to affect the timing of release of information and, thus, to affect the allocation of the total risk premium for future cash flows over time.

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<sup>1</sup>Theoretical studies include Easley and O'Hara (2004), Hughes et al. (2007), Lambert et al. (2007, 2012), Christensen et al. (2010), Armstrong et al. (2011), and Bloomfield and Fischer (2011), while empirical studies include Botosan (1997), Botosan and Plumlee (2002), Easley et al. (2002), and Francis et al. (2008), among many others.

<sup>2</sup>Although this intuition may seem simple and straightforward, one has to be careful in interpreting these results in multi-period models in which any interim period has elements of both *ex post* and *ex ante* effects (see the discussion in Christensen et al. 2010). In a standard continuous-time model, Veronesi (2000) shows that more precise public signals about economic growth tend to increase conditional equity premia through a higher equilibrium covariance between current consumption and stock returns.

Is a low *ex ante* cost of equity capital and, thus, high *ex ante* stock prices good or bad? In a partial equilibrium analysis focusing on a single firm and its shareholders, the answer is clearly “good.” This is merely a cousin of the familiar *value maximization principle* for competitive markets, cf. Debreu (1959). However, financial reporting regulation (and other mandated disclosure requirements) is about choosing information systems for the economy at large. In such settings, a general equilibrium analysis is in order and, in general, welfare consequences of policy changes cannot be assessed directly through stock market values.

For example, how is the other component of the cost of equity capital, i.e., the riskless interest rate, affected by changes in the information system in the economy? In competitive exchange economies with homogeneous prior beliefs, time-additive preferences, and public information, the *ex ante* riskless interest rates will not be affected by changes in the information system (see, e.g., Christensen et al. 2010 and the references therein). We show that even for an exchange economy, but with heterogeneous prior beliefs, the *ex ante* equilibrium interest rate is affected by the informativeness of the public information system. In particular, the *ex ante* equilibrium interest rate is a linear increasing function of the growth in the investors’ certainty equivalents. More efficient dynamic trading opportunities based on the heterogeneity in prior beliefs and public information increase the growth in certainty equivalents, while (in our particular model) the *ex ante* risk premium is unaffected by the public information system. In other words, from a general equilibrium perspective, the preferred public information system is the system, which enjoys the highest *ex ante* cost of equity capital and, thus, the lowest *ex ante* stock prices.

Our analysis focusses on a competitive *exchange economy* and, thus, a relevant question is whether the higher *ex ante* cost of capital due to more efficient dynamic trading opportunities based on the heterogeneity in prior beliefs and public information comes with a *negative real effect* due to costlier financing of firms production in a more general *production economy*. Interestingly, introducing a riskless standard convex production technology into the setting of this paper, a higher *ex ante* cost of capital is associated with *positive real effects*. A higher *ex ante* cost of capital is a consequence of a higher growth in certainty equivalents and, thus, the intertemporal trade-off between current and future aggregate consumption changes such that it becomes optimal to invest less in production (and, thus, consume more) now and consume less in the future. Such changes in production choices would then reduce the *ex ante* cost of capital, in equilibrium, but not fully back to the level with less efficient dynamic trading opportunities.

Our model is a two-period extension of the classical single-period capital asset pricing model with heterogeneous beliefs of Lintner (1969). For simplicity, we assume there is a single risky asset in non-zero net-supply paying a known dividend at  $t = 0$  and a normally

distributed dividend at  $t = 2$ . The investors have time-additive exponential utility, and we assume, for simplicity, that they have identical time-preference rates and risk aversion parameters. However, their prior beliefs at  $t = 0$  for the dividend at  $t = 2$  can differ with regard to both the mean and the precision (i.e., the inverse variance or confidence).

It is well known that Pareto efficient allocations in settings with heterogeneous beliefs require not only an efficient sharing of the risks, but also an efficient side-betting arrangement (see, e.g., Wilson 1968). If the investors' prior precisions are identical, then the Pareto efficient side-betting (or speculative positions) based on their disagreements about the mean can be achieved by trading in the risky asset and the zero-coupon bond at  $t = 0$ : The optimistic (pessimistic) investors hold more (less) than their efficient risk sharing fraction of the risky asset.

If the investors have different prior precisions, trading in the risky asset and the zero-coupon bond at  $t = 0$  does not facilitate efficient side-betting: An investor with a low (high) prior precision would like to have a payoff at  $t = 2$  which is a convex (concave) function of the dividend.<sup>3</sup> The key is that investors with low precisions value a convex payoff more than investors with higher precisions and, thus, trading gains can be achieved with non-linear payoffs. Based on the seminal paper, Wilson (1968), we show that if a derivative security in zero net-supply with a payoff at  $t = 2$  equal to the square of the dividend on the risky asset is also available for trade at  $t = 0$ , then the market is effectively complete such that both Pareto efficient risk sharing and side-betting are achieved (see also Brennan and Cao 1996). On the other hand, if this dividend derivative specifically targeted towards the heterogeneity in the investors' prior precisions is not available for trade, then it can be valuable to have public information and another round of trading at the interim date  $t = 1$ .

We consider a simple public information system with a public signal at  $t = 1$  equal to the  $t = 2$  dividend on the risky asset plus independent noise. The investors have identical normally distributed beliefs for the noise in the signal, i.e., a zero mean and a common signal precision, such that the investors' posterior precisions for the dividend are equal to their heterogeneous prior dividend precisions plus the common signal precision. This specification allows us to measure the informativeness of the public information system by the signal precision. Hence, while we assume the investors may disagree about the fundamentals in the economy (i.e., the dividends), we assume the investors have homogeneous beliefs about the noise in the information system, i.e., the investors have so-called *concordant beliefs* (Milgrom and Stokey 1982) or *homogeneous information beliefs* (Hakansson, Kunkel,

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<sup>3</sup>Note that this is similar to so-called Gamma strategies in derivatives pricing and risk management (see, e.g., Hull 2009, Chapter 17). However, while the Black-Scholes model can accommodate differences in expected returns, it does not allow for heterogeneous volatilities among investors on the underlying asset.

and Ohlson 1982).<sup>4</sup> This is in contrast to the growing so-called differences-of-opinion literature in which the investors have homogeneous beliefs about the fundamentals in the economy, but disagree on how to interpret common public signals.<sup>5</sup> This literature is mainly targeted towards explaining empirical stylized facts for the relationship between trading volume and stock returns, whereas our model allows us to investigate the relationship between the informativeness of the public information system and the equilibrium asset prices and investor welfare (in addition to trading volume).

If the investors have homogeneous prior dividend precisions, there will be no equilibrium trading at  $t = 1$  contingent on the public signal. If they also have an identical prior mean, they hold on to the efficient risk sharing fraction of the risky asset after trading at  $t = 0$ , while disagreements about the mean and the associated efficient side-betting is facilitated by trading at  $t = 0$  (as noted above). However, if the investors have heterogeneous prior dividend precisions, they update their posterior beliefs differently, and this gives the basis for additional trading gains contingent on the public signal. In particular, the equilibrium investor demand for the risky asset at  $t = 1$  is an increasing (decreasing) function of the public signal for investors with a lower (higher) prior dividend precision than the investors' average prior dividend precision. Since the public signal is equal to the dividend plus noise, investors with low (high) prior dividend precisions will, in equilibrium, achieve a payoff at  $t = 2$  which is a convex (concave) function of the dividend on the risky asset. Hence, another round of trading in the risky asset (and the zero-coupon bond) contingent on the public information at  $t = 1$  *partly* facilitates the efficient side-betting based on the heterogeneity in prior dividend precisions.

However, the investors' equilibrium payoffs at  $t = 2$  are also affected by the independent noise in the public signal, which implies that the additional side-betting opportunities come with a cost. Moreover, reducing the variance of the noise in the public signal (and, thus, increasing the signal precision) reduces the heterogeneity in the investors' posterior beliefs as well as the risk premium in the equilibrium price of the risky asset. In the limit with a perfect public signal, there will be no equilibrium trading at  $t = 1$ , since the risky asset and the zero-coupon bond become perfect substitutes. Consequently, the trading gains decrease if the signal precision becomes too high. We show that the trading gains are maximized with an imperfect public information system with a signal precision equal to the investors'

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<sup>4</sup>This assumption ensures that Pareto efficient allocations will only include side-betting on the public signal to the extent that it is informative about the fundamentals and not because it is informative about payoff-irrelevant events (see, e.g., the discussion in Christensen and Feltham 2003, Appendix 4A).

<sup>5</sup>This literature includes Harrison and Kreps (1978), Varian (1985, 1989), Harris and Raviv (1993), Kandel and Pearson (1995), Scheinkman and Xiong (2003), Cao and Ou-Yang (2009), Banerjee and Kremer (2010), and Bloomfield and Fischer (2011), among others.



average prior dividend precision. This is also the information system which has the maximum expected abnormal trading volume at  $t = 1$ .

The trading gains following from an imperfect public signal at  $t = 1$  translate directly into higher *ex ante* certainty equivalents of the investors'  $t = 2$  consumption, and this reduces the demand for the zero-coupon bond at  $t = 0$  and, thus, increases the equilibrium interest rate from  $t = 0$  to  $t = 2$ .<sup>6</sup> Hence, the equilibrium interest rate is also maximized for the public information system with a signal precision equal to the investors' average prior dividend precision. Since the aggregate consumption at  $t = 0$  is equal to the exogenous  $t = 0$  dividend on the risky asset, and the investors' trading gains are maximized for this information system, this is also the *unconstrained* Pareto preferred public information system.

However, the investors may not *unanimously* prefer this system over public information systems with different signal precisions. Of course, it is voluntary for the investors to refrain from trading at  $t = 1$ , for example, an investor with a prior dividend precision equal to the investors' average prior dividend precision does not engage in signal-contingent trading at  $t = 1$ . However, the equilibrium interest rate affects the equilibrium asset prices at  $t = 0$  and, therefore, the equilibrium value of the investors' endowments. A low asset price due to a high equilibrium interest rate is of course good if the investor wants to increase the holding of the asset at  $t = 0$ , but it is bad if the investor wants to reduce the holding of the asset. Hence, the individual investors' preferences over public information systems depend on their trading gains (which in turn depend on the absolute difference between their personal prior dividend precision and the investors' average prior dividend precision), and on their endowments of the zero-coupon bond and the risky asset relative to their equilibrium holdings of these assets at  $t = 0$ . We show how the investors' endowments can be re-allocated (for example, due to a prior round of trading) such that all investors unanimously support the unconstrained Pareto efficient public information system.

In this paper, the heterogeneous prior beliefs are specified exogenously, and it is common knowledge that investors have different beliefs. However, our analysis can be extended to certain Hellwig-type noisy rational expectations equilibrium settings in which the heterogeneous beliefs are *equilibrium* posterior beliefs resulting from an initial trading round based on homogeneous prior dividend beliefs, diverse private signals for a continuum of rational investors, and a noisy supply of the risky asset (see, e.g., Grundy and McNichols 1989, Kim and Verrecchia 1991a, Kim and Verrecchia 1991b, and Brennan and Cao 1996). It is well known that these models have a multiplicity of linear equilibria (while our model

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<sup>6</sup>We assume, for simplicity, that there is no consumption at the interim date  $t = 1$  and, thus, only the equilibrium interest rate from  $t = 0$  to  $t = 2$  has any substance (and not how that interest rate is divided between the two periods).

has a unique equilibrium). Some of these equilibria are fully revealing following subsequent trading rounds based on independent public signals given the dividend (and, thus, do not involve any trading), while there is one linear equilibrium which is only partially revealing and, thus, involves non-trivial trading among rational investors. Of course, the former type of equilibria are deemed “unappealing” if trading volume is the subject under investigation and, thus, this literature focus on the latter.

The key property of the linear partially revealing rational expectations equilibrium is that the rational investors cannot make better inferences about the private information/noise relationship in the equilibrium price of the risky asset as subsequent public signals are released (since, otherwise, the equilibrium price would be fully revealing).<sup>7</sup> This means that the investors react parametrically on equilibrium prices in subsequent trading rounds. Hence, it makes no difference for the impact of public information whether the heterogeneous prior beliefs are specified exogenously (as in our model) or these beliefs are equilibrium posterior beliefs following an initial trading round based on diverse private signals and a noisy supply. Consequently, the results we obtain for the impact of public information for efficient side-betting on trading volume are very similar to the corresponding results in this noisy rational expectations equilibrium literature.

The noisy rational expectations equilibrium literature relies on the introduction of unmodelled noise/liquidity trading. As pointed out by Cao and Ou-Yang (2009, page 303), a “potential problem with this approach is that the argument to explain trading volume is circular: it essentially requires new exogenous supply shocks to the stock to generate trading volume. In this sense, trading is imposed onto the economy rather than endogenously generated.” Furthermore, since these models are single-date consumption models, public information has no impact on *ex ante* risk premia and interest rates and, thus, no impact on the *ex ante* cost of capital and the *ex ante* stock price.

The rest of the paper is organized as follows. Section 2 presents the model and derives the equilibrium asset prices and asset demands in the incomplete market economy with the zero-coupon bond and the single risky asset as the only marketed securities. Section 3 establishes the relationship between the informativeness of the public information system and the equilibrium asset prices, the *ex ante* cost of capital, the expected abnormal trading volume, and the investors’ welfare in the incomplete market economy. The effectively complete market is introduced in Section 4. Section 5 concludes with some brief remarks on the empirical

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<sup>7</sup>This condition requires that the independent noise terms in the subsequent public signals must all be independent of the noise terms in the investors’ diverse private signals. Hence, these models do not allow for subsequent public signals being sufficient statistics for earlier private information with respect to the dividend as in the Grossman and Stiglitz type model of Demski and Feltham (1994), which in turn leads to homogeneous posterior beliefs and an efficient risk sharing following the public signal.

and policy implications of our analysis.

## 2 The Model

In our basic incomplete market model, we examine the impact of heterogeneity in prior beliefs and signal precision on equilibrium asset prices, trading volume, and investor welfare for a two-period economy in which investors have identical preferences but differ in their prior beliefs about the dividends on a single risky asset. The following two subsections describe the model and the equilibrium, respectively.

### 2.1 Investor Beliefs and Preferences

There are two consumption dates,  $t = 0$  and  $t = 2$ , and there are  $I$  investors who are endowed at  $t = 0$  with a portfolio of securities, potentially receive public information at  $t = 1$ , and receive terminal normally-distributed dividends from their portfolio of securities at  $t = 2$ . The trading of the marketed securities takes place at  $t = 0$  and  $t = 1$  based on heterogeneous prior and posterior beliefs, respectively. There are two securities available for trade at  $t = 0$  and  $t = 1$ : a zero-coupon bond that pays one unit of consumption at  $t = 2$  and is in zero net-supply, and the shares of a single risky asset that has a fixed non-zero net-supply  $Z$  throughout. The investors are endowed with  $\bar{\gamma}_i$  units of the  $t = 2$  zero-coupon bond and  $\bar{z}_i$  shares of the risky asset,  $i = 1, 2, \dots, I$ . In addition, the investors are endowed with  $\bar{\kappa}_i$  units of a zero-coupon bond, also in zero net-supply, paying one unit of consumption at  $t = 0$ . Let  $\gamma_{it}$  and  $x_{it}$  represent the units held by investor  $i$  of the  $t = 2$  zero-coupon bond and the risky asset after trading at date  $t$ , respectively. The market clearing conditions at date  $t$  are

$$\sum_{i=1}^I \gamma_{it} = 0, \quad \sum_{i=1}^I x_{it} = Z, \quad t = 0, 1.$$

A share of the risky asset pays a dividend  $d_0$  at date  $t = 0$  and a dividend  $d$  at date  $t = 2$ . We assume the investors have heterogeneous prior beliefs with respect to the  $t = 2$  dividend represented by  $\varphi_i(d) \sim N(m_i, \sigma_i^2)$ ,  $i = 1, \dots, I$ , where  $m_i$  is the expected dividend per share and  $\sigma_i^2$  is the variance of the dividend per share for investor  $i$ .

At  $t = 1$ , all investors receive a public signal  $y$  from an information system  $\eta$ , which is jointly normally distributed with the dividend paid by the risky asset at  $t = 2$ . The public signal is given as the dividend plus noise, i.e.,  $y = d + \varepsilon$ , where  $\varepsilon$  and  $d$  are independent and  $\varphi(\varepsilon) \sim N(0, \sigma_\varepsilon^2)$ . We refer to  $h_\varepsilon \equiv 1/\sigma_\varepsilon^2$  as the common signal precision, and we use  $h_{(\cdot)} \equiv 1/\sigma_{(\cdot)}^2$  throughout to denote precisions for the associated variances. Hence, while

the investors may disagree about the fundamentals in the economy (i.e., the dividends), we assume the investors have homogeneous beliefs about the noise in the information system, i.e., the investors have *concordant beliefs* (Milgrom and Stokey 1982) or *homogeneous information beliefs* (Hakansson, Kunkel, and Ohlson 1982). As noted in the Introduction, this is in contrast to, for example, Cao and Ou-Yang (2009), Banerjee and Kremer (2010), and Bloomfield and Fischer (2011), who assume that the investors have homogeneous beliefs about the fundamentals, i.e., dividends and earnings, but disagree on how to interpret public disclosures about these fundamentals. Our specification of the heterogeneity in beliefs allows us to ask how the informativeness of the public information, i.e., the signal precision  $h_\varepsilon$ , affects the equilibrium asset prices, the trading volume, and the investors' welfare.

The prior beliefs of investor  $i$  for the public signal and the dividend is  $\varphi_i(y, d) \sim N(\mu_i, \Sigma_i)$ , where

$$\mu_i = \begin{pmatrix} m_i \\ m_i \end{pmatrix}, \quad \Sigma_i = \begin{pmatrix} \sigma_i^2 + \sigma_\varepsilon^2 & \sigma_i^2 \\ \sigma_i^2 & \sigma_i^2 \end{pmatrix}.$$

Hence, conditional on the public signal, the posterior beliefs of investor  $i$  at  $t = 1$  about the dividend is  $\varphi_{i1}(d | y) \sim N(m_{i1}, \sigma_{i1}^2)$ , where

$$m_{i1} = \omega_i y + (1 - \omega_i) m_i, \quad \omega_i = \frac{\sigma_i^2}{\sigma_i^2 + \sigma_\varepsilon^2}, \quad (1a)$$

$$\sigma_{i1}^2 = \omega_i \sigma_\varepsilon^2, \quad h_{i1} = h_i + h_\varepsilon. \quad (1b)$$

The posterior mean is a linear function of the investors' signal, while the posterior variance only depends on the informativeness of the information system and not on the specific signal. Investor  $i$ 's prior distribution with respect to the posterior mean  $m_{i1}$ , i.e., the pre-posterior beliefs, is a normal distribution with a mean equal to the prior mean  $m_i$  of the dividend and variance  $\sigma_{i0}^2 = \sigma_i^2 - \sigma_{i1}^2$ , i.e.,  $\varphi(m_{i1}) \sim N(m_i, \sigma_{i0}^2)$ .

The investors trade in the zero-coupon bond with equilibrium price  $\beta_0$  at  $t = 0$  and  $\beta_1$  at  $t = 1$ . We assume without loss of generality that  $\beta_1 = 1$  since there is no consumption at  $t = 1$ . The equilibrium price of the risky asset at  $t = 0$  is denoted  $p_0(\eta)$ , which reflects the fact that the *ex ante* price at  $t = 0$  may be affected by the public information system  $\eta$ . The *ex post* equilibrium price of the risky asset at  $t = 1$  given the public signal  $y$  is denoted  $p_1(y)$ .

Investor  $i$ 's consumption at date  $t = 0$  and  $t = 2$  is denoted  $c_{it}$  and we assume the investors have time-additive utility. The investors have common period-specific exponential utility functions, i.e.,  $u_{i0}(c_{i0}) = -\exp[-rc_{i0}]$  and  $u_{i2}(c_{i2}) = -\exp[-\delta] \exp[-rc_{i2}]$ , where  $r > 0$  is the investors' common constant absolute risk aversion parameter, and  $\delta$  is the common utility discount rate for date  $t = 2$  consumption. Our results are qualitatively

unaffected by allowing investors to have different risk aversion parameters and different utility discount rates.

