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TAIL ASYMPTOTICS FOR THE SUM OF TWO HEAVY-TAILED DEPENDENT RISKS

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Abstract

Let X_1, X_2 denote positive exchangable heavy-tailed random variables with continuous marginal distribution function F. The asymptotic behavior of the tail of $X_1 + X_2$ is studied in a general copula framework and some bounds and extremal properties are provided. For more specific assumptions on F and the underlying dependence structure of X_1 and X_2, we survey explicit asymptotic results available in the literature and add several new cases.

Keywords: copula, dependence, mean excess function, regular variation, subexponential distribution, tail dependence

1 Introduction and background

A qualitative and quantitative understanding of the probability of an overshoot of a sum of heavy-tailed risks over a large threshold is of major importance in applied probability and its applications in risk management, such as the determination of risk measures for given portfolios of risks, evaluation of credit risk etc. Under the assumption of independence among the risks, the situation is well understood. In particular, from the very definition of subexponential distributions, given identical marginal distributions, the maximum among the involved risks determines the distribution of the sum and, on the other hand, for non-identical marginals the distribution of the sum is determined by the component with the heaviest tail (see e.g. Asmussen [3, Ch.IX]).
However, for practical purposes the independence assumption is often too restrictive and there is a need for an understanding of the sensitivity of the distribution of sums of risks on the dependence structure between them.
Over the last few years, several results in this direction have been developed. The purpose of this paper is to analyze the particular setting of two risk components with identical marginal distribution and symmetric dependence structure, both collecting relevant material from the literature under the same umbrella and adding some additional explicit results in this direction. Some of the observations are of basic

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nature, but may help to get a clearer intuitive picture of the matter. For clarity of exposition, we focus on the sum of two exchangeable random variables, although some of the cited results are available in more general settings.
That is, for positive exchangable random variables X_1, X_2 with continuous marginal distribution function F, we are interested in the asymptotic behavior of

$$\mathbb{P}(X_1 + X_2 > x) \tag{1}$$

for large x for given (heavy-tailed) F and given type of dependence among X_1 and X_2. Particularly interesting questions are when (1) is of the same order $2\,\mathbb{P}(X_1 > x)$ as for the independent case, and more generally, when the asymptotics of (1) are of the order $c\,\mathbb{P}(X_1 > x)$ for some $c \in (0, \infty)$.

If the joint distribution function of X_1 and X_2 can be bounded below by some distribution function $G(x_1, x_2)$ for any $x_1, x_2 \geq 0$, then Denuit *et al.* [8] gave the following bounds:

$$1 - \inf_{y \geq 0}(F(y) + F(x-y) - G(y, x-y)) \leq \mathbb{P}(X_1 + X_2 > x) \leq 1 - \sup_{y \geq 0} G(y, x-y).$$

For each x, these bounds are best possible, although neither the lower nor the upper bound is the distribution tail of a sum of random variables with marginal distribution F (in particular, the comonotone and counter-monotone copula do in general not provide bounds for the tail of $X_1 + X_2$, contrary to what one might expect at a first glance, see [8] for details). For positive quadrant dependence (i.e. $\mathbb{P}(X_1 > x, X_2 > x) \geq \mathbb{P}(X_1 > x)\,\mathbb{P}(X_2 > x)$ for all $x \geq 0$) we have $G(x_1, x_2) = F(x_1)F(x_2)$. On the other hand, without any knowledge of the underlying dependence structure, $G(x_1, x_2)$ has to be replaced by the counter-monotone copula $C_W(F(x_1), F(x_2)) = \max\{F(x_1) + F(x_2) - 1, 0\}$, in which form the above result is due to Makarov [18]. For an extension to best-possible bounds on the distribution of general non-decreasing functions of n dependent risks, see for instance Cossette *et al.* [7] and Embrechts & Puccetti [10].
However, the above approach is not well-suited for asymptotic considerations and does not make use of the heavy-tail assumption directly. Moreover, one can get more explicit results by specifying classes of dependence structures.

Let $\mathcal{S}$ and $\mathcal{L}$ denote the class of subexponential and long-tailed distributions, respectively. We will use the notation $F \in \mathcal{R}$ if $\overline{F}(x) = 1 - F(x)$ is regularly varying at infinity with some index $-\alpha < 0$ ($\overline{F} \in \mathcal{R}_{-\alpha}$). Recall that $\mathcal{R} \subset \mathcal{S} \subset \mathcal{L}$ (see e.g. Embrechts *et al.* [9]). The tail probability of weighted sums of independent random variables with regularly varying tails, where the weights are dependent random variables, was studied in Goovaerts et al. [12].

