Research Report

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Abstract

A risk process with constant premium rate c and Poisson arrivals of claims is considered. A threshold r is defined for claim interarrival times, such that if k consecutive interarrival times are larger than r, then the next claim has distribution G. Otherwise, the claim size distribution is F. Asymptotic expressions for the infinite horizon ruin probabilities are given both for the light- and the heavy-tailed case. A basic observation is that the process regenerates at each G-claim. Also an approach via Markov additive processes is outlined, and heuristics are given for the distribution of the time to ruin.

Keywords: Ruin theory, Subexponential distribution, Large deviations, Markov additive process, Finite horizon ruin

1 Introduction

In a recent paper, Biard, Lefevre, Loisel & Nagaraja [8] considered risk processes with constant premium inflow at rate c, claims arriving according to a Poisson(λ) process N and a certain dependence between claim interarrival times and claims. That is, the claim surplus process S ([4, p. 1]) is given by

$$S(t) = \sum_{i=1}^{N(t)} U_i - ct \qquad (1.1)$$

where a claim U_i arriving at time t has a distribution depending on $\{N(s)\}_{s \leq t}$.

In one model of [8], a threshold r is defined for claim interarrival times, such that if k consecutive interarrival times are larger than r, then the claim has distribution G. Otherwise, the claim size distribution is F. A motivating examples in [8] is situations where claims are generated by earthquakes. The size of a claim is then positively correlated to the size of the earthquake, and one expects earthquakes to the more severe the less earthquakes have recently occurred so that a G-claim should be larger than an F claim in some appropriate sense. Another model in [8] has flooding as motivating example. Here a close succession of claims is expected to be followed by

a more important catastrophe. For more details on the model, see [8] and Section 8. We focus here on the first model, since the treatment of the second is largely very similar (see again the discussion in Section 8). We also take $k = 2$ for simplicity; the needed extensions to $k > 2$ are outlined, and note also that even the case $k = 1$ is non-trivial (but of course simpler than $k \geq 2$).

The results of [8] are asymptotic relations for finite-ruin probabilities $\psi(u, T)$ when u is large and T is fixed. The present paper complement these results by giving the asymptotics for the infinite horizon ruin probability $\psi(u)$ with initial reserve u as $u \to \infty$. We also heuristically look into the time $\tau(u)$ of ruin, identify the most likely deterministic time where ruin occurs, which leads to expansions for the finite horizon ruin probability $\psi(u, T)$ when both u and T are large. Here

$$\tau(u) = \inf\big\{t > 0 : S(t) > u\big\}, \quad \psi(u) = \mathbb{P}\big(\tau(u) < \infty\big), \quad \psi(u, T) = \mathbb{P}\big(\tau(u) \leq T\big).$$

Whereas these topics are more mainstream ruin theory than [8], we also need a restriction: it is assumed that all claim sizes are independent. Note, however, that some of the dependence modeling in [8] does not easily generalize to an infinite horizon setting.

Our basic point of view is that of regenerative processes, [3, Ch. VI]. The process regenerates at a G claim in the sense that the process forgets about the preceding large claims. Before the next G claim can occur, it has to wait until k long (short) interarrival times have occured. See Section 2 for formalities. Asmussen, Schmidli & Schmidt [6] used such structure to get ruin probability asymptotics in some non-standard risk processes with heavy tails (regeneration is also used in Schmidli [17] but at process levels rather than time instants).

The regenerative set-up is presented in Section 2, with a key calculation (of the mean cycle length) given in Section 3. The first application to the present model is presented in Section 4, where we show that the classical heavy-tailed asymptotics for the ruin probability $\psi(u)$ as $u \to \infty$ holds here as well. Section 5 deals with light tails. Using large deviations techniques from Glynn & Whitt [14] and Duffield & O'Connell [10], we show that the first order decay of $\psi(u)$ is exponential and we identify the rate γ (the adjustment coefficient). Also here the details use the regenerative structure. Section 6 presents an alternative point of view, to view the process as a Markov-modulated risk process where the modulating process is the relevant information on the last two interarrival intervals. This point of view, though not fully exploited here, has the potential of giving stronger results than the large deviations approach, and we demonstrate this by deriving a proper Cramér-Lundberg approximation $\psi(u) \sim Ce^{-\gamma u}$. ln Section 7, we then (in part heuristically) derive the conditional distribution of the ruin time given that ruin occurs. Finally Section 8 contains a number of concluding remarks, and in Appendices A–C, we give the details of some of the longer technical calculations.

2 Preliminaries

We recall the basic regenerative set-up of [6]:

Assumption 2.1. *There exists random times $\omega_0 = 0, \omega_1, \omega_2, \ldots$ such that the post-ω_k process $\left\{S(t+\omega_k) - S(\omega_k)\right\}_{t\geq 0}$ is independent of the pre-ω_k process $\left\{S(t)\right\}_{0\leq t \leq \omega_k}$ and its distribution does not depend on k.*

In our setting, $\omega_1, \omega_2, \ldots$ are the times of G claims. Except for independence of $\left\{S(t+\omega_1) - S(\omega_1)\right\}_{t\geq 0}$, we will typically not need strong conditions on the distribution of the initial cycle $\left\{S(t)\right\}_{0\leq t < \omega_1}$. However, some weak technical conditions are given in the respective Theorems. When $\omega_1 = 0$, we write $\mathbb{P}_0, \mathbb{E}_0$. We often use notation like $\mathbb{P}(\omega \leq x)$ instead of $\mathbb{P}_0(\omega_2 \leq x)$. Define further

$$M_k^* = \sup_{\omega_k \leq t < \omega_{k+1}} \left(S(t) - S(\omega_k)\right), \quad S_k^* = S(\omega_{k+1}), \quad m^* = \left|\mathbb{E}_0 S_1^*\right| = \left|\mathbb{E}S(\omega)\right|.$$

Under Assumption 2.1, $M_1, M_2, \ldots$ are i.i.d. and the the $S_k^* - S_1^*$ form a random walk. A basic assumption of [6] is (note that trivially $\mathbb{P}_0(S_1^* > x) \leq \mathbb{P}_0(M_1^* > x)$):

Assumption 2.2. $\mathbb{P}_0(S_1^* > x) \sim \mathbb{P}_0(M_1^* > x)$, $x \to \infty$.