## 2.2 Equilibrium with Public Information and Heterogeneous Beliefs

In this section, we derive the equilibrium in the economy with heterogeneous beliefs, public information and trading in the zero-coupon bond and the single risky asset. There are two rounds of trading: one round of trading at  $t = 0$  prior to the release of information, and a second round of trading subsequent to the release of the public signal at  $t = 1$ . We solve for the equilibrium by first deriving the equilibrium prices at  $t = 1$ , and given this equilibrium, we can subsequently derive the equilibrium prices at  $t = 0$ .

### 2.2.1 Equilibrium prices at date $t = 1$

From the perspective of  $t = 1$ , date  $t = 2$  consumption for investor  $i$  is

$$c_{i2} = x_{i1}d + \gamma_{i1},$$

and is thus normally distributed given the public signal  $y$  at  $t = 1$ . Investor  $i$  maximizes his certainty equivalent of  $t = 2$  consumption subject to his budget constraint, and given period-specific exponential utility this can be expressed as

$$\begin{aligned} \max_{x_{i1}, \gamma_{i1}} \text{CE}_{i2} (x_{i1}, \gamma_{i1} \mid y, \gamma_{i0}, x_{i0}) \\ = \max_{x_{i1}, \gamma_{i1}} \gamma_{i1} + m_{i1}x_{i1} - \frac{1}{2}r\sigma_{i1}^2x_{i1}^2, \\ \text{subject to } \gamma_{i1} + p_1(y)x_{i1} \leq \gamma_{i0} + p_1(y)x_{i0}. \end{aligned}$$

The first-order conditions imply that the optimal portfolio at  $t = 1$  given investor  $i$ 's posterior beliefs is

$$x_{i1}(y) = \rho h_{i1} (m_{i1}(y) - p_1(y)), \quad (2a)$$

$$\gamma_{i1}(y) = \gamma_{i0} + p_1(y)x_{i0} - p_1(y)x_{i1}(y), \quad (2b)$$

where  $\rho \equiv 1/r$  is the investors' common risk tolerance and  $h_{i1} = h_i + h_\varepsilon$  is investor  $i$ 's

posterior precision for the terminal dividend. Market clearing at date  $t = 1$  implies that

$$\begin{aligned} \sum_{i=1}^I \rho h_{i1} (m_{i1}(y) - p_1(y)) &= Z \Leftrightarrow \\ p_1(y) &= \bar{m}_1^h(y) - r\bar{\sigma}_1^2 Z/I, \end{aligned} \quad (3)$$

where  $\bar{m}_1^h(y)$  is the precision weighted average of the investors' posterior means, i.e.,

$$\bar{m}_1^h(y) \equiv \frac{1}{I} \sum_{i=1}^I \frac{h_{i1}}{\bar{h}_1} m_{i1}(y), \quad \bar{h}_1 \equiv \frac{1}{I} \sum_{i=1}^I h_{i1},$$

and  $\bar{\sigma}_1^2$  is the inverse of the average posterior precision, i.e.,  $\bar{\sigma}_1^2 \equiv 1/\bar{h}_1$ .

Inserting the equilibrium price of the risky asset into investor  $i$ 's demand function in (2a) yields

$$x_{i1}^*(y) = \rho h_{i1} (m_{i1}(y) - [\bar{m}_1^h(y) - r\bar{\sigma}_1^2 Z/I]). \quad (4)$$

The posterior beliefs, i.e.,  $m_{i1}(y)$  and  $h_{i1}$ , are functions of the priors and the signal precision. Hence, the equilibrium price of the risky asset and the equilibrium demand functions at date  $t = 1$  are affected by both the priors, the signal precision and, moreover, they are linear functions of the public signal (through the posterior means,  $m_{i1} = \omega_i y + (1 - \omega_i) m_i$ ), which implies that, in general, there is non-trivial trading at  $t = 1$  in equilibrium. Note, however, that if the investors have homogeneous prior precisions (such that  $\omega_i = \omega$  and  $h_{i1} = \bar{h}_1$  for all  $i$ ), the equilibrium demand is independent of the public signal.

Consider the two extreme cases for the signal precision separately. If the public signal is a perfect signal of the dividend, i.e.,  $h_\varepsilon \rightarrow \infty$  ( $\sigma_\varepsilon^2 = 0$ ), the investors get to know the realization of the dividend already at  $t = 1$  before any second-round trading can occur. In this case, no arbitrage implies that

$$p_1(y) = y = d.$$

That is, the equilibrium asset price at  $t = 1$  is equal to the dividend and, thus, independent of the prior beliefs (recall that the equilibrium interest rate from  $t = 1$  to  $t = 2$  is normalized to zero).

When the signal tends to be uninformative, i.e.,  $h_\varepsilon = 0$  ( $\sigma_\varepsilon^2 \rightarrow \infty$ ), the posterior beliefs are equal to the prior beliefs and, thus,

$$p_1(y) = \bar{m}^h - r\bar{\sigma}^2 Z/I,$$

where

$$\bar{m}^h \equiv \frac{1}{I} \sum_{i=1}^I \frac{h_i}{\bar{h}} m_i, \quad \bar{h} \equiv \frac{1}{I} \sum_{i=1}^I h_i, \quad \bar{\sigma}^2 \equiv \frac{1}{\bar{h}}.$$

In this case, the *ex post* asset price is, of course, independent of the signal but is a function of the priors; and it is given as a precision weighted average of the investors' prior mean minus a risk premium determined by the average prior precision. Moreover, with homogeneous prior beliefs, i.e.,  $m_i = m$ , and  $\sigma_i^2 = \sigma^2$ , for  $i = 1, 2, \dots, I$ , we have

$$p_1(y) = m - r\sigma^2 Z/I,$$

which is the standard no-information exponential-utility/normal-distribution version of the CAPM.

### 2.2.2 Equilibrium prices at date $t = 0$

We now determine the equilibrium *ex ante* prices and demand functions at  $t = 0$ , taking the equilibrium at  $t = 1$  characterized by Equations (3) and (4) as given. From the perspective of  $t = 0$ , investor  $i$ 's date  $t = 2$  consumption is

$$c_{i2} = [d - p_1(y)] x_{i1}^*(y) + p_1(y)x_{i0} + \gamma_{i0},$$

and investor  $i$ 's date  $t = 0$  consumption is

$$c_{i0} = [p_0(\eta) + d_0] \bar{z}_i + \beta_0 \bar{\gamma}_i + \bar{\kappa}_i - p_0(\eta)x_{i0} - \beta_0 \gamma_{i0}.$$

Conditional on the public signal at  $t = 1$ , investor  $i$ 's  $t = 1$  certainty equivalent of  $t = 2$  consumption is

$$\text{CE}_{i2}(x_{i0}, \gamma_{i0}, x_{i1}^*(y) | y) = \gamma_{i0} + p_1(y)x_{i0} + [m_{i1} - p_1(y)] x_{i1}^*(y) - \frac{1}{2} r \sigma_{i1}^2 (x_{i1}^*(y))^2. \quad (5)$$

Note that from the perspective of  $t = 0$ , the second term in  $\text{CE}_{i2}(x_{i0}, \gamma_{i0}, x_{i1}^*(y) | y)$  is a normally-distributed variable, while the last two terms contain products of normally-distributed variables if  $x_{i1}^*(y)$  varies with the public signal at  $t = 1$  (in which case it is a non-degenerate normally distributed variable). Substituting in the equilibrium demand functions and the equilibrium price of the risky asset at  $t = 1$ , i.e., equations (4) and (3), yields the following result.

**Remark 1** *Investor  $i$ 's  $t = 0$  certainty equivalent of  $t = 2$  consumption is given by*

$$\text{CE}_{i2}(x_{i0}, \gamma_{i0}) = \gamma_{i0} + U_{1i} + U_{2i} + M_i x_{i0} - \frac{1}{2} r V_i x_{i0}^2, \quad (6)$$

where

$$U_{1i} = \frac{1}{2} \rho \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_\varepsilon}{(\bar{h} + h_\varepsilon)^2} \right], \quad (7a)$$

$$U_{2i} = \frac{1}{2} \rho \frac{h_i [m_i \bar{h} - \bar{h} \bar{m}^h + r Z / I]^2}{\bar{h}^2 + h_i h_\varepsilon}, \quad (7b)$$

$$M_i = \frac{h_\varepsilon h_i m_i + \bar{h}^2 \bar{m}^h - r \bar{h} Z / I}{\bar{h}^2 + h_\varepsilon h_i}, \quad (7c)$$

$$V_i = \frac{h_\varepsilon}{\bar{h}^2 + h_\varepsilon h_i}. \quad (7d)$$

The certainty equivalent  $\text{CE}_{i2}(x_{i0}, \gamma_{i0})$  can be expressed as a constant, i.e.,  $\gamma_{i0} + U_{1i} + U_{2i}$ , plus the certainty equivalent of  $x_{i0}$  units of a normally-distributed dividend with mean  $M_i$  and variance  $V_i$ . Since there are no wealth effects with exponential utility, the investor's demand at  $t = 0$  for the risky asset is the same as in a single-period model with this prior mean and variance of a normally distributed dividend. However, note that these priors reflect that there will be a second round of trading at  $t = 1$  based on the public signal. The term  $U_{1i}$  is a function of the signal precision, but as we shall see below (as part of Proposition 2), in equilibrium, the term  $U_{2i} + M_i x_{i0} - \frac{1}{2} r V_i x_{i0}^2$  is independent of the signal precision. Thus, the signal precision affects the equilibrium prices and the equilibrium investor welfare only through the terms  $U_{1i}$ .

With the investors'  $t = 0$  certainty equivalent of their  $t = 2$  consumption determined, investor  $i$ 's decision problem at  $t = 0$  can be stated as follows

$$\max_{\gamma_{i0}, x_{i0}} - \exp(-r \text{CE}_{i0}(x_{i0}, \gamma_{i0})) - \exp(-\delta) \exp(-r \text{CE}_{i2}(x_{i0}, \gamma_{i0})),$$

where

$$\text{CE}_{i0}(x_{i0}, \gamma_{i0}) = [p_0(\eta) + d_0] \bar{z}_i + \beta_0 \bar{\gamma}_i + \bar{\kappa}_i - p_0(\eta) x_{i0} - \beta_0 \gamma_{i0}.$$

The first-order condition for investments in the zero-coupon bond is

$$\begin{aligned} -r \exp(-r \text{CE}_{i0}(x_{i0}, \gamma_{i0})) \beta_0 + r \exp(-\delta) \exp(-r \text{CE}_{i2}(x_{i0}, \gamma_{i0})) &= 0 \Leftrightarrow \\ \iota &= \delta + r (\text{CE}_{i2}(x_{i0}, \gamma_{i0}) - \text{CE}_{i0}(x_{i0}, \gamma_{i0})), \end{aligned} \quad (8)$$



where  $\iota \equiv -\ln \beta_0$  is the zero-coupon interest rate from  $t = 0$  to  $t = 2$ . The first-order condition for investments in the risky asset is

$$-r \exp(-r\text{CE}_{i0}(x_{i0}, \gamma_{i0})) p_0(\eta) + r \exp(-\delta) \exp(-r\text{CE}_{i2}(x_{i0}, \gamma_{i0})) [M_i - rV_i x_{i0}] = 0.$$

Hence, the *ex ante* price and the demand for the risky asset at  $t = 0$  can be expressed as

$$p_0(\eta) = \beta_0 [M_i - rV_i x_{i0}], \quad (9a)$$

$$x_{i0} = \rho \frac{M_i - R_0 p_0(\eta)}{V_i}, \quad (9b)$$

where  $R_0 = 1/\beta_0$ . Thus, the market clearing condition for the risky asset implies that its equilibrium price at  $t = 0$  is

$$\begin{aligned} \sum_{i=1}^I x_{i0} &= Z \Leftrightarrow \\ p_0(\eta) &= \beta_0 [\bar{M}^v - r\bar{V}Z/I], \end{aligned} \quad (10)$$

where

$$\bar{M}^v \equiv \frac{1}{I} \sum_{i=1}^I \frac{v_i}{\bar{v}} M_i, \quad v_i \equiv V_i^{-1}, \quad \bar{v} \equiv \frac{1}{I} \sum_{i=1}^I v_i, \quad \bar{V} \equiv \bar{v}^{-1}.$$

In other words, the equilibrium price of the risky asset is equal to its discounted “risk-adjusted expected dividend,” where the latter is defined as

$$\mathbb{E}^Q[d] \equiv \bar{M}^v - r\bar{V}Z/I.$$

The following proposition shows properties of the risk-adjusted expected dividend.

**Proposition 1** *The ex ante equilibrium price of the risky asset at  $t = 0$  is equal to the equilibrium riskless discount factor times the risk-adjusted expected dividend, i.e.,*

$$p_0(\eta) = \beta_0 \mathbb{E}^Q[d]. \quad (11)$$

*The risk-adjusted expected dividend is independent of the information system, and it can be*

expressed as a function of the prior means and variances, i.e.,<sup>8</sup>

$$E^Q[d] = \bar{m}^h - r\bar{\sigma}^2 Z/I. \quad (12)$$

Hence, given the priors, the risk-adjusted expected dividend is independent of the information system at  $t = 1$  and, in particular, it is determined entirely by the prior beliefs *as if* there would be no second round of trading at  $t = 1$ . In other words, the informativeness of the public signal at  $t = 1$  affects the *ex ante* equilibrium asset price only through the impact on the equilibrium interest rate.

Substituting the *ex ante* equilibrium price of the risky asset (11) into the demand functions (9b), we obtain the investors' equilibrium demand for the risky asset at  $t = 0$ :

$$x_{i0}^* = \rho V_i^{-1} [M_i - E^Q[d]]. \quad (13)$$

Substitution of  $M_i$ ,  $V_i$  and  $E^Q[d]$ , and simplifying yield the following result.

**Remark 2** *In equilibrium, investor  $i$ 's  $t = 0$  equilibrium demand for the risky asset is given by*

$$x_{i0}^* = \rho h_i [m_i - E^Q[d]]. \quad (14)$$

Note that the equilibrium demand for the risky asset is the same as in an otherwise identical economy in which there is no public information at  $t = 1$ . In other words, the investors' equilibrium demands are *myopic*, independently of the informativeness of the forthcoming public signal. The equilibrium demand is increasing in the investors' prior mean and in the prior dividend precision such that the more optimistic and confident investors invest more in the risky asset than the more pessimistic and less confident investors. This result is a consequence of the investors' incentive to take speculative positions based on their heterogeneous prior beliefs and, thus, the equilibrium entails "side-betting." With homogeneous priors, however, all investors hold the same efficient risk sharing equilibrium positions in the risky asset, i.e.,  $x_{i0}^* = Z/I$ .

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<sup>8</sup>This means that we can define the risk-adjusted probability measure  $Q$  explicitly such that under  $Q$ , the terminal dividend is normally distributed as  $d \sim N(\bar{m}^h - r\bar{\sigma}^2 Z/I, \bar{\sigma}^2)$ , and the noise  $\varepsilon$  is normally distributed as  $\varepsilon \sim N(0, \sigma_\varepsilon^2)$ . Note that while the expected dividend under  $Q$  is uniquely determined in equilibrium, the variance of the dividend under  $Q$  is not uniquely determined due to the market incompleteness and, thus, we just take it to be  $\bar{\sigma}^2$ . Fortunately, the lack of the uniqueness of the variance has no consequences in the subsequent analysis.

Substituting the equilibrium portfolios into the certainty equivalents, we get

$$\text{CE}_{i0}^* = [p_0(\eta) + d_0] \bar{z}_i + \beta_0 \bar{\gamma}_i + \bar{\kappa}_i - p_0(\eta) x_{i0}^* - \beta_0 \gamma_{i0}^*, \quad (15a)$$

$$\text{CE}_{i2}^* = \gamma_{i0}^* + U_{1i} + U_{2i} + M_i x_{i0}^* - \frac{1}{2} r V_i (x_{i0}^*)^2. \quad (15b)$$

Substituting the equilibrium certainty equivalents into the expression for the interest rate (8), we obtain

$$\iota = \delta + r (\text{CE}_{i2}^* - \text{CE}_{i0}^*). \quad (16)$$

Using the market clearing conditions for the riskless and risky asset, and simplifying yield the equilibrium interest rate.

**Proposition 2** *The equilibrium interest rate is given by*

$$\iota = \delta + r \bar{U}_1 + \Phi \left( \{m_i, \sigma_i^2\}_{i=1, \dots, I} \right), \quad (17)$$

where

$$\bar{U}_1 \equiv \frac{1}{I} \sum_{i=1}^I U_{1i} = \frac{1}{2} \rho \frac{1}{I} \sum_{i=1}^I \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_\varepsilon}{(\bar{h} + h_\varepsilon)^2} \right], \quad (18)$$

and  $\Phi(\cdot)$  is a function of the priors but independent of the signal precision,

$$\Phi \left( \{m_i, \sigma_i^2\}_{i=1, \dots, I} \right) \equiv r [\bar{m}^h - d_0] Z/I - \frac{1}{2} r^2 \bar{\sigma}^2 (Z/I)^2 + \frac{1}{2} \frac{1}{I} \sum_{i=1}^I h_i m_i^2 - \frac{1}{2} (\bar{m}^h)^2 \bar{h}. \quad (19)$$

If the investors have homogeneous prior expected dividends, i.e.,  $m_i = m$ , then

$$\Phi \left( \{m, \sigma_i^2\}_{i=1, \dots, I} \right) = r [m - d_0] Z/I - \frac{1}{2} r^2 \bar{\sigma}^2 (Z/I)^2. \quad (20)$$

If the investors have homogeneous prior dividend precisions, the equilibrium interest rate is independent of the signal precision.

The equilibrium interest rate is equal to the utility discount rate plus a function of the signal precision and the priors. The function  $\Phi(\cdot)$  is a function of the priors only and, thus, independent of the information system. Hence, the signal precision only affects the equilibrium interest rate and, thus, the equilibrium price of the risky asset (since  $\text{E}^Q[d]$  is independent of  $h_\varepsilon$  by Proposition 1), through the logarithmic terms  $\{U_{1i}\}_{i=1, \dots, I}$ .

If the investors hold homogenous prior precisions (i.e.,  $h_i = \bar{h}$  for all  $i$ ), the logarithmic terms are all equal to zero. Thus, in this case the signal precision does not affect the

equilibrium prices at  $t = 0$ . Moreover, when  $m_i = m$ , and  $\sigma_i^2 = \sigma^2$ , for  $i = 1, 2, \dots, I$ , the equilibrium interest rate can be expressed as

$$\iota = \delta + r(m - d_0)Z/I - \frac{1}{2}r^2\sigma^2(Z/I)^2.$$

Hence, in a benchmark setting with homogeneous prior beliefs, the equilibrium interest rate is given as the utility discount rate plus a risk-adjusted expected dividend growth minus a risk premium for the uncertainty in the dividend growth. Of course, this is the standard expression for the equilibrium interest rate in effectively complete markets with time-additive HARA utilities and homogeneous prior beliefs (see, e.g., Christensen and Feltham 2009). On the other hand, if the investors have homogeneous prior expected dividends, but heterogeneous prior dividend precisions, then there is an additional component to the equilibrium interest rate, i.e.,

$$\iota = \delta + r\bar{U}_1 + r[m - d_0]Z/I - \frac{1}{2}r^2\bar{\sigma}^2(Z/I)^2.$$

This additional component, i.e.,  $r\bar{U}_1$ , depends on the signal precision, and it plays a key role in the following analysis.