For fixed continuous marginals, a copula representation of (1) may be considered as a natural tool to analyze the impact of dependence, and we will take up this approach in what follows. For background reading on copulae and their properties, we refer to Joe [14] or Nelsen [20]. Intuitively, there is a trade-off between dependence in the tail and heaviness of F: the heavier F is, the stronger the dependence in the tail has to be in order to affect the tail behavior of $X_1 + X_2$. In the paper, this relationship is formalized to some extent.

Recall that the (upper) tail dependence coefficient is defined by

$$\lambda := \lim_{u \to 1} \mathbb{P}(F(X_2) > u | F(X_1) > u).$$

If $\lambda = 0$, then X_1 and X_2 are called tail-independent. λ is a frequently used measure of extremal dependence (for estimation procedures, see Frahm *et al.* [11]). For a comparison of various tail dependence measures with a view towards financial time series, see Malevergne and Sornette [19].
In fact, many available joint tail dependence models have been developed in the framework of bivariate extreme value theory and are based on max-stability (for estimation procedures in this context we refer to Abdous *et al.* [1]). Except for the independent case, all bivariate extreme value distributions have a $\lambda > 0$. On the other hand, as pointed out in Coles *et al.* [6], several classical estimation procedures for λ from a data set might lead to the conclusion $\lambda > 0$ where in fact tail independence is present (see [6] for details and suggestions to overcome this problem). For another model of joint tail dependence in extreme value theory that allows for asymptotic independence, see Ledford & Tawn [17]. An alternative extremal dependence measure feasible for multivariate regularly varying tails is discussed in Resnick [23]. The approach pursued in this paper is related to, but not contained in the extreme value framework. We are rather interested in the question: Given F, what types of asymptotic behavior of $\mathbb{P}(X_1 + X_2 > x)$ are possible and what assumptions on F and the underlying dependence structure admit an explicit description of that behavior?

Although tail dependence provides a rather restrictive description of the dependence in the tail (for identical marginals, one basically looks at the dependence behavior along the line $X_1 = X_2$ in the tail), due to the exchangability assumption, λ already gives some crude information about the distribution of the sum. Moreover, as will be shown in Section 3.1.2, for $F \in \mathcal{R}$ tail independence is a sufficient condition for insensitivity of tail asymptotics of the sum with respect to dependence, whereas for lognormal marginals this is not true, as we will show by explicitly constructing a counter-example (cf. Section 3.2).
For certain classes of copulae among X_1 and X_2 (including those of Archimedean type), Juri & Wüthrich [15, 16] established a distributional limit result of conditional dependence in the tail, which in particular refines the description through the coefficient λ. For Archimedean copulae this result could be exploited in Wüthrich [26] and Alink *et al.* [2] to give sharp asymptotics of the tail of $X_1 + X_2$, see Section 3.3. Another related refinement of the coefficient λ based on so-called tail copulae is discussed in Schmidt & Stadtmüller [25].

In Section 2, some general bounds and a copula representation of $\mathbb{P}(X_1+X_2 > x)$ are discussed. Section 3 then gives explicit results under more specific assumptions on F and the underlying dependence structure. This should be viewed as an outline of several partial answers to the question raised above, setting the stage for further research towards a full understanding of the matter, including the extension to sums of arbitrarily many and non-exchangeable risks.

2 Some general considerations

Let us first collect some preliminary facts:

Lemma 2.1. *(a)* $\mathbb{P}(\max(X_1, X_2) > x) \sim (2-\lambda)\overline{F}(x)$

(b) $\lim_{x\to\infty} \mathbb{P}(X_1 > x | \max(X_1, X_2) > x) = \frac{1}{2-\lambda}$