Then ([6]; for background on subexponential distributions, see, e.g., [11], [4, X.1] or [12]):

Theorem 2.3. *Assume in addition to Assumptions 2.1 and 2.2 that $\mathbb{E}_0 S_1^* < 0$, that both $\overline{H}(x) = \mathbb{P}_0(S_1^* > x)$ and the integrated tail $\overline{H}_I(x) = \int_x^\infty \overline{H}(y) \, \mathrm{d}y$ are subexponential tails, and that $\mathbb{P}(M_0^* > x) = \mathrm{o}\big(\overline{H}_I(x)\big)$. Then*

$$\psi(u) \sim \frac{1}{m^*} \overline{H}_I(u), \quad u \to \infty.$$

3 The mean cycle length

As usual when applying regenerative processes, a main step is computation of the mean cycle length $\mu = \mathbb{E}\omega = \mathbb{E}_0\omega_2$. To this end, let V denote the generic claim interarrival time and define

$$p^+ = \mathbb{P}(V > r) = \mathrm{e}^{-\lambda r}, \quad p^- = \mathbb{P}(V \leq r) = 1 - \mathrm{e}^{-\lambda r}, \tag{3.1}$$

$$m^+ = \mathbb{E}[V \mid V > r] = \frac{1}{p^+} \int_r^\infty x\lambda \mathrm{e}^{-\lambda x} \, \mathrm{d}x = r + \frac{1}{\lambda}, \tag{3.2}$$

$$m^- = \mathbb{E}[V \mid V \leq r] = \frac{1}{p^-} \int_0^r x\lambda \mathrm{e}^{-\lambda x} \, \mathrm{d}x = \frac{1}{\lambda} - \frac{r\mathrm{e}^{-\lambda r}}{1 - \mathrm{e}^{-\lambda r}}. \tag{3.3}$$

Recall that a cycle starts just after a G claim and ends at the next. A cycle is initiated by K short claims where K is geometric, $\mathbb{P}(K = k) = p^{-k}p^+$; this part $\omega^{(1)}$ of the cycle has mean $(p^-/p^+)\,m^-$. After that comes a long part $\omega^{(2)}$ with mean m^+. For the final part $\omega^{(3)}$, two possibilities occur: The next interarrival time is long and the cycle terminates. This gives an additional contribution to the mean to of $p^+ m^+$.

Or the next interarrival time is short and the cycle starts afresh after that. Putting things together gives

$$\mu = \frac{p^-}{p^+}m^- + m^+ + p^+m^+ + p^-(m^- + \mu)$$

so that we have shown

Theorem 3.1. *The mean cycle length $\mu = \mathbb{E}\omega = \mathbb{E}_0\omega_2$ is given by*

$$\mu = \frac{1}{p^+}\left[\frac{p^-}{p^+}m^- + m^+ + p^+m^+ + p^-m^-\right]$$

$$= \frac{1}{p^+}\left[(1 + \frac{1}{p^+})(p^+m^+ + p^-m^-)\right] = \frac{1}{\lambda}\frac{1 + p^+}{p^{+2}} = \frac{1}{\lambda}e^{\lambda r}(1 + e^{\lambda r}). \qquad (3.4)$$

4 Heavy tails and regeneration

Having computed μ, the criterion for stability follows easily. Let m_F, m_G denote the means of F, resp. G.

Corollary 4.1. *Define*

$$\eta = 1 - \frac{(\lambda\mu - 1)m_F + m_G}{c\mu}.$$

If $\eta > 0$, then $\psi(u) < 1$ for all u, whereas $\psi(u) \equiv 1$ if $\eta \leq 0$.

Proof. Let M denote the number of claims in a cycle. By regenerative process theory, $\lambda = \mathbb{E}M/\mu$ and $\lim_{t\to\infty} S(t)/t$ exists and equals $\mathbb{E}S(\omega)/\mu$. But

$$\mathbb{E}S(\omega) = \mathbb{E}(M - 1)\mu_F + \mu_G - c\mu = (\lambda\mu - 1)\mu_F + \mu_G - c\mu. \qquad (4.1)$$

Thus $\lim S(t)/t$ is < 0 for $\eta > 0$ and > 0 for $\eta < 0$, which immediately gives the conclusion. If $\eta = 0$, we need in addition to remark that $S(\omega_1), S(\omega_2), \ldots$ is a mean zero random walk. Hence $\limsup_k S(\omega_k) = \infty$ so that $\tau(u) < \infty$ for all u. $\qquad \square$

Note that, as the proof reveals, η plays the role of the safety loading (the relative amount by which the premium rate c exceeds the expected inflow of claims), cf. [4, p. 3]. In the following, it is throughout assumed that $\eta > 0$.

We are now ready to state and prove our main result for the heavy-tailed case. From e.g. [12], subexponentiality of G implies that there exists a non-decreasing function δ with $\delta(x) \to \infty$ as $x \to \infty$, such that

$$\overline{G}(x \pm \delta(x)) \sim \overline{G}(x) \quad \text{as } x \to \infty. \qquad (4.2)$$

Theorem 4.2. *Assume that $\overline{F}(x) = 1 - F(x)$, $\overline{G}(x) = 1 - G(x)$ and the integrated tail $\overline{G}_I(x) = \int_x^\infty \overline{G}(y)\,\mathrm{d}y$ are all subexponential tails. Assume further that $\mathrm{e}^{-g\delta(x)} = \mathrm{o}(\overline{G}_I(x))$ for all $g > 0$, that $\overline{F}(x) = \mathrm{o}(\overline{G}(x))$ and that $\mathbb{P}(M_0^* > x) = \mathrm{o}(\overline{G}_I(x))$. Then*

$$\psi(u) \sim \frac{1}{\eta c\mu}\overline{G}_I(u), \quad u \to \infty. \qquad (4.3)$$

Proof. Define $S^* = S(\omega)$, $M^* = \sup_{t \leq \omega} S(t)$. We shall show that

$$\mathbb{P}(S^* > x) \sim \overline{G}(x), \qquad (4.4)$$

$$\mathbb{P}(M^* > x) \sim \overline{G}(x) \qquad (4.5)$$

as $x \to \infty$, from which the result follows from Theorem 3.1, using (4.1) to identify the constant $\left|\mathbb{E}S(\omega)\right|$ in front of $\overline{G}(u)$.

Write $S^* = S(\omega) = \sum_{i=1}^{M} U_i - c\omega$ where as in Section 3 M is the number of claims in a cycle. We have

$$\mathbb{P}\Big(\sum_{i=1}^{N(\omega)} U_i > x\Big) = \mathbb{P}\Big(\sum_{i=1}^{M} U_i > x\Big) = \mathbb{P}\Big(U_M + \sum_{i=1}^{M-1} U_i > x\Big).$$

Since M is light-tailed and independent of the U_i, we know from, e.g., [11] that

$$\mathbb{P}\Big(\sum_{i=1}^{M-1} U_i > x\Big) \sim \mathbb{E}\big[M-1\big]\overline{F}(x) = \mathrm{o}\big(\overline{G}(x)\big).$$

Thus, from $\mathbb{P}(U_M > x) = \overline{G}(x)$ and, e.g., [12], we have

$$\mathbb{P}\Big(\sum_{i=1}^{M} U_i > x\Big) \sim \overline{G}(x).$$

For $x > 0$, let us decompose $\mathbb{P}(S^* > x)$ as

$$\mathbb{P}\big(S^* > x, \, \omega \leq \delta(x)\big) + \mathbb{P}\big(S^* > x, \, \omega > \delta(x)\big).$$

