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Abstract

This paper considers a number of structural properties of reflected Lévy processes, where both one-sided reflection (at 0) and two-sided reflection (at both 0 and $K > 0$) are examined. With V_t being the position of the reflected process at time t, we focus on the analysis of $\zeta(t) := \mathbb{E}V_t$ and $\xi(t) := \mathbb{V}\text{ar}V_t$. We prove that for the one- and two-sided reflection we have $\zeta(t)$ is increasing and concave, whereas for the one-sided reflection we also show that $\xi(t)$ is increasing. In most proofs we first establish the claim for the discrete-time counterpart (that is, a reflected random walk), and then we use a limiting argument. A key step in our proofs for the two-sided reflection is a new representation of the position of the reflected process in terms of the driving Lévy process.

Keywords Complete monotonicity, Lévy processes, One/Two-sided reflection, Mean function, Variance function, Stationary increments, concordance.

Mathematics Subject Classification (2000) Primary 60K25 Secondary 60F05 90B22

1 Introduction

In this paper we consider structural properties of reflected Lévy processes, where both one-sided reflection (at 0) and two-sided reflection (at both 0 and $K > 0$) are examined. We assume throughout that the reflected process is started at 0, and we have that in the case of one-sided reflection, the position of the reflected process V_t is given by $S_t + L_t$, where $\{S_t\}_{t\geq 0}$ is the driving Lévy process, and $\{L_t\}_{t\geq 0}$ is the local time at 0, which can be written as $-\inf_{0\leq s\leq t} S_s$. In case of two-sided reflection, we have a similar construction in the sense that V_t can be decomposed as $S_t + L_t - \bar{L}_t$, with $\bar{L}_t$ the local time at K, given as part of the solution to a Skorokhod problem, but finding explicit solutions for V_t, in terms of S_s with $0 \leq s \leq t$, is rather involved; recently, such expressions have appeared in (10) and (11).

More precisely, we focus in this work on the analysis of two objects, viz. $\zeta(t) := \mathbb{E}V_t$ and $\xi(t) := \mathbb{V}\text{ar}V_t$. Our goal is to prove a number of structural properties

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regarding the shape of these two functions. For the one-sided reflection, the function $\zeta(\cdot)$ was already examined in detail before. In Kella (7) it was shown that $\zeta(\cdot)$ is concave as long as the underlying Lévy process does not have any positive jumps, relying on martingale techniques. This result was generalized by Kella and Sverchkov (9) to general Lévy processes (in fact even just stationary increments are needed), with an elementary proof that uses stochastic monotonicity. To our best knowledge, however, there are no results for the two-sided counterpart, nor any results for the variance function $\xi(\cdot)$.

The contributions of this paper are the following. In the first part of the paper we consider the case of one-sided reflection.

- In Section 3 we consider the special case of a spectrally-positive Lévy process, that is, a Lévy process without negative jumps. We present an elementary proof of the fact that the expected value of the position at time t is concave in t. Although this result was already covered by (7), we included it because we believe the proof technique is interesting, and may be of use in other situations as well. More particularly, the proof relies on the concept of complete monotonicity to show that the desired property holds in the special case of a compound Poisson Lévy process, and then uses a limiting argument (approximating any spectrally-positive Lévy process by a suitable sequence of compound Poisson processes).

- Section 4 focuses on one-sided reflection, but now we treat the case of general Lévy input, roughly as follows. First we prove the desired result for the discrete-time version of the Lévy process (which is a random walk), by means of an extremely short and insightful argument. Then a limiting procedure ensures that the concavity is preserved in continuous time, thus reestablishing the result by (9). Importantly, the same method (that is, first proving the desired property for the random walk, and then a limiting argument) can be followed to prove the new result that the variance curve, i.e., $\xi(t)$, is increasing in t; the proof relies on the concept of 'concordance'.

The second part of the paper concentrates on similar issues, but now in the setting of a two-sided reflected Lévy process.

- As mentioned above, new explicit formulae for V_t (in terms of S_s for $0 \leq s \leq t$) have appeared recently. We derive in Section 5 a new explicit representation, which is similar to the one found in (10), but somewhat shorter. This alternative representation carries over to continuous time, as argued in Section 6.

- Relying on the new representation for V_t for the case of two-sided reflection, as presented in Section 6, in Section 7 we prove the new result that $\zeta(t)$ is an increasing concave function of t. We do this by first proving the desired result for the discrete-time counterpart, that is, a random walk reflected at 0 and K, and then we use a limiting argument. We finish this second part with the observation that the results carry over to the situation in which we just assume stationary increments (rather than stationary independent increments).

The paper now continues with a section in which the model and some preliminaries are given.

2 Model, notation, and preliminaries

In this paper we study reflected versions of the Lévy process $\{S_t\}_{t\geq 0}$. We distinguish between one-sided and two-sided reflection.

- *One-sided reflection* (at 0). The reflection of $\{S_t\}_{t\geq 0}$ at 0, which we denote by $\{V_t\}_{t\geq 0}$, can be formally introduced as follows, see for instance (2, Ch. IX). Define the increasing process $\{L_t\}_{t\geq 0}$ by $L_t = -\inf_{0\leq s\leq t} S_s$; this process is commonly referred to as the local time at 0. Then the reflected process (or: workload process, queueing process) $\{V_t\}_{t\geq 0}$ is given through

  $$V_t := S_t + \max\{L_t, V_0\};$$

  observe that $V_t \geq 0$ for all $t \geq 0$. Throughout this paper the focus lies on the special case that $V_0 = 0$, and hence $V_t = S_t + L_t$. It is straightforward that $\zeta(t)$ increases in t, using Proposition 3 p. 158 in (4).

- *Two-sided reflection* (at 0 and $K > 0$). Again starting off at 0, we now have that the position of the reflected process at time t, i.e., V_t, is given by $V_t = S_t + L_t - \bar{L}_t$, with the increasing process $\{\bar{L}_t\}_{t\geq 0}$ denoting the local time at K, given as part of the solution to a Skorokhod problem. In (10) an explicit expression for L_t and $\bar{L}_t$ (in terms of S_s with $0 \leq s \leq t$) is given. In particular,

  $$V_t = S_t - \sup_{s\in[0,t]} \Big[\Big((S_s - K) \vee \inf_{u\in[0,t]} S_u\Big) \wedge \inf_{u\in[s,t]} S_u\Big].$$

We recall that we denote $\zeta(t) := \mathbb{E}V_t$ and $\xi(t) := \mathbb{V}\mathrm{ar}V_t$.

In Section 3 we consider the case in which the underlying Lévy process does not have negative jumps (i.e., is spectrally-positive), and in which there is just reflection at 0. Assuming stability (i.e., $\mathbb{E}S_1 < 0$), the Laplace exponent $\varphi(\alpha) := \log \mathbb{E}e^{-\alpha S_1}$ is given by a function $\varphi(\cdot) : [0,\infty) \mapsto [0,\infty)$ that is increasing and convex on $[0,\infty)$, with slope $\varphi'(0) = -\mathbb{E}S_1$ in the origin. Therefore the inverse $\psi(\cdot)$ of $\varphi(\cdot)$ is well-defined on $[0,\infty)$. In the sequel we rule out the trivial case that $\{S_t\}_{t\geq 0}$ is a (downward) *subordinator*, i.e., a monotone (decreasing) process. We throughout assume that $\varphi''(0)$ is finite (unless stated otherwise).

Important examples of spectrally-positive Lévy processes are: (1) *Brownian motion with drift*, where $\varphi(\alpha) = -\alpha\mu + \frac{1}{2}\alpha^2\sigma^2$. (2) *Compound Poisson with drift.* Jobs arrive according to a Poisson process of rate λ; the jobs $B_1, B_2, \ldots$ are i.i.d. samples from a distribution with Laplace transform $\beta(\alpha) := \mathbb{E}e^{-\alpha B}$; the storage system is continuously depleted at a rate $-M < 0$ (where M is often referred to as the *drift*). It can be verified that $\varphi(\alpha) = M\alpha - \lambda + \lambda\beta(\alpha)$.

Using (2, Thm. IX.3.10) or (8), it is straightforward to prove that, as long as the Lévy process is spectrally positive, $\mu_V := \mathbb{E}V_\infty = \varphi''(0)/(2\varphi'(0))$, and

$$\rho(\vartheta) := \int_0^\infty e^{-\vartheta t}\mathbb{E}V_t \mathrm{d}t = \int_0^\infty e^{-\vartheta t}\zeta(t)\mathrm{d}t = -\frac{\varphi'(0)}{\vartheta^2} + \frac{1}{\vartheta\psi(\vartheta)}. \tag{2.1}$$

3 One-sided reflection: spectrally-positive input

This section focuses on establishing a number of structural properties of $\zeta(\cdot)$ for the case of spectrally-positive Lévy input. As mentioned above, it is evident that $\zeta(\cdot)$ is positive and increasing; in this section we prove that it is concave as well. We do this by extensively using the concept of completely monotonous functions (3; 13). The desired result is first proven for the case of compound Poisson input; then we show how to construct a sequence of compound Poisson processes approximating any spectrally-positive Lévy process arbitrarily closely, which allows us to prove the claim. The class $\mathscr{C}$ of completely monotone functions is defined as follows.

Definition 3.1. *A function $f(\alpha)$ on $[0,\infty)$ is completely monotone if for all $n \in \mathbb{N}$*

$$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\alpha^n} f(\alpha) \geq 0.$$

We write $f(\alpha) \in \mathscr{C}$.

The following deep and powerful result is due to Bernstein (3). It says that there is equivalence between $f(\alpha)$ being completely monotone, and the possibility of writing $f(\alpha)$ as a Laplace transform. For more background and basic properties of completely monotone functions, see (6, pp. 439-442).