### 3 The Impact of Signal Precision on *Ex Ante* Asset Prices, Trading Volume, and Investor Welfare

We are interested in how the informativeness of the public information system, i.e., the signal precision, affects the *ex ante* equilibrium prices, the trading volume, and the investors' *ex ante* expected utilities at  $t = 0$  when the investors hold heterogeneous beliefs including heterogeneous prior means and/or heterogeneous prior dividend precisions. We first examine the impact on the *ex ante* equilibrium prices and the trading volume.

#### 3.1 *Ex ante* Equilibrium Prices and Trading Volume

Proposition 1 establishes that the equilibrium asset prices at  $t = 0$  are only affected by the signal precision through the equilibrium interest rate. Furthermore, Proposition 2 establishes that the equilibrium interest rate is also independent of the signal precision if the investors hold homogeneous prior dividend precisions. This is due to the fact that in this case there is no equilibrium trading at  $t = 1$  based on the public signal (see, e.g., Grundy and McNichols 1989 for a similar result).

**Proposition 3** *When the investors hold identical prior dividend precisions, i.e.,  $h_i = h, i = 1, \dots, I$ , the date  $t = 1$  equilibrium portfolios are independent of the information system and*

equal to the date  $t = 0$  equilibrium portfolios, i.e.,

$$x_{i1}^*(y) = x_{i0}^*, \quad \gamma_{i1}^* = \gamma_{i0}^*.$$

With heterogeneous prior dividend precisions, however, the signal precision plays a key role in determining the equilibrium interest rate and, thus, the equilibrium price of the risky asset at  $t = 0$ . As noted above, the impact of the signal precision on the equilibrium interest rate is only through the logarithmic terms in (17). The following proposition characterizes the equilibrium interest rate as a function of the signal precision.

**Proposition 4** *Assume the investors have heterogeneous prior dividend precisions. The equilibrium interest rate is bell-shaped with respect to the signal precision  $h_\varepsilon$ .<sup>9</sup> The unique maximum for the equilibrium interest rate is attained when  $h_\varepsilon = \bar{h}$ , and its minimum is attained for uninformative information ( $h_\varepsilon = 0$ ) and for perfect information ( $h_\varepsilon \rightarrow \infty$ ).*

The intuition for the result in Proposition 4 can be obtained from equation (16), in which the interest rate is expressed as a linear increasing function of the growth in the investors' certainty equivalents,  $CE_{i2}^* - CE_{i0}^*$ . In equilibrium, all investors have the same growth in certainty equivalents. For the two extreme values of the signal precision ( $h_\varepsilon = 0$  and  $h_\varepsilon \rightarrow \infty$ ) there is no trading at  $t = 1$  based on the public signal: (a) for  $h_\varepsilon = 0$ , no new information is released at  $t = 1$  and, thus, the equilibrium portfolios after trading at  $t = 0$  remain equilibrium portfolios; and (b) when the signal precision increases, the investors' posterior beliefs converge and the risk premium in the equilibrium price of the risky asset decreases, and in the limit for  $h_\varepsilon \rightarrow \infty$  all uncertainty is resolved at  $t = 1$  and, thus, there is no basis for additional trading. On the other hand, for intermediate values of the signal precision ( $h_\varepsilon \in (0, \infty)$ ) there is non-trivial trading based on the public signal at  $t = 1$  if the investors have heterogeneous prior dividend precisions. The source of this trading is that the investors can achieve improved side-betting based on their heterogeneously updated posterior beliefs. These gains to trade translate directly into increased certainty equivalents of  $t = 2$  consumption and, thus, a higher growth in their certainty equivalents, *ceteris paribus*. A highly informative or an almost uninformative public signal at  $t = 1$  yields only limited side-betting benefits and, thus, the highest growth in certainty equivalents is obtained for a unique interior signal precision  $h_\varepsilon = \bar{h}$ . The equilibrium price of the risky asset is the

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<sup>9</sup>Note the equilibrium interest rate looks bell-shaped after the transformation  $x = \ln(1 + h_\varepsilon \cdot 1.5E+07)$ . However, the equilibrium interest rate has actually only one inflection point with respect to the signal precision, i.e.,  $h_{ip} = 2\bar{h}$ . When  $h_\varepsilon \leq h_{ip}$  ( $h_\varepsilon \geq h_{ip}$ ), the second derivative of the equilibrium interest rate with respect to the signal precision is negative (positive) and, thus, the equilibrium interest rate is concave (convex) with respect to the signal precision as the signal precision increases.

product of the equilibrium riskless discount factor and the risk-adjusted expected dividend (which is independent of  $h_\varepsilon$  by Proposition 1) and, thus, the equilibrium price of the risky asset is inverted bell-shaped as a function of  $h_\varepsilon$  with a minimum point at  $h_\varepsilon = \bar{h}$ .

### ***Ex ante* cost of capital**

The *ex ante* cost of capital defined as the expected rate of return on the risky asset is an ambiguous concept in a setting in which the investors have heterogeneous prior means for the dividend on the risky asset. However, we can define the *ex ante* cost of capital as the (continuously compounded) expected rate of return  $\mu^{xa}(\eta)$  using the beliefs implicit in the unambiguous *ex ante* equilibrium price of the risky asset, i.e.,  $\varphi^h(d) \sim N(\bar{m}^h, \bar{\sigma}^2)$ ,

$$\exp(\mu^{xa}(\eta)) \equiv \frac{\bar{m}^h}{p_0(\eta)}. \quad (21)$$

Inserting the *ex ante* equilibrium price of the risky asset (11), and using Proposition 1 we get that

$$\mu^{xa}(\eta) = \iota + \varpi^{xa},$$

where the risk premium  $\varpi^{xa}$  is given by

$$\varpi^{xa} = \ln \left( 1 + \frac{r\bar{\sigma}^2 Z/I}{\bar{m}^h - r\bar{\sigma}^2 Z/I} \right).$$

Hence, the *ex ante* cost of capital for the risky asset  $\mu^{xa}(\eta)$  is equal to the equilibrium interest rate plus a risk premium  $\varpi^{xa}$ , which is independent of the informativeness of the public signal.<sup>10</sup> Propositions 2 and 4 then imply that the *ex ante* cost of capital is minimized for no public information ( $h_\varepsilon = 0$ ) and for perfect public information ( $h_\varepsilon \rightarrow \infty$ ), while it is maximized for a unique interior signal precision  $h_\varepsilon = \bar{h}$ . Is a low *ex ante* cost of capital good or bad for the investors? We address this question in the following subsection.

In order to illustrate our results we use the following three-investor example throughout with the parameters given in Table 1.

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<sup>10</sup>Similarly, if we define investor-specific expected rates of returns based on their prior dividend beliefs, i.e.,  $\exp(\mu_i^{xa}(\eta)) \equiv m_i/p_0(\eta)$ , then the investor-specific “equity premiums,” i.e.,  $\varpi_i^{xa} \equiv \mu_i^{xa}(\eta) - \iota = \ln(m_i/[\bar{m}^h - r\bar{\sigma}^2 Z/I])$ , are also independent of the informativeness of the public signal.

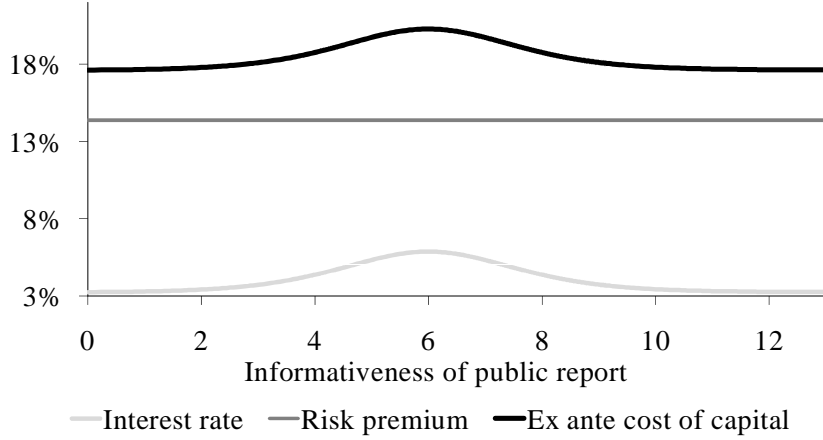


Figure 1: Equilibrium interest rate, risk premium, and *ex ante* cost of capital as functions of the signal precision  $h_\varepsilon$  given the parameters in Table 1. The scale on the horizontal axis is  $x = \ln(1 + h_\varepsilon \cdot 1.5\text{E}+07)$ .

	Investor 1	Investor 2	Investor 3	Aggregate
Risk aversion ( $r$ )	0.01%	0.01%	0.01%	
Utility discount rate ( $\delta$ )	0.00%	0.00%	0.00%	
Prior mean ( $m_i$ )	800	1,000	1,200	
Prior variance ( $\sigma_i^2$ )	25,000	37,500	75,000	
Initial dividend ( $d_0$ )				950
Supply ( $Z$ )				100

Table 1: Investor and risky asset parameters of the running example.

Figure 1 illustrates the equilibrium interest rate, the risk premium, and the *ex ante* cost of capital as functions of the signal precision for the parameters in Table 1. Note that these quantities are independent of the investors' individual endowments and, thus, even though the investors have heterogeneous beliefs, the equilibrium admits aggregation as with HARA utilities in effectively complete markets with homogeneous beliefs.

## Trading Volume

The source of the increased growth in the investors' certainty equivalents is the trading gains based on the investors' heterogeneously updated posterior beliefs. In this section we demonstrate that the signal precision, which maximizes the trading gains and, thus, the equilibrium interest rate, also maximizes the expected abnormal trading volume.

Investor  $i$ 's equilibrium net-trade in the risky asset at  $t = 1$  is  $\tau_i^*(y) \equiv x_{i1}^*(y) - x_{i0}^*$ , where  $x_{i1}^*(y)$  is given in (4) and  $x_{i0}^*$  is given in (14). Inserting the definitions of the posterior means

and precisions in (4), and simplifying yield the following result (see Kim and Verrecchia 1991a and 1991b for a similar result).

**Remark 3** *Investor  $i$ 's equilibrium net-trade in the risky asset at  $t = 1$  is given by*

$$\tau_i^*(y) = \rho \frac{h_\varepsilon (\bar{h} - h_i)}{\bar{h} + h_\varepsilon} [y - \mathbb{E}^Q[d]], \quad (22)$$

*and the risk-adjusted expected net-trade is equal to zero, i.e.,*

$$\mathbb{E}^Q[\tau_i^*(y)] = 0.$$

Hence, the sensitivity of the investor's equilibrium net-trade increases with the difference between the investor's prior dividend precision  $h_i$  and the average prior dividend precision  $\bar{h}$ . Furthermore, the equilibrium net-trade of the investors with lower (higher) prior dividend precisions than the average is increasing (decreasing) in  $y = d + \varepsilon$  and, thus, their  $t = 2$  consumption is a convex (concave) function of  $d$ . This relationship between the public signal and  $t = 2$  consumption is the source of the improved side-betting opportunities following from the fact that investors with low prior dividend precisions value convex payoffs more than investors with high prior dividend precisions.

Even though the risk-adjusted expected net-trade is equal to zero, the investors' expected net-trade is *not* equal to zero, and it depends on their personal prior dividend beliefs. Therefore, in order to investigate the impact of the signal precision on the expected trading volume in the securities market as a whole, we define the *abnormal* net-trade of investor  $i$  as the difference between the net-trade and the expected net-trade conditional on the  $t = 2$  dividend, i.e.,

$$a\tau_i^*(y) \equiv \tau_i^*(y) - \mathbb{E}[\tau_i^*(y) | d] = \rho \frac{h_\varepsilon (\bar{h} - h_i)}{\bar{h} + h_\varepsilon} \varepsilon.$$

Since the investors have concordant beliefs, they have homogeneous beliefs about their abnormal net-trades and, in particular, the abnormal net-trades are normally distributed with a zero mean. Recognizing that some investors are selling while others are buying, the abnormal trading volume per investor is defined as

$$T^* \equiv \frac{1}{2} \frac{1}{I} \sum_{i=1}^I |a\tau_i^*(y)|.$$

The following proposition characterizes the expected abnormal trading volume.



**Proposition 5** *The expected abnormal trading volume is*

$$\mathbb{E}[T^*] = \frac{\sqrt{h_\varepsilon}}{\bar{h} + h_\varepsilon} \frac{\rho}{\sqrt{2\pi}} \frac{1}{I} \sum_{i=1}^I |\bar{h} - h_i|.$$

*Assume the investors have heterogeneous prior dividend precisions. The expected abnormal trading volume is bell-shaped with respect to the signal precision  $h_\varepsilon$ . The unique maximum for the expected abnormal trading volume is attained when  $h_\varepsilon = \bar{h}$ , and its minimum is attained for uninformative information ( $h_\varepsilon = 0$ ) and for perfect information ( $h_\varepsilon \rightarrow \infty$ ).*

The proposition establishes that the expected abnormal trading volume has the same comparative statics as the equilibrium interest rate with respect to the signal precision, cf. Proposition 4.<sup>11</sup> Of course, the key empirical implication is that there is a direct positive relationship between the empirically unobservable growth in certainty equivalents and the expected (average) abnormal trading volume.

### 3.2 *Ex ante* Expected Utilities

The investors' *ex ante* expected utilities are affected in two ways by changes in the signal precision. First, changes in the signal precision affects the gains to trade based on heterogeneously updated posterior beliefs and, thus, the growth in their certainty equivalents as illustrated in the preceding subsection. Secondly, the signal precision affects the *ex ante* equilibrium asset prices through the equilibrium interest rate and, thus, affects the value of the investors' individual endowments. The latter may affect the investors in different ways depending on their individual endowments relative to their equilibrium portfolio at  $t = 0$ . The following lemma (partly) characterizes the impact of changing the signal precision on the investors' *ex ante* expected utilities.

**Lemma 1** *The derivative of the investors' ex ante expected utilities with respect the signal precision  $h_\varepsilon$  is given by*

$$\frac{\partial}{\partial h_\varepsilon} \mathbb{E}U_{i0}^* = r \exp(-r\mathbb{C}\mathbb{E}_{i0}^*) \left\{ \beta_0 \frac{\partial}{\partial h_\varepsilon} U_{1i} + [\gamma_{i0}^* - \bar{\gamma}_i + \mathbb{E}^Q[d](x_{i0}^* - \bar{z}_i)] \frac{\partial}{\partial h_\varepsilon} \beta_0 \right\}, \quad (23)$$

where

$$\frac{\partial}{\partial h_\varepsilon} \beta_0 = -r\beta_0 \frac{\partial}{\partial h_\varepsilon} \bar{U}_1.$$

*All the investors' ex ante expected utilities have a stationary point at  $h_\varepsilon = \bar{h}$ .*

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<sup>11</sup>Consequently, we do not present a figure of the expected abnormal trading volume as a function of the signal precision for the numerical example.

The gains to trade based on the public signal is reflected in the first term in the braces of (23),  $\frac{\partial}{\partial h_\varepsilon} U_{1i}$ , where  $U_{1i}$  is investor  $i$ 's *ex ante* value of the trading gains at  $t = 1$  and is given in (7a). Of course, the investors can always refuse to engage in any second-round trading at  $t = 1$  and, thus, the *ex ante* value of trading gains is always non-negative, and it is maximized for  $h_\varepsilon = \bar{h}$ . Note that an investor with a prior dividend precision  $h_i$  equal to the average dividend precision  $\bar{h}$  has an *ex ante* value of the trading gains equal to zero, i.e.,  $U_{1i} = 0$ . This is due to the fact that the investor does not engage in signal-contingent trading at  $t = 1$  if  $h_i = \bar{h}$  (see equation (22)). On the other hand, all “more confident” and “less confident” investors than the average have a strictly positive *ex ante* value of trading gains, and these *ex ante* values are maximized for all investors exactly at the signal precision at which the equilibrium interest rate is maximized,  $h_\varepsilon = \bar{h}$ .

The positive *ex ante* values of trading gains (which will be realized when consuming at  $t = 2$ ) shift the investors' incentive to consume more at  $t = 0$  (in order to smooth consumption over time) and, thus, increase the equilibrium interest rate. This reduces the value of the investors' (positive) endowments of the  $t = 2$  zero-coupon bond and the risky asset through a reduction in the zero-coupon bond price,  $\beta_0$  (recall that the risk-adjusted expected dividend of the risky asset is independent of the signal precision). This reduction is maximized when the average *ex ante* value of the trading gains  $\bar{U}_1$  is maximized at  $h_\varepsilon = \bar{h}$ . However, the reduction in the asset prices also makes it cheaper to buy these assets. Hence, the impact of the changed equilibrium prices on the investors' consumption possibilities at  $t = 0$  depends on whether the investor wants to increase or decrease the holdings of the assets, i.e., whether  $(\gamma_{i0}^* - \bar{\gamma}_i)$  and  $(x_{i0}^* - \bar{z}_i)$  are positive or negative. These effects are reflected in the second term in the braces of (23). Hence, in general, even though the signal precision has a unique impact on the *ex ante* value of trading gains and on the equilibrium interest rate, the investors' *ex ante* expected utilities may be affected in different directions by changes in the signal precision. In other words, changes in the information system may not result in Pareto improvements or Pareto inferior allocations—some investors may be better off while other investors may be worse off depending on their endowments relative to their equilibrium asset holdings at  $t = 0$ .

In order to illustrate these effects of changes in the signal precision on the *ex ante* expected utilities, we define investor  $i$ 's equilibrium *ex ante* certainty equivalent  $\text{CE}_i^{xa}(h_\varepsilon)$  as

$$-\exp(-r\text{CE}_i^{xa}(h_\varepsilon)) \equiv -\exp(-r\text{CE}_{i0}^*) - \exp(-\delta) \exp(-r\text{CE}_{i2}^*).$$

Of course, there is a positive one-to-one relationship between investor  $i$ 's *ex ante* expected utility and his equilibrium *ex ante* certainty equivalent  $\text{CE}_i^{xa}(h_\varepsilon)$ . Hence, the change in the

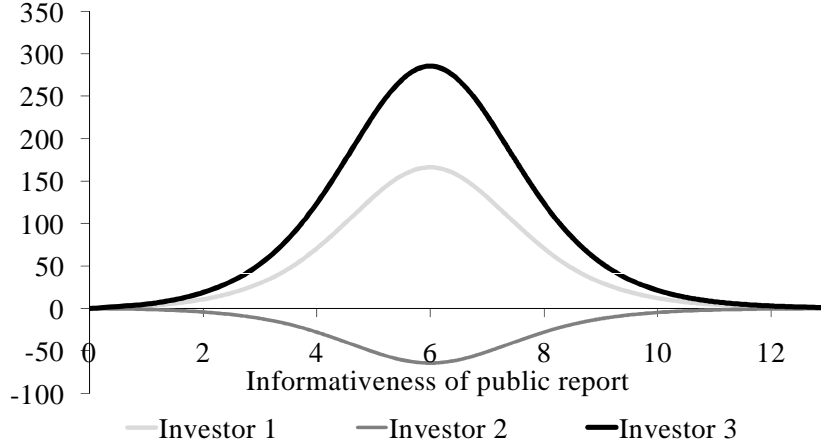


Figure 2: Changes in equilibrium *ex ante* certainty equivalents  $\Delta\text{CE}_i^{xa}(h_\varepsilon)$  as functions of the signal precision  $h_\varepsilon$  given the parameters in Table 1 and endowments  $z_i = Z/3$ ,  $\bar{\gamma}_i = 0$ ,  $i = 1, 2, 3$ , and  $\bar{\kappa}_1 = \bar{\kappa}_3 = 5,000$ ,  $\bar{\kappa}_2 = -10,000$ . The scale on the horizontal axis is  $x = \ln(1 + h_\varepsilon \cdot 1.5\text{E}+07)$ .

equilibrium *ex ante* certainty equivalent relative to the no public information case  $\text{CE}_i^{xa}(h_\varepsilon = 0)$ , i.e.,

$$\Delta\text{CE}_i^{xa}(h_\varepsilon) \equiv \text{CE}_i^{xa}(h_\varepsilon) - \text{CE}_i^{xa}(h_\varepsilon = 0),$$

is a measure of investor  $i$ 's increased *ex ante* expected utility from changing the informativeness of the public information system from being uninformative to having a signal precision of  $h_\varepsilon$ . Figure 2 illustrates these changes in investor welfare for the three-investor example given in Table 1 assuming that the investors have efficient risk sharing endowments of the risky asset and zero endowments of the  $t = 2$  zero-coupon bond, i.e.,  $z_i = Z/3$  and  $\bar{\gamma}_i = 0$ ,  $i = 1, 2, 3$ , while the endowments of the  $t = 0$  zero-coupon bond are  $\bar{\kappa}_1 = \bar{\kappa}_3 = 5,000$  and  $\bar{\kappa}_2 = -10,000$ .