Since ω has finite exponential moments, we have, for some constant $\gamma > 0$,

$$\mathbb{P}\big(S^* > x, \, \omega > \delta(x)\big) \leq \mathbb{P}\big(\omega > \delta(x)\big) \leq \mathrm{e}^{-\gamma\delta(x)} = \mathrm{o}\big(\overline{G}_I(x)\big). \qquad (4.6)$$

Moreover,

$$\mathbb{P}\Big(\sum_{i=1}^{M} U_i > x + c\delta(x)\Big) \leq \mathbb{P}\big(S^* > x, \, \omega \leq \delta(x)\big) \leq \mathbb{P}\Big(\sum_{i=1}^{M} U_i > x\Big).$$

From (4.2) and (4.6) we get

$$\mathbb{P}(S^* > x) \sim \mathbb{P}\big(S^* > x, \, \omega \leq \delta(x)\big) \sim \mathbb{P}\Big(\sum_{i=1}^{M} U_i > u\Big) \sim \overline{G}(x),$$

proving (4.4).

We have also $\mathbb{P}(M^* > x) \sim \mathbb{P}\big(S^* > x\big)$ since

$$S^* \leq M^* \leq \sum_{i=1}^{M} U_i,$$

and thus

$$\mathbb{P}(S^* > x) \sim \mathbb{P}\Big(\sum_{i=1}^{M} U_i > x\Big). \qquad \square$$

5 Light tails and regeneration

We now assume both G and F are light-tailed and write $\widehat{F}[\cdot]$ for the m.g.f. of F and $\widehat{G}[\cdot]$ for the m.g.f. of G. To obtain the first order asymptotics of the ruin probability $\psi(u)$ as $u \to \infty$, the theory of large deviations ([4, XIII.1]) suggest that we first establish that in a suitable α-range,

$$\frac{1}{t} \log \mathbb{E} \mathrm{e}^{\alpha S(t)} \to \kappa(\alpha) \tag{5.1}$$

for some function κ. If $\kappa(\gamma) = 0$ for some $\gamma > 0$, it then holds under weak additional conditions that $\psi(u)$ is of rough order $\mathrm{e}^{-\gamma u}$ in the sense that

$$\frac{1}{u} \log \psi(u) \to -\gamma \,. \tag{5.2}$$

The precise additional conditions are listed in Glynn & Whitt [14] and Duffield & O'Connor [10]. Note that the set-up of [14] is discrete time, whereas we here, as in [10], have a continuous-time problem. The conditions of [10] are somewhat more technical to verify than those of [14], but luckily, the path structure (1.1) essentially reduces the problem to discrete time; see Section 7.2 for details. Thus, we will omit the easy verification of conditions of [14] beyond (5.1) and the existence of γ.

We proceed in two steps, of which the first is:

Proposition 5.1. $\varphi(\alpha, \beta) = \mathbb{E}\big[\mathrm{e}^{\alpha S(\omega)}\mathrm{e}^{\beta \omega}\big]$ *is given by*

$$\varphi(\alpha, \beta) = \frac{g(\alpha, \beta)^2}{1 - k(\alpha, \beta)\big[1 + g(\alpha, \beta)\big]} \,, \tag{5.3}$$

where

$$g(\alpha, \beta) = \frac{\lambda}{\lambda + c\alpha - \beta} \, \widehat{G}[\alpha]\mathrm{e}^{(\beta - c\alpha - \lambda)r} \,,$$

$$k(\alpha, \beta) = \frac{\lambda}{\lambda + c\alpha - \beta} \, \widehat{F}[\alpha]\big[1 - \mathrm{e}^{(\beta - c\alpha - \lambda)r}\big] \,.$$

Proof. The argument is similar, though more elaborate, than for (3.4). We will need the formulas

$$\widehat{A}^{+}[\alpha] = \mathbb{E}[\mathrm{e}^{\alpha V} \,|\, V > r] = \frac{\lambda}{\lambda - \alpha}\mathrm{e}^{\alpha r} \,, \tag{5.4}$$

$$\widehat{A}^{-}[\alpha] = \mathbb{E}[\mathrm{e}^{\alpha V} \,|\, V \le r] = \frac{\lambda}{\lambda - \alpha}\Big(\frac{1 - \mathrm{e}^{-(\lambda - \alpha)r}}{1 - \mathrm{e}^{-\lambda r}}\Big) \,. \tag{5.5}$$

Let $S_1^{\#} = S(\omega^{(1)})$, $S_2^{\#} = S(\omega^{(1)} + \omega^{(2)}) - S(\omega^{(1)})$, $S_3^{\#} = S(\omega^{(1)} + \omega^{(2)} + \omega^{(3)}) - S(\omega^{(1)} + \omega^{(2)})$ with $\omega^{(1)}, \omega^{(2)}, \omega^{(3)}$ as in Section 3. Then

$$\varphi(\alpha, \beta) = \prod_{i=1}^{3} \mathbb{E}\big[\mathrm{e}^{\alpha S_i^{\#}} \mathrm{e}^{\beta \omega^{(i)}}\big] \,.$$

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The easy case is $i = 2$ where we can write $S_2^\# = U_G - cV^+$, $\omega^{(2)} = V^+$ where U_G has distribution G and V^+ the conditional distribution of V given $V > r$. This gives

$$\mathbb{E}\left[e^{\alpha S_2^\#} e^{\beta \omega^{(2)}}\right] = \widehat{G}[\alpha]\widehat{A}^+[\beta - c\alpha].$$

For $i = 3$, considering the same two possibilities as for the mean gives by a similar sample path representation the expression

$$p^+ \widehat{G}[\alpha]\widehat{A}^+[\beta - c\alpha] + p^- \widehat{A}^-[\beta - c\alpha]\widehat{F}[\alpha]\varphi(\alpha, \beta)$$

for $\mathbb{E}\left[e^{\alpha S_3^\#} e^{\beta \omega^{(3)}}\right]$. Finally, $\mathbb{E}\left[e^{\alpha S_1^\#} e^{\beta \omega^{(1)}}\right]$ is given by

$$\mathbb{E}\left[\widehat{F}[\alpha]^K \widehat{A}^-[\beta - c\alpha]^K\right] = \frac{p^+}{1 - p^- \widehat{F}[\alpha]\widehat{A}^-[\beta - c\alpha]}.$$

Putting things together proves the proposition. □

To obtain (5.1) from (5.3), we use renewal theory. Let $Z(t) = \mathbb{E}_0 e^{\alpha S(t)}$. Conditioning on ω then yields the renewal equation

$$Z(t) = \mathbb{E}\left[e^{\alpha S(t)}; \omega > t\right] + \int_0^t Z(t - y)\mathbb{E}\left[e^{\alpha S(\omega)}; \omega \in \mathrm{d}y\right] \tag{5.6}$$