Theorem 3.2. [Bernstein] *A function $f(\alpha)$ on $[0,\infty)$ is the Laplace transform of a non-negative random variable if and only if* (i) *$f(\alpha) \in \mathscr{C}$, and* (ii) *$f(0) = 1$.*

Let us consider the transforms of $\zeta'(t)$ and $\zeta''(t)$. Using integration by parts, it is readily checked that

$$\int_0^\infty e^{-\vartheta t} \zeta'(t) \mathrm{d}t = -\frac{\varphi'(0)}{\vartheta} + \frac{1}{\psi(\vartheta)}.$$

In the M/G/1 setting we have that $\psi(\vartheta) = \lambda + \vartheta - \lambda\pi(\vartheta)$, where $\pi(\vartheta)$ is the Laplace transform of the busy period, and the deterministic service rate has value 1; it is assumed that $-\lambda\beta'(0) < 1$. Applying integration by parts once again yields that

$$\int_0^\infty e^{-\vartheta t} \zeta''(t) \mathrm{d}t = -\zeta'(0) - \varphi'(0) + \frac{\vartheta}{\psi(\vartheta)} = -\left(1 - \frac{\vartheta}{\psi(\vartheta)}\right),$$

using that

$$\zeta'(0) = \lim_{\vartheta \to \infty} \int_0^\infty \vartheta e^{-\vartheta t} \zeta'(t) \mathrm{d}t = 1 - \varphi'(0);$$

notice that the transform of $\zeta''(t)$ is only well defined when $\vartheta/\psi(\vartheta)$ has a finite limit as $\vartheta \to \infty$, which is indeed the case for compound Poisson input. The transform can further be simplified to

$$-\frac{\lambda(1 - \pi(\vartheta))}{\lambda(1 - \pi(\vartheta)) + \vartheta}. \tag{3.2}$$

Observe that Lemma 4.1 (item 1) of (13) entails that the negative of (3.2) is in $\mathscr{C}$, thus proving that indeed in the M/G/1 context $\zeta''(t)$ is negative, i.e., $\zeta(t)$ is concave. We have proved the following.

Lemma 3.3. *$\zeta(t)$ is concave for compound Poisson input processes with negative drift (with one-sided reflection).*

We now consider the context of a general spectrally-positive Lévy process, and use Lemma 3.3 to prove that also in this setting $\zeta(t)$ is concave. We first recall that the Laplace exponent $\varphi(\alpha)$ of a spectrally positive Lévy process can be written as (4, Section VII.1), with $M \in \mathbb{R}$, $\sigma^2 > 0$, and measure $\Pi_\varphi(\cdot)$ such that $\int_{(0,\infty)} \min\{1, x^2\}\Pi_\varphi(\mathrm{d}x) < \infty$,

$$\varphi(\alpha) = \alpha M + \frac{1}{2}\alpha^2\sigma^2 + \int_{(0,\infty)} (e^{-\alpha x} - 1 + \alpha x 1_{(0,1)})\Pi_\varphi(\mathrm{d}x).$$

The idea is now to approximate the spectrally-positive Lévy process arbitrarily closely by a sequence of compound Poisson processes. To this end, let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of numbers in $(0, 1]$, such that $\varepsilon_n \downarrow 0$. Then we can rewrite $\varphi(\alpha) = \varphi_n(\alpha) + \bar{\varphi}_n(\alpha)$, with

$$\begin{aligned}
\varphi_n(\alpha) &= \left(M + \int_{\varepsilon_n}^1 x\Pi_\varphi(\mathrm{d}x) + \frac{\sigma^2}{\varepsilon_n}\right)\alpha + \frac{\sigma^2}{\varepsilon_n^2}\left(e^{-\alpha\varepsilon_n} - 1\right) \\
&\quad + \int_{\varepsilon_n}^\infty \left(e^{-\alpha x} - 1\right)\Pi_\varphi(\mathrm{d}x); \\
\bar{\varphi}_n(\alpha) &= \sigma^2\left(\frac{1}{2}\alpha^2 - \frac{e^{-\alpha\varepsilon_n} + \alpha\varepsilon_n - 1}{\varepsilon_n^2}\right) + \int_0^{\varepsilon_n} \left(e^{-\alpha x} - 1 + \alpha x\right)\Pi_\varphi(\mathrm{d}x).
\end{aligned}$$

Let $\psi_n(\cdot)$ denote the inverse of $\varphi_n(\cdot)$.

Lemma 3.4. (i) *For all $\alpha \geq 0$, $\varphi_n(\alpha) \to \varphi(\alpha)$ as $n \to \infty$.*
(ii) *For all $\alpha \geq 0$, $\varphi_n'(\alpha) \to \varphi'(\alpha)$ as $n \to \infty$.*
(iii) *For all $n \in \mathbb{N}$, $\varphi_n'(0) = \varphi'(0)$.*

Proof. Straightforward calculations. □

It is important to notice that, for any $n \in \mathbb{N}$, $\varphi_n(\cdot)$ can be interpreted as the Laplace exponent of a compound Poisson process (with negative drift), say $\{S_{n,t}\}_{t\geq 0}$. This is seen as follows. The drift term is

$$\left(M + \int_{\varepsilon_n}^1 x\Pi_\varphi(\mathrm{d}x) + \frac{\sigma^2}{\varepsilon_n}\right),$$

which is positive for n sufficiently large. Then, the term $(\sigma^2/\varepsilon_n^2) \cdot (e^{-\alpha\varepsilon_n} - 1)$ can be interpreted as the contribution of a Poisson stream (arrival rate σ^2/ε_n^2) of jobs of deterministic size ε_n. Also,

$$\int_{\varepsilon_n}^\infty \left(e^{-\alpha x} - 1\right)\Pi_\varphi(\mathrm{d}x) = \Pi_\varphi([\varepsilon_n, \infty)) \int_{\varepsilon_n}^\infty \left(e^{-\alpha x} - 1\right)\frac{\Pi_\varphi(\mathrm{d}x)}{\Pi_\varphi([\varepsilon_n, \infty))},$$

which is the contribution of a Poisson stream (arrival rate $\Pi_\varphi([\varepsilon_n, \infty))$) of jobs, whose sizes are i.i.d. samples from a 'truncated distribution' with density $\Pi_\varphi(\mathrm{d}x)/\Pi_\varphi([\varepsilon_n, \infty))$, for $x \geq \varepsilon_n$.

Just as we introduced the reflected version $\{V_t\}_{t\geq 0}$ of $\{S_t\}_{t\geq 0}$, we can construct the reflected version $\{V_{n,t}\}_{t\geq 0}$ of $\{S_{n,t}\}_{t\geq 0}$. Analogously to $\zeta(t)$, we denote $\zeta_n(t) := \mathbb{E}V_{n,t}$. Note that, due to Lemma 3.4.(iii), the queueing processes $\{V_{n,t}\}_{t\geq 0}$ are stable (recall that we assumed $\varphi'(0) > 0$). From (2.1), we have that for any $n \in \mathbb{N}$,

$$\rho_n(\vartheta) := \int_0^\infty e^{-\vartheta t}\,\zeta_n(t)\mathrm{d}t = -\frac{\varphi'(0)}{\vartheta^2} + \frac{1}{\vartheta\psi_n(\vartheta)}. \tag{3.3}$$

Corollary 3.5. *For all $n \in \mathbb{N}$, $\zeta_n(t)$ is positive (that is, larger than or equal to 0), increasing (non-strictly), and concave (non-strictly).*

Lemma 3.6. *For all $\vartheta \geq 0$, $\psi_n(\vartheta) \to \psi(\vartheta)$ as $n \to \infty$.*

Proof. First observe that $\varphi_n(\alpha) \to \varphi(\alpha)$ (Lemma 3.4) entails that, as $n \to \infty$,

$$|\psi_n(\varphi(\alpha)) - \psi_n(\varphi_n(\alpha))| \leq \Big|\sup_{\vartheta\geq 0}\psi_n'(\vartheta)\Big| \cdot |\varphi(\alpha) - \varphi_n(\alpha)| \;\to\; 0,$$

where we used that $\psi_n(\cdot)$ is concave with slope $1/\varphi'(0)$ in 0. Hence it also holds that $\psi_n(\varphi(\alpha))$ converges, as $n \to \infty$, to $\alpha = \psi(\varphi(\alpha))$. But as $\varphi(\alpha)$ is a bijection of $[0,\infty)$ onto $[0,\infty)$, this proves the claim. □

Proposition 3.7. *$\zeta(t)$ is concave for spectrally-positive Lévy processes (with one-sided reflection).*

Proof. Our proof consists of the following steps.

(1) Using (3.3) and Lemma 3.6, we see that, for all $\vartheta \geq 0$,

$$\lim_{n\to\infty}\rho_n(\vartheta) = \rho(\vartheta) = \int_0^\infty e^{-\vartheta t}\zeta(t)\mathrm{d}t.$$

(2) Realize that, as $\zeta_n(\cdot)$ is positive (that is, larger than or equal to 0), increasing (non-strictly), and concave (non-strictly) due to Lemma 3.3, $\lim_{n\to\infty}\zeta_n(\cdot)$ (given it exists) inherits these properties.

(3) Because of dominated convergence (use that $\zeta_n(t)$ increases in t, and that $\mu_{V,n} := \zeta_n(\infty) \to \zeta(\infty) = \mu_V$ as $n \to \infty$; these observations immediately yield an integrable majorizing function),

$$\lim_{n\to\infty}\rho_n(\vartheta) = \lim_{n\to\infty}\int_0^\infty e^{-\vartheta t}\zeta_n(t)\mathrm{d}t = \int_0^\infty e^{-\vartheta t}\lim_{n\to\infty}\zeta_n(t)\mathrm{d}t.$$

(4) The uniqueness of the Laplace transform, together with Steps (1) and (3), now implies that we have $\lim_{n\to\infty}\zeta_n(t) = \zeta(t)$. Then Step (2) yields the stated.