Proof. Assertion (a) follows from

$$\begin{aligned}\mathbb{P}(\max(X_1, X_2) > x) &= \mathbb{P}(X_1 > x) + \mathbb{P}(X_2 > x) - \mathbb{P}(X_1 > x, X_2 > x)\\ &= 2\,\overline{F}(x) - \overline{F}(x)\,\mathbb{P}(X_2 > x|X_1 > x)\end{aligned}$$

and (b) is a direct consequence of (a). □

The following trivial bounds can be given:

$$\mathbb{P}(\max(X_1, X_2) > x) \leq \mathbb{P}(X_1 + X_2 > x) \leq \mathbb{P}(\max(X_1, X_2) > x/2),$$

leading to

$$2-\lambda \leq \liminf_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} \quad \text{and} \quad \limsup_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x/2)} \leq 2-\lambda. \qquad (2)$$

These bounds are determined by the dependence structure through λ. If there is no information on λ available, one is left with the "worst case" bounds

$$\overline{F}(x) \ll \mathbb{P}(X_1 + X_2 > x) \ll 2\overline{F}(x/2).$$

At the same time, the bounds (2) cannot be improved without any further assumptions, since for very heavy tails with $\overline{F}(x/2) \sim \overline{F}(x)$ we have

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} = 2-\lambda,$$

so that both bounds are attained (for any value of λ). For such heavy tails, the dependence structure obviously only affects the tail behaviour through the tail dependence coefficient and the sum X_1+X_2 is essentially determined by the maximum of the two random variables.

But also for distributions with lighter tails than above, the bounds (2) are sharp: The upper bound is attained for comonotone dependence and arbitrary marginals (note that in this case $\lambda = 1$), whereas the lower bound is attained for independence and subexponential marginals.

2.1 A copula representation

Proposition 2.2. *Let the random variables* X_1 *and* X_2 *be dependent according to an arbitrary absolutely continuous copula function* $C(a,b)$ *with partial derivative* $c_a(a,b) := \frac{\partial C(a,b)}{\partial a}$. *Then*

$$\frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} = 1 + \int_0^x \frac{1 - c_a(F(z), F(x-z))}{\overline{F}(x)} F(dz). \tag{3}$$

Proof. From the identity

$$\begin{aligned}\frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} &= 1 + \frac{F(x) - \mathbb{P}(X_1 + X_2 \le x)}{\overline{F}(x)} \\ &= 1 + \int_0^x \frac{1 - \mathbb{P}(X_2 \le x - z | X_1 = z)}{\overline{F}(x)} F(dz),\end{aligned}$$

relation (3) follows from the copula representation of the conditional distribution function

$$\mathbb{P}(X_2 \le x_2 | X_1 = x_1) = \mathbb{P}(F(X_2) \le F(x_2) | F(X_1) = F(x_1)) = c_a(F(x_1), F(x_2)).$$

□

Note that due to exchangibility, we have $C(a,b) = C(b,a)$. Formula (3) can also be interpreted geometrically:

$$\mathbb{P}(X_1 + X_2 > x) = \mathbb{P}(\max(X_1, X_2) > x) + \mathbb{P}(X_1 + X_2 > x, \max(X_1, X_2) \le x), \tag{4}$$

where the second summand is the integral of the copula density function $c_{ab}(a,b) = \frac{\partial^2 C(a,b)}{\partial a \partial b}$ over the shaded area in Figure 1, so that one obtains

$$\begin{aligned}&\mathbb{P}(X_1 + X_2 > x) \\ &\quad = 1 - C(F(x), C(F(x))) + \int_{u_1=0}^{F(x)} \int_{u_2=F(x-F^{-1}(u_1))}^{F(x)} c_{ab}(u_1, u_2)\, du_2\, du_1, \\ &\quad = 1 - \int_0^{F(x)} c_a(u_1, F(x - F^{-1}(u_1)))\, du_1,\end{aligned}$$

which is equivalent to (3).

Since the first summand in (4) is given by Lemma 2.1(a), it suffices to study the contribution of the shaded area in Figure 1 for the tail behavior of the sum. Note also that the lower bound in (2) is sharp whenever the contribution from the shaded area is asymptotically negligible compared to the probability mass in the two stripes to the right and above of it. On the other hand, the upper bound in (2) is sharp whenever the area between the two dashed lines and the lower-bounding curve of the shaded area in Figure 1 is asymptotically negligible to the contribution of the domain above that curve. The latter is in particular fulfilled for comonotonicity, since then there is only probability mass along the diagional. Note that the latter does not imply that comonotonicity provides an upper bound for the tail of the sum among all possible dependence structures, see Section 3.1 for a counter-example.