Choose $\kappa = \kappa(\alpha)$ as the solution of

$$1 = \varphi(\alpha, \kappa) = \int_0^\infty e^{-\kappa y}\mathbb{E}\left[e^{\alpha S(\omega)}; \omega \in \mathrm{d}y\right]. \tag{5.7}$$

Then by standard renewal theory, $e^{-\kappa(\alpha)t} Z(t)$ has limit

$$\int_0^\infty e^{-\kappa(\alpha)v}\mathbb{E}\left[e^{\alpha S(v)}; \omega > v\right] \mathrm{d}v \bigg/ \int_0^\infty v e^{-\kappa(\alpha)v}\mathbb{E}\left[e^{\alpha S(v)}; \omega \in \mathrm{d}v\right] \tag{5.8}$$

as $t \to \infty$. This implies (5.1). Taking $\gamma > 0$ as solution of $\kappa(\gamma) = 0$ gives the ruin probability approximation (5.2) in the zero-delayed case. Putting things together and involving the further technical conditions of [14] gives:

Theorem 5.2. *Assume there exists $\gamma > 0$ such that (5.1) holds in a neighborhood of γ, that $\kappa(\gamma) = 0$ and that $\mathbb{P}(M_0^* > x) = \mathrm{o}(e^{-\gamma x})$. Then (5.2) holds.*

6 The Markov additive point of view

The large deviations approach of Section 5 is quick and applicable to more general models, but also somewhat imprecise by only giving logarithmic asymptotics. In this section, we outline how a Markov additive point of view ([4, III.4, VI.3b, VII, IX.5] can lead to more precise asymptotics, in particular a Cramér-Lundberg approximation of the form $\psi(u) \sim C e^{-\gamma u}$.

A special case of a Markov additive model for the claim surplus process S is Markov-modulated Poisson process arrivals and -claims. It asserts that there is a

driving background Markov process J with state space E, such that the claim arrival intensity is λ_i when $J(t) = i$ and the claim size distribution is F_i. Cf. [4, III.4, Ch. VII]. This covers the model of the present paper as will next be explained.

The relevant background information for our Poisson gap generated risk process at time t is at a first sight (b, a) where a is the time since the last arrival and b is the length of the previous interarrival interval. However, this description is somewhat redundant since what matters is really comparisons of b, a to the threshold length r. Accordingly, we define $J(t) = -, a$ if $b, a \leq r$, $J(t) = -, +$ if $b \leq r, a > r$, $J(t) = +, a$ if $b > r, a \leq r$ and $J(t) = +, +$ if $a, b > r$. Thus, the state space of J is

$$E = \{-\} \times [0, r] \cup \{-, +\} \cup \{+\} \times [0, r] \cup \{+, +\}$$

and $\lambda_i \equiv \lambda$, $F_i = G$ when $i = +, +$, $F_i = F$ for all other $i \in E$. Note that after a G-claim, J is reset to $-, 0$. An important point is that E is uncountable, which presents a complication compared to the most studied finite case, cf. [4, Ch. VII].

For the analysis, we proceed according to the program outlined for the infinite case in [4, Remark III.4.8 p. 58] (the examples treated in [4] is periodic environment in VII.6 and the renewal risk processes in VI.3b). For a fixed θ, we need to determine $\kappa(\theta)$ and $h^\theta : E \to (0, \infty)$ such that

$$\mathbb{E}_i e^{\theta S(t)} \sim e^{\kappa(\theta)} h^\theta(i), \quad t \to \infty. \tag{6.1}$$

where $\mathbb{E}_i$ refers to the case $J(0) = i$. Noting that $J(0) = -, 0$ corresponds to the zero-delayed regenerative case, we know from Section 5 that this holds for $i = -, 0$ with the same $\kappa(\theta)$ as computed there and $h^\theta(-, 0)$ given by (5.8). For a general i, we have

$$\mathbb{E}_i e^{\theta S(t)} = \int_0^t \mathbb{E}_i \big[e^{\theta S(\omega_1)}; \omega_1 \in ds \big] \mathbb{E}_{-,0} e^{\theta S(t-s)} + \mathbb{E}_i \big[e^{\theta S(t)}; \omega_1 > t \big].$$

Letting $t \to \infty$, the second term vanishes, and using (6.1) with $i = -, 0$ gives (6.1) for a general i with

$$h^\theta(i) = \mathbb{E}_i e^{\theta S(\omega_1) - \kappa(\theta)\omega_1}. \tag{6.2}$$

Here, as will be shown in Appendix C, we have with g, h as in Proposition 5.1 and with

$$g^{(a)}(\alpha, \beta) = \frac{\lambda}{\lambda + c\alpha - \beta} \widehat{G}[\alpha] e^{(\beta - c\alpha - \lambda)(r-a)},$$

$$k^{(a)}(\alpha, \beta) = \frac{\lambda}{\lambda + c\alpha - \beta} \widehat{F}[\alpha] \left(1 - e^{(\beta - c\alpha - \lambda)(r-a)} \right),$$

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that

$$h^\theta(-,0) = \frac{g(\theta, -\kappa(\theta))^2}{1 - k(\theta, -\kappa(\theta))(1 + g(\theta, -\kappa(\theta)))} \,, \tag{6.3}$$

$$h^\theta(+,+) = \frac{\lambda}{\lambda + c\theta + \kappa(\theta)} \widehat{G}[\theta] \,, \tag{6.4}$$

$$h^\theta(+,a) = \frac{k^{(a)}(\theta, -\kappa(\theta)) g(\theta, -\kappa(\theta))}{1 - k(\theta, -\kappa(\theta)) \left[1 + g(\theta, -\kappa(\theta))\right]} + g^{(a)}(\theta, -\kappa(\theta)) \,, \tag{6.5}$$

$$h^\theta(-,+) = \frac{\lambda}{\lambda + c\theta + \kappa(\theta)} \widehat{G}[\theta] \frac{g(\theta, -\kappa(\theta))(1 - k(\theta, -\kappa(\theta)))}{1 - k(\theta, -\kappa(\theta)) \left[1 + g(\theta, -\kappa(\theta))\right]} \,, \tag{6.6}$$

$$h^\theta(-,a) = \frac{1}{1 - k\big(\theta, -\kappa(\theta)\big) \left[1 + g\big(\theta, -\kappa(\theta)\big)\right]} \Big\{ k^{(a)}(\theta, -\kappa(\theta)) g^2(\theta, -\kappa(\theta))$$

$$+ \frac{\widehat{F}[\theta]}{\widehat{G}[\theta]} g^{(a)}\big(\theta, -\kappa(\theta)\big) g\big(\theta, -\kappa(\theta)\big) \left[1 - k\big(\theta, -\kappa(\theta)\big)\right] \Big\} \,. \tag{6.7}$$