This finishes the proof. □

4 One-sided reflection: general Lévy input

In this section we prove that for the one-sided reflection we have $\zeta(t)$ is increasing and concave, and that $\xi(t)$ is increasing.

4.1 Discrete-time case

Let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables, and define $S_0 := 0$, $S_n := X_1 + X_2 + \cdots + X_n$, its associated random walk. Define the convex function $\Psi(x) := \max(0, x) = x^+$ and let $\{V_n\}_{n=0}^{\infty}$ denote the *reflected version* of $\{S_n\}_{n=0}^{\infty}$, that is, V_n is given by the Lindley recursion $V_{n+1} := \Psi(V_n + X_{n+1})$, initialized by $V_0 := 0$. By (2, Cor. III.6.4), $V_n =_{\mathscr{D}} M_n$, where M_n denotes the 'running maximum', i.e., $\max_{0 \le k \le n} S_k$.

We say a sequence $(a_n)_{n=0}^{\infty}$ is *concave* if $a_{n+2} + a_n \le 2a_{n+1}$ $\forall n$, that is, if $a_{n+1} - a_n$ is decreasing. We now give an extremely short proof of the fact that $\zeta(n) := \mathbb{E}V_n$ is a concave sequence.

Proposition 4.1. *$\zeta(n)$ is concave for random walks (with one-sided reflection).*

Proof. According to (2, Prop. VIII.4.5), we have that $\zeta(n) - \zeta(n-1) = \mathbb{E}S_n^+/n$. Furthermore, using $(X_i, S_n) =_{\mathscr{D}} (X_1, S_n)$ we have

$$S_n = \mathbb{E}[S_n \mid S_n] = \sum_{i=1}^{n} \mathbb{E}[X_i \mid S_n] = \sum_{i=1}^{n} \mathbb{E}[X_1 \mid S_n] = n\mathbb{E}[X_1 \mid S_n] \qquad a.s.$$

which implies $\mathbb{E}[X_1 \mid S_n] = S_n/n$ *a.s*, which in turn implies $\mathbb{E}[S_n/n \mid S_{n+1}] = S_{n+1}/(n+1)$ *a.s.* and applying the conditional Jensen's inequality to the convex function $\Psi(\cdot)$, we conclude

$$\frac{S_{n+1}^+}{n+1} = \Psi\left(\mathbb{E}\left[\frac{S_n}{n} \mid S_{n+1}\right]\right) \le \mathbb{E}\left[\Psi\left(\frac{S_n}{n}\right) \mid S_{n+1}\right] = \mathbb{E}\left[\frac{S_n^+}{n} \mid S_{n+1}\right] \quad a.s.,$$

and taking means on both sides yields the desired result. □

Our next goal is to prove that for the random walk introduced above $\xi(n) = \mathbb{V}\mathrm{ar}(S_n)$ increases in n. We do so by using the concept of *concordance*, cf. the results of (12). Here, a pair of random variables (X, Y) or its distribution function F is said to be *positively quadrant dependent* if

$$\mathbb{P}(X \le x, Y \le y) \ge \mathbb{P}(X \le x)\mathbb{P}(Y \le y) \qquad \forall x, y.$$

According to Lemma 3 of (12) it holds that positively quadrant dependence implies that the covariance between X and Y is non-negative: $\mathbb{C}\mathrm{ov}(X, Y) \ge 0$. Furthermore, we define a two real-valued functions r, s to be *concordant* for the i-th coordinate if, considered as functions of the i-th coordinate (with all other coordinates held fixed) they are either both non-decreasing or both non-increasing. The main result in (12), which we will use below, is the following.

Theorem 4.2. [Lehmann] *Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independent with distribution functions $F_1, \ldots, F_n$, and let F_i be concordant for the i-th coordinate. Set*

$$X := r(X_1, \ldots, X_n), \qquad Y := s(Y_1, \ldots, Y_n).$$

Then (X, Y) is positively quadrant dependent.

In particular, since (X, X) is positively quadrant dependent, we have

$$\mathbb{C}\mathrm{ov}(r(X_1, X_2, \ldots, X_n), s(X_1, X_2, \ldots, X_n)) \geq 0$$

if the X_i's are independent and r and s are concordant for all coordinates. Using this insight, we can prove the following result.

Theorem 4.3. *$\xi(n)$ is increasing for random walks (with one-sided reflection).*

Proof. Using the identity

$$\mathbb{V}\mathrm{ar}(X+Y) = \mathbb{C}\mathrm{ov}(X, X)+2\mathbb{C}\mathrm{ov}(X, Y)+\mathbb{C}\mathrm{ov}(Y, Y) = \mathbb{C}\mathrm{ov}(X, X)+\mathbb{C}\mathrm{ov}(2X+Y, Y)$$

with $X \equiv M_{n-1}$ and $Y \equiv (S_n - M_{n-1}) \cdot I(M_{n-1} < S_n)$ (where $I(A)$ is the indicator function of the event A), we obtain that $\mathbb{V}\mathrm{ar}(M_n)$ equals $\mathbb{V}\mathrm{ar}(M_{n-1}) + j_n$, where

$$j_n := \mathbb{C}\mathrm{ov}(2M_{n-1} + (S_n - M_{n-1}) \cdot I(M_{n-1} < S_n), (S_n - M_{n-1}) \cdot I(M_{n-1} < S_n)),$$

and therefore the proof is complete if we can show that $j_n \geq 0$. For $\underline{x} := (x_1, \ldots, x_n) \in \mathbb{R}^n$, we set $s_n \equiv s_n(\underline{x}) := x_1 + \cdots + x_n$ and $m_n \equiv m_n(\underline{x}) = \max(0, s_1, \ldots, s_n)$, and we define functions

$$r_n(\underline{x}) = 2m_{n-1} + (s_n - m_{n-1}) \cdot I(s_n > m_{n-1}), \quad t_n(\underline{x}) = (s_n - m_{n-1}) \cdot I(s_n > m_{n-1}),$$

so that we have that $j_n = \mathbb{C}\mathrm{ov}(r_n(\underline{X}), t_n(\underline{X}))$. Hence, we wish to prove that r_n and t_n are concordant in all coordinates. We shall show that both functions are increasing in all their coordinates. To this end first rewrite $t(\underline{x})$ as $\max(\hat{t}(\underline{x}), 0)$, where

$$\hat{t}(\underline{x}) = \min(x_1 + \cdots + x_{n-1}, x_2 + \cdots + x_{n-1}, \ldots, 0) + x_n,$$

which is evidently increasing in all its coordinates. Finally, regarding r_n, we notice that since $r_n(\underline{x}) = 2m_{n-1} + t_n(\underline{x})$ and the fact that the term $2m_{n-1} = 2\max(x_1 + \cdots + x_{n-1}, \ldots, 0)$ is increasing, we see that so is r_n, and we are done. □

4.2 Continuous-time case

We now consider a Lévy process $\{S_t\}_{t\geq 0}$, as well as its reflection at 0, denoted by $\{V_t\}_{t\geq 0}$. We wish to extend Prop. 4.1 and Th. 4.3 to Lévy processes, that is, we wish to prove that $\zeta(\cdot)$ is, and $\xi(\cdot)$ is increasing. We prove the former by showing that for given $0 \leq x < y < z$ we have

$$\frac{\zeta(y) - \zeta(x)}{y - x} \geq \frac{\zeta(z) - \zeta(x)}{z - x},$$

which is an alternative characterization of concavity. We throughout assume that $\mathbb{E}S_1 < 0$ and $\mathbb{E}S_1^2 < \infty$, which is a natural assumption, since it implies that $\lim_{t\to\infty} \zeta(t) < \infty$, as proven in (1, Cor. 4.2).

Let $0 \leq x < y < z$ be given, and let $T \in \mathbb{R}$ be any number larger than z. In the sequel we use bold fonts to denote the corresponding process between 0 and T; for instance, $\boldsymbol{S} := \{S_t\}_{0\le t\le T}$. Define the one-sided reflection mapping $\mathscr{S} : D[0,T] \to D[0,T]$ by

$$\mathscr{S}[\boldsymbol{x}](t) := x(t) - \inf_{s\le t} x(s) \qquad \text{for } \boldsymbol{x} \in D[0,T].$$

This means that the value of the reflected process at time t, that is, V_t, is alternatively written as $\mathscr{S}[\boldsymbol{S}](t)$.

We define the sequence $\boldsymbol{S}^n := \{S_t^n\}_{t\ge 0}$ by $S_t^n = S_{\lfloor nt\rfloor/n}$, $n \in \mathbb{N}, 0 \le t \le T$, which, as shown below, approximates the Lévy process $\boldsymbol{S}$ sufficiently well for our purposes. We also introduce the reflected version $V_t^n = \mathscr{S}[\boldsymbol{S}^n](t)$ of the elements of the sequence $\boldsymbol{S}^n$. Let $\zeta^n(\cdot)$ and $\xi^n(\cdot)$ be defined in a self-evident manner as piecewise constant functions.

We prove our claims on $\zeta(\cdot)$ and $\xi(\cdot)$ by first showing that $\mathscr{S}[\boldsymbol{S}^n]$ converges weakly to $\mathscr{S}[\boldsymbol{S}]$ in the Skorokhod topology, by which we mean the J_1-topology on $D[0,T]$; see (15) for background on the J_1-topology. This result will be used to prove uniform convergence of the $\zeta^n(\cdot)$ and $\xi^n(\cdot)$ functions, which is needed in order to extend our discrete-time results to continuous time.