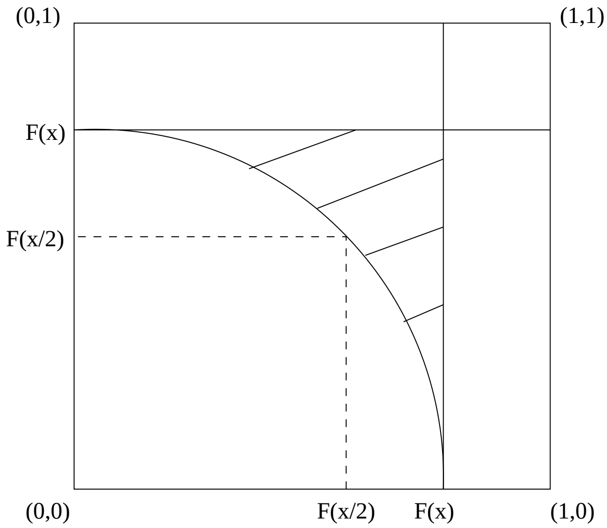

Figure 1: The domain of the copula density function

Proposition 2.3. *Let $F \in \mathcal{L}$ be absolutely continuous and X_1 and X_2 be dependent according to an arbitrary absolutely continuous copula function $C(a, b)$. Then*

$$\liminf_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} \geq 1 + c_b(1,1) - c_b(0,1). \tag{5}$$

Proof. From (3), Fatou's lemma and de l'Hopital, we obtain

$$\begin{aligned}\liminf_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} &\geq 1 + \int_0^\infty \liminf_{x\to\infty} \frac{1_{\{z\leq x\}}\,(1 - c_a\,(F(z), F(x-z)))}{\overline{F}(x)}\, F(dz) \\ &= 1 + \int_0^\infty c_{ab}(F(z), 1)\; \liminf_{x\to\infty} \frac{f(x-z)}{f(x)}\, F(dz).\end{aligned}$$

From the definition of a long-tailed distribution, it immediately follows for its density f that $\lim_{x\to\infty} \frac{f(x-z)}{f(x)} = 1$ for all $z > 0$, so that we are left with

$$\liminf_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} \geq 1 + \int_0^\infty c_{ab}(F(z), 1)\, F(dz) = 1 + c_b(1,1) - c_b(0,1).$$

□

From the definition of the tail-dependence coefficient, we have in terms of (absolutely continuous) copulae

$$\lambda = \lim_{u\to 1} \frac{1 - 2u + C(u,u)}{1-u} = 2 - 2\, c_b(1,1),$$

which, together with the trivial lower bound (2) and Proposition 2.3 implies:

Corollary 2.4. *Under the assumptions of Proposition 2.3,*

$$\begin{aligned}\liminf_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} &\geq c_b(1,1) + \max\{c_b(1,1),\, 1 - c_b(0,1)\} \\ &= 2 - \min\Big\{\lambda,\, \frac{\lambda}{2} + c_b(0,1)\Big\}.\end{aligned}$$

Remark 2.1. The minimum in the above expression can be attained for either term (see for instance Section 3.3), so this lower bound represents an improvement over the trivial bound $2 - \lambda$. In particular, $c_b(0,1) < \lambda/2$ is a sufficient condition for the fact that the distribution of the sum is asymptotically not determined by the distribution of the maximum.

If interchanging limits in (3) is justified, then the r.h.s. of (5) even gives the correct asymptotic behavior of the limit. This is in particular the case for independence, where $c_b(a,b) = a$ yields the constant 2 on the r.h.s. of (5). The latter gives rise to a sufficient criterion for interchanging limits in (3):

Proposition 2.5. *Let $F \in \mathcal{L}$ be absolutely continuous and X_1 and X_2 be dependent according to an absolutely continuous copula function $C(a,b)$ with*

$$c_{ab}(a,0) > 0 \quad \forall a \in [0,1]. \tag{6}$$

Then

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} = 1 + c_b(1,1) - c_b(0,1). \tag{7}$$

Proof. Consider representation (3). For the independent case we obviously have

$$\lim_{x\to\infty} \int_0^\infty \frac{1_{\{z\le x\}}\overline{F}(x-z)}{\overline{F}(x)}\, F(dz) = 1.$$

Since, for any copula, $c_a(a,b)$ is non-decreasing in b for all $a \in [0,1]$, condition (6) is equivalent to the fact that

$$c_a(a,b) \ge k\, b \quad \forall\, (a,b) \in [0,1]^2$$

for some $k > 0$, and the latter implies $c_a(F(z), F(x-z)) \ge k\, F(x-z)$ for all $z \le x$. Hence the above integrand for the independent case serves as an upper bound for which interchanging limits is justified. The assertion then follows by virtue of Pratt's Lemma (cf. [21]). □

Remark 2.2. In situations where condition (6) does not apply, one can try to evaluate the asymptotic behavior of (3) directly. Example 3.2 in Section 3.3 illustrates that this program can actually be carried out in some cases.