From [4, Remark III.4.8 p. 58], we now have the representation

$$\psi_i(u) = \mathrm{e}^{-\gamma u} h^\gamma(i) \widetilde{\mathbb{E}}_i \big[\mathrm{e}^{-\gamma \xi(u)} / h^\gamma \big(J(\tau(u))\big) \big] \,, \tag{6.8}$$

where $\psi_i(u) = \mathbb{P}_i\big(\tau(u) < \infty\big)$, $\xi(u) = S\big(\tau(u)\big) - u$ and $\widetilde{\mathbb{E}}$ refers to a suitably exponentially tilted probability measure. Now the process $\big\{ \big(\xi(u), J(\tau(u))\big) \big\}_{u>0}$ is regenerative (say, regeneration occurs at u-values with $\xi(u) = 0$, $J(\tau(u)) = +, +$). Hence a limit $\big(\xi(\infty), J^*(\infty)\big)$ in distribution exists and is independent of i, and (6.8) gives:

Theorem 6.1. *In the light-tailed case,*

$$\psi_i(u) \sim \mathrm{e}^{-\gamma u} h^\gamma(i) C \,, \tag{6.9}$$

where $C = \mathbb{E} \big[\mathrm{e}^{-\gamma \xi(\infty)} / h^\gamma \big(J^*(\infty)\big) \big]$.

We note that whereas $h^\gamma(i)$ is explicitly computable for all i, so is hardly the case for C (even in the finite case, the evaluation leads into ladder height problems with no explicit solution). One should also note that the Markov additive point of view hardly brings any strengthenings or simplications for the heavy-tailed case: we are not aware of any general state space results in this setting.

7 The time to ruin

Let $\mathbb{P}^{(u)}$ denote the conditional distribution given $\tau(u) < \infty$. Our goal here is to determine the asymptotic form of the $\mathbb{P}^{(u)}$-distribution of the ruin time $\tau(u)$. Once this has been obtained, asymptotic expansions of the finite horizon ruin probabilities follow from

$$\psi(u, T) = \psi(u) \mathbb{P}^{(u)} \big(\tau(u) \le T\big) \,,$$

but we shall not spell out the easy details.

7.1 The heavy-tailed case

We here state a result that we strongly believe to be true, with a sketch (but only sketch!) of proof. Results of this type have their origin in Asmussen & Klüppelberg [5]). The setting is that of the general regenerative structure in Section 2. which, of course, cover our Poisson gap model as a special case . We will need the *mean excess function* $e(x)$ of H given by $e(x) = \overline{H}_I(x)/\overline{H}(x)$. There are two main cases (cf. [4, pp. 300–301]):

(a) $\overline{H}$ is a regularly varying tail with index $\alpha > 1$, i.e. $\overline{H}(x) = L(x)/x^\alpha$ with L slowly varying at $+\infty$. Then $\overline{H}_I(x) \sim L(x)/x^{\alpha-1}(\alpha - 1)$, $e(x) \sim x/(\alpha - 1)$ and $\overline{H}(x + we(x)) \sim \overline{H}(x)\mathbb{P}(W > w)$ where W is Pareto with mean 1 and the same α, i.e. with tail $(1 + w/(\alpha - 1))^{-\alpha}$;

(b) $e(x)$ is self-neglecting in the sense that for all y, $e(x + ye(x))/e(x) \to 1$ as $x \to \infty$. Then $\overline{H}(x + we(x)) \sim \overline{H}(x)\mathbb{P}(W > w)$ where W is standard exponential.

Note that the self-neglecting property holds in standard examples of subexponential distributions with lighter tails than regularly varying, say the lognormal or the DFR Weibull. For further discussion and background, see [5], [7], [16].

Recall that $m^* = |\mathbb{E}S(\omega)|$.

Conjecture 7.1. *In addition to the conditions of Theorem 2.3, assume that either (a) or (b) holds and that $\mathbb{E}_0[\omega_2 \,|\, M_2^* > x] = \mathrm{o}(e(x))$. Then $\tau(u)/e(u) \to W\mu/m^*$ in $\mathbb{P}^{(u)}$-distribution*

Proof. It is contained in the proof of [6] that $\psi(u) \sim \mathbb{P}(M^* > u) = \mathbb{P}(\tau^*(u) < \infty)$ where $M^* = \sup(S^*, S_2^*, \ldots)$, $\tau^*(u) = \inf\{k : S_k^* > u\}$. It follows (using more general results by Foss, Palmowski & Zachary [13] in the first step) that

$$\mathbb{P}(\tau^*(u) \le ae(u)) \sim \sum_{k=1}^{ae(u)} \overline{H}(km^* + u) \sim \int_0^{ae(u)} \overline{H}(zm^* + u)\,\mathrm{d}z$$
$$= \left[\overline{H}_I(u) - \overline{H}_I(u + am^*e(u))\right]/m^*$$
$$= \left[\overline{H}(u)e(u) - \overline{H}(u + am^*e(u))e(u + am^*e(u))\right]/m^*. \quad (7.1)$$

In case (a), the asymptotics of $[\cdot]$ is

$$\frac{L(u)}{u^\alpha} \cdot \frac{u}{\alpha - 1} - \frac{L(u + am^*u/(\alpha - 1))}{(u + am^*e(u))^\alpha} \cdot \frac{u + am^*u/(\alpha - 1)}{\alpha - 1}$$
$$\sim \frac{L(u)}{u^\alpha} \cdot \frac{u}{\alpha - 1} - \frac{L(u)}{(u + am^*u/(\alpha - 1))^\alpha} \cdot \frac{u + am^*u/(\alpha - 1)}{\alpha - 1}$$
$$= \frac{L(u)}{(\alpha - 1)u^{\alpha-1}}\left[1 - \frac{1}{(1 + am^*/(\alpha - 1))^{\alpha-1}}\right] = \overline{H}_I(u)\mathbb{P}(W/m^* \le a)$$

It is further strongly suggested by the proof of [6] that cycles prior to the one with ruin are 'typical' and so should have approximate mean μ. Thus one expects that (recall the assumption $\mathbb{E}_0[\omega_2 \,|\, M_2^* > x] = \mathrm{o}(e(x))$)

$$\tau(u) \approx (\tau^*(u) - 1)\mu + \mathbb{E}_0[\omega_2 \,|\, M_2^* > u] \approx \tau^*(u)\mu$$

on the set where $\tau(u) < \infty$. Therefore, by Theorem 2.3,

$$\mathbb{P}^{(u)}\big(\tau(u)/e(u) \le b\big) = \frac{\mathbb{P}\big(\mu\tau(u)/e(u \le b)}{\psi(u)}$$

$$\approx \frac{\overline{H}_I(u)\mathbb{P}(\mu W/m^* \le b)/m^*}{\overline{H}_I(u)/m^*} = \mathbb{P}(\mu W/m^* \le b)\,,$$

showing the claim in case (a).