Lemma 4.4. *$V_t^n \to_{\mathscr{D}} V_t$, as $n \to \infty$.*

Proof. First we prove $\boldsymbol{S}^n \to_{\mathscr{D}} \boldsymbol{S}$, $n \to \infty$ in $D[0,T]$ equipped with the Skorokhod topology, under the assumption that $\mathbb{E}S_1 = 0$ (which we later generalize to any value of $\mathbb{E}S_1 \neq 0$). To this end, we need to prove convergence of the corresponding finite-dimensional distributions, as well as tightness. We notice that there is pointwise convergence, i.e., $S_t^n = S_{\lfloor nt\rfloor/n} \to S_t$ as $n \to \infty$, as a direct consequence of the fact that $\boldsymbol{S}$ is right-continuous. Furthermore, for $s < t$,

$$(S_t^n - S_s^t, S_s^n) = (S_{\lfloor nt\rfloor/n} - S_{\lfloor ns\rfloor/n}, S_{\lfloor ns\rfloor/n}) \to_{\mathscr{D}} (S_t - S_s, S_s),$$

applying (i) $S_t^n \to S_t$, (ii) independence of the components of this random vector, and (iii) (5, Thm. 3.2). The case with more than two time points is dealt with analogously, and we have thus proved convergence of the finite-dimensional distributions. Regarding tightness, we have, for $t_1 \le t \le t_2$ and $\sigma^2 := \mathbb{V}\mathrm{ar}(S_1)$,

$$\mathbb{E}(S_t^n - S_{t_1}^n)^2(S_{t_2}^n - S_t^n)^2 = \frac{\sigma^4}{n^2}(\lfloor nt\rfloor - \lfloor nt_1\rfloor)(\lfloor nt_2\rfloor - \lfloor nt\rfloor) \le \sigma^4(t_2 - t_1)^2\,,$$

where the last inequality is due to (5, Eqns. (16.4)-(16.5)). Tightness now follows as a direct application of (5, Thm. 15.6).

The case where $\mathbb{E}S_1 =: \mu \neq 0$ follows by defining processes $\hat{\boldsymbol{S}}^n$ through

$$\hat{S}_t^n := S_{\lfloor nt\rfloor/n} - \frac{\mu\lfloor nt\rfloor}{n},$$

and using the above to conclude that $\hat{\boldsymbol{S}}^n \to_{\mathscr{D}} \hat{\boldsymbol{S}}$. Furthermore, $\mu\lfloor nt\rfloor/n \to \mu t$ uniformly, and therefore also in the Skorokhod topology. Since $\{\mu t\}$ is continuous, the functions $\{S_t + \mu t\}$ and $\{\mu t\}$ have no common discontinuity points and therefore

$$\boldsymbol{S}^n = \left\{S_{\lfloor nt\rfloor/n} - \frac{\mu\lfloor nt\rfloor}{n}\right\} + \left\{\frac{\mu\lfloor nt\rfloor}{n}\right\} \to_{\mathscr{D}} \{S_t - \mu t\} + \{\mu t\} = \boldsymbol{S}.$$

This completes to proof of the weak convergence $\boldsymbol{S}^n \to_{\mathscr{D}} \boldsymbol{S}$.

Next, we use the Skorokhod Representation Theorem, i.e., (15, Thm. 3.2.2), to construct a sequence of processes

$$\tilde{\boldsymbol{S}}^n = \{\tilde{S}^n_s\}_{s\geq 0}, \quad n \in \mathbb{N},$$

with $\tilde{\boldsymbol{S}}^n =_{\mathscr{D}} \boldsymbol{S}^n$ such that

$$\lim_{n\to\infty} \tilde{\boldsymbol{S}}^n = \tilde{\boldsymbol{S}} \quad a.s. \text{ in the Skorokhod topology on } D[0,T],$$

where $\tilde{\boldsymbol{S}} =_{\mathscr{D}} \boldsymbol{S}$. Since $\mathscr{S}$ is continuous in the Skorokhod topology (15, Thm. 13.5.1), we have

$$\lim_{n\to\infty} \{\mathscr{S}[\tilde{\boldsymbol{S}}^n]\} = \{\mathscr{S}[\tilde{\boldsymbol{S}}]\} \quad a.s.$$

Furthermore, since $\mathbb{P}(\Delta\mathscr{S}[\tilde{\boldsymbol{S}}^n](t) \neq 0) \leq \mathbb{P}(\Delta\tilde{S}_t \neq 0) = 0$ for all $t \geq 0$ we conclude, relying on (5, p. 121), that $\mathscr{S}[\tilde{\boldsymbol{S}}^n](t) \to \mathscr{S}[\tilde{\boldsymbol{S}}](t)$ as $n \to \infty$, a.s., $0 \leq t \leq T$. Since almost sure convergence implies weak convergence, it holds that $\mathscr{S}[\tilde{\boldsymbol{S}}^n](t) \to_{\mathscr{D}} \mathscr{S}[\tilde{\boldsymbol{S}}](t)$ which together with $\tilde{\boldsymbol{S}} =_{\mathscr{D}} \boldsymbol{S}$ implies $\mathscr{S}[\boldsymbol{S}^n](t) \to_{\mathscr{D}} \mathscr{S}[\boldsymbol{S}](t)$, $0 \leq t \leq T$, or, in other words, $V^n_t \to_{\mathscr{D}} V_t$ for all $0 \leq t \leq T$. □

Lemma 4.5. *As $n \to \infty$,*

$$\sup_{0\leq y<\infty} |\zeta^n(y) - \zeta(y)| \to 0.$$

As $n \to \infty$, for $a, b \geq 0$,

$$\sup_{a\leq y\leq b} |\xi^n(y) - \xi(y)| \to 0$$

Proof. $\zeta^n(t) \to \zeta(t)$ follows from Lemma 4.4 and dominated convergence, using that $V^n_t \leq \sup_{t\leq T} S_t - \inf_{t\leq T} S_t$; here realize that $\mathbb{E}[\sup_{t\leq T} S_t] < \infty$ (because of the fact that $\mathbb{E}[\sup_{t\leq T} S_t] \leq \lim_{t\to\infty} \mathbb{E}V_t < \infty$), and also $\mathbb{E}[-\inf_{t\leq T} S_t] < \infty$ (due to (2, Lemma IX.3.3)).

The stated uniform convergence is now a consequence of Corollary A.2, after extending the function $\zeta^n(t)$ and $\zeta(t)$ to the negative half-line by equating them to 0 for $t < 0$, and by noticing that $\mathbb{E}V^n_t \leq \mathbb{E}V_t \leq \lim_{t\to\infty} \mathbb{E}V_t < \infty$.

Similarly, the result for $\xi(t)$ is a consequence of Corollary A.2, after noting that both $\mathbb{E}(\sup_{t\leq T} S_t)^2$ and $\mathbb{E}[(\inf_{t\leq T} S_t)^2]$ are finite, as follows from (2, Lemma IX.3.3). □

To prove that the function $\zeta(\cdot)$ is concave, we have to circumvent the difficulty that the functions $\zeta_n(\cdot)$, being piecewise constant, *themselves* are not concave. This is done by defining a linear interpolation, which *is* concave, see (4.4) below.

Note that, with a slight abuse of notation, from now on we allow the one-sided reflection mapping to be applied to sequences, so that

$$\mathscr{S}[\boldsymbol{a}](n) = a_n - \min_{0\leq i\leq n} a_i \qquad \boldsymbol{a} = (a_i)_{i=0}^\infty \in \mathbb{R}^\infty.$$

Theorem 4.6. *$\zeta(t)$ is concave, and $\xi(t)$ is increasing for Lévy processes (with one-sided reflection).*

Proof. We start by proving the claimed concavity of $\zeta(\cdot)$. Let $n \in \mathbb{N}$ be fixed and consider a sequence of i.i.d. random variables $\{Y_i^n\}_{i=1}$ such that $Y_1^n =_{\mathscr{D}} S_{1/n}$. Then

$$\boldsymbol{S}^n =_{\mathscr{D}} \left\{ \sum_{i=1}^{\lfloor nt \rfloor} Y_i^n \right\}_{t \geq 0},$$

defining empty sums as 0. Now consider the random walk $T_m^n := \sum_{i=1}^m Y_i^n$, and its reflected version $(\mathscr{S}[\boldsymbol{T}^n](k))_{k=1}^\infty$, and set $s^n(k) := \mathbb{E}[\mathscr{S}[\boldsymbol{T}^n](k)]$. We know from Prop. 4.1 that the sequence $(s^n(m))_{m=1}^\infty$ is concave, and hence so is the following function (which linearly interpolates):

$$\bar{\zeta}^n(t) := n(s^n(\lfloor nt \rfloor + 1) - s^n(\lfloor nt \rfloor))t + (\lfloor nt \rfloor + 1)s^n(\lfloor nt \rfloor) - s^n(\lfloor nt \rfloor + 1)\lfloor tn \rfloor. \quad (4.4)$$

Note that $\bar{\zeta}^n(\lfloor nt \rfloor/n) = s^n(\lfloor nt \rfloor) = \mathbb{E}[V_t^n]$ and $\bar{\zeta}^n((\lfloor nt \rfloor + 1)/n) = s^n(\lfloor nt \rfloor + 1) = \mathbb{E}[V_{t+\frac{1}{n}}^n]$, the latter being seen by realizing that

$$\begin{aligned} s^n(\lfloor nt \rfloor + 1) &= \mathbb{E}\mathscr{S}[\boldsymbol{T}^n](\lfloor nt \rfloor + 1) = \mathbb{E}\mathscr{S}[\boldsymbol{T}^n](\lfloor n(t + 1/n) \rfloor) \\ &= \mathbb{E}\mathscr{S}\left[\boldsymbol{S}^n\right](t + 1/n) = \mathbb{E}[V_{t+\frac{1}{n}}^n]. \end{aligned}$$

By concavity of $\bar{\zeta}^n(\cdot)$, we have, for $x < y < z$ and any $n \in \mathbb{N}$,

$$\frac{\bar{\zeta}^n(y) - \zeta^n(x)}{y - x} \geq \frac{\bar{\zeta}^n(z) - \bar{\zeta}^n(x)}{z - x}.$$

Since n was arbitrary, we may let n approach infinity to obtain

$$\frac{\zeta(y) - \zeta(x)}{y - x} \geq \frac{\zeta(z) - \zeta(x)}{z - x},$$

using $\zeta^n(t) = \bar{\zeta}^n(\lfloor nt \rfloor)/n) \leq \bar{\zeta}^n(t) \leq \bar{\zeta}^n(\lceil nt \rceil)/n) = \zeta^n(t + 1/n)$ and the uniform convergence established in Lemma 4.5.