2.2 A remark on the role of the mean excess function

As already mentioned, the tail dependence coefficient λ is a rather rough measure of the dependence in the tail. The following result uses a somewhat finer criterion of conditional exceedances and can be applied for any type of dependence structure between X_1 and X_2. Recall the definition of the mean excess function $e(x)$ of F given by

$$e(x) = \mathbb{E}(X - x \,|\, X > x) = \int_x^\infty \frac{\overline{F}(u)}{\overline{F}(x)}\, du$$

and note that $e(x) \to \infty$ for $x \to \infty$ for every $F \in \mathcal{S}$ (see e.g. [9]).

Proposition 2.6. *If the mean-excess function $e(x)$ is self-neglecting, i.e.*

$$\lim_{x\to\infty} \frac{e(x + a\, e(x))}{e(x)} = 1 \quad \forall\, a \geq 0, \tag{8}$$

and if

$$\inf_{a>0} \liminf_{x\to\infty} \mathbb{P}(X_2 > a\, e(x) \,|\, X_1 > x) > 0, \tag{9}$$

then

$$\liminf_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} = \infty.$$

Proof. The self-neglecting property (8) implies

$$\lim_{x\to\infty} \frac{\overline{F}(x + a\, e(x))}{\overline{F}(x)} = \mathrm{e}^{-a}$$

(see e.g. [3, p.258]) and we have

$$\begin{aligned} \frac{\overline{F}(x)}{\overline{F}(x - a\, e(x))} &\sim \frac{\overline{F}(x + a\, e(x))}{\overline{F}(x + a\, e(x) - a\, e(x + a\, e(x)))} \\ &\sim \frac{\overline{F}(x + a\, e(x))}{\overline{F}(x)}, \end{aligned}$$

also due to (8). Hence, together with (9),

$$\begin{aligned} \mathbb{P}(X_1 + X_2 > x) &\geq \mathbb{P}(X_1 > x - a\, e(x), X_2 > a\, e(x)) \\ &= \mathbb{P}(X_1 > x - a\, e(x))\, \mathbb{P}(X_2 > a\, e(x) \,|\, X_1 > x - a\, e(x)) \\ &\sim \mathbb{P}(X_1 > x - a\, e(x))\, \mathbb{P}(X_2 > a\, e(x) \,|\, X_1 > x) \\ &\geq \varepsilon\, \mathbb{P}(X_1 > x - a\, e(x)) \\ &\sim \varepsilon\, \mathbb{P}(X_1 > x)\, \mathrm{e}^{a} \end{aligned}$$

for some $\varepsilon > 0$ and any $a > 0$. Hence

$$\liminf_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} \geq \varepsilon\, \mathrm{e}^{a}$$

and the latter is unbounded for $a \to \infty$. □

Remark 2.3. A sufficient condition for (9) to hold is

$$\liminf_{x\to\infty} \mathbb{P}(X_2 > e^*(x) \,|\, X_1 > x) > 0$$

for any $e^*(x)$ with $e^*(x)/e(x) \to \infty$. Condition (8) is satisfied for all standard subexponential distributions with a tail lighter than regularly varying (like the lognormal and the Weibull distribution).