Note that all three investors in Figure 2 have a stationary point for their equilibrium *ex ante* certainty equivalent at  $h_\varepsilon = \bar{h}$  (see Lemma 1). However, while the equilibrium *ex ante* certainty equivalents of investors 1 and 3 are both maximized at  $h_\varepsilon = \bar{h}$ , investor 2's equilibrium *ex ante* certainty equivalent is minimized at that point. Hence, investors 1 and 3 are both better off with an interior signal precision, whereas investor 2 is better off with no public information ( $h_\varepsilon = 0$ ) or perfect information ( $h_\varepsilon \rightarrow \infty$ ). The parameters in Table 1 are such that investor 2 has a prior dividend precision equal to the average prior dividend precision, i.e.,  $h_2 = \bar{h}$ . As noted above, this implies that investor 2 does not engage in signal-contingent trading at  $t = 1$  and, thus, his *ex ante* value of trading gains  $U_{12}$  is equal to zero. Therefore, his equilibrium *ex ante* certainty equivalent is only affected by the changes in the *ex ante* equilibrium asset prices. With the assumed efficient risk sharing endowments of the

risky asset and his negative endowment of the  $t = 0$  zero-coupon bond, investor 2 is a “net seller” of assets (for all levels of  $h_\varepsilon$ ) at  $t = 0$  (i.e.,  $(\gamma_{i0}^* - \bar{\gamma}_i) + E^Q[d](x_{i0}^* - \bar{z}_i) < 0$ ) and, thus, he is hurt by the lower equilibrium asset prices with interior signal precisions. On the other hand, both investor 1 and investor 3 have strictly positive *ex ante* values of trading gains, which exceed any loss from selling assets at the lower equilibrium asset prices for interior signal precisions.

It is well known that even though a Pareto improvement can be achieved by changing the public information system, the change of information system may leave some investors better off and others worse off if the implementation of the equilibrium consumption plans requires trading of assets at equilibrium prices which depend on the information system (see Christensen and Feltham 2003, Chapter 7). The above example in Figure 2 illustrates such a setting. In order to investigate whether there exists a Pareto superior information system, consider a setting in which the investors have “equilibrium endowments.” The growth in certainty equivalents is the same for all investors in equilibrium, and is maximized exactly for the signal precision at which the equilibrium interest rate and, thus, the *ex ante* cost of capital, is maximized, i.e., for  $h_\varepsilon = \bar{h}$ . Hence, this level of signal precision is the obvious candidate for a Pareto efficient information system in a setting in which the endowments of the three assets can be re-allocated among the investors, and this is indeed the case.

In order to see why, consider the investors’ equilibrium portfolios at  $t = 0$ . It follows from (14) that the investors’ equilibrium demand for the risky asset at  $t = 0$ , i.e.,  $x_{i0}^* = \rho h_i [m_i - E^Q[d]]$ , depends on the prior beliefs but is independent of the informativeness of the public signal at  $t = 1$ . However, the investors’ equilibrium demand for the  $t = 2$  zero-coupon bond at  $t = 0$  varies with the signal precision. Consider any given signal precision different from the average prior dividend precision, i.e.,  $h_\varepsilon \neq \bar{h}$ , and the associated equilibrium certainty equivalents,  $\{CE_{i0}^*, CE_{i2}^*\}_{i=1,\dots,I}$ , and equilibrium prices,  $\beta_0, p_0(\eta)$ . We want to show that this system cannot be a Pareto efficient information system. The equilibrium demand for the  $t = 2$  zero-coupon bond  $\gamma_{i0}^*$  is determined by

$$CE_{i0}^* = d_0 \bar{z}_i + p_0(\eta) [\bar{z}_i - x_{i0}^*] + \beta_0 [\bar{\gamma}_i - \gamma_{i0}^*] + \bar{\kappa}_i.$$

As noted above, the equilibrium prices are independent of the investors’ individual endowments. Hence, a re-allocation of the endowments of the three assets defined by

$$\widehat{z}_i \equiv x_{i0}^*, \quad \widehat{\gamma}_i \equiv \gamma_{i0}^*, \quad \widehat{\kappa}_i \equiv CE_{i0}^* - d_0 x_{i0}^*, \quad (24)$$

implies that the investors do not trade at  $t = 0$  given these endowments. That is, the investors have equilibrium endowments relative to the signal precision  $h_\varepsilon$ , and they achieve

the same certainty equivalents as with the original endowments. It then follows from Lemma 1 that

$$\begin{aligned}
\frac{\partial}{\partial h_\varepsilon} \text{EU}_{i0}^* &= r \exp(-r \text{CE}_{i0}^*) \left\{ \beta_0 \frac{\partial}{\partial h_\varepsilon} U_{1i} + \left[ (\gamma_{i0}^* - \widehat{\gamma}_i) + \text{E}^Q[d] (x_{i0}^* - \widehat{z}_i) \right] \frac{\partial}{\partial h_\varepsilon} \beta_0 \right\} \\
&= r \exp(-r \text{CE}_{i0}^*) \beta_0 \frac{\partial}{\partial h_\varepsilon} U_{1i} \\
&= \frac{1}{2} \exp(-r \text{CE}_{i0}^*) \beta_0 \frac{\partial}{\partial h_\varepsilon} \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_\varepsilon}{(\bar{h} + h_\varepsilon)^2} \right].
\end{aligned}$$

Since the common term for all investors  $h_\varepsilon / (\bar{h} + h_\varepsilon)^2$  is a concave function of the signal precision, which is maximized for  $h_\varepsilon = \bar{h}$ , all investors are weakly better off by marginally increasing (decreasing) the signal precision for  $h_\varepsilon < \bar{h}$  ( $h_\varepsilon > \bar{h}$ ). Hence, for any  $h_\varepsilon \neq \bar{h}$  and heterogeneous prior dividend precisions (such that there are investors with  $h_i \neq \bar{h}$ ), there is an allocation of the endowments such that there exists a Pareto superior equilibrium with a marginal change in the signal precision. If  $h_\varepsilon = \bar{h}$ , no such Pareto improvements can be obtained, since  $\frac{\partial}{\partial h_\varepsilon} U_{1i} = 0$  for all investors in this case. These arguments establish the following result.

**Proposition 6** *Assume the investors have heterogeneous prior dividend precisions.*

- (a) *The information system with signal precision  $h_\varepsilon = \bar{h}$  is the unique Pareto efficient public information system, and it enjoys the maximum equilibrium ex ante cost of capital and the maximum expected abnormal trading volume.*
- (b) *For given endowments, some investors may be worse off with information system  $h_\varepsilon = \bar{h}$  than with  $h_\varepsilon \neq \bar{h}$ , but at least one investor is better off with information system  $h_\varepsilon = \bar{h}$  than with  $h_\varepsilon \neq \bar{h}$ .*

The trading gains are maximized with an intermediate level of signal precision  $h_\varepsilon = \bar{h}$ , and this yields a superior Pareto efficient allocation with the maximum growth in certainty equivalents and, thus, the maximum *ex ante* cost of capital and the maximum expected abnormal trading volume. However, as demonstrated above, this level of signal precision may leave some (but not all) investors worse off depending on their endowments and their incentives to trade at  $t = 0$ . Table 2 illustrates a setting in which the endowments for the example in Figure 2 are re-allocated as in (24) in order to achieve equilibrium endowments relative to signal precision  $h_\varepsilon = 0$ .

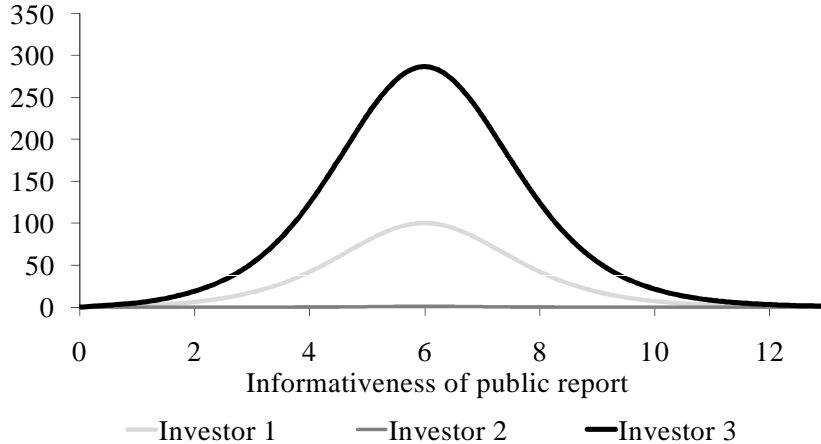


Figure 3: Changes in equilibrium *ex ante* certainty equivalents  $\Delta CE_i^{xa}(h_\varepsilon)$  as functions of the signal precision  $h_\varepsilon$  given the parameters in Table 1 and the endowments in Table 2. The scale on the horizontal axis is  $x = \ln(1 + h_\varepsilon \cdot 1.5\text{E}+07)$ .

Endowments	Investor 1	Investor 2	Investor 3	Aggregate
$\bar{z}_i$	-3.33	51.11	52.22	100
$\bar{\gamma}_i$	34,736	-19,376	-15,360	0
$\bar{\kappa}_i$	34,898	-22,043	-12,856	0

Table 2: Equilibrium endowments relative to  $h_\varepsilon = 0$ .

With the endowments in Table 2, all three investors do not trade at  $t = 0$  if  $h_\varepsilon = 0$ , and they achieve the same certainty equivalents as with the endowments in Figure 2. As the signal precision is increased, the investors continue not to trade at  $t = 0$  in the risky asset (since  $x_{i0}^*$  does not depend on  $h_\varepsilon$ ), but investor 2, who has a prior dividend precision  $h_2$  equal to the average prior dividend precision  $\bar{h}$ , starts to increase his holdings of the  $t = 2$  zero-coupon bond due to its lower equilibrium price. The other two investors reduce their holdings of the  $t = 2$  zero-coupon bond, i.e., sell units of the bond, even though its equilibrium price decreases. This is due to the fact that increasing the signal precision increases their future trading gains (which are realized at  $t = 2$ ) and, thus, they also want to consume more at  $t = 0$  in order to smooth consumption across the two consumption dates. Hence, all three investors gain from a higher signal precision (up to  $h_\varepsilon = \bar{h}$ ) as illustrated in Figure 3 (although the gains to investor 2 are hardly visible in the graph). Of course, a prior round of trading at  $t = -1$  based on the belief that the signal precision will be  $h_\varepsilon = 0$ , such that the investors have equilibrium endowments at  $t = 0$  relative to  $h_\varepsilon = 0$ , ensures that the investors at  $t = 0$  *unanimously* support a public information system change to a system with  $h_\varepsilon = \bar{h}$ .

## 4 Effectively Complete Market

The analyses in the preceding sections show that the public signal at  $t = 1$  plays a key role in determining the *ex ante* cost of capital, the expected abnormal trading volume, and the investors' welfare. If the signal is uninformative or a perfect signal about the risky asset's dividend, there is no trading at  $t = 1$ . However, if the signal is imperfect and the investors have heterogeneous prior dividend precisions, there will be signal-contingent trading at  $t = 1$ , and the *ex ante* trading gains will be reflected in both a higher *ex ante* cost of capital, a higher expected abnormal trading volume, and higher investor welfare. Hence, the key role of the public signal is to facilitate improved dynamic trading opportunities based on heterogeneously updated posterior beliefs in order to take advantage of the disagreements and the differences in confidence among the investors. However, does the dynamic trading in the zero-coupon bond and the risky asset based on the public signal allow the investors to take *full advantage* of the differences in their beliefs? This is the question addressed in this section, and the answer is, in general, no. Additional assets, such as various forms of derivative securities, or more trading rounds based on a sequence of public signals may lead to more efficient side-betting based on the heterogeneous beliefs (see Brennan and Cao 1996 for a similar model with several trading dates). In the extreme, additional assets of the right type may even eliminate the need for dynamic trading based on public signals. We examine only this extreme case in this paper, and we leave the intermediate cases for future research.

The initial question is what would be the right types of derivative assets to facilitate fully efficient side-betting? Wilson (1968) gives almost immediately the answer to this question. Wilson shows that with exponential utility and heterogeneous beliefs, a fully Pareto efficient risk-sharing and side-betting contract is such that each investor gets his risk tolerance fraction (i.e.,  $v_i = \rho_i/\rho_o$ , where  $\rho_o \equiv \sum_i \rho_i$ ) of the aggregate outcome (as with homogeneous beliefs) plus a term, which depends on the state of nature. The fraction  $\rho_i/\rho_o$  is a constant independent of the state and, thus, in a sense, the fully efficient contract is still a linear contract in terms of the aggregate outcome as it is in the homogeneous beliefs case. The key is that the efficient risk sharing and the efficient side-betting can be separated into two additive terms. In our setting, the dividend at  $t = 2$  on the risky asset,  $Zd$ , constitutes the aggregate outcome. Moreover, the  $t = 2$  dividend is also a unique outcome-adequate representation of the state and, thus, the dividend also enters into the efficient side-betting term. The efficient side-betting term has the following form with homogeneous risk aversion (see Christensen and Feltham 2003, Appendix 4A):

$$f_i(d) = \rho \left( f_o(d) - \ln \left( \frac{\rho}{\lambda_i \varphi_i(d)} \right) \right), \quad i = 1, \dots, I,$$

where

$$f_o(d) \equiv \frac{1}{I} \sum_{i=1}^I \ln \left( \frac{\rho}{\lambda_i \varphi_i(d)} \right),$$

and  $\lambda_i$  is the positive expected utility weight in a central planner's efficient risk-sharing and side-betting program. Note that  $\sum_i f_i(d) = 0$  and, thus, the side-betting terms constitute a zero-sum game. Furthermore, with a normally distributed dividend,  $\varphi_i(d) \sim N(m_i, \sigma_i^2)$ ,  $i = 1, \dots, I$ , the efficient side-betting terms  $f_i(d)$  can be expressed as quadratic functions of the dividend  $d$ . Hence, any fully efficient risk-sharing and side-betting allocation of the  $t = 2$  dividend can be expressed as

$$c_{i2}^\dagger = \alpha_{i0} + \alpha_{id}d + \alpha_{is}d^2, \quad i = 1, \dots, I, \quad (25)$$

where  $\alpha_{i0}, \alpha_{id}, \alpha_{is}$  are investor-specific constants with  $\sum_i \alpha_{i0} = \sum_i \alpha_{is} = 0$  and  $\sum_i \alpha_{id} = Z$ . This suggests that a  $t = 2$  zero-coupon bond, the risky asset itself, and a derivative security in zero net-supply with a payoff  $d^2$  would be sufficient to trade at  $t = 0$  to a fully efficient risk-sharing and side-betting allocation in a decentralized market setting without any need for an additional round of trading at  $t = 1$  based on the public signal.<sup>12</sup> The following analysis shows that this is indeed the case.<sup>13,14</sup>

Compared to the market structure in the preceding sections, we now add an additional asset, denoted the *dividend derivative*, in zero net-supply with payoff  $d^2$  at  $t = 2$ , and prices  $\pi_0(\eta)$  and  $\pi_1(y)$  at  $t = 0$  and  $t = 1$ , respectively. The investors have endowments  $\bar{\theta}_i$  of this asset at  $t = 0$ , and let  $\theta_{it}$  be the units of the asset held after trading at date  $t$  satisfying the

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<sup>12</sup>Note that if the investors have identical prior dividend precisions, then  $\alpha_{is} = 0, i = 1, \dots, I$ , such that the efficient allocation is linear in  $d$  and, thus, the zero-coupon bond and risky asset are sufficient to implement the efficient allocation without any trading at  $t = 1$ . Hence, in this setting, the market structure examined in the preceding sections yields fully efficient allocations independently of the public information system (compare to Proposition 3).

<sup>13</sup>Brennan and Cao (1996) were the first to introduce a quadratic derivative as a means to achieve (*ex post*) Pareto optimality in their noisy rational expectations equilibrium setting with heterogeneous posterior equilibrium beliefs.

<sup>14</sup>As noted in the Introduction (footnote 4), our assumption of concordant beliefs ensures that Pareto efficient allocations only include side-betting on the public signal to the extent that it is informative about the fundamentals, i.e., the payoff-relevant events. In their differences-of-opinion model, Cao and Ou-Yang (2009) also introduce the  $d^2$  derivative security in order to effectively complete the market. In their model, however, there is dynamic information-contingent trading in both the risky asset(s) and the derivative security. This is because the side-betting in their model pertains to *payoff-irrelevant* events determining the differences in how the investors interpret the public signal(s). Recall that "differences of opinion make a horse race," cf. Harris and Raviv (1993). In fact, in the Cao and Ou-Yang (2009) model with exponential utilities and normally distributed dividends (like ours), a fully Pareto efficient allocation based on the prior homogeneous dividend beliefs can be achieved through trading only at  $t = 0$  in the zero-coupon bond and the risky asset(s). It is not obvious why the investors should chose subsequently to take speculative positions in the financial markets based on public signals they disagree on how to interpret. Any other zero-sum betting game would achieve the same result.



market clearing conditions

$$\sum_{i=1}^I \bar{\theta}_i = \sum_{i=1}^I \theta_{it} = 0, \quad i = 1, \dots, I, t = 0, 1.$$

Investor  $i$ 's  $t = 2$  consumption is now given as

$$c_{i2} = \theta_{i1}d^2 + x_{i1}d + \gamma_{i1}.$$

Given the public signal at  $t = 1$ , the investor's certainty equivalent of  $t = 2$  consumption can be calculated (using Lemma 2 in the appendix) to be<sup>15</sup>

$$\text{CE}_{i2}(\theta_{i1}, x_{i1}, \gamma_{i1} \mid y) = \gamma_{i1} + \frac{1}{2}\rho h_{i1}m_{i1}^2 - \frac{1}{2}\rho \frac{[rx_{i1} - h_{i1}m_{i1}]^2}{2r\theta_{i1} + h_{i1}} + \frac{1}{2}\rho \ln \left[ \frac{2r\theta_{i1} + h_{i1}}{h_{i1}} \right]$$

and, thus, investor  $i$ 's decision problem at  $t = 1$  given the public signal  $y$  is

$$\begin{aligned} & \max_{\theta_{i1}, x_{i1}, \gamma_{i1}} \text{CE}_{i2}(\theta_{i1}, x_{i1}, \gamma_{i1} \mid y) \\ & \text{subject to } \gamma_{i1} + p_1(y)x_{i1} + \pi_1(y)\theta_{i1} \leq \gamma_{i0} + p_1(y)x_{i0} + \pi_1(y)\theta_{i0}. \end{aligned}$$

The first-order conditions imply that the optimal portfolio at  $t = 1$  given investor  $i$ 's posterior beliefs is

$$x_{i1}(y) = \rho h_{i1} \left[ m_{i1} - \frac{p_1(y)}{h_{i1} [\pi_1(y) - p_1(y)^2]} \right], \quad (26a)$$

$$\theta_{i1}(y) = \frac{1}{2}\rho \left[ \frac{1}{\pi_1(y) - p_1(y)^2} - h_{i1} \right], \quad (26b)$$

$$\gamma_{i1}(y) = \gamma_{i0} + p_1(y)x_{i0} + \pi_1(y)\theta_{i0} - p_1(y)x_{i1}(y) - \pi_1(y)\theta_{i1}. \quad (26c)$$