Next, we define $\xi^n(t) := \mathbb{V}\mathrm{ar} V_t^n$, and $v^n(k) = \mathbb{V}\mathrm{ar}(\mathscr{S}[T^n])$. From Prop. 4.1 we have, for $t_1 \leq t_2$, $\xi^n(t_1) = v^n(\lfloor nt_1 \rfloor) \leq v^n(\lfloor nt_2 \rfloor) = \xi^n(t_2)$, and letting n tend to infinity and invoking the convergence of ξ^n, as given by Lemma 4.5, we conclude that $\xi(t_1) \leq \xi(t_2)$. □

5 Two-sided reflection: solution of Lindley recursion in discrete time

Let, as before, $\boldsymbol{X} = (X_n)_{n=1}^\infty \in \mathbb{R}^\infty$ be an i.i.d. sequence, and $S_0 := 0$, $S_n = X_1 + \cdots + X_n$, for $n \geq 1$. Where the previous sections studied the one-sided Lindley recursion, we now consider a variant in which there is reflection at $K > 0$ as well:

$$V_{n+1} = 0 \vee (V_n + X_{n+1}) \wedge K;$$

we say that the random walk has two reflecting barriers, viz. 0 and K. We write the V_n obtained through this procedure as $\mathscr{D}[\boldsymbol{S}](n)$ (analogously to $\mathscr{S}[\boldsymbol{S}](n)$).

In the discrete-time, one-sided case, as mentioned before, the Lindley recursion was solved through

$$\mathscr{S}[\boldsymbol{s}](n) = s_n - \min_{0\le i\le n} s_i,$$

for $\boldsymbol{s} = (s_i)_{i=0}^{\infty} \in \mathbb{R}^{\infty}$. Our first goal is to find the counterpart of this solution for the case of two-sided reflection. This is done in the following result. We denote, for a finite index set A,

$$\min_{j\in A}(a_j, b_k) := \min_{j\in A} a_j \wedge b_k.$$

Proposition 5.1. *The solution of the two-sided reflection is given by*

$$\mathscr{D}[\boldsymbol{s}](n) = \max_{k\in\{0,\dots,n\}}\Big(\min_{j\in\{k,\dots,n\}}(s_n - s_k, K + s_n - s_j)\Big) \tag{5.5}$$

Proof. We prove the claim by induction. For $n = 1$ we indeed have

$$\begin{aligned}&\max_{k\in\{0,1\}}\big(\min_{j\in\{k,1\}}(s_1 - s_k \wedge K + s_1 - s_j)\big)\\ &\qquad = \max(\min(s_1, K + s_1, K), \min(0, K)) = 0 \vee x_1 \wedge K = \mathscr{D}[\boldsymbol{s}](1).\end{aligned}$$

Now, assume (5.5) holds for some n. We first focus on the case $x_{n+1} \le 0$. Then we have that

$$\begin{aligned}\mathscr{D}[\boldsymbol{s}](n+1) &= v_{n+1} = 0 \vee (v_n + x_{n+1}) \wedge K = 0 \vee (v_n + x_{n+1})\\ &= 0 \vee \big(\max_{k\in\{0,\dots,n\}}\big(\min_{j\in\{k,\dots,n\}}(s_n - s_k, K + s_n - s_j)\big) + x_{n+1}\big)\\ &= 0 \vee \big(\max_{k\in\{0,\dots,n\}}\big(\min_{j\in\{k,\dots,n\}}(s_{n+1} - s_k, K + s_{n+1} - s_j)\big)\big).\end{aligned} \tag{5.6}$$

Since $x_{n+1} \le 0$, we have

$$\min_{j\in\{k,\dots,n+1\}} s_{n+1} - s_j = \min_{j\in\{k,\dots,n\}} s_{n+1} - s_j,$$

so that (5.6) equals

$$\begin{aligned}&0 \vee \big(\max_{k\in\{0,\dots,n\}}\big(\min_{j\in\{k,\dots,n+1\}}(s_{n+1} - s_k, K + s_{n+1} - s_j)\big)\big)\\ &\qquad = \max_{k\in\{0,\dots,n+1\}}\big(\min_{j\in\{k,\dots,n+1\}}(s_{n+1} - s_k, K + s_{n+1} - s_j)\big),\end{aligned} \tag{5.7}$$

as desired. Similarly, when $x_{n+1} > 0$ we have:

$$\begin{aligned}v_{n+1} &= 0 \vee (v_n + x_{n+1}) \wedge K = (v_n + x_{n+1}) \wedge K\\ &= \big(\max_{k\in\{0,\dots,n\}}\big(\min_{j\in\{k,\dots,n\}}(s_n - s_k, K + s_n - s_j)\big) + x_{n+1}\big) \wedge K\\ &= \max_{k\in\{0,\dots,n\}}\big(\min_{j\in\{k,\dots,n\}}(s_{n+1} - s_k, K + s_{n+1} - s_j) \wedge K\big),\end{aligned}$$

which equals (5.7) as well, as desired. This finishes the proof. □

Remark 5.2. To see why the doubly-reflected has the particular form (5.5), we may, for $n \geq k$, define w_n^k to be the value obtained by applying the recursion $w_{n+1}^k = (w_n^k + x_{n+1}) \wedge K$ to the increments $x_{k+1}, x_{k+2}, \ldots$, with $w_k^k = 0$. Let v_n be the sequence of outcomes of the two-sided reflection.

Then $w_n^k = \min_{j\in\{k,\ldots,n\}}(s_n - s_k, K + s_n - s_j)$, and obviously $w_n^k \leq v_n$. But v_n has to be one of the w_n^k for some $k \in \{0, \ldots, n\}$, namely the largest i such that $v_i = 0$. Therefore $v_n = \max_{k\in\{0,\ldots,n\}} w_n^k$, so that we obtain (5.5). This explains why this specific expression comes out. ♢

Next, we present an alternative expression for $\mathscr{D}[\boldsymbol{s}]$, which we will need when we treat the continuous-time case.

Proposition 5.3. *The solution of the two-sided reflection is given by*

$$\mathscr{D}[\boldsymbol{s}](n) = \min_{k\in\{0,\ldots,n\}} \Big[\big((s_n - s_k + K) \wedge \max_{i\in\{0,\ldots,n\}} (s_n - s_i)\big) \vee \max_{i\in\{k,\ldots,n\}} (s_n - s_i)\Big] \tag{5.8}$$

Proof. The proof is again by induction. The case $n = 1$ is a matter of straightforward verification. Next, assume the stated holds for some n. Then we have

$$\begin{aligned}
\mathscr{D}[\boldsymbol{s}](n+1) &= 0 \vee (v_n + x_{n+1}) \wedge K \\
&= 0 \vee \Big(\min_{k\in\{0,\ldots,n\}} \Big[\big((s_n - s_k + K) \wedge \max_{i\in\{0,\ldots,n\}} (s_n - s_i)\big) \\
&\qquad\qquad \vee \max_{i\in\{k,\ldots,n\}} (s_n - s_i)\Big] + x_{n+1}\Big) \wedge K \\
&= 0 \vee \min_{k\in\{0,\ldots,n\}} \Big[\big((s_{n+1} - s_k + K) \wedge \max_{i\in\{0,\ldots,n\}} (s_{n+1} - s_i)\big) \\
&\qquad\qquad \vee \max_{i\in\{k,\ldots,n\}} (s_{n+1} - s_i)\Big] \wedge K \\
&= \min_{k\in\{0,\ldots,n\}} \Big[\big((s_{n+1} - s_k + K) \wedge \max_{i\in\{0,\ldots,n\}} ((s_{n+1} - s_i) \vee 0)\big) \\
&\qquad\qquad \vee \max_{i\in\{k,\ldots,n\}} ((s_{n+1} - s_i) \vee 0)\Big] \wedge K \\
&= \min_{k\in\{0,\ldots,n\}} \Big[\big((s_{n+1} - s_k + K) \wedge \max_{i\in\{0,\ldots,n+1\}} (s_{n+1} - s_i)\big) \\
&\qquad\qquad \vee \max_{i\in\{k,\ldots,n+1\}} (s_{n+1} - s_i)\Big] \wedge K.
\end{aligned} \tag{5.9}$$

We notice that

$$\begin{aligned}
&\big((s_{n+1} - s_k + K) \wedge \max_{i\in\{0,\ldots,n+1\}} (s_{n+1} - s_i)\big) \vee \max_{i\in\{k,\ldots,n+1\}} (s_{n+1} - s_i) \\
&\quad = \begin{cases} \max_{i\in\{0,\ldots,n+1\}} (s_{n+1} - s_i) & \text{if } k = 0; \\ \max_{i\in\{0,\ldots,n+1\}} (s_{n+1} - s_i) \wedge K & \text{if } k = n+1, \end{cases}
\end{aligned}$$

so that (5.9) equals

$$\min_{k\in\{0,\ldots,n+1\}} \Big[\big((s_{n+1} - s_k + K) \wedge \max_{i\in\{0,\ldots,n+1\}} (s_{n+1} - s_i)\big) \vee \max_{i\in\{k,\ldots,n+1\}} (s_{n+1} - s_i)\Big]$$

This proves the claim. □

The expressions (5.5) and (5.8) provide two solutions to the two-sided Lindley recursion. Since the latter is a discrete-time analogue of the two-sided reflection mapping found in (10), Proposition 5.1 suggests an alternative expression for the two-sided reflection mapping. Our next goal is to formulate and prove this. We do this in the next section.