In case (b) the asymptotics of (7.1) is

$$\big[\overline{H}(u)e(u) - \overline{H}\big(u + am^*e(u)\big)e(u)\big]/m^* \sim \overline{H}_I(u)\big[1 - \mathbb{P}(W/m^* > a)\big]/m^*\,.$$

Now just proceed as in case (a). $\qquad\qquad\square$

7.2 The light-tailed case

With light tails, we have

Proposition 7.2. *For each $\delta > 0$,* $\dfrac{1}{u}\log\mathbb{P}\Big(\Big|\dfrac{\tau(u)}{u} - \dfrac{1}{\kappa'(\gamma)}\Big| \le \delta\Big) \to -\gamma\,.$

Proof. A discrete time version of the results is in Glynn & Whitt [14] (see also [4, p. 404]), but is less explicit in the continuous-time setting of Duffield & O'Connel [10]. However, with the path structure (1.1) the continuous-time version is easily derived from the discrete-time one. Let $\tau_0(u) = \inf\{n \in \mathbb{N} : S(n) > u\}$. Then clearly $\tau(u) \le \tau_0(u)$, whereas (1.1) implies $\tau(u - c) \le \tau(u) + 1$. Further, by [14],

$$\frac{1}{u}\log\mathbb{P}\Big(\Big|\frac{\tau_0(u)}{u} - \frac{1}{\kappa'(\gamma)}\Big| \le \delta\Big) \to -\gamma \qquad\qquad (7.2)$$

which together with (5.2) gives

$$\frac{1}{u}\log\mathbb{P}\Big(\Big|\frac{\tau_0(u)}{u} - \frac{1}{\kappa'(\gamma)}\Big| > \delta,\, \tau_0(u) < \infty\Big) \to 0\,. \qquad\qquad (7.3)$$

Now write

$$\mathbb{P}\Big(\Big|\frac{\tau(u)}{u} - \frac{1}{\kappa'(\gamma)}\Big| \le \delta\Big) \qquad\qquad (7.4)$$

$$= \mathbb{P}\Big(\frac{\tau(u)}{u} - \frac{1}{\kappa'(\gamma)} \le \delta\Big) - \mathbb{P}\Big(\frac{\tau(u)}{u} - \frac{1}{\kappa'(\gamma)} < -\delta\Big)\,. \qquad\qquad (7.5)$$

Recalling $\tau(u) \le \tau_0(u)$, the first term in (7.5) can be written

$$\mathbb{P}\Big(\frac{\tau_0(u)}{u} - \frac{1}{\kappa'(\gamma)} \le \delta\Big) + \mathbb{P}\Big(\frac{\tau_0(u)}{u} - \frac{1}{\kappa'(\gamma)} > \delta,\, \frac{\tau(u)}{u} - \frac{1}{\kappa'(\gamma)} \le \delta\Big)\,.$$

Using (7.2), (7.3) shows that the first term dominates the second in the logarithmic sense. Similarly, the second term in (7.5) can be written

$$\mathbb{P}\Big(\frac{\tau_0(u)}{u} - \frac{1}{\kappa'(\gamma)} < -\delta\Big) - \mathbb{P}\Big(\frac{\tau_0(u)}{u} - \frac{1}{\kappa'(\gamma)} > -\delta,\, \frac{\tau(u)}{u} - \frac{1}{\kappa'(\gamma)} < -\delta\Big)\,.$$

Recalling $\tau(u-c) \leq \tau(u) + 1$, the second term can be bounded by

$$\mathbb{P}\Big(\frac{\tau_0(u-c)-1}{u} - \frac{1}{\kappa'(\gamma)} \leq -\delta\Big)$$

$$= \mathbb{P}\Big(\frac{\tau_0(u-c)}{u-c} - \frac{1}{\kappa'(\gamma)} \leq -\delta\frac{u}{u-c} + \frac{1+c}{\kappa'(\gamma)(u-c)}\Big)$$

which is at most

$$\mathbb{P}\Big(\frac{\tau_0(u-c)}{u-c} - \frac{1}{\kappa'(\gamma)} \leq -\delta/2\Big)$$

for large u. Collecting terms shows that the logarithmic order of (7.4) is the same as that of

$$\mathbb{P}\Big(\Big|\frac{\tau_0(u)}{u} - \frac{1}{\kappa'(\gamma)}\Big| \leq \delta\Big).$$

Using (7.2) completes the proof. □

For discussion of Proposition 7.2, see Section 8.

8   Remarks and extensions

1. Of further recent studies of risk models with dependence between claim sizes and interarrival times, we mention in particular Albrecher & Boxma [2] and Boudreault, Cossette, Landriault & Marceau [9]. A survey of different types of risk models with dependence and references is in [4, Ch. XIII].

2. In Biard, Lefevre, Loisel & Nagaraja [8], a flooding model is also considered which has a regenerative structure as well. The difference between the two models is the cycle structure. In the earthquake model the process regenerates at a G claim, so after a sequence of k-successive interarrival times larger than r. In the flooding model, the process regenerates after a sequence of at least k-successive interarrival times smaller than r followed by one larger than r.

   To substantiate the claim made in Section 1 that the regenerative analysis carries over to this case, we will as an example derive the mean cycle length for also $k=2$. Let ω be the cycle length and μ its mean. A cycle in initiated by K long interarrival times where K is geometric, $\mathbb{P}(K=k) = p^{+k}p^-$; this part $\omega^{(1)}$ of the cycle has mean $(p^+/p^-)\,m^+$. After that comes a short part $\omega^{(2)}$ with mean m^-. For the final part $\omega^{(3)}$, two possibilities occur: The next interarrival time is short and the cycle terminates after K' short interarrival times where K' is geometric plus one other long time. This gives an additional contribution to the mean to of $p^-(m^- + \frac{p^-}{p^+}m^- + m^+)$. Or the next interarrival time is long and the cycle starts afresh after that. Putting things together gives

$$\mu = \frac{p^+}{p^-}m^+ + m^- + p^-\Big(m^- + \frac{p^-}{p^+}m^- + m^+\Big) + p^+(m^+ + \mu)$$

so that

$$\mu = \frac{1}{p^-}\left[\frac{p^+}{p^-}m^+ + m^- + p^-\left(m^- + \frac{p^-}{p^+}m^- + m^+\right) + p^+ m^+\right]$$

$$= \frac{1}{p^-}\left[\frac{1}{p^-}m^+ + \frac{1}{p^+}m^-\right] = \frac{1}{p^+ p^{-2}}\frac{1}{\lambda} = \frac{1}{\lambda}\frac{\mathrm{e}^{3\lambda r}}{(1-\mathrm{e}^{\lambda r})^2}\,.$$

3. In the earthquake interpretation of the model, one could argue that an earthquake triggers not only a single claim but a whole cascade due to the effects of afterquakes. This is conveniently modeled by a shot-noise process with the same characteristics of the primary process of initiation of cascades as in this paper. Many of our conclusions carry over in an easy matter to this case. The basic fact is that replacing cascades by a single claim of the time of initiation does not change neither the exact asymptotics in the heavy-tailed case nor the logarithmic asymptotics in the light-tailed case, cf. Albrecher & Asmussen [1].