The market clearing conditions for the two risky assets then imply that

$$p_1(y) = \bar{m}_1^h - r\bar{\sigma}_1^2 Z/I, \quad (27a)$$

$$\pi_1(y) = \bar{\sigma}_1^2 + [\bar{m}_1^h - r\bar{\sigma}_1^2 Z/I]^2. \quad (27b)$$

Note that the equilibrium price of the risky asset at  $t = 1$  is not affected by the addition of the dividend derivative to the marketed assets (compare to (3)). Inserting the equilibrium prices of the two risky assets into the investors' demand functions (26) and simplifying yield

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<sup>15</sup>Here we assume that  $2r\theta_{i1} + h_{i1} > 0$ . We later verify that this is indeed the case, in equilibrium.

the equilibrium demands for the two risky assets,<sup>16</sup>

$$x_{i1}^\dagger(y) = \rho [h_i m_i - \overline{hm}] + Z/I, \quad \overline{hm} \equiv \frac{1}{I} \sum_{i=1}^I h_i m_i, \quad i = 1, \dots, I, \quad (28a)$$

$$\theta_{i1}^\dagger(y) = \frac{1}{2}\rho [\overline{h} - h_i], \quad i = 1, \dots, I. \quad (28b)$$

Of course, the equilibrium prices of the two risky assets at  $t = 1$  both depend on the public signal at  $t = 1$  and its informativeness. However, the equilibrium demands for the two risky assets do neither depend on the specific public signal  $y$  at  $t = 1$  nor on the informativeness of the public information system—equilibrium demands at  $t = 1$  only depend on the investors' prior beliefs.<sup>17</sup> Hence, there will be no signal-contingent trading at  $t = 1$  and, therefore, the equilibrium demands for the two risky assets will be the same at both  $t = 1$  and  $t = 0$ , i.e.,

$$x_{i0}^\dagger = x_{i1}^\dagger(y), \quad \theta_{i0}^\dagger = \theta_{i1}^\dagger(y), \quad \gamma_{i0}^\dagger = \gamma_{i1}^\dagger(y), \quad (29)$$

and the equilibrium prices at  $t = 0$  are the same as in (27) except that they are based on the prior beliefs rather than the posterior beliefs and that they are discounted by the riskless discount factor, i.e.,

$$p_0(\eta) = \beta_0 \mathbf{E}^Q [d], \quad (30a)$$

$$\pi_0(\eta) = \beta_0 \left[ \overline{\sigma}^2 + (\mathbf{E}^Q [d])^2 \right]. \quad (30b)$$

Inserting these equilibrium demands and equilibrium prices into investor  $i$ 's  $t = 2$  certainty equivalent and simplifying yield his equilibrium *ex ante* certainty equivalent of  $t = 2$  consumption,

$$\text{CE}_{i2}^\dagger = \gamma_{i0}^\dagger + \frac{1}{2}\rho h_i \left[ m_i^2 - (\mathbf{E}^Q [d])^2 \right] + \frac{1}{2}\rho (\mathbf{E}^Q [d])^2 (h_i - \overline{h}) + \Upsilon_i,$$

where

$$\Upsilon_i \equiv \frac{1}{2}\rho \ln \left[ \frac{\overline{h}}{h_i} \right].$$

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<sup>16</sup>Note that  $\overline{h} - h_i = \overline{h}_1 - h_{i1}$  and, thus,  $2r\theta_{i1}^\dagger(y) + h_{i1} = \overline{h}_1 > 0$  which was assumed in the derivation of the investors'  $t = 2$  certainty equivalent.

<sup>17</sup>In equilibrium, less confident investors than the average, i.e.,  $h_i < \overline{h}$ , take long positions in the dividend derivative, while more confident investors than the average take short positions. Note that the dividend derivative resembles a “smooth” straddle and, thus, investors, who think the variance of the dividend is high, like the convexity of its payoff, while investors, who think the variance is low, take a short position to get a concave payoff profile. This result suggests that straddles, i.e., long positions in both a call and a put option with the same strike price, play an important role in incomplete market settings with heterogeneous beliefs about the risks on the underlying assets.

Comparing this certainty equivalent to the equilibrium certainty equivalent in the setting without the dividend derivative (see equation (15)) yields that

$$\text{CE}_{i0}^\dagger = d_0 \bar{z}_i + \bar{\kappa}_i + \pi_0(\eta) \left[ \bar{\theta}_i - \theta_{i0}^\dagger \right] + p_0(\eta) \left[ \bar{z}_i - x_{i0}^\dagger \right] + \beta_0 \left[ \bar{\gamma}_i - \gamma_{i0}^\dagger \right], \quad (31a)$$

$$\text{CE}_{i2}^\dagger = \text{CE}_{i2}^* + \left( \gamma_{i0}^\dagger - \gamma_{i0}^* \right) + \frac{1}{2} \rho \left( \mathbb{E}^Q [d] \right)^2 \left( h_i - \bar{h} \right) + \Upsilon_i - U_{1i}. \quad (31b)$$

The equilibrium interest rate is, as in the setting without the dividend derivative, determined as the utility discount rate plus the risk aversion parameter times the growth in certainty equivalents, i.e.,

$$\iota = \delta + r \left( \text{CE}_{i2}^\dagger - \text{CE}_{i0}^\dagger \right).$$

Using the market clearing conditions for the riskless and risky asset, and simplifying yield the equilibrium interest rate.

**Proposition 7** *Consider the setting in which the investors can trade in the dividend derivative in addition to the zero-coupon bond and the risky asset.*

(a) *The equilibrium interest rate is given by*

$$\iota = \delta + r \bar{\Upsilon} + \Phi \left( \{m_i, \sigma_i^2\}_{i=1, \dots, I} \right), \quad (32)$$

where

$$\bar{\Upsilon} \equiv \frac{1}{I} \sum_{i=1}^I \Upsilon_i = \frac{1}{2} \rho \frac{1}{I} \sum_{i=1}^I \ln \left[ \frac{\bar{h}}{h_i} \right].$$

(b) *The equilibrium interest rate is independent of the signal precision  $h_\varepsilon$ .*

(c) *The equilibrium interest rate is strictly higher than the equilibrium interest rate in the setting without the dividend derivative, irrespectively of the signal precision, i.e.,  $\bar{\Upsilon} - \bar{U}_1 > 0, \forall h_\varepsilon \in (0, \infty)$ , if, and only if, the investors have heterogeneous prior dividend precisions.*

The higher equilibrium interest rate reflects that the investors can achieve more efficient side-betting based on their heterogeneous beliefs by being able to trade also in the dividend derivative instead of having to rely on dynamic trading in the risky asset alone. Of course, since there is no trading based on the public signal at  $t = 1$  when the investors can also trade in the dividend derivative, the equilibrium interest rate is independent of the informativeness of the public signal.

The *ex ante* equilibrium prices of the risky asset are in both settings, with and without the dividend derivative, determined as the equilibrium discount factor times the risk-adjusted

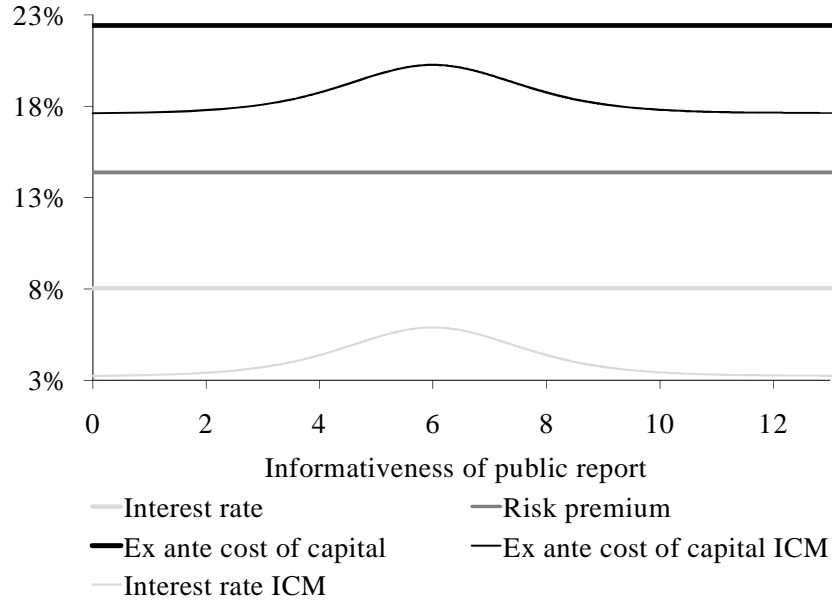


Figure 4: Equilibrium interest rate, risk premium, and *ex ante* cost of capital as functions of the signal precision  $h_\varepsilon$  given the parameters in Table 1 for the settings with the dividend derivative and without the dividend derivative (i.e., the incomplete market (ICM) setting). The scale on the horizontal axis is  $x = \ln(1 + h_\varepsilon \cdot 1.5\text{E}+07)$ .

expected dividend on the risky asset. Since the latter is the same in both settings, the equilibrium *ex ante* cost of capital,  $\mu^{xa}(\eta) = \iota + \varpi^{xa}$ , is independent of the informativeness of the public signal in the setting with the dividend derivative, but it is uniformly higher than in the setting without the dividend derivative (as a function of  $h_\varepsilon$ ). Figure 4 illustrates these differences for the three-investor example in Table 1.

Inserting the investors'  $t = 0$  (and  $t = 1$ ) equilibrium portfolios yields that the investors'  $t = 2$  equilibrium consumption is

$$\begin{aligned} c_{i2}^\dagger &= \theta_{i0}^\dagger d^2 + x_{i0}^\dagger d + \gamma_{i0}^\dagger \\ &= f_i(d) + v_i Z d, \end{aligned}$$

where

$$f_i(d) = \frac{1}{2}\rho [\bar{h} - h_i] d^2 + \rho [h_i m_i - \overline{hm}] d + \gamma_{i0}^\dagger, \quad v_i = 1/I.$$

Note that

$$\sum_{i=1}^I f_i(d) = 0, \quad \sum_{i=1}^I v_i = 1,$$

such that the consumption allocation can be expressed as a linear function of the aggregate

outcome with a fixed slope but a state-dependent intercept.<sup>18</sup> Wilson (1968) shows that this is both a necessary and sufficient condition for a fully efficient risk-sharing and side-betting allocation (see also Amershi and Stoeckenius 1983). Hence, the equilibrium in the setting with the dividend derivative constitutes a fully Pareto efficient risk sharing and side-betting allocation. Since the growth in certainty equivalents is higher in the setting with the dividend derivative than in the setting without it (as reflected by the higher equilibrium interest rate), the addition of the dividend derivative yields a more efficient market structure independently of informativeness of the public signal. Of course, some investors are made better off by the addition of the dividend derivative, but again some investors might be worse off depending on the impact of the higher equilibrium interest rate on the value of their endowments.

**Proposition 8** *Assume the investors have heterogeneous prior dividend precisions.*

- (a) *The market structure with the dividend derivative strictly dominates (in a Pareto efficiency sense) the market structure without the dividend derivative, and it enjoys a strictly higher equilibrium ex ante cost of capital, irrespectively of the informativeness of the public information system.*
- (b) *For given endowments, some investors may be worse off with the addition of the dividend derivative to the marketed assets, but at least one investor is made better off by the addition of this asset.*

Figure 5 shows the changes in the investors' *ex ante* certainty equivalents as functions of the signal precision  $h_\varepsilon$  relative to the *ex ante* certainty equivalents without the dividend derivative and a zero signal precision,  $h_\varepsilon = 0$ , i.e.,

$$\Delta \text{CE}_i^{xa}(h_\varepsilon) \equiv \text{CE}_i^{xa}(h_\varepsilon) - \text{CE}_i^{xa}(h_\varepsilon = 0),$$

where the *ex ante* certainty equivalents for  $h_\varepsilon > 0$  are defined as

$$\begin{aligned} -\exp(-r\text{CE}_i^{xa}(h_\varepsilon)) &\equiv -\exp(-r\text{CE}_{i0}^\dagger) - \exp(-\delta) \exp(-r\text{CE}_{i2}^\dagger), \\ -\exp(-r\text{CE}_i^{xa}(h_\varepsilon)) &\equiv -\exp(-r\text{CE}_{i0}^*) - \exp(-\delta) \exp(-r\text{CE}_{i2}^*), \end{aligned}$$

for the setting with the dividend derivative and the setting without this asset (the incomplete market (ICM) setting in the preceding sections), respectively, while the base *ex ante* certainty

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<sup>18</sup>Note that the state-dependent intercept is comprised of (a) a fixed component,  $\gamma_{i0}^*$ , which depends on the endowments (as in a homogeneous beliefs setting), (b) a side-betting term due to differences between the precision-weighted prior means,  $\rho [h_i m_i - \bar{h} m] d$ , and (c) a side-betting term due to differences in the prior dividend precisions,  $\frac{1}{2} \rho [\bar{h} - h_i] d^2$ .

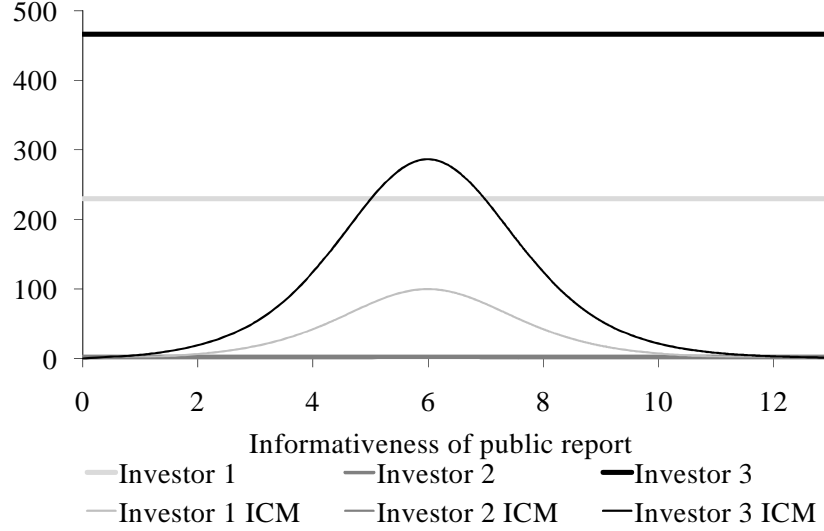


Figure 5: Changes in equilibrium *ex ante* certainty equivalents  $\Delta CE_i^{xa}(h_\varepsilon)$  as functions of the signal precision  $h_\varepsilon$  given the parameters in Table 1, the endowments in Table 2, and zero endowments of the dividend derivative,  $\bar{\theta}_i = 0, i = 1, 2, 3$ . The scale on the horizontal axis is  $x = \ln(1 + h_\varepsilon \cdot 1.5E+07)$ .

equivalents,  $CE_i^{xa}(h_\varepsilon = 0)$ , are defined by

$$-\exp(-rCE_i^{xa}(h_\varepsilon = 0)) \equiv -\exp(-rCE_{i0}^*(h_\varepsilon = 0)) - \exp(-\delta) \exp(-rCE_{i2}^*(h_\varepsilon = 0))$$

for both settings. The investors' endowments of the zero-coupon bonds and the risky asset are the same as in Figure 3, and it is assumed that the investors have zero endowments of the dividend derivative. Hence, Figure 5 shows the *ex ante* value to the investors of having the public information system in the incomplete market setting versus the *ex ante* value to the investors of the addition of the dividend derivative to the marketed assets at  $t = 0$ . In this example, all three investors benefit from imperfect public information in the incomplete market setting, but they benefit even more from the introduction of the dividend derivative (again the gains to investor 2 are hardly visible in the graph—note that his equilibrium holdings of the dividend derivative is equal to zero).

## 5 Concluding Remarks

We have developed a simple analytical model of public information and heterogeneous prior beliefs allowing us to study the relationship between the informativeness of the public information system and the investors' welfare in an incomplete market setting. The source

of the welfare improvements due to public information is the trading gains following from the investors' speculative positions based on the differences in the precision of their prior beliefs. These trading gains are reflected in an additional positive component in the equilibrium interest rate (in addition to the usual utility discount rate, the risk-adjusted aggregate consumption growth and the risk premium for aggregate consumption risk). Moreover, the model provides a direct positive relationship between welfare improvements and the expected abnormal trading volume and, thus, it provides an empirical measure for the relationship between public information systems and investor welfare. This result is in contrast to the extant literature on the impact of public information on trading volume based on noisy rational expectations models (see, e.g., Kim and Verrecchia 1991a, McNichols and Trueman 1994, and Demski and Feltham 1994). Due to the unmodelled "noise traders," these models do not allow welfare comparisons of market structures with different public information systems. Similarly, the differences-of-opinion literature is also to a large extent silent about the relationship between trading volume and investor welfare, mainly because trading volume in these models is generated based on unmodelled heterogeneous beliefs about payoff-irrelevant events as opposed to about the fundamentals of the economy.

Our analysis of the incomplete market setting shows that the public information must be imperfect to be valuable. No information and perfect information rule out valuable dynamic trading strategies to take advantage of the differences in prior dividend precisions. In this sense, our results are related to the so-called information-risk problem due to Hirshleifer (1971), and to the literature on dynamically completing markets by trading long-lived securities (see, e.g., Ohlson and Buckman 1981, Duffie and Huang 1985, and Christensen and Feltham 2003, Chapter 7). Hence, the model provides an argument for the gradual release of information by firms through, for example, earnings forecasts and quarterly earnings announcements, such that the news at the annual audited earnings announcements is limited.<sup>19</sup> In the limit, in which the public information about the normally distributed dividend is generated by a continuous arithmetic Brownian motion and trading in the risky asset and the zero-coupon bond is continuous, a dynamically complete market is achieved and, thus, a fully Pareto efficient Arrow-Debreu equilibrium can be implemented irrespectively of any heterogeneity in prior beliefs, preferences, and wealth distributions (see Duffie and Huang 1985).

A continuous public information flow may be considered an extreme model of information flows in actual financial markets. We show that a dividend derivative specifically targeted towards the investors' incentive to take advantage of their differences in dividend precisions

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<sup>19</sup>Lev (1989) shows that earnings and earnings-related information only has an explanatory power of about 5% of the cross-sectional and time-series variability of stock returns for medium-size windows.

completely eliminates the need for dynamic trading based on public signals. This result, however, relies heavily on our assumptions of exponential utilities (such that efficient risk sharing and side-betting arrangements are additively separable) and normally distributed dividends and, thus, it cannot be expected to carry over to more general settings. On the other hand, the result does show that derivative markets (i.e., securities with non-linear payoffs) may play an important welfare enhancing role in incomplete markets with “jumps” in public information flows. Careful modeling and investigations of the intimate relationship between public information flows and the need for derivative markets could be a fruitful topic for future research.

Finally, our model provides a direct positive relationship between welfare improvements and the *ex ante* cost of capital, i.e., the Pareto efficient public information system is the system enjoying the maximum *ex ante* cost of capital and, thus, the lowest equilibrium *ex ante* price of the risky asset. This (maybe at first counterintuitive) result shows the importance of using a general equilibrium analysis in the evaluation of public information systems, which have consequences for the economy at large. In our model, the impact of the public information system on investor welfare is reflected through the equilibrium interest rate, i.e., the expected marginal rate of substitution between current and future consumption.

The *ex ante* risk premium, however, is not affected by the public information system, neither in the incomplete nor in the effectively complete market setting. While the impact on the equilibrium interest can be expected to carry over to more general settings as a measure of changes in investor welfare (through the expected marginal utility of future consumption), the lack of an impact on the *ex ante* risk premium is quite likely specific to our particular model with exponential utilities and normally distributed dividends.<sup>20</sup> Qin (2011) shows numerically that adding a call option to our incomplete market setting yields an *ex ante* risk premium which (as a function of the signal precision) is not aligned with investor welfare. The equilibrium interest rate and the investor welfare, however, are still perfectly aligned and attain their unique maximum for a common interior signal precision. Hence, it is probably wise to be cautious in making policy statements about, for example, financial reporting regulation, based on empirical measures of equity premia (which are hard to measure reliably anyway).