3 Some specific cases

3.1 Regularly varying marginal distribution

3.1.1 An upper bound

Proposition 3.1. *Let* $\overline{F} \in \mathcal{R}_{-\alpha}$ *with* $\alpha > 0$*. Then*

$$\limsup_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} \leq \begin{cases} \left(\lambda^{\frac{1}{\alpha+1}} + (2-2\lambda)^{\frac{1}{\alpha+1}}\right)^{\alpha+1}, & 0 \leq \lambda \leq \frac{2}{3} \\ 2^\alpha(2-\lambda), & \frac{2}{3} < \lambda \leq 1. \end{cases} \tag{10}$$

Proof. For any $0 < \delta < 1/2$ we have

$$\begin{aligned} &\mathbb{P}(X_1 + X_2 > x) \\ &\quad \leq \mathbb{P}(\{X_1 > (1-\delta)x\} \cup \{X_2 > (1-\delta)x\} \cup (\{X_1 > \delta x\} \cap \{X_2 > \delta x\})) \\ &\quad \leq 2\overline{F}((1-\delta)x) + \mathbb{P}(X_1 > \delta x, X_2 > \delta x) - 2\mathbb{P}(X_1 > (1-\delta)x, X_2 > (1-\delta)x) \end{aligned}$$

so that

$$\begin{aligned} &\limsup_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} \\ &\qquad \leq \limsup_{x\to\infty} \left((2-2\lambda)\,\frac{\overline{F}((1-\delta)x)}{\overline{F}(x)} + \frac{\overline{F}(\delta x)}{\overline{F}(x)}\,\mathbb{P}(X_2 > \delta x \,|\, X_1 > \delta x) \right) \\ &\qquad = \frac{2-2\lambda}{(1-\delta)^\alpha} + \frac{\lambda}{\delta^\alpha}. \end{aligned}$$

Within the defined range of δ, this upper bound is minimized for

$$\delta^* = \begin{cases} \dfrac{1}{1 + \left(\frac{2}{\lambda} - 2\right)^{\frac{1}{\alpha+1}}}, & 0 \leq \lambda \leq \frac{2}{3} \\ \dfrac{1}{2}, & \frac{2}{3} < \lambda \leq 1, \end{cases}$$

which yields (10). □

Note that this upper bound is sharp for both independence and comonotone dependence. In particular, together with assertion (a) of Lemma 2.1, we obtain

Corollary 3.2. *If* $F \in \mathcal{R}$ *and* $\lambda = 0$*, then* $\mathbb{P}(X_1 + X_2 > x) \sim 2\overline{F}(x)$*.*

Thus for regularly varying tails of the marginals, tail independence suffices to guarantee that the tail of the dependent sum behaves asymptotically as if X_1 and X_2 were independent. From the proof of Proposition 3.1, it becomes clear that this also holds true for any $F \in \mathcal{S}$ with heavier tail than regularly varying. On the other hand, for light-tailed distributions tail independence clearly does not imply such an insensitivity (for instance, consider a bivariate normal distribution, where the dependence is described by the (tail independent) Gaussian copula; in this case, the variance of the sum is a function of the correlation coefficient ρ and the value of

ρ does affect the asymptotic behavior of the sum). This gives rise to the question of "how heavy" the marginal tails have to be in order to dominate the "dependence effect" in the tail of the sum, given $\lambda = 0$. In Section 3.2 it will be shown that lognormal marginals are not a sufficient condition to that end.

For fixed marginals, it was already pointed out by Denuit *et al.* [8] that, unlike the case of stop-loss premiums, the comonotone dependence structure does not always provide an extremal case for the asymptotic behavior of the sum of the tail. The following simple example demonstrates this fact:

Example 3.1. Let $\overline{F} \in \mathcal{R}_{-\alpha}$ with $\alpha > 0$. Then for independence between X_1 and X_2, by standard subexponential theory, $\lim_{x\to\infty} \mathbb{P}(X_1 + X_2 > x)/\overline{F}(x) = 2$. On the other hand, for comonotone X_1 and X_2 (which due to identical marginals is equivalent to $X_1 = X_2$ a.s.), we have $\lim_{x\to\infty} \mathbb{P}(X_1 + X_2 > x)/\overline{F}(x) = 2^\alpha$. Thus, for $\alpha < 1$ the comonotone case does not provide an upper bound.

Intuitively, if the marginal distribution tail is heavy enough, then the two random sources for a possibility of a large sum caused by one of the summands outweighs the effect of summing two large components from one random source.