6 Two-sided reflection: solution of Lindley recursion in continuous time

The starting point of two-sided reflection in 0 and $K > 0$ in the continuous time case, is the Skorokhod problem. Given $\boldsymbol{\psi} \in D[0,\infty)$ there exists a functional $\mathscr{D}[\boldsymbol{\psi}]$ taking only values in $[0,K]$ and non-decreasing functions $\boldsymbol{\eta}_\ell$ and $\boldsymbol{\eta}_u$ such that

$$\mathscr{D}[\boldsymbol{\psi}] = \boldsymbol{\psi}+\boldsymbol{\eta}_\ell-\boldsymbol{\eta}_u, \int_0^\infty I(\mathscr{D}[\boldsymbol{\psi}](s) > 0)\mathrm{d}\eta_\ell(s) = 0, \int_0^\infty I(\mathscr{D}[\boldsymbol{\psi}](s) < K)\mathrm{d}\eta_u(s) = 0.$$

The triple $(\mathscr{D}[\boldsymbol{\psi}], \eta_\ell, \eta_u)$ is said to *solve the Skorokhod problem* for $\boldsymbol{\psi}$ on $[0,K]$, and we think of $\mathscr{D}[\boldsymbol{\psi}]$ as $\boldsymbol{\psi}$ reflected at 0 and K. The existence and uniqueness of such a triple was established in (14), and explicit solutions were given in (11) and (10), the simplest of which is

$$\mathscr{D}[\boldsymbol{\psi}](t) = \psi(t) - \sup_{s\in[0,t]} \left[\left((\psi(s) - K)\right) \vee \inf_{u\in[0,t]} \psi(u)\right) \wedge \inf_{u\in[s,t]} \psi(u)\right], \tag{6.10}$$

where we assume $\psi(0) = 0$; notice that this is the continuous-time counterpart of (5.8). In view of Props. 5.1 and 5.3 it seems reasonable to conjecture that $\mathscr{D} = \mathscr{M}$, where

$$\mathscr{M}[\psi](t) := \sup_{s\in[0,t]} \left[(\psi(t) - \psi(s)) \wedge \inf_{u\in[s,t]} (K + \psi(t) - \psi(u))\right]. \tag{6.11}$$

We prove this by first showing that $\mathscr{M}$ is Lipschitz-continuous in the J_1 topology.

Lemma 6.1. *The mapping $\mathscr{M}$ is Lipschitz-continuous in the uniform and J_1 metrics as a mapping from $D[0,T]$ for $T \in [0,\infty]$, with constant 2.*

Proof. We follow the proof of Corollary 1.5 in (11) closely. Fix $T < \infty$. We start by proving Lipschitz-continuity in the uniform metric. Define

$$R_t[\boldsymbol{\psi}](s) := \left[(-\psi(s)) \wedge \inf_{u\in[s,t]}(K - \psi(u))\right]; \quad S[\boldsymbol{\psi}](t) := \sup_{s\in[0,t]} R_t[\boldsymbol{\psi}](s). \tag{6.12}$$

For $\psi_1, \psi_2 \in D[0,T]$ we have

$$\begin{aligned} S[\boldsymbol{\psi_1}](t) - S[\boldsymbol{\psi_2}](t) &\leq \sup_{s\in[0,t]} (R_t[\boldsymbol{\psi_1}](s) - R_t[\boldsymbol{\psi_2}](s)) \\ &\leq \sup_{s\in[0,t]} \left[|-\psi_1(s) - (-\psi_2(s))| \vee \left|\inf_{u\in[s,t]}(K - \psi_1(u)) - \inf_{u\in[s,t]}(K - \psi_2(u))\right|\right] \\ &\leq \| \boldsymbol{\psi_1} - \boldsymbol{\psi_2} \|_T . \end{aligned}$$

The same inequality applies to $S[\boldsymbol{\psi_2}](t) - S[\boldsymbol{\psi_2}](t)$, so that taking the supremum leads to

$$\| S[\boldsymbol{\psi_1}] - S[\boldsymbol{\psi_2}] \|_T \leq \| \boldsymbol{\psi_1} - \boldsymbol{\psi_2} \|_T ,$$

and this proves Lipschitz-continuity, with constant 2:

$$\| \mathscr{M}[\boldsymbol{\psi_1}] - \mathscr{M}[\boldsymbol{\psi_2}] \|_T \ \leq \| \boldsymbol{\psi_1} - \boldsymbol{\psi_2} \| + \| S[\boldsymbol{\psi_1}] - S[\boldsymbol{\psi_t}] \|_T \leq 2 \| \boldsymbol{\psi_1} - \boldsymbol{\psi_2} \|_T .$$

We now turn to the J_1-metric, and we let $\mathcal{M}$ denote the class of strictly increasing continuous functions from $[0, T]$ onto itself with continuous inverse. An elementary verification yields that for $\boldsymbol{\psi} \in D[0, T]$ and $\boldsymbol{\lambda} \in \mathcal{M}$ we have $\mathscr{M}[\boldsymbol{\psi} \circ \boldsymbol{\lambda}] = \mathscr{M}[\boldsymbol{\psi}] \circ \boldsymbol{\lambda}$. With $\boldsymbol{e}$ being the identity, this leads to

$$\begin{aligned} d_{J_1}(\mathscr{M}[\boldsymbol{\psi_1}], \mathscr{M}[\boldsymbol{\psi_2}]) &= \inf_{\boldsymbol{\lambda} \in \mathcal{M}} \{ \| \mathscr{M}[\boldsymbol{\psi_1}] \circ \boldsymbol{\lambda} - \mathscr{M}[\boldsymbol{\psi_2}] \|_T \vee \| \boldsymbol{\lambda} - \boldsymbol{e} \|_T \} \\ &= \inf_{\boldsymbol{\lambda} \in \mathcal{M}} \{ \| \mathscr{M}[\boldsymbol{\psi_1} \circ \boldsymbol{\lambda}] - \mathscr{M}[\boldsymbol{\psi_2}] \|_T \vee \| \boldsymbol{\lambda} - \boldsymbol{e} \|_T \} \\ &\leq \inf_{\boldsymbol{\lambda} \in \mathcal{M}} \{ 2 \| \boldsymbol{\psi_1} \circ \boldsymbol{\lambda} - \boldsymbol{\psi_2} \|_T \vee \| \boldsymbol{\lambda} - \boldsymbol{e} \|_T \} \leq 2 d_{J_1}(\boldsymbol{\psi_1}, \boldsymbol{\psi_2}), \end{aligned}$$

where we used the Lipschitz-continuity in the uniform metric. This proves Lipschitz-continuity in the J_1 metric, again with constant 2; it is valid for every $T < \infty$ and hence also for $T = \infty$. □

We are now ready to prove that $\mathscr{D} = \mathscr{M}$.

Theorem 6.2. *For* $\boldsymbol{\psi} \in D[0, \infty)$ *we have* $\mathscr{D}[\boldsymbol{\psi}](t) = \mathscr{M}[\boldsymbol{\psi}](t)$.

Proof. Let $\boldsymbol{\psi} \in D[0, \infty)$ be given, and define $\boldsymbol{\gamma}_n$ and $\boldsymbol{\psi}_n$ by $\gamma_n(t) := \lfloor nt \rfloor / n$, $\psi_n(t) := \psi(\gamma_n(t))$. Since $\boldsymbol{\gamma}_n \to \boldsymbol{e}$ in the uniform topology, we have $\boldsymbol{\gamma}_n \to_{d_{J_1}} \boldsymbol{e}$ and hence $(\boldsymbol{\psi}, \boldsymbol{\gamma}_n) \to (\boldsymbol{\psi}, \boldsymbol{e})$ in the strong version of the J_1 topology (see p. 83 in (15)). Since $\boldsymbol{e}$ is strictly increasing we may apply Theorem 13.2.2 in (15) to obtain $\boldsymbol{\psi}_n \to_{d_{J_1}} \boldsymbol{\psi}$. Fix $t < T$, and consider $\boldsymbol{\psi}$ as element of $D[0, T]$. Since the image $\psi_n([0, T])$ is finite, we may apply Props. 5.1 and 5.3, in conjunction with (6.10), to obtain $\mathscr{D}[\boldsymbol{\psi}_n] = \mathscr{M}[\boldsymbol{\psi}_n]$. Next, we let $n \to \infty$ and use the J_1-continuity of the $\mathscr{D}$ mapping proved in (11), and the J_1-continuity of $\mathscr{M}$ proved to obtain Lemma 6.1. We thus establish the stated. □

Remark 6.3. Letting $K \to \infty$ yields $\sup_{s \in [0,t]} [(\psi(t) - \psi(s))]$, which is indeed the standard one-sided reflection, $\mathscr{S}$. ♢

7 Two-sided reflection: structural properties

In this section, we use the results proved in Sections 5 – 6 to prove that the mean value of the position of a reflected Lévy process, on which a double reflection is imposed, is an increasing and concave function. We thus establish the 'two-sided counterpart' of the result presented in (9).