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<sup>20</sup>See, however, Krueger and Lustig (2010), who in a related setting with power utilities gives sufficient conditions for market incompleteness being irrelevant for the equity premium.



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## Appendix: Proofs

**Proof of Remark 1:** Substituting the  $t = 1$  demand functions (2a) into investor  $i$ 's certainty equivalent (5) yields

$$\begin{aligned} \text{CE}_{i2}(x_{i0}, \gamma_{i0}, x_{i1}(y)) \\ &= \gamma_{i0} + x_{i0}p_1(y) + \rho h_{i1}(m_{i1} - p_1(y))(m_{i1} - p_1(y)) - \frac{1}{2}r(\rho h_{i1}(m_{i1} - p_1(y)))^2 \sigma_{i1}^2 \\ &= \gamma_{i0} + x_{i0}p_1(y) + \frac{1}{2}\rho h_{i1}(m_{i1} - p_1(y))^2, \end{aligned}$$

where  $p_1(y)$  is given by (3). Investor  $i$ 's posterior mean in (1a) for the  $t = 2$  dividend can be expressed as weighted average of the prior mean and the public signal, i.e.,  $m_{i1} = \omega_i y + (1 - \omega_i)m_i$ , where  $\omega_i = \sigma_i^2 / (\sigma_i^2 + \sigma_\varepsilon^2)$ . Thus, the precision weighted average of the investors' posterior can be written as

$$\begin{aligned} \bar{m}_1^h &= \frac{1}{I} \sum_{i=1}^I \frac{h_{i1}}{\bar{h}_1} m_{i1} \\ &= \frac{1}{I} \sum_{i=1}^I \frac{1}{\bar{h}_1} [h_\varepsilon y + h_i m_i] \\ &= \bar{\sigma}_1^2 [h_\varepsilon y + \bar{h} \bar{m}^h], \end{aligned}$$

and, consequently, the  $t = 1$  equilibrium price of the risky asset is

$$p_1(y) = \bar{\sigma}_1^2 [h_\varepsilon y + \bar{\sigma}^2 \bar{m}^h - rZ/I].$$

Inserting in the above expression for the certainty equivalent yields

$$\begin{aligned}
& \text{CE}_{i2}(x_{i0}, \gamma_{i0}, x_{i1}(y)) \\
&= \gamma_{i0} + x_{i0}\bar{\sigma}_1^2 [h_\varepsilon y + \bar{h}\bar{m}^h - rZ/I] \\
&\quad + \frac{1}{2}\rho h_{i1} ([\omega_i y + (1 - \omega_i) m_i] - \bar{\sigma}_1^2 [h_\varepsilon y + \bar{h}\bar{m}^h - rZ/I])^2 \\
&= \gamma_{i0} + x_{i0}\bar{\sigma}_1^2 [h_\varepsilon y + \bar{h}\bar{m}^h - rZ/I] \\
&\quad + \frac{1}{2}\rho h_{i1} (\sigma_{i1}^2 [h_\varepsilon y + h_i m_i] - \bar{\sigma}_1^2 [h_\varepsilon y + \bar{h}\bar{m}^h - rZ/I])^2 \\
&= \gamma_{i0} + x_{i0}\bar{\sigma}_1^2 h_\varepsilon y + x_{i0}\bar{\sigma}_1^2 [\bar{h}\bar{m}^h - rZ/I] \\
&\quad + \frac{1}{2}\rho h_{i1} ([\sigma_{i1}^2 - \bar{\sigma}_1^2] h_\varepsilon y + \sigma_{i1}^2 h_i m_i - \bar{\sigma}_1^2 [\bar{h}\bar{m}^h - rZ/I])^2.
\end{aligned}$$

For notational simplicity, let

$$\begin{aligned}
E_{1i} &\equiv [\sigma_{i1}^2 - \bar{\sigma}_1^2] h_\varepsilon, \\
E_{2i} &\equiv \bar{\sigma}_1^2 h_\varepsilon, \\
E_{3i} &\equiv \bar{\sigma}_1^2 [\bar{h}\bar{m}^h - rZ/I], \\
E_{4i} &\equiv \sigma_{i1}^2 h_i m_i - E_{3i},
\end{aligned}$$

and substituting into the certainty equivalent yields

$$\begin{aligned}
& \text{CE}_{i2}(x_{i0}, \gamma_{i0}, x_{i1}(y)) \\
&= \gamma_{i0} + x_{i0}E_{2i}y + x_{i0}E_{3i} + \frac{1}{2}\rho h_{i1} (E_{1i}y + E_{4i})^2 \\
&= \gamma_{i0} + x_{i0}E_{3i} + \frac{1}{2}\rho h_{i1} E_{4i}^2 + [x_{i0}E_{2i} + \rho h_{i1} E_{1i}E_{4i}] y + \frac{1}{2}\rho h_{i1} E_{1i}^2 y^2.
\end{aligned}$$

Note that  $\text{CE}_{i2}(x_{i0}, \gamma_{i0}, x_{i1}(y))$  is a quadratic function of  $y$ , and from the perspective of investor  $i$ ,  $y \sim N(m_i, \sigma_i^2 + \sigma_\varepsilon^2)$ . Hence, we can apply the following lemma (see Christensen and Feltham 2003, Appendix 3A, for proof) to calculate investor  $i$ 's certainty equivalent.

**Lemma 2** *Let  $w$  be a normally distributed variable with distribution  $w \sim N(q, \bar{s})$  and precision  $\bar{d} = \bar{s}^{-1}$ , and let a quadratic function of  $w$  be given by  $Q(w) = \bar{a} + \bar{b}w + \frac{1}{2}\bar{c}w^2$ , where  $\bar{c} > 0$ . The certainty equivalent  $\text{CE}_{Q(w)}$  of the quadratic function as defined by*

$$-\exp[-r\text{CE}_{Q(w)}] = \text{E}[-\exp[-rQ(w)]],$$

is given by

$$\text{CE}_{Q(w)} = \bar{a} + \frac{1}{2}\rho\bar{d}q^2 - \frac{1}{2}\rho\frac{[r\bar{b} - \bar{d}q]^2}{r\bar{c} + \bar{d}} + \frac{1}{2}\rho[\ln[r\bar{c} + \bar{d}] - \ln[\bar{d}]].$$

In our model,

$$\begin{aligned}
w &= y, \quad q = m_i, \quad \bar{d} = (\sigma_i^2 + \sigma_\varepsilon^2)^{-1}, \\
\bar{a} &= \gamma_{i0} + x_{i0}E_{3i} + \frac{1}{2}\rho h_{i1}E_{4i}^2, \\
\bar{b} &= x_{i0}E_{2i} + \rho h_{i1}E_{1i}E_{4i}, \\
\bar{c} &= \rho h_{i1}E_{1i}^2.
\end{aligned}$$

Hence, investor  $i$ 's  $t = 0$  certainty equivalent of  $t = 2$  consumption can be expressed as

$$\begin{aligned}
CE_{i2}(x_{i0}, \gamma_{i0}) &= \gamma_{i0} + x_{i0}E_{3i} + \frac{1}{2}\rho h_{i1}E_{4i}^2 + \frac{1}{2}\frac{\rho m_i^2}{\sigma_i^2 + \sigma_\varepsilon^2} \\
&\quad - \frac{1}{2}\rho \frac{[rx_{i0}E_{2i} + h_{i1}E_{1i}E_{4i} - \frac{m_i}{\sigma_i^2 + \sigma_\varepsilon^2}]^2}{h_{i1}E_{1i}^2 + h_i} \\
&\quad + \frac{1}{2}\rho \left[ \ln \left[ h_{i1}E_{1i}^2 + \frac{1}{\sigma_i^2 + \sigma_\varepsilon^2} \right] + \ln[\sigma_i^2 + \sigma_\varepsilon^2] \right] \\
&= \gamma_{i0} + \frac{1}{2}\rho \ln \left[ 1 + h_{i1}E_{1i}^2[\sigma_i^2 + \sigma_\varepsilon^2] \right] + \frac{1}{2}\rho h_{i1}E_{4i}^2 + \frac{1}{2}\rho \frac{m_i^2}{\sigma_i^2 + \sigma_\varepsilon^2} \\
&\quad + x_{i0}E_{3i} - \frac{1}{2}\rho \frac{(\sigma_i^2 + \sigma_\varepsilon^2) [rx_{i0}E_{2i} + h_{i1}E_{1i}E_{4i} - \frac{m_i}{\sigma_i^2 + \sigma_\varepsilon^2}]^2}{1 + h_{i1}E_{1i}^2(\sigma_i^2 + \sigma_\varepsilon^2)}.
\end{aligned}$$

Collecting terms yields that

$$CE_{i2} = \gamma_{i0} + U_{i1} + U_{i2} + M_i x_{i0} - \frac{1}{2}rV_i x_{i0}^2,$$

where

$$\begin{aligned}
U_{i1} &\equiv \frac{1}{2}\rho \ln \left[ 1 + h_{i1}E_{1i}^2[\sigma_i^2 + \sigma_\varepsilon^2] \right], \\
U_{i2} &\equiv \frac{1}{2}\rho h_{i1}E_{4i}^2 + \frac{1}{2}\rho \frac{m_i^2}{\sigma_i^2 + \sigma_\varepsilon^2} - \frac{1}{2}\rho \frac{(\sigma_i^2 + \sigma_\varepsilon^2) \left( h_{i1}E_{1i}E_{4i} - \frac{m_i}{\sigma_i^2 + \sigma_\varepsilon^2} \right)^2}{1 + h_{i1}E_{1i}^2(\sigma_i^2 + \sigma_\varepsilon^2)}, \\
M_i &\equiv E_{3i} - \frac{(\sigma_i^2 + \sigma_\varepsilon^2) E_{2i} \left( h_{i1}E_{1i}E_{4i} - \frac{m_i}{\sigma_i^2 + \sigma_\varepsilon^2} \right)}{1 + h_{i1}E_{1i}^2(\sigma_i^2 + \sigma_\varepsilon^2)}, \\
V_i &\equiv \frac{(\sigma_i^2 + \sigma_\varepsilon^2) E_{2i}^2}{1 + h_{i1}E_{1i}^2(\sigma_i^2 + \sigma_\varepsilon^2)}.
\end{aligned}$$

Using the definition of  $E_{1i} = [\sigma_{i1}^2 - \bar{\sigma}_1^2] h_\varepsilon$ , we get

$$E_{1i} = \left[ \frac{1}{h_i + h_\varepsilon} - \frac{1}{\bar{h} + h_\varepsilon} \right] h_\varepsilon = h_\varepsilon \frac{\bar{h} - h_i}{(h_\varepsilon + h_i)(\bar{h} + h_\varepsilon)}.$$

This implies that

$$\begin{aligned}
A_i &\equiv 1 + h_{i1} E_{1i}^2 [\sigma_i^2 + \sigma_\varepsilon^2] \\
&= 1 + [h_\varepsilon + h_i] h_\varepsilon^2 \left[ \frac{\bar{h} - h_i}{(h_\varepsilon + h_i)(h_\varepsilon + \bar{h})} \right]^2 \left[ \frac{1}{h_i} + \frac{1}{h_\varepsilon} \right] \\
&= 1 + \frac{1}{h_\varepsilon + h_i} h_\varepsilon^2 \left[ \frac{\bar{h} - h_i}{h_\varepsilon + \bar{h}} \right]^2 \frac{h_\varepsilon + h_i}{h_\varepsilon h_i} \\
&= 1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_\varepsilon}{(\bar{h} + h_\varepsilon)^2}
\end{aligned}$$

and, thus,

$$U_{i1} = \frac{1}{2} \rho \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_\varepsilon}{(\bar{h} + h_\varepsilon)^2} \right],$$

which establishes (7a).

Turning to  $V_i$  and using the definitions of  $E_{2i} = \bar{\sigma}_1^2 h_\varepsilon$  and of  $A_i = 1 + h_{i1} E_{1i}^2 [\sigma_i^2 + \sigma_\varepsilon^2]$ , we get that

$$\begin{aligned}
V_i &= \frac{(\sigma_i^2 + \sigma_\varepsilon^2) (\bar{\sigma}_1^2 h_\varepsilon)^2}{A_i} = \frac{\left( \frac{1}{h_i} + \frac{1}{h_\varepsilon} \right) \left( \frac{h_\varepsilon}{\bar{h} + h_\varepsilon} \right)^2}{1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_\varepsilon}{(\bar{h} + h_\varepsilon)^2}} \\
&= \frac{(h_\varepsilon + h_i) h_\varepsilon}{h_i (\bar{h} + h_\varepsilon)^2 + (\bar{h} - h_i)^2 h_\varepsilon} = \frac{h_\varepsilon}{\bar{h}^2 + h_\varepsilon h_i},
\end{aligned}$$

which establishes (7d).

Turning to  $M_i$  and using the definition of  $V_i$ , we can express  $M_i$  as

$$\begin{aligned}
M_i &= E_{3i} - \frac{(\sigma_i^2 + \sigma_\varepsilon^2) E_{2i} \left( h_{i1} E_{1i} E_{4i} - \frac{m_i}{\sigma_i^2 + \sigma_\varepsilon^2} \right)}{1 + h_{i1} E_{1i}^2 (\sigma_i^2 + \sigma_\varepsilon^2)} \\
&= E_{3i} - \frac{h_{i1} E_{1i} E_{4i} - \frac{m_i}{\sigma_i^2 + \sigma_\varepsilon^2}}{E_{2i}} V_i \\
&= V_i \left[ \frac{E_{3i}}{V_i} - h_{i1} \frac{E_{1i} E_{4i}}{E_{2i}} + \frac{m_i h_i h_\varepsilon}{E_{2i} (h_\varepsilon + h_i)} \right].
\end{aligned}$$

Inserting the expressions for  $E_{ji}, j = 1, 2, 3, 4$ , and simplifying yield

$$\begin{aligned}
M_i &= V_i \left[ \frac{\bar{\sigma}_1^2 [\bar{h}\bar{m}^h - rZ/I]}{V_i} - h_{i1} \frac{E_{1i} (\sigma_{i1}^2 h_i m_i - \bar{\sigma}_1^2 [\bar{h}\bar{m}^h - rZ/I])}{\bar{\sigma}_1^2 h_\varepsilon} + \frac{m_i h_i h_\varepsilon}{\bar{\sigma}_1^2 h_\varepsilon (h_\varepsilon + h_i)} \right] \\
&= V_i \left[ \frac{\bar{\sigma}_1^2 [\bar{h}\bar{m}^h - rZ/I]}{V_i} - \frac{h_\varepsilon \frac{\bar{h}-h_i}{(h_\varepsilon+h_i)(\bar{h}+h_\varepsilon)} (h_i m_i - h_{i1} \bar{\sigma}_1^2 [\bar{h}\bar{m}^h - rZ/I])}{\frac{h_\varepsilon}{\bar{h}+h_\varepsilon}} + \frac{m_i h_i h_\varepsilon}{\frac{h_\varepsilon}{\bar{h}+h_\varepsilon} (h_\varepsilon + h_i)} \right] \\
&= V_i \left[ \frac{\bar{\sigma}_1^2 [\bar{h}\bar{m}^h - rZ/I]}{V_i} - \frac{\bar{h} - h_i}{h_\varepsilon + h_i} (h_i m_i - h_{i1} \bar{\sigma}_1^2 [\bar{h}\bar{m}^h - rZ/I]) + \frac{m_i h_i (\bar{h} + h_\varepsilon)}{h_\varepsilon + h_i} \right] \\
&= V_i \left[ \frac{\bar{\sigma}_1^2 [\bar{h}\bar{m}^h - rZ/I]}{V_i} + h_i m_i + (\bar{h} - h_i) \bar{\sigma}_1^2 [\bar{h}\bar{m}^h - rZ/I] \right] \\
&= V_i \left[ h_i m_i + \left[ \frac{1}{V_i} + (\bar{h} - h_i) \right] \bar{\sigma}_1^2 [\bar{h}\bar{m}^h - rZ/I] \right].
\end{aligned}$$

Substituting the expression for  $V_i$  back in yields

$$\begin{aligned}
M_i &= \frac{h_\varepsilon \left[ h_i m_i + \left[ \frac{\bar{h}^2 + h_\varepsilon h_i}{h_\varepsilon} + (\bar{h} - h_i) \right] \bar{\sigma}_1^2 [\bar{h}\bar{m}^h - rZ/I] \right]}{\bar{h}^2 + h_\varepsilon h_i} \\
&= \frac{h_\varepsilon \left[ h_i m_i + \left[ \frac{\bar{h}^2}{h_\varepsilon} + \bar{h} \right] \frac{1}{\bar{h} + h_\varepsilon} [\bar{h}\bar{m}^h - rZ/I] \right]}{\bar{h}^2 + h_\varepsilon h_i} \\
&= \frac{h_\varepsilon h_i m_i + \bar{h} [\bar{h}\bar{m}^h - rZ/I]}{\bar{h}^2 + h_\varepsilon h_i} = \frac{h_\varepsilon h_i m_i + \bar{h}^2 \bar{m}^h - r\bar{h}Z/I}{\bar{h}^2 + h_\varepsilon h_i},
\end{aligned}$$

which establishes (7c).