4. As the two previous items suggest, the techniques of the paper carry far beyond our specific model, so that in a wider perspective, one could view the paper as a case study in regenerative methods.

5. A different approach to the heavy-tailed case is *reduced load equivalence*, which is a way to formalize the frequent occuring phenomenon that if a risk process S' with heavy tails is perturbed by a process S'' with drift zero and lighter tails, then the ruin probability asymptotics for $S = S' + S''$ remains the same as for S'. See for example Jelenković, Momcilović & Zwart [15]. In our setting, S'' consists of the centered F claims,

$$S''(t) = \sum_{i=1}^{N(t)} U_i^F - \frac{m^* - m_G}{\mu}t$$

where $U_i^F = U_i$ if the ith claim is a F-claim and $U_i^F = 0$ if it is a G-claim, and S' is the renewal risk model with interarrival times distributed as ω, claim size distribution G, and premium rate $c' = m_G/\mu$. However, verification of the technical conditions of [15] is messy, and one should also note that the results obtained in this way only covers so-called square-root insensitivity, excluding for example that $\overline{G}(x) = \mathrm{e}^{-x^\beta}$ with $1/2 < \beta < 1$.

6. The analysis of Section 4 easily generalizes to the renewal model, where the time between claims is not necessarily exponential but follows a general distribution. Since one of us has repeatedly questioned how worthwhile such a generalization is (e.g. [4, pp. 12, 152–153]), we have chosen not to give the details!

7. Proposition 7.2 strongly suggest that, as in many other risk models, the most likely time of ruin is $u/\kappa'(\gamma)$. However, because of the logarithmic form the result is not entirely satisfying. For example, it does not exclude that

$$\mathbb{P}^{(u)}\left(\left|\frac{\tau(u)}{u} - \frac{1}{\kappa'(\gamma)}\right| \le \delta\right) \to 0\,. \tag{8.1}$$

Classical theory ([4, V.4]) suggests that

$$\frac{\tau(u) - u/\kappa'(\gamma)}{\left[u\kappa''(\gamma)/\kappa'(\gamma)^3\right]^{1/2}}$$

has a standard normal limit in $\mathbb{P}^{(u)}$-distribution, which shows that the limit in (8.1) is 1 as one would expect. A rigorous verification of this CLT could, e.g., proceed via the Markov additive approach but requires several steps which are easy for a finite Markov additive process but at best tedious in the general case. First one needs to identify the evolvment of the process under the γ-tilted distribution $\widetilde{\mathbb{P}}$, next to derive a CLT for S under $\widetilde{\mathbb{P}}$ and finally to justify an application of Anscombe's theorem together with Stam's lemma.

A   Regenerative calculations for $k > 2$

Now, let us prove that for $k \geq 2$,

$$\mu^{(k)} = \frac{\mathrm{e}^{\lambda r}}{\lambda} \frac{\mathrm{e}^{\lambda k r} - 1}{\mathrm{e}^{\lambda r} - 1}, \tag{A.1}$$

where $\mu^{(k)}$ is the mean of the cycle length in the k-case, which we denote a k-cycle. For $k = 2$, we can easily check that

$$\frac{1}{\lambda}\mathrm{e}^{\lambda r}(1 + \mathrm{e}^{\lambda r}) = \frac{\mathrm{e}^{\lambda r}}{\lambda} \frac{\mathrm{e}^{\lambda 2 r} - 1}{\mathrm{e}^{\lambda r} - 1}.$$

We assume now the result holds for some $k \geq 2$. After a k-cycle, we have two possibilities. The next interarrival time is long and we get a $(k+1)$-cycle. This gives an additional contribution to the mean of a $(k+1)$-cycle length to of $p^+ m^+$. Or the next interarrival time is short and the $(k+1)$-cycle starts afresh after that. Putting things together gives

$$\mu^{(k+1)} = \mu^{(k)} + p^+ m^+ + p^- (m^- + \mu^{(k+1)})$$

so that

$$\mu^{(k+1)} = \frac{1}{p^+}\left[\mu^{(k)} + p^+ m^+ + p^- m^-\right] = \mathrm{e}^{\lambda r}\left[\frac{\mathrm{e}^{\lambda r}}{\lambda}\frac{\mathrm{e}^{\lambda k r} - 1}{\mathrm{e}^{\lambda r} - 1} + \frac{1}{\lambda}\right],$$
$$= \frac{\mathrm{e}^{\lambda r}}{\lambda}\frac{\mathrm{e}^{\lambda(k+1)r} - 1}{\mathrm{e}^{\lambda r} - 1},$$

which ends the proof.

B   Light-tailed calculations for $k > 2$

Here, let us prove that, for $k \geq 2$,

$$\varphi^{(k)}(\alpha, \beta) = \frac{g(\alpha, \beta)^k}{1 - k(\alpha, \beta)\left[1 - g(\alpha, \beta)^k\right]/\left[1 - g(\alpha, \beta)\right]},$$

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where

$$g(\alpha, \beta) = \frac{\lambda}{\lambda + c\alpha - \beta} \widehat{G}(\alpha) \mathrm{e}^{(\beta - c\alpha - \lambda)r} \,,$$

$$k(\alpha, \beta) = \frac{\lambda}{\lambda + c\alpha - \beta} \widehat{F}(\alpha) \left(1 - \mathrm{e}^{(\beta - c\alpha - \lambda)r}\right) \,,$$

and $\varphi^{(k)}(\alpha, \beta) = \mathbb{E}\left[\mathrm{e}^{\alpha S(\omega)} \mathrm{e}^{\beta \omega}\right]$ in the k-case. For $k = 2$, we can easily check that

$$\frac{g(\alpha, \beta)^2}{1 - k(\alpha, \beta)\left[1 + g(\alpha, \beta)\right]} = \frac{g(\alpha, \beta)^2}{1 - k(\alpha, \beta)\left[1 - g(\alpha, \beta)^2\right] / \left[1 - g(\alpha, \beta)\right]} \,.$$