3.1.2 Multivariate regularly varying tails

A well-known specific way to couple regularly varying marginals is by multivariate regular variation. In our bivariate setting it can be defined as follows: The vector $\mathbf{X} = (X_1, X_2)$ is regularly varying with index $-\alpha < 0$, if there exists a probability measure S on $\mathbb{S}^1$ (the unit sphere in $\mathbb{R}^2$ with respect to the Euclidean norm $|\cdot|$) such that for all $t > 0$

$$\frac{\mathbb{P}(|\mathbf{X}| > tu, \mathbf{X}/|\mathbf{X}| \in \cdot)}{\mathbb{P}(|\mathbf{X}| > u)} \stackrel{v}{\to} t^{-\alpha}\, S(\cdot) \quad \text{as } u \to \infty,$$

where $\stackrel{v}{\to}$ stands for vague convergence in $\mathbb{S}^1$ (see for instance Resnick [24]). S is often referred to as the *spectral measure* of $\mathbf{X}$.
With positive random variables X_1, X_2, an equivalent formulation is that there exists a probability measure $S(\cdot)$ on $\mathbb{S}^1_+$ (the restriction of $\mathbb{S}^1$ to the first quadrant) and a function $b(x) \to \infty$ such that

$$b^{-1}(x)\, \mathbb{P}\left(\left(\frac{|\mathbf{X}|}{x}, \frac{\mathbf{X}}{|\mathbf{X}|}\right) \in \cdot\right) \stackrel{v}{\to} c\, \nu_\alpha \times S \tag{11}$$

in the space of positive Radon measures on $\big((0,\infty] \times \mathbb{S}^1_+\big)$, where $c > 0$ and $\nu_\alpha(t,\infty] = t^{-\alpha}$, $(t > 0, \alpha > 0)$ (cf. Resnick [23]).
The above implies in particular that on every ray from (0,0) into the positive quadrant, we have a regularly varying tail with index $-\alpha$. Moreover, the tail of $X_1 + X_2$ is also regularly varying with the same index (in fact the relationship between regular variation of $\mathbf{X} = (X_1, X_2)$ and one-dimensional regular variation of linear combinations of its components is much deeper, see Basrak *et al.* [5]).
So for this specific dependence structure among regularly varying marginals, the asymptotic behavior of the sum can be given explicitly. To that end, considering in

(11) the events $|\mathbf{X}|/x > t$ for $t = \frac{1}{\cos\varphi + \sin\varphi}$ and $t = \frac{1}{\cos\varphi}$, with $\varphi \in [0, \pi/2]$ denoting the angle corresponding to $\mathbf{X}/|\mathbf{X}|$, we obtain

$$b^{-1}(x)\,\mathbb{P}(X_1 + X_2 > x) \to c \int_0^{\pi/2} (\cos\varphi + \sin\varphi)^\alpha \, S(d\varphi)$$

(where in an obvious way we have identified $\mathbb{S}^1_+$ with $[0, \pi/2]$) and

$$b^{-1}(x)\,\mathbb{P}(X_1 > x) \to c \int_0^{\pi/2} \cos^\alpha \varphi \, S(d\varphi),$$

so that

$$\mathbb{P}(X_1 + X_2 > x) \sim \overline{F}(x)\, \frac{\int_0^{\pi/2} (\cos\varphi + \sin\varphi)^\alpha \, S(d\varphi)}{\int_0^{\pi/2} \cos^\alpha \varphi \, S(d\varphi)}.$$

Since X_1 and X_2 are assumed to be exchangable, we have $S(d\varphi) = S(d(\pi/2 - \varphi))$ and hence

$$\mathbb{P}(X_1 + X_2 > x) \sim 2\,\overline{F}(x)\, \frac{\int_0^{\pi/2} (\cos\varphi + \sin\varphi)^\alpha \, S(d\varphi)}{\int_0^{\pi/2} (\cos^\alpha \varphi + \sin^\alpha \varphi) \, S(d\varphi)}.$$

In particular, the quotient on the right hand side is larger than 1 for $\alpha > 1$, smaller than 1 for $\alpha < 1$ and equal to 1 for $\alpha = 1$ (irrespective of the value of λ). The comonotone case is retrieved when S is concentrated at $\varphi = \pi/4$ which indeed gives $\mathbb{P}(X_1 + X_2 > x) \sim 2^\alpha \overline{F}(x)$. For asymptotic independence, S is concentrated on the two axes, so that $\mathbb{P}(X_1 + X_2 > x) \sim 2\,\overline{F}(x)$. A natural extremal dependence measure in this setting is

$$\rho := 1 - \frac{1}{(\pi/4)^2} \int_0^{\pi/2} \left(\varphi - \frac{\pi}{4}\right)^2 S(d\varphi),$$

see Resnick [23]. Finally, the tail dependence coefficient λ as defined in Section 1 can in this case be obtained by considering the event $|\mathbf{X}|/x > t$ for $t = \frac{1}{\min\{\cos\varphi, \sin\varphi\}}$ in (11), yielding

$$\lambda = \lim_{x \to \infty} \frac{\mathbb{P}(X_1 > x, X_2 > x)}{\mathbb{P}(X_1 > x)} = \frac{2 \int_0^{\pi/4} \sin^\alpha \varphi \, S(d\varphi)}{\int_0^{\pi/4} (\sin^\alpha \varphi + \cos^\alpha \varphi) \, S(d\varphi)}.$$