Finally, turning to  $U_{i2}$  we get

$$\begin{aligned}
2rU_{i2} &= h_{i1} E_{4i}^2 + \frac{m_i^2}{\sigma_i^2 + \sigma_\varepsilon^2} - \frac{(\sigma_i^2 + \sigma_\varepsilon^2) \left( h_{i1} E_{1i} E_{4i} - \frac{m_i}{\sigma_i^2 + \sigma_\varepsilon^2} \right)^2}{1 + h_{i1} E_{1i}^2 (\sigma_i^2 + \sigma_\varepsilon^2)} \\
&= h_{i1} E_{4i}^2 + \frac{1}{2} \frac{m_i^2}{\sigma_i^2 + \sigma_\varepsilon^2} - \frac{1}{2} \frac{h_{i1}^2 E_{1i}^2 E_{4i}^2 (\sigma_i^2 + \sigma_\varepsilon^2) + \frac{m_i^2}{\sigma_i^2 + \sigma_\varepsilon^2} - 2h_{i1} E_{1i} E_{4i} m_i}{1 + h_{i1} E_{1i}^2 (\sigma_i^2 + \sigma_\varepsilon^2)} \\
&= \frac{h_{i1} E_{4i}^2 + \frac{m_i^2}{\sigma_i^2 + \sigma_\varepsilon^2} (1 + h_{i1} E_{1i}^2 (\sigma_i^2 + \sigma_\varepsilon^2))}{1 + h_{i1} E_{1i}^2 (\sigma_i^2 + \sigma_\varepsilon^2)} - \frac{\frac{m_i^2}{\sigma_i^2 + \sigma_\varepsilon^2} - 2h_{i1} E_{1i} E_{4i} m_i}{1 + h_{i1} E_{1i}^2 (\sigma_i^2 + \sigma_\varepsilon^2)} \\
&= \frac{h_{i1} E_{4i}^2 + \frac{m_i^2}{\sigma_i^2 + \sigma_\varepsilon^2} + m_i^2 h_{i1} E_{1i}^2 - \frac{m_i^2}{\sigma_i^2 + \sigma_\varepsilon^2} + 2h_{i1} E_{1i} E_{4i} m_i}{1 + h_{i1} E_{1i}^2 (\sigma_i^2 + \sigma_\varepsilon^2)} \\
&= h_{i1} \frac{E_{4i}^2 + m_i^2 E_{1i}^2 + 2E_{1i} E_{4i} m_i}{1 + h_{i1} E_{1i}^2 (\sigma_i^2 + \sigma_\varepsilon^2)} = h_{i1} \frac{(E_{4i} + m_i E_{1i})^2}{1 + h_{i1} E_{1i}^2 (\sigma_i^2 + \sigma_\varepsilon^2)}.
\end{aligned}$$



Substituting the expressions for  $E_{1i}$  and  $E_{4i}$  in, we obtain

$$\begin{aligned}
2rU_{i2} &= (h_\varepsilon + h_i) \frac{\left( \frac{1}{h_i+h_\varepsilon} h_i m_i - \bar{\sigma}_1^2 [\bar{h}\bar{m}^h - rZ/I] + m_i [\sigma_{i1}^2 - \bar{\sigma}_1^2] h_\varepsilon \right)^2}{1 + [h_\varepsilon + h_i] [[\sigma_{i1}^2 - \bar{\sigma}_1^2] h_\varepsilon]^2 (\sigma_i^2 + \sigma_\varepsilon^2)} \\
&= \frac{\left( \frac{1}{h_i+h_\varepsilon} h_i m_i - \frac{1}{\bar{h}+h_\varepsilon} [\bar{h}\bar{m}^h - rZ/I] + m_i \frac{\bar{h}-h_i}{(h_\varepsilon+h_i)(\bar{h}+h_\varepsilon)} h_\varepsilon \right)^2}{\frac{1}{h_\varepsilon+h_i} + \left[ \frac{\bar{h}-h_i}{(h_\varepsilon+h_i)(\bar{h}+h_\varepsilon)} h_\varepsilon \right]^2 (\sigma_i^2 + \sigma_\varepsilon^2)} \\
&= \frac{\frac{1}{(h_\varepsilon+h_i)^2 (h_\varepsilon+\bar{h})^2} (h_i m_i (h_\varepsilon + \bar{h}) + m_i h_\varepsilon (\bar{h} - h_i) - [\bar{h}\bar{m}^h - rZ/I] [h_\varepsilon + h_i])^2}{\frac{1}{(h_\varepsilon+h_i)^2 (h_\varepsilon+\bar{h})^2} h_\varepsilon^2 (\bar{h} - h_i)^2 (\sigma_i^2 + \sigma_\varepsilon^2) + (h_\varepsilon + h_i) (h_\varepsilon + \bar{h})^2}}{=} \\
&= \frac{(h_i m_i (h_\varepsilon + \bar{h}) + m_i h_\varepsilon (\bar{h} - h_i) - [\bar{h}\bar{m}^h - rZ/I] [h_\varepsilon + h_i])^2}{h_\varepsilon^2 (\bar{h} - h_i)^2 (\sigma_i^2 + \sigma_\varepsilon^2) + (h_\varepsilon + h_i) (h_\varepsilon + \bar{h})^2} \\
&= \frac{((h_i + h_\varepsilon) m_i \bar{h} - [\bar{h}\bar{m}^h - rZ/I] [h_\varepsilon + h_i])^2}{h_\varepsilon^2 (\bar{h} - h_i)^2 (\sigma_i^2 + \sigma_\varepsilon^2) + (h_\varepsilon + h_i) (h_\varepsilon + \bar{h})^2} \\
&= \frac{h_i (h_i + h_\varepsilon) [m_i \bar{h} - \bar{h}\bar{m}^h + rZ/I]^2}{h_\varepsilon (\bar{h} - h_i)^2 + h_i (h_\varepsilon + \bar{h})^2} = \frac{h_i [m_i \bar{h} - \bar{h}\bar{m}^h + rZ/I]^2}{\bar{h}^2 + h_i h_\varepsilon}.
\end{aligned}$$

Hence, we have that

$$U_{i2} = \frac{\rho h_i [m_i \bar{h} - \bar{h}\bar{m}^h + rZ/I]^2}{2 (\bar{h}^2 + h_i h_\varepsilon)},$$

which establishes (7b). ■

**Proof of Proposition 1:** First we calculate

$$\bar{v} \equiv \frac{1}{I} \sum_{i=1}^I v_i = \frac{1}{I} \sum_{i=1}^I \frac{\bar{h}^2 + h_\varepsilon h_i}{h_\varepsilon} = \frac{\bar{h}^2}{h_\varepsilon} + \bar{h},$$

and

$$\begin{aligned}
\bar{M}^v &\equiv \frac{1}{I} \sum_{i=1}^I \frac{v_i}{\bar{v}} M_i = \frac{1}{I} \frac{1}{\frac{\bar{h}^2}{h_\varepsilon} + \bar{h}} \sum_{i=1}^I \frac{\bar{h}^2 + h_\varepsilon h_i}{h_\varepsilon} \frac{h_\varepsilon h_i m_i + \bar{h} [\bar{h}\bar{m}^h - rZ/I]}{\bar{h}^2 + h_\varepsilon h_i} \\
&= \frac{1}{I} \frac{h_\varepsilon}{\bar{h}^2 + h_\varepsilon \bar{h}} \sum_{i=1}^I \frac{1}{h_\varepsilon} [h_\varepsilon h_i m_i + \bar{h} [\bar{h}\bar{m}^h - rZ/I]] \\
&= \frac{1}{I} \frac{1}{\bar{h}^2 + h_\varepsilon \bar{h}} \left( I h_\varepsilon \bar{h}\bar{m}^h + I \bar{h}^2 \bar{m}^h - \bar{h} r Z \right) = \frac{h_\varepsilon \bar{h}\bar{m}^h + \bar{h}^2 \bar{m}^h - \bar{h} r Z / I}{\bar{h}^2 + h_\varepsilon \bar{h}}.
\end{aligned}$$

Hence, the risk-adjusted expected dividend is

$$\begin{aligned}
\mathbb{E}^Q[d] &\equiv \bar{M}^v - r\bar{V}Z/I = \frac{h_\varepsilon \bar{h} \bar{m}^h + \bar{h} \bar{h} \bar{m}^h - \bar{h} r Z/I}{\bar{h}^2 + h_\varepsilon \bar{h}} - r \frac{h_\varepsilon}{\bar{h}^2 + h_\varepsilon \bar{h}} Z/I \\
&= \frac{h_\varepsilon \bar{h} \bar{m}^h + \bar{h} \bar{h} \bar{m}^h - \bar{h} r Z/I - h_\varepsilon r Z/I}{\bar{h}^2 + h_\varepsilon \bar{h}} \\
&= \frac{(h_\varepsilon + \bar{h}) \bar{h} \bar{m}^h - (\bar{h} + h_\varepsilon) r Z/I}{(\bar{h} + h_\varepsilon) \bar{h}} = \bar{m}^h - r \bar{\sigma}^2 Z/I,
\end{aligned}$$

which shows the claim in the proposition that the risk-adjusted expected dividend is independent of the signal precision. ■

**Proof of Remark 2:** Substituting the expressions for  $M_i$  and  $V_i$ , i.e., (7c) and (7d), into (13) and simplifying yield

$$x_{i0}^* = \rho \frac{\frac{h_\varepsilon h_i m_i + \bar{h} [\bar{h} \bar{m}^h - r Z/I]}{\bar{h}^2 + h_\varepsilon h_i} - \mathbb{E}^Q[d]}{\frac{h_\varepsilon}{\bar{h}^2 + h_\varepsilon h_i}} = \rho \frac{1}{h_\varepsilon} \left[ h_\varepsilon h_i m_i + \bar{h} [\bar{h} \bar{m}^h - r Z/I] - (\bar{h}^2 + h_\varepsilon h_i) \mathbb{E}^Q[d] \right].$$

Using the expression for  $\mathbb{E}^Q[d]$ , i.e., (12), yields

$$x_{i0}^* = \rho \frac{1}{h_\varepsilon} \left[ h_\varepsilon h_i m_i + \bar{h}^2 \mathbb{E}^Q[d] - (\bar{h}^2 + h_\varepsilon h_i) \mathbb{E}^Q[d] \right] = \rho h_i \{ m_i - \mathbb{E}^Q[d] \}.$$

■

**Proof of Proposition 2:** Summing (16) across investors and using the market clearing conditions yield

$$\begin{aligned}
\iota &= \delta + r \frac{1}{I} \sum_{i=1}^I (\text{CE}_{i2}^* - \text{CE}_{i0}^*) \\
&= \delta + r \frac{1}{I} \sum_{i=1}^I U_{1i} + r \frac{1}{I} \sum_{i=1}^I \left( U_{2i} + M_i x_{i0}^* - \frac{1}{2} r V_i (x_{i0}^*)^2 \right) - r d_0 Z/I.
\end{aligned}$$

From (7a) we get

$$r \frac{1}{I} \sum_{i=1}^I U_{1i} = \frac{1}{2} \frac{1}{I} \sum_{i=1}^I \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_\varepsilon}{(\bar{h} + h_\varepsilon)^2} \right].$$

Substituting (14) yields

$$\begin{aligned}
U_{2i} + M_i x_{i0}^* - \frac{1}{2} r V_i (x_{i0}^*)^2 &= U_{2i} + M_i \rho \frac{M_i - \mathbf{E}^Q[d]}{V_i} - \frac{1}{2} r V_i \left( \rho \frac{M_i - \mathbf{E}^Q[d]}{V_i} \right)^2 \\
&= U_{2i} + \rho \frac{M_i^2 - M_i \mathbf{E}^Q[d]}{V_i} - \frac{1}{2} \rho \frac{M_i^2 + (\mathbf{E}^Q[d])^2 - 2M_i \mathbf{E}^Q[d]}{V_i} \\
&= U_{2i} + \frac{1}{2} \rho \frac{M_i^2 - (\mathbf{E}^Q[d])^2}{V_i}.
\end{aligned}$$

Substituting in  $U_{2i}$ ,  $M_i$  and  $V_i$  yields

$$\begin{aligned}
&2r \left[ U_{2i} + \frac{1}{2} \rho \frac{M_i^2 - (\mathbf{E}^Q[d])^2}{V_i} \right] \\
&= \frac{h_i [m_i \bar{h} - \bar{h} \bar{m}^h + rZ/I]^2}{\bar{h}^2 + h_i h_\varepsilon} + \frac{\left[ \frac{h_\varepsilon h_i m_i + \bar{h} [\bar{h} \bar{m}^h - rZ/I]}{\bar{h}^2 + h_\varepsilon h_i} \right]^2 - (\mathbf{E}^Q[d])^2}{\frac{h_\varepsilon}{\bar{h}^2 + h_\varepsilon h_i}}.
\end{aligned}$$

Let  $B \equiv \bar{h} \bar{m}^h - rZ/I$  such that

$$\begin{aligned}
&2r \left[ U_{2i} + \frac{1}{2} \rho \frac{M_i^2 - (\mathbf{E}^Q[d])^2}{V_i} \right] \\
&= \frac{h_i [m_i \bar{h} - B]^2}{\bar{h}^2 + h_i h_\varepsilon} + \frac{[h_\varepsilon h_i m_i + \bar{h} B]^2}{h_\varepsilon (\bar{h}^2 + h_\varepsilon h_i)} - \frac{\bar{h}^2 + h_\varepsilon h_i}{h_\varepsilon \bar{h}^2} B^2 \\
&= \frac{h_\varepsilon h_i [m_i \bar{h} - B]^2}{h_\varepsilon (\bar{h}^2 + h_i h_\varepsilon)} + \frac{[h_\varepsilon h_i m_i + \bar{h} B]^2}{h_\varepsilon (\bar{h}^2 + h_\varepsilon h_i)} - \frac{(1 + h_\varepsilon \frac{h_i}{\bar{h}^2}) (\bar{h}^2 + h_\varepsilon h_i)}{h_\varepsilon (\bar{h}^2 + h_\varepsilon h_i)} B^2 \\
&= \frac{1}{h_\varepsilon (\bar{h}^2 + h_i h_\varepsilon)} \left[ h_\varepsilon h_i [m_i \bar{h} - B]^2 + [h_\varepsilon h_i m_i + \bar{h} B]^2 - \frac{1}{\bar{h}^2} (\bar{h}^2 + h_\varepsilon h_i) (\bar{h}^2 + h_\varepsilon h_i) B^2 \right] \\
&= \frac{h_\varepsilon h_i}{h_\varepsilon (\bar{h}^2 + h_i h_\varepsilon)} \left[ m_i^2 \bar{h}^2 - B^2 + h_\varepsilon h_i m_i^2 - \frac{1}{\bar{h}^2} h_\varepsilon h_i B^2 \right] \\
&= \frac{h_\varepsilon h_i}{h_\varepsilon (\bar{h}^2 + h_i h_\varepsilon)} \left[ m_i^2 [\bar{h}^2 + h_\varepsilon h_i] - B^2 \frac{\bar{h}^2 + h_\varepsilon h_i}{\bar{h}^2} \right] \\
&= h_i \left[ m_i^2 - \frac{B^2}{\bar{h}^2} \right].
\end{aligned}$$

Thus, we obtain that

$$U_{2i} + \frac{1}{2}\rho \frac{M_i^2 - (\mathbb{E}^Q[d])^2}{V_i} = \frac{1}{2}\rho h_i \left[ m_i^2 - \frac{[\bar{h}\bar{m}^h - rZ/I]^2}{\bar{h}^2} \right] = \frac{1}{2}\rho h_i \left[ m_i^2 - (\mathbb{E}^Q[d])^2 \right].$$

Hence,

$$\begin{aligned} \iota &= \delta + r \frac{1}{I} \sum_{i=1}^I U_{1i} + r \frac{1}{I} \sum_{i=1}^I \left( U_{2i} + M_i x_{i0}^* - \frac{1}{2} r V_i (x_{i0}^*)^2 \right) - r d_0 Z/I \\ &= \delta + \frac{1}{2} \frac{1}{I} \sum_{i=1}^I \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_\varepsilon}{(\bar{h} + h_\varepsilon)^2} \right] + \Phi \left( \{m_i, \sigma_i^2\}_{i=1, \dots, I} \right), \end{aligned}$$

where

$$\Phi \left( \{m_i, \sigma_i^2\}_{i=1, \dots, I} \right) \equiv \frac{1}{2} \frac{1}{I} \sum_{i=1}^I h_i \left[ m_i^2 - (\mathbb{E}^Q[d])^2 \right] - r d_0 Z/I.$$

Inserting the expression for  $\mathbb{E}^Q[d]$  in (12) yields

$$\begin{aligned} \Phi \left( \{m_i, \sigma_i^2\}_{i=1, \dots, I} \right) &= \frac{1}{2} \frac{1}{I} \sum_{i=1}^I h_i \left[ m_i^2 - (\bar{m}^h - r\bar{\sigma}^2 Z/I)^2 \right] - r d_0 Z/I \\ &= \frac{1}{2} \frac{1}{I} \sum_{i=1}^I h_i m_i^2 - \frac{1}{2} (\bar{m}^h - r\bar{\sigma}^2 Z/I)^2 \bar{h} - r d_0 Z/I \\ &= r [\bar{m}^h - d_0] Z/I - \frac{1}{2} r^2 \bar{\sigma}^2 (Z/I)^2 + \frac{1}{2} \frac{1}{I} \sum_{i=1}^I h_i m_i^2 - \frac{1}{2} (\bar{m}^h)^2 \bar{h}. \end{aligned}$$

If the investors have homogeneous prior dividend means, i.e.,  $m_i = m$ , then

$$\begin{aligned} \Phi \left( \{m, \sigma_i^2\}_{i=1, \dots, I} \right) &= r [m - d_0] Z/I - \frac{1}{2} r^2 \bar{\sigma}^2 (Z/I)^2 + \frac{1}{2} \frac{1}{I} \sum_{i=1}^I h_i m_i^2 - \frac{1}{2} \left( \frac{1}{I} \sum_{i=1}^I \frac{h_i}{\bar{h}} m_i \right)^2 \frac{1}{\bar{h}} \\ &= r [m - d_0] Z/I - \frac{1}{2} r^2 \bar{\sigma}^2 (Z/I)^2 + \frac{1}{2} m^2 \bar{h} - \frac{1}{2} m^2 \bar{h}^2 \frac{1}{\bar{h}} \\ &= r [m - d_0] Z/I - \frac{1}{2} r^2 \bar{\sigma}^2 (Z/I)^2. \end{aligned}$$

If the investors have homogeneous prior dividend precisions, i.e.,  $h_i = \bar{h}$ , then

$$\iota = \delta + \Phi \left( \{m_i, \sigma^2\}_{i=1, \dots, I} \right).$$

Hence, the equilibrium interest rate is independent of the signal precision, since the function  $\Phi(\cdot)$  only depends on the prior beliefs. ■

**Proof of Proposition 3:** With identical prior dividend precisions, let  $h_i = h$ , for  $i = 1, \dots, I$ . Hence,

$$\bar{h} = h, \quad \bar{m}^h = \frac{1}{I} \sum_{i=1}^I m_i,$$

and

$$m_{i1} = \frac{h_\varepsilon}{h + h_\varepsilon} y + \left( \frac{h}{h + h_\varepsilon} \right) m_i = \frac{1}{h + h_\varepsilon} (h_\varepsilon y + h m_i).$$

This implies that

$$\begin{aligned} x_{i1}^*(y) &= \rho h_{i1} (m_{i1} - [\bar{m}_1^h - r\bar{\sigma}_1^2 Z/I]) \\ &= \rho [h_\varepsilon + h] \left( m_{i1} - \frac{1}{\rho [h_\varepsilon + h] I} \left( \sum_{i=1}^I \rho [h_\varepsilon + h] m_{i1} - Z \right) \right) \\ &= \rho (h_\varepsilon y + h m_i) - \frac{1}{I} \left( \sum_{i=1}^I \rho [h_\varepsilon + h] \frac{1}{h + h_\varepsilon} (h_\varepsilon y + h m_i) - Z \right) \\ &= \rho (h_\varepsilon y + h m_i) - \frac{1}{I} \left( \sum_{i=1}^I \rho (h_\varepsilon y + h m_i) - Z \right) \\ &= \rho h m_i - \rho h \bar{m}^h + Z/I = \rho h \{ m_i - [\bar{m}^h - \sigma^2 Z/I] \}. \end{aligned}$$

Using the expression for  $\mathbb{E}^Q[d]$ , i.e., (12), we have that

$$\mathbb{E}^Q[d] = \bar{m}^h - \sigma^2 Z/I,$$

and Remark 2 yields that

$$x_{i1}^*(y) = \rho h \{ m_i - \mathbb{E}^Q[d] \} = x_{i0}^*.$$

Therefore, the demand for riskless asset is also time- and signal-invariant due to

$$\gamma_{i1}^* = \gamma_{i0}^* + p_1(y) x_{i0}^* - p_1(y) x_{i1}^*(y) = \gamma_{i0}^*.$$

■

**Proof of Proposition 4:** By Proposition 2 the equilibrium interest rate attains its maximum value when the logarithmic term in (17),

$$\frac{1}{2} \frac{1}{I} \sum_{i=1}^I \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_\varepsilon}{(\bar{h} + h_\varepsilon)^2} \right],$$

is maximized with respect to  $h_\varepsilon$ . This term is maximized whenever the common term for all investors  $h_\varepsilon / (\bar{h} + h_\varepsilon)^2$  is maximized. The first-order condition is

$$\begin{aligned} \frac{(\bar{h} + h_\varepsilon)^2 - 2h_\varepsilon(\bar{h} + h_\varepsilon)}{(\bar{h} + h_\varepsilon)^4} = 0 &\Leftrightarrow \bar{h}^2 + h_\varepsilon^2 + 2\bar{h}h_\varepsilon - 2h_\varepsilon\bar{h} - 2h_\varepsilon^2 = 0 \Leftrightarrow \\ &\bar{h}^2 - h_\varepsilon^2 = 0 \Leftrightarrow (\bar{h} - h_\varepsilon)(\bar{h} + h_\varepsilon) = 0 \Leftrightarrow \bar{h} = h_\varepsilon, \end{aligned}$$

where the last equation follows from the fact that  $\bar{h} + h_\varepsilon > 0$ . Similarly,  $h_\varepsilon / (\bar{h} + h_\varepsilon)^2$  is minimized (and is equal to zero) for  $h_\varepsilon = 0$  and  $h_\varepsilon \rightarrow \infty$ .

Note the second derivative of the equilibrium interest rate with respect to the signal precision is

$$-4(\bar{h} + h_\varepsilon)^{-3} + 6h_\varepsilon(\bar{h} + h_\varepsilon)^{-4} = 2(\bar{h} + h_\varepsilon)^{-3} \left( -2 + 3h_\varepsilon(\bar{h} + h_\varepsilon)^{-1} \right).$$

Hence, the equilibrium interest rate has only one inflection point with respect to the signal precision, i.e.,  $h_{ip} = 2\bar{h}$ . Since  $(\bar{h} + h_\varepsilon)^{-3} > 0$ , thus, when  $h_\varepsilon \geq h_{ip}$  ( $h_\varepsilon \leq h_{ip}$ ), the second derivative of the equilibrium interest rate with respect to the signal precision is positive (negative), thus, the equilibrium interest rate is convex (concave) with respect to the signal precision. ■

**Proof of Remark 3:** Using that (see the proof of Remark 1)

$$\bar{m}_1^h = \bar{\sigma}_1^2 [h_\varepsilon y + \bar{h}\bar{m}^h] = \frac{1}{\bar{h} + h_\varepsilon} [h_\varepsilon y + \bar{h}\bar{m}^h],$$

(4) can be re-written as

$$\begin{aligned} x_{i1}^*(y) &= \rho h_{i1} (m_{i1} - [\bar{m}_1^h - r\bar{\sigma}_1^2 Z/I]) \\ &= \rho h_{i1} \left( \omega_i y + (1 - \omega_i) m_i - \left[ \frac{1}{\bar{h} + h_\varepsilon} [h_\varepsilon y + \bar{h}\bar{m}^h] - r \frac{1}{\bar{h} + h_\varepsilon} Z/I \right] \right) \\ &= \rho \left( h_\varepsilon y + h_i m_i - \frac{h_i + h_\varepsilon}{\bar{h} + h_\varepsilon} [h_\varepsilon y + \bar{h}\bar{m}^h - rZ/I] \right) \\ &= \rho \left( \frac{h_\varepsilon(\bar{h} - h_i)}{\bar{h} + h_\varepsilon} y + h_i m_i - \frac{\bar{h}(h_i + h_\varepsilon)}{\bar{h} + h_\varepsilon} [\bar{m}^h - r\bar{\sigma}^2 Z/I] \right) \\ &= \rho \left( \frac{h_\varepsilon(\bar{h} - h_i)}{\bar{h} + h_\varepsilon} y + h_i m_i - \frac{\bar{h}(h_i + h_\varepsilon)}{\bar{h} + h_\varepsilon} \mathbf{E}^Q[d] \right). \end{aligned}$$

It then follows from (14) that

$$\begin{aligned}
\tau_i^*(y) &= \rho \left( \frac{h_\varepsilon (\bar{h} - h_i)}{\bar{h} + h_\varepsilon} y + h_i m_i - \frac{\bar{h} (h_i + h_\varepsilon)}{\bar{h} + h_\varepsilon} \mathbb{E}^Q[d] \right) - \rho h_i [m_i - \mathbb{E}^Q[d]] \\
&= \rho \left( \frac{h_\varepsilon (\bar{h} - h_i)}{\bar{h} + h_\varepsilon} y - \left[ \frac{\bar{h} (h_i + h_\varepsilon)}{\bar{h} + h_\varepsilon} - h_i \right] \mathbb{E}^Q[d] \right) \\
&= \rho \left( \frac{h_\varepsilon (\bar{h} - h_i)}{\bar{h} + h_\varepsilon} y - \left[ \frac{\bar{h} h_\varepsilon - h_i h_\varepsilon}{\bar{h} + h_\varepsilon} \right] \mathbb{E}^Q[d] \right) = \rho \frac{h_\varepsilon (\bar{h} - h_i)}{\bar{h} + h_\varepsilon} (y - \mathbb{E}^Q[d]).
\end{aligned}$$

The risk-adjusted expected equilibrium net-trade is

$$\begin{aligned}
\mathbb{E}^Q[\tau_i^*(y)] &= \rho \frac{h_\varepsilon (\bar{h} - h_i)}{\bar{h} + h_\varepsilon} (\mathbb{E}^Q[y] - \mathbb{E}^Q[d]) = \rho \frac{h_\varepsilon (\bar{h} - h_i)}{\bar{h} + h_\varepsilon} (\mathbb{E}^Q[d + \varepsilon] - \mathbb{E}^Q[d]) \\
&= \rho \frac{h_\varepsilon (\bar{h} - h_i)}{\bar{h} + h_\varepsilon} \mathbb{E}^Q[\varepsilon] = 0.
\end{aligned}$$

■

**Proof of Proposition 5:** Using that the expected value of the absolute value of a zero-mean normally distributed variable  $X$  is

$$\mathbb{E}[|X|] = \sqrt{2/\pi} \sqrt{\text{Var}[X]},$$

the expected abnormal trading volume is

$$\begin{aligned}
\mathbb{E}[T^*] &= \frac{1}{2} \frac{1}{I} \sum_{i=1}^I \mathbb{E}[|a\tau_i^*(y)|] = \frac{1}{2} \frac{1}{I} \sum_{i=1}^I \sqrt{2/\pi} \sqrt{\text{Var}[a\tau_i^*(y)]} \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{I} \sum_{i=1}^I \sqrt{\rho^2 \frac{h_\varepsilon^2 (\bar{h} - h_i)^2}{(\bar{h} + h_\varepsilon)^2} \sigma_\varepsilon^2} = \frac{\sqrt{h_\varepsilon}}{\bar{h} + h_\varepsilon} \frac{\rho}{\sqrt{2\pi}} \frac{1}{I} \sum_{i=1}^I |\bar{h} - h_i|.
\end{aligned}$$

As a function of the signal precision, the expected abnormal trading volume can be expressed as

$$\mathbb{E}[T^*] = a \sqrt{\frac{h_\varepsilon}{(\bar{h} + h_\varepsilon)^2}}$$

with  $a$  being a positive constant. The comparative statics stated in the proposition then follows from the proof of Proposition 4, and the fact that the square-root function is a strictly increasing function. ■

**Proof of Lemma 1:** Using (8), the equilibrium expected utility of investor  $i$  can be

expressed as

$$\begin{aligned} EU_0^* &= -\exp(-rCE_{i0}^*) [1 + \exp(-\delta) \exp(-r\{CE_{i2}^* - CE_{i0}^*\})] \\ &= -\exp(-rCE_{i0}^*) [1 + \exp(-\delta) \exp(-[\iota - \delta])] = -\exp(-rCE_{i0}^*) [1 + \beta_0]. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\partial}{\partial h_\varepsilon} EU_0^* &= r \exp(-rCE_{i0}^*) [1 + \beta_0] \frac{\partial}{\partial h_\varepsilon} CE_{i0}^* - \exp(-rCE_{i0}^*) \frac{\partial}{\partial h_\varepsilon} \beta_0 \\ &= r \exp(-rCE_{i0}^*) \left\{ [1 + \beta_0] \frac{\partial}{\partial h_\varepsilon} CE_{i0}^* - \rho \frac{\partial}{\partial h_\varepsilon} \beta_0 \right\}. \end{aligned}$$

Investor  $i$ 's  $t = 0$  certainty equivalent is given by

$$CE_{i0}^* = H(\beta_0) - \beta_0 \gamma_{i0}^*,$$

where

$$H(\beta_0) \equiv [p_0(\eta) + d_0] \bar{z}_i + \beta_0 \bar{\gamma}_i + \bar{\kappa}_i - p_0(\eta) x_{i0}^*$$

is the value of the endowments minus the investment in the risky asset. The equilibrium investment in the zero-coupon bond is given by its first-order condition (8), i.e.,

$$\iota = \delta + r \left( \gamma_{i0}^* + U_{1i} + U_{2i} + M_i x_{i0}^* - \frac{1}{2} r V_i (x_{i0}^*)^2 - H(\beta_0) + \beta_0 \gamma_{i0}^* \right).$$

Solving for  $\gamma_{i0}^*$  and using Proposition 2 yield that

$$\begin{aligned} \gamma_{i0}^* &= \frac{\rho(\iota - \delta) - (U_{1i} + U_{2i} + M_i x_{i0}^* - \frac{1}{2} r V_i (x_{i0}^*)^2 - H(\beta_0))}{1 + \beta_0} \\ &= \frac{\bar{U}_1 - U_{1i} + \rho f(\{m_i, \sigma_i^2\}_{i=1, \dots, I}) - (U_{2i} + M_i x_{i0}^* - \frac{1}{2} r V_i (x_{i0}^*)^2) + H(\beta_0)}{1 + \beta_0}. \end{aligned}$$

Note by the proof of Proposition 2 that all except for the first two and the last term in the numerator are independent of the signal precision. Hence,

$$\begin{aligned} \frac{\partial \gamma_{i0}^*}{\partial h_\varepsilon} &= \frac{\left[ \frac{\partial}{\partial h_\varepsilon} \bar{U}_1 - \frac{\partial}{\partial h_\varepsilon} U_{1i} + H'(\beta_0) \frac{\partial}{\partial h_\varepsilon} \beta_0 \right] (1 + \beta_0) - \gamma_{i0}^* (1 + \beta_0) \frac{\partial}{\partial h_\varepsilon} \beta_0}{(1 + \beta_0)^2} \\ &= \frac{\frac{\partial}{\partial h_\varepsilon} \bar{U}_1 - \frac{\partial}{\partial h_\varepsilon} U_{1i} + [H'(\beta_0) - \gamma_{i0}^*] \frac{\partial}{\partial h_\varepsilon} \beta_0}{1 + \beta_0}. \end{aligned}$$



This implies that

$$\begin{aligned}
[1 + \beta_0] \frac{\partial}{\partial h_\varepsilon} \text{CE}_{i0}^* &= [1 + \beta_0] \frac{\partial}{\partial h_\varepsilon} [H(\beta_0) - \beta_0 \gamma_{i0}^*] \\
&= [1 + \beta_0] \left[ [H'(\beta_0) - \gamma_{i0}^*] \frac{\partial}{\partial h_\varepsilon} \beta_0 - \beta_0 \frac{\partial \gamma_{i0}^*}{\partial h_\varepsilon} \right] \\
&= [1 + \beta_0] [H'(\beta_0) - \gamma_{i0}^*] \frac{\partial}{\partial h_\varepsilon} \beta_0 - \beta_0 \left[ \frac{\partial}{\partial h_\varepsilon} \bar{U}_1 - \frac{\partial}{\partial h_\varepsilon} U_{1i} + [H'(\beta_0) - \gamma_{i0}^*] \frac{\partial}{\partial h_\varepsilon} \beta_0 \right] \\
&= [H'(\beta_0) - \gamma_{i0}^*] \frac{\partial}{\partial h_\varepsilon} \beta_0 - \beta_0 \left[ \frac{\partial}{\partial h_\varepsilon} \bar{U}_1 - \frac{\partial}{\partial h_\varepsilon} U_{1i} \right].
\end{aligned}$$

Furthermore, it follows from (8) and Proposition 2 that

$$\begin{aligned}
\frac{\partial}{\partial h_\varepsilon} \beta_0 &= \frac{\partial}{\partial h_\varepsilon} \exp(-(\delta + r(\text{CE}_{i2}^* - \text{CE}_{i0}^*))) \\
&= -r \exp(-(\delta + r(\text{CE}_{i2}^* - \text{CE}_{i0}^*))) \frac{\partial}{\partial h_\varepsilon} [\text{CE}_{i2}^* - \text{CE}_{i0}^*] \\
&= -r \exp(-(\delta + r(\text{CE}_{i2}^* - \text{CE}_{i0}^*))) \frac{\partial}{\partial h_\varepsilon} \bar{U}_1 = -r \beta_0 \frac{\partial}{\partial h_\varepsilon} \bar{U}_1.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\partial}{\partial h_\varepsilon} \text{EU}_0^* &= r \exp(-r \text{CE}_{i0}^*) \left\{ [1 + \beta_0] \frac{\partial}{\partial h_\varepsilon} \text{CE}_{i0}^* - \rho \frac{\partial}{\partial h_\varepsilon} \beta_0 \right\} \\
&= r \exp(-r \text{CE}_{i0}^*) \left\{ [H'(\beta_0) - \gamma_{i0}^*] \frac{\partial}{\partial h_\varepsilon} \beta_0 - \beta_0 \left[ \frac{\partial}{\partial h_\varepsilon} \bar{U}_1 - \frac{\partial}{\partial h_\varepsilon} U_{1i} \right] + \beta_0 \frac{\partial}{\partial h_\varepsilon} \bar{U}_1 \right\} \\
&= r \exp(-r \text{CE}_{i0}^*) \left\{ \beta_0 \frac{\partial}{\partial h_\varepsilon} U_{1i} + [\gamma_{i0}^* - H'(\beta_0)] \frac{\partial}{\partial h_\varepsilon} \beta_0 \right\} \\
&= r \exp(-r \text{CE}_{i0}^*) \beta_0 \left\{ \frac{\partial}{\partial h_\varepsilon} U_{1i} + r [\gamma_{i0}^* - H'(\beta_0)] \frac{\partial}{\partial h_\varepsilon} \bar{U}_1 \right\}.
\end{aligned}$$

Using the fact that the risk-adjusted expected dividend of the risky asset is independent of  $h_\varepsilon$ , it follows that

$$H'(\beta_0) = \text{E}^Q[d] [\bar{z}_i - x_{i0}^*] + \bar{\gamma}_i.$$

Hence,

$$\frac{\partial}{\partial h_\varepsilon} \text{EU}_0^* = r \exp(-r \text{CE}_{i0}^*) \left\{ \beta_0 \frac{\partial}{\partial h_\varepsilon} U_{1i} + [(\gamma_{i0}^* - \bar{\gamma}_i) + \text{E}^Q[d] (x_{i0}^* - \bar{z}_i)] \frac{\partial}{\partial h_\varepsilon} \beta_0 \right\}.$$

Furthermore, it follows from (8) and Proposition 2 that

$$\begin{aligned}
\frac{\partial}{\partial h_\varepsilon} \beta_0 &= \frac{\partial}{\partial h_\varepsilon} \exp(-(\delta + r(\text{CE}_{i2}^* - \text{CE}_{i0}^*))) \\
&= -r \exp(-(\delta + r(\text{CE}_{i2}^* - \text{CE}_{i0}^*))) \frac{\partial}{\partial h_\varepsilon} [\text{CE}_{i2}^* - \text{CE}_{i0}^*] \\
&= -r \exp(-(\delta + r(\text{CE}_{i2}^* - \text{CE}_{i0}^*))) \frac{\partial}{\partial h_\varepsilon} \bar{U}_1 = -r\beta_0 \frac{\partial}{\partial h_\varepsilon} \bar{U}_1.
\end{aligned}$$

Since both  $U_{1i}$  and  $\bar{U}_1$  have a unique maximum for  $h_\varepsilon = \bar{h}$ , all investors' expected utilities have a stationary point for  $h_\varepsilon = \bar{h}$ . ■

**Proof of Proposition 7:** (a) Inserting (31b) into the expression for the equilibrium interest rate and using the market clearing conditions yield

$$\begin{aligned}
\iota &= \delta + r \frac{1}{I} \sum_{i=1}^I [\text{CE}_{i2}^\dagger - \text{CE}_{i0}^\dagger] \\
&= \delta + r \frac{1}{I} \sum_{i=1}^I \left[ \text{CE}_{i2}^* + (\gamma_i^\dagger - \gamma_i^*) + \frac{1}{2} \rho (\mathbf{E}^Q[d])^2 (h_i - \bar{h}) + \Upsilon_i - U_{1i} - \text{CE}_{i0}^* \right] \\
&= \delta + r \frac{1}{I} \sum_{i=1}^I [\text{CE}_{i2}^* + \Upsilon_i - U_{1i} - \text{CE}_{i0}^*] \\
&= \delta + r\bar{\Upsilon} - r\bar{U}_1 + r \frac{1}{I} \sum_{i=1}^I [\text{CE}_{i2}^* - \text{CE}_{i0}^*] = \delta + r\bar{\Upsilon} - r\bar{U}_1 + r [\text{CE}_{i2}^* - \text{CE}_{i0}^*],
\end{aligned}$$

where the second equality follows from the fact that  $\sum_i \text{CE}_{i0}^\dagger = \sum_i \text{CE}_{i0}^* = d_0$ . It then follows from (16) and (17) that

$$\iota = \delta + r\bar{\Upsilon} - r\bar{U}_1 + r\bar{U}_1 + \Phi\left(\{m_i, \sigma_i^2\}_{i=1, \dots, I}\right) = \delta + r\bar{\Upsilon} + \Phi\left(\{m_i, \sigma_i^2\}_{i=1, \dots, I}\right).$$

(b) follows from the fact that both  $\bar{\Upsilon}$  and  $\Phi(\cdot)$  only depend on the prior dividend beliefs.

(c) Comparing (17) and (32), we must show that  $\bar{\Upsilon} > \bar{U}_1$  for any signal precision  $h_\varepsilon$ . Note that

$$\begin{aligned}
\max_{h_\varepsilon} U_{1i} &= \max_{h_\varepsilon} \frac{1}{2} \rho \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_\varepsilon}{(\bar{h} + h_\varepsilon)^2} \right] \\
&= \frac{1}{2} \rho \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{4\bar{h}h_i} \right] = \frac{1}{2} \rho \ln \left[ \frac{\bar{h}^2 + h_i^2 + 2\bar{h}h_i}{4\bar{h}h_i} \right].
\end{aligned}$$

Hence, for investors  $i$  with  $h_i \neq \bar{h}$

$$\begin{aligned}
\Upsilon_i - U_{1i} &= \frac{1}{2}\rho \left\{ \ln \left[ \frac{\bar{h}}{h_i} \right] - \ln \left[ 1 + \frac{(\bar{h} - h_i)^2}{h_i} \frac{h_\varepsilon}{(\bar{h} + h_\varepsilon)^2} \right] \right\} \\
&> \frac{1}{2}\rho \left\{ \ln \left[ \frac{\bar{h}}{h_i} \right] - \ln \left[ \frac{\bar{h}^2 + h_i^2 + 2\bar{h}h_i}{4\bar{h}h_i} \right] \right\} = \frac{1}{2}\rho \ln \left[ \frac{\bar{h}}{h_i} \frac{4\bar{h}h_i}{\bar{h}^2 + h_i^2 + 2\bar{h}h_i} \right] \\
&= \frac{1}{2}\rho \ln \left[ 4 \frac{\bar{h}^2}{(\bar{h} + h_i)^2} \right] = \rho \ln \left[ \frac{2\bar{h}}{\bar{h} + h_i} \right].
\end{aligned}$$

This implies that in settings with heterogeneous prior dividend precisions

$$\begin{aligned}
\bar{\Upsilon} - \bar{U}_1 &> \frac{1}{I} \sum_{i=1}^I \ln \left[ \frac{2\bar{h}}{\bar{h} + h_i} \right] = \frac{1}{I} \sum_{i=1}^I [\ln [2\bar{h}] - \ln [\bar{h} + h_i]] \\
&= \ln [2\bar{h}] - \frac{1}{I} \sum_{i=1}^I \ln [\bar{h} + h_i] > \ln [2\bar{h}] - \ln \left[ \frac{1}{I} \sum_{i=1}^I [\bar{h} + h_i] \right] \\
&= \ln [2\bar{h}] - \ln [2\bar{h}] = 0,
\end{aligned}$$

where the second inequality follows from Jensen's inequality and the fact that  $\ln[\cdot]$  is a concave function. This establishes (c). ■

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