We assume now the result holds for some $k \geq 2$. After a k-cycle, we have two possibilities. The next interarrival time is long and we get a $(k + 1)$-cycle. Or the next interarrival time is short and the $(k + 1)$-cycle starts afresh after that. Putting things together gives

$$\varphi^{(k+1)} = \varphi^{(k)}(p^+ \widehat{G}[\alpha]\widehat{A}^+[\beta - c\alpha] + p^- \widehat{A}^-[\beta - c\alpha]\widehat{F}[\alpha]\varphi^{(k+1)}(\alpha, \beta)) \,,$$

$$= \varphi^{(k)}(g(\alpha, \beta) + k(\alpha, \beta)\varphi^{(k+1)}(\alpha, \beta)) \,.$$

so that

$$\varphi^{(k+1)}(\alpha, \beta) = \frac{\varphi^{(k)} g(\alpha, \beta)}{1 - \varphi^{(k)}(\alpha, \beta)k(\alpha, \beta)} \,,$$

$$= \frac{\frac{g(\alpha,\beta)^k}{1 - k(\alpha,\beta)\left[1 - g(\alpha,\beta)^k\right] / \left[1 - g(\alpha,\beta)\right]} g(\alpha, \beta)}{1 - \frac{g(\alpha,\beta)^k}{1 - k(\alpha,\beta)\left[1 - g(\alpha,\beta)^k\right] / \left[1 - g(\alpha,\beta)\right]} k(\alpha, \beta)} \,,$$

$$= \frac{g(\alpha, \beta)^k g(\alpha, \beta)}{1 - k(\alpha, \beta)\left[1 - g(\alpha, \beta)^k\right] / \left[1 - g(\alpha, \beta)\right] - g(\alpha, \beta)^k k(\alpha, \beta)} \,,$$

$$= \frac{g(\alpha, \beta)^{k+1}}{1 - k(\alpha, \beta)\left[1 - g(\alpha, \beta)^{k+1}\right] / \left[1 - g(\alpha, \beta)\right]} \,,$$

which ends the proof.

C   Proof of (6.3)–(6.7)

The expression (6.3) for $h^\theta(-, 0)$ is just Proposition 5.1.

To compute $h^\theta(i)$ for all i, we first introduce some notation, with V an exponential random variable with rate λ:

$$p^{(a)+} = \mathbb{P}\left(V > r - a\right) = \mathrm{e}^{-\lambda(r-a)} \,,$$

$$p^{(a)-} = \mathbb{P}\left(V \leq r - a\right) = 1 - \mathrm{e}^{-\lambda(r-a)} \,,$$

$$\widehat{A}[\alpha] = \mathbb{E}\left[\mathrm{e}^{\alpha V}\right] = \frac{\lambda}{\lambda + \alpha} \,,$$

$$\widehat{A}^{(a)+}[\alpha] = \mathbb{E}\left[\mathrm{e}^{\alpha V} \mid V > r - a\right] = \frac{\lambda}{\lambda - \alpha} \mathrm{e}^{\alpha(r-a)} \,,$$

$$\widehat{A}^{(a)-}[\alpha] = \mathbb{E}\left[\mathrm{e}^{\alpha V} \mid V \leq r - a\right] = \frac{\lambda}{\lambda - \alpha}\left(\frac{1 - \mathrm{e}^{-(\lambda - \alpha)(r-a)}}{1 - \mathrm{e}^{-\lambda(r-a)}}\right) \,,$$

and we are using p^+, p^-, $A^+[\alpha]$ and $A^-[\alpha]$ defined in Sections 3, 5.

The easiest case is the $(+,+)$-one. Actually, after an exponential time, a G-claim occurs which ends the cycle, so

$$h^\theta(+,+) = \mathbb{E}_{(+,+)}e^{\theta S(\omega_1)-\kappa(\theta)\omega_1} = \mathbb{E}e^{-(c\theta+\kappa(\theta))V}e^{U_G}$$

$$= \widehat{A}[-c\theta - \kappa(\theta)]\widehat{G}[\theta] = \frac{\lambda}{\lambda + c\theta + \kappa(\theta)}\widehat{G}[\theta].$$

For the $(+,a)$-case, two cases have to be considered. If the exponential time before the first claim is larger than $r-a$ then a G-claim occurs, otherwise, a F-claim occurs and the state becomes $(-,0)$:

$$h^\theta(+,a) = \mathbb{E}_{(+,a)}e^{\theta S(\omega_1)-\kappa(\theta)\omega_1},$$

$$= p^{(a)-}\widehat{A}^{(a)-}[-c\theta - \kappa(\theta)]\widehat{F}[\theta]h^\theta(-,0) + p^{(a)+}\widehat{A}^{(a)+}[-c\theta - \kappa(\theta)]\widehat{G}[\theta]$$

$$= \frac{k^{(a)}(\theta,-\kappa(\theta))g(\theta,-\kappa(\theta))}{1 - k(\theta,-\kappa(\theta))(1 + g(\theta,-\kappa(\theta)))} + g^{(a)}(\theta,-\kappa(\theta)).$$

The particular case $(+,0)$ gives

$$h^\theta(+,0) = k^{(0)}(\theta,-\kappa(\theta))h^\theta(-,0) + g^{(0)}(\theta,-\kappa(\theta))$$

$$= \frac{g(\theta,-\kappa(\theta))(1 - k(\theta,-\kappa(\theta)))}{1 - k(\theta,-\kappa(\theta))(1 + g(\theta,-\kappa(\theta)))}.$$

The $(-,+)$-case can be easily solved using the $(+,0)$ one. Actually, after an exponential time, a F-claim occurs and the state becomes $(+,0)$:

$$h^\theta(-,+) = \mathbb{E}_{(-,+)}e^{\theta S(\omega_1)-\kappa(\theta)\omega} = \widehat{A}[-c\theta - \kappa(\theta)]\widehat{F}[\theta]h^\theta(+,0)$$

$$= \frac{\lambda}{\lambda + c\theta + \kappa(\theta)}\widehat{G}[\theta]\frac{g(\theta,-\kappa(\theta))(1 - k(\theta,-\kappa(\theta)))}{1 - k(\theta,-\kappa(\theta))(1 + g(\theta,-\kappa(\theta)))}.$$

We solve the $(-,a)$-case using both $(+,0)$ and $(-,0)$ ones. If the exponential time before the first claim is larger than $r-a$ then a F-claim occurs and the state becomes $(+,0)$, otherwise, a F-claim occurs and the state becomes $(-,0)$:

$$h^\theta(-,a) = \mathbb{E}_{(-,a)}e^{\theta S(\omega_1)-\kappa(\theta)\omega_1},$$

$$= p^{(a)-}\widehat{A}^{(a)-}[-c\theta - \kappa(\theta)]\widehat{F}[\theta]h^\theta(-,0)$$

$$+ p^{(a)+}\widehat{A}^{(a)+}[-c\theta - \kappa(\theta)]\widehat{F}[\theta]h^\theta(+,0),$$

$$= \frac{1}{1 - k(\theta,-\kappa(\theta))[1 + g(\theta,-\kappa(\theta))]}\Big\{k^{(a)}(\theta,-\kappa(\theta))g^2(\theta,-\kappa(\theta))$$

$$+ \frac{\widehat{F}[\theta]}{\widehat{G}[\theta]}g^{(a)}(\theta,-\kappa(\theta))g(\theta,-\kappa(\theta))[1 - k(\theta,-\kappa(\theta))]\Big\}.$$

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