Lemma 7.1. *Let $\boldsymbol{x} \in \mathbb{R}^\infty$ be a sequence of real numbers, with cumulative sums $\boldsymbol{s} \in \mathbb{R}^\infty$. Define, for a given $m \in \mathbb{N}$, $\boldsymbol{s}_m = (s_{n,m})_{n\geq 0}$, where $s_{n,m} := s_{m+n} - s_m$. For $n \in \mathbb{N}$ we have*

$$\mathscr{D}[\boldsymbol{s}](m+n) - \mathscr{D}[\boldsymbol{s}_m](n) \geq 0, \tag{7.13}$$

and for $n_1, n_2 \in \mathbb{N}$,with $n_1 \leq n_2$,

$$\mathscr{D}[\boldsymbol{s}](m+n_2) - \mathscr{D}[\boldsymbol{s}_m](n_2) \leq \mathscr{D}[\boldsymbol{s}](m+n_1) - \mathscr{D}[\boldsymbol{s}_m](n_1). \tag{7.14}$$

Proof. By (5.5) we have

$$\begin{aligned}
\mathscr{D}[\boldsymbol{s}_m](n) &= \max_{k\in\{0,\dots,n\}} \big(\min_{j\in\{k,\dots,n\}} (s_{n,m} - s_{k,m}, K + s_{n,m} - s_{j,m}) \big) \\
&= s_{n+m} + \max_{k\in\{0,\dots,n\}} \big(\min_{j\in\{k,\dots,n\}} (-s_{k+m}, K - s_{j+m}) \big) \\
&= s_{n+m} + \max_{k\in\{0,\dots,n\}} \big(\min_{j\in\{k+m,\dots,n+m\}} (-s_{k+m}, K - s_j) \big) \\
&= s_{n+m} + \max_{m\leq k\leq n+m} \big(\min_{j\in\{k,\dots,n+m\}} (-s_k, K - s_j) \big),
\end{aligned}$$

so that $\mathscr{D}[\boldsymbol{s}](m+n) - \mathscr{D}[\boldsymbol{s}_m](n)$ equals

$$\begin{aligned}
&\max_{k\in\{0,\dots,n+m\}} \big(\min_{j\in\{k,\dots,n+m\}} (-s_k, K - s_j)\big) - \max_{k\in\{m,\dots,n+m\}} \big(\min_{j\in\{k,\dots,n+m\}} (-s_k, K - s_j)\big) \\
&\quad \geq 0,
\end{aligned}$$

which proves (7.13). Turning to (7.14), we first notice that it is enough to prove the statement for $n_2 = n_1 + 1$, and using the notation $v_n := \mathscr{D}[\boldsymbol{s}](n)$, $v_n^m := \mathscr{D}[\boldsymbol{s_m}](n)$ we find

$$\begin{aligned}
&v_{m+n_1+1} - v^m_{n_1+1} - (v_{n_1+m} - v^m_{n_1}) \\
&\quad = 0 \vee v_{m+n_1} + x_{m+n_1+1} \wedge K - 0 \vee v^m_{n_1} + x_{m+n+1} \wedge K - (v_{n_1+m} - v^m_{n_1}),
\end{aligned}$$

which equals

$$\begin{cases}
-(v_{n_1+m} - v^m_{n_1}) & \text{if } v_{m+n_1} + x_{m+n_1+1} < 0 \\
\begin{aligned} &v_{m+n_1} + x_{m+n_1+1} - (0 \vee v^m_{n_1} + x_{m+n_1+1}) \\ &\quad - (v_{n_1+m} - v^m_{n_1}) = (v^m_{n_1} + x_{m+n_1+1}) \wedge 0 \end{aligned} & \text{if } v_{m+n_1} + x_{m+n_1+1} \in [0, K] \\
\begin{aligned} &K - ((v^m_{n_1} + x_{m+n_1+1}) \wedge K) - (v_{n_1+m} - v^m_{n_1}) \\ &\quad = K + (-(v^m_{n_1} - x_{m+n_1+1}) \vee (-K)) \\ &\qquad - (v_{n_1+m} - v^m_{n_1}) \\ &\quad = K + (-x_{m+n_1+1} \vee -K + v^m_{n_1}) - v_{n_1+m} \\ &\quad = (K - x_{m+n_1+1} \vee v^m_{n_1}) - v_{n_1+m} \\ &\quad = (K - x_{m+n_1+1} - v_{n_1+m}) \vee (v^m_{n_1} - v_{n_1+m}) \end{aligned} & \text{if } v_{m+n_1} + x_{m+n_1+1} > K.
\end{cases}$$

Now (7.14) follows, since $-(v_{n_1+m} - v^m_{n_1}) \leq 0$. □

The results of Lemma 7.1 are easily extended to a class of piecewise constant functions.

Lemma 7.2. *Let $\boldsymbol{\psi} \in D[0,\infty)$ be of the form*

$$\psi(t) = \sum_{i=0}^{\infty} s_i I([ai, a(i+1)))(t)$$

for $\boldsymbol{s} := (s_i)_{i=0}^{\infty} \in \mathbb{R}^{\infty}$, with $s_0 \equiv 0$, $a > 0$. Define $\boldsymbol{\psi}_r \in D[0,\infty)$ by $\psi_r(t) := \psi(r+t) - \psi(r)$. Then

$$\mathscr{D}[\boldsymbol{\psi}](r+t) - \mathscr{D}[\boldsymbol{\psi}_r](t) \geq 0, \tag{7.15}$$

and, for $t_1 \leq t_2$,

$$\mathscr{D}[\boldsymbol{\psi}](r+t_2) - \mathscr{D}[\boldsymbol{\psi}_r](t_2) \leq \mathscr{D}[\boldsymbol{\psi}](r+t_1) - \mathscr{D}[\boldsymbol{\psi}_r](t_1). \tag{7.16}$$

Proof. Assume $a = 1$, and write $r = m + q$ for $q \in [0,1)$ and $m = \lfloor r \rfloor$. Recall from Lemma 7.1 the definition of $\boldsymbol{s}_m$, viz. $s_{n,m} := s_{m+n} - s_m$. Then $\psi(t) = s_{\lfloor t \rfloor}$ and

$$\begin{aligned}\psi_r(t) &= \psi(\lfloor r+t \rfloor) - \psi(\lfloor r \rfloor) = \psi(\lfloor q+t \rfloor + m) - \psi(m) \\ &= s_{m+\lfloor q+t \rfloor} - s_m = s_{\lfloor t+q \rfloor, m}\,,\end{aligned}$$

so that $\mathscr{D}[\boldsymbol{\psi}](t) = \mathscr{D}[\boldsymbol{s}](\lfloor t \rfloor)$ and $\mathscr{D}[\boldsymbol{\psi}_r](t) = \mathscr{D}[\boldsymbol{s}_m](\lfloor t+q \rfloor)$ where $m = \lfloor r \rfloor$ (which can be verified by making an elementary picture). Using that $\lfloor r+t \rfloor = \lfloor r \rfloor + \lfloor t+q \rfloor$, we find that

$$\mathscr{D}[\boldsymbol{\psi}](r+t) - \mathscr{D}[\boldsymbol{\psi}_r](t) = \mathscr{D}[\boldsymbol{s}](\lfloor r \rfloor + \lfloor t+q \rfloor) - \mathscr{D}[\boldsymbol{s}_m](\lfloor t+q \rfloor) \geq 0$$

and

$$\begin{aligned}\mathscr{D}[\boldsymbol{\psi}](r+t_2) - \mathscr{D}[\boldsymbol{\psi}_r](t_2) &= \mathscr{D}[\boldsymbol{s}](\lfloor r \rfloor + \lfloor t_2+q \rfloor) - \mathscr{D}[\boldsymbol{s}_m](\lfloor t_2+q \rfloor) \\ &\leq \mathscr{D}[\boldsymbol{s}](\lfloor r \rfloor + \lfloor t_1+q \rfloor) - \mathscr{D}[\boldsymbol{s}_m](\lfloor t_1+q \rfloor) = \mathscr{D}[\boldsymbol{\psi}](r+t_1) - \mathscr{D}[\boldsymbol{\psi}_r](t_1).\end{aligned}$$

Now choose an $a \neq 1$ arbitrarily. Define $\tilde{\psi}(t) := \psi(at)$. Then $\tilde{\psi}_r(t) = \psi_{ar}(at)$, and $\mathscr{D}[\tilde{\boldsymbol{\psi}}](t) = \mathscr{D}[\boldsymbol{\psi}](at)$, and $\mathscr{D}[\tilde{\boldsymbol{\psi}}_r](t) = \mathscr{D}[\boldsymbol{\psi}_{ar}](at)$. Since (7.15) and (7.16) hold for $\tilde{\boldsymbol{\psi}}$ for any $r, t \geq 0$ we find for the given $r \geq 0$ that

$$\mathscr{D}[\boldsymbol{\psi}](r+t) - \mathscr{D}[\boldsymbol{\psi}_r](t) = \mathscr{D}[\tilde{\boldsymbol{\psi}}](r/a + t/a) - \mathscr{D}[\tilde{\boldsymbol{\psi}}_{r/a}](t/a) \geq 0$$

and similarly

$$\begin{aligned}\mathscr{D}[\boldsymbol{\psi}](r+t_2) - \mathscr{D}[\boldsymbol{\psi}_r](t_2) &= \mathscr{D}[\tilde{\boldsymbol{\psi}}](r/a + t_2/a) - \mathscr{D}[\tilde{\boldsymbol{\psi}}_{r/a}](t_2/a) \\ &\leq \mathscr{D}[\tilde{\boldsymbol{\psi}}](r/a + t_1/a) - \mathscr{D}[\tilde{\boldsymbol{\psi}}_{r/a}](t_1/a) = \mathscr{D}[\boldsymbol{\psi}](r+t_1) - \mathscr{D}[\boldsymbol{\psi}_r](t_1).\end{aligned}$$

This proves the claim. □

We can now prove the continuous-time version of Lemma 7.1.