Under further restrictions on the shape of $\mathbf{X}$, the spectral measure S may be explicitly computable (for instance, in case of elliptical distributions with regularly varying tail, see Hult & Lindskog [13]; however, the latter class is not relevant for the present purpose due to our restriction to positive random variables).

Remark 3.1. While in this specific setting, clearly λ is a rougher measure for dependence in the tail than ρ, both measures identify the same distributions as asymptotically independent, i.e. $\rho = \lambda = 0$. In the latter case there are refinements for the study of multivariate regularly varying distributions available, cf. Resnick [22].

3.2 Lognormal marginal distribution

Asmussen & Rojas-Nandayapa [4] considered $X_1 + \cdots + X_n$ where $X_1, \ldots, X_n$ are lognormal with a multivariate Gaussian copula. That is, $X_i = \mathrm{e}^{Y_i}$ where $Y_1, \ldots, Y_n$ are jointly multivariate Gaussian(μ, Σ) for some mean vector μ and some covariance matrix Σ; exchangability is not required. Their results state that the tail of the sum is asymptotically the same as for the independent case $\Sigma = (\sigma_i^2)_{\mathrm{diag}}$. When specialized to the present setting, this means:

Proposition 3.3. *Let X_1, X_2 be bivariate normal with the same mean μ, the same variance σ^2 and covariance $\rho \in [-1, 1)$. Then*

$$\mathbb{P}(X_1 + X_2 > x) \sim 2\,\mathbb{P}(X_1 > x) \;\sim\; \frac{\sqrt{2/\pi}}{\sigma \log x} \exp\left\{-(\log x - \mu)^2/2\sigma^2\right\}.$$

A short heuristical argument (different from the rigorous, more technical proof of [4]) supporting this result goes as follows. We take $\mu = 0$, $\sigma^2 = 1$, $\rho > 0$ for simplicity. Then we can write

$$Y_1 \;=\; U + V_1\,, \quad Y_2 \;=\; U + V_2\,,$$

where U, V_1, V_2 are independent univariate Gaussian with mean zero and variances a^2, b^2, b^2, respectively, where $a^2 + b^2 = 1$, $a^2 = \rho$. Given $U = u$, X_1 and X_2 are independent lognormals with log-variance b^2, so by subexponential limit theory

$$\begin{aligned}\mathbb{P}\big(X_1 + X_2 > x \,\big|\, U = u\big) &= \mathbb{P}(\mathrm{e}^{V_1} + \mathrm{e}^{V_2} > x\mathrm{e}^{-u}) \\ &\sim \frac{\sqrt{2/\pi}}{b(\log x - u)} \exp\left\{-(\log x - u)^2/2b^2\right\}.\end{aligned}$$

We make the guess

$$\mathbb{P}(X_1 + X_2 > x) \;\approx\; \max_u \frac{1}{a\sqrt{2\pi}} \mathrm{e}^{-u^2/2a^2} \mathbb{P}\big(X_1 + X_2 > x \,\big|\, U = u\big) \tag{12}$$

and ignore everything not in the exponent and constants. Then we have to find the u minimizing

$$\frac{u^2}{2a^2} - \frac{u \log x}{b^2} + \frac{u^2}{2b^2}$$

which (using $a^2 + b^2 = 1$) is easily seen to be $u = a^2 \log x$. Substituting back in (12), we get

$$\begin{aligned}\mathbb{P}(X_1 + X_2 > x) &\approx \exp\left\{-a^4 \log^2 x/2a^2 \;-\; (1 - a^2)^2 \log^2 x/2b^2\right\} \\ &= \exp\left\{-\log^2 x/2\right\} \qquad (13)\end{aligned}$$

in agreement with Proposition 3.3 (we have used $\approx$ to indicate aymptotics at a