Lemma 7.3. *Let $\boldsymbol{\psi} \in D[0,\infty)$ and define $\boldsymbol{\psi}_r \in D[0,\infty)$ by $\psi_r(t) := \psi(r+t) - \psi(r)$. Then*

$$\mathscr{D}[\boldsymbol{\psi}](r+t) - \mathscr{D}[\boldsymbol{\psi}_r](t) \geq 0, \tag{7.17}$$

and, for $t_1 \leq t_2$,

$$\mathscr{D}[\boldsymbol{\psi}](r+t_2) - \mathscr{D}[\boldsymbol{\psi}_r](t_2) \leq \mathscr{D}[\boldsymbol{\psi}](r+t_1) - \mathscr{D}[\boldsymbol{\psi}_r](t_1). \tag{7.18}$$

Proof. Define $\gamma^n(t) := \lfloor nt \rfloor / n$, and $\psi^n(t) = \psi(\gamma^n(t))$. Then Lemma 7.2 applies to $\boldsymbol{\psi}^n$, and hence

$$\mathscr{D}[\boldsymbol{\psi}^n](r+t) - \mathscr{D}[\boldsymbol{\psi}_r^n](t) \geq 0, \tag{7.19}$$

$$\mathscr{D}[\boldsymbol{\psi}^n](r+t_2) - \mathscr{D}[\boldsymbol{\psi}_r^n](t_2) \leq \mathscr{D}[\boldsymbol{\psi}^n](r+t_1) - \mathscr{D}[\boldsymbol{\psi}_r^n](t_1). \tag{7.20}$$

Using the same argument as in the proof of Theorem 6.2, we have $\boldsymbol{\psi}^n \to_{d_{J_1}} \boldsymbol{\psi}$. We also have $\boldsymbol{\psi}_r^n \to_{d_{J_1}} \boldsymbol{\psi}_r$, since $\psi_r^n(t) = \psi^n(t+r) - \psi^n(r)$ and we regard $-\psi^n(r)$ as a constant function, which converges uniformly, and hence in the J_1-topology as well, to $-\psi(r)$. In general, addition is not continuous in the J_1 topology, but since $-\psi(r)$ is a constant function, $t \mapsto \psi(r+t)$ and $-\psi(r)$ have no common discontinuity points (where both are considered as function of t), and we have $\boldsymbol{\psi}_r^n \to_{d_{J_1}} \boldsymbol{\psi}_r$. We wish to let n tend to infinity in (7.19) and (7.20), and we therefore assume that $r + t_1$ and $r + t_1$ are both continuity points for ψ, which implies that they are continuity points for $\mathscr{D}[\boldsymbol{\psi}]$, and also that t_1 and t_2 are continuity points for $\mathscr{D}[\boldsymbol{\psi}_r]$. Under this assumption, we let $n \to \infty$, and thus obtain (7.17) and (7.18) when r, t_1, t_2 are continuity points. However, since $\mathscr{D}$ maps càdlàg functions to càdlàg functions, we have that (7.17) and (7.18) hold for all r, t_1, t_2 whenever $\boldsymbol{\psi} \in D[0, \infty)$, as claimed. □

We can now prove the main results.

Theorem 7.4. *$\zeta(n)$ is increasing and concave for random walks (with two-sided reflection).*

Proof. Set $S_n^1 := S_{n+1} - S_1$, $\boldsymbol{S}^1 := \{S_n^1\}_{n=0}^{\infty}$, and $V_n^1 = \mathscr{D}[\boldsymbol{S}^{\mathbf{1}}](n)$. By stationarity of the increments we have $\{S_n\} =_{\mathscr{D}} \{S_n^1\}$ and $\{V_n\} =_{\mathscr{D}} \{V_n^1\}$. Using (7.13) with $m = 1$ we have $V_{n+1} - V_n^1 \geq 0$, and we see that $\zeta(n)$ is increasing by taking means. Furthermore, by (7.14) we have $n \mapsto V_{n+1} - V_n^1$ is decreasing, and taking means implies that $\zeta(n)$ is concave. □

Theorem 7.5. *$\zeta(t)$ is increasing and concave for Lévy processes (with two-sided reflection).*

Proof. Define $\boldsymbol{S}^r$ by $S_t^r = S_{t+r} - S_r$. By the stationary increments we have $\boldsymbol{S}^r =_{\mathscr{D}} \boldsymbol{S}$ and $\mathscr{D}[\boldsymbol{S}^r] =_{\mathscr{D}} \mathscr{D}[\boldsymbol{S}]$. Set $V_t^r = \mathscr{D}[\boldsymbol{S}^r](t)$ and $V_t = \mathscr{D}[\boldsymbol{S}](t)$. According to (7.17) and (7.18) we have that $V_{r+t} - V_t^r \geq 0$ and also that $t \mapsto V_{r+t} - V_t^r$ is decreasing. Taking means yields the desired result. □

The following statement follows immediately from the facts that $V_{n+1} - V_n^1 \geq 0$ and $V_{s+t} - V_t^s \geq 0$.

Corollary 7.6. *For any $q \geq 0$, we have $n \mapsto \mathbb{E}V_n^q$ and $t \mapsto V_t^q$ are increasing, both for random walks and Lévy processes (with two-sided reflection).*

Remark 7.7. The reader can verify that in the argumentation of this section, we *did not use that increments are independent* — in fact all results, in particular Thms. 7.4 and 7.5, hold under the assumption of just stationary increments. We conclude that we have, in passing, extended the result by Kella and Sverchkov (9), who considered processes with stationary increments reflected at 0, to the case of two-sided reflection. ♢

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A Appendix

In this appendix we prove a number of results on uniform convergence.

Lemma A.1. *Let $F_n(\cdot)$, $n = 1, 2, \ldots$, be a sequence of uniformly bounded increasing functions, such that $F_n(x) \to F_0(x)$ $\forall x \in \mathbb{R}$, where F_0 is continuous, and $\lim_{x\to-\infty} F_0(x) =: F_0(-\infty) \le F_n(x)$ and $F_n(x) \le F(\infty) := \lim_{x\to\infty} F_0(x)$ for all n and x. Then*

$$\sup_{-\infty<y<\infty} |F_n(y) - F_0(y)| \to 0$$

Proof. Without loss of generality, we may assume that $0 \le F_n(x) \le 1$ for all x and n. F_0 is increasing, so the limits $a := F(-\infty)$ and $b := F(\infty)$ exist, and are finite, and we may assume $a = 0$ and $b = 1$. Set $F_0^{-1}(y) := \inf\{x \in \mathbb{R} \mid F(x) = y\}$ for $0 < y < 1$ and $F_0^{-1}(0) = -\infty$ and $F_0^{-1}(1) = \infty$. Let $k \in \mathbb{N}$, and set $x_j^k := F_0^{-1}(j/k)$ $j = 0, 1, \ldots k$. Then for $0 \le j < k$ and $x_j^k < x < x_{j+1}^k$

$$\begin{aligned} F_n(x_j^k) - F_0(x_j^k) - \tfrac{1}{k} &= F_n(x_j^k) - F_0(x_{j+1}^k) \le F_n(x) - F_0(x) \\ &\le F_n(x_{j+1}^k) - F_0(x_j^k) = F_n(x_{j+1}^k) - F_0(x_{j+1}^k) + \tfrac{1}{k}, \end{aligned}$$

since F_n and F are increasing, and F is continuous. Continuing our calculation, we obtain

$$\begin{aligned} |F_n(x) - F(x)| &= \max(F_n(x) - F_0(x), F_0(x) - F_n(x)) \\ &\le \max_{j\in\{0,\ldots,k-1\}} \big(F_n(x_{j+1}^k) - F_0(x_{j+1}^k) + \tfrac{1}{k}, F_0(x_j^k) - F_n(x_j^k) + \tfrac{1}{k}\big) \\ &= \tfrac{1}{k} + \max_{j\in\{0,\ldots,k-1\}} \big(F_n(x_{j+1}^k) - F_0(x_{j+1}^k), F_0(x_j^k) - F_n(x_j^k)\big) \\ &\le \tfrac{1}{k} + \max_{j\in\{0,\ldots,k\}} |F_n(x_j^k) - F_0(x_j^k)|, \end{aligned}$$

and therefore

$$\sup_{-\infty<x<\infty} |F_n(x) - F(x)| \le \tfrac{1}{k} + \max_{j\in\{0,\ldots,k\}} |F_n(x_j^k) - F_0(x_j^k)|.$$

Using that $0 \le \lim_n F_n(-\infty) \le F(y)$ for all $y \in \mathbb{R}$ we see that $\lim_n F_n(-\infty) = 0$, and similarly, that $\lim_n F_n(\infty) = 1$, we obtain

$$\lim_{n\to\infty} \sup_{-\infty\le j\le\infty} |F_n(x) - F(x)| \le \tfrac{1}{k}$$

Since k was arbitrary, the proof is complete. □

Corollary A.2. *Let $F_n(\cdot)$, $n = 1, 2, \ldots$, be a sequence of increasing functions, such that for some $K > 0, a, b \in \mathbb{R} : \sup_{x\in[a,b]} |F_n(x)| \le K$ for all n, and $F_n(x) \to F_0(x)$ $\forall x \in [a, b]$, where F_0 is continuous. Then*

$$\sup_{a\le y\le b} |F_n(y) - F_0(y)| \to 0$$

Proof. Define $\tilde{F}_n$ for $n = 0, 1, \ldots$ by $\tilde{F}_n(t) := F_n(t)$ for $t \in [a, b]$, $\tilde{F}_n(t) = F_n(a)$ for $t < a$, and $\tilde{F}_n(t) = F_n(b)$ for $t > b$. By applying Lemma A.1, we obtain

$$\sup_{a\le y\le b} |F_n(y) - F_0(y)| = \sup_{a\le y\le b} |\tilde{F}_n(y) - \tilde{F}_0(y)| \le \sup_{-\infty\le y\le\infty} |\tilde{F}_n(y) - \tilde{F}_0(y)| \to 0$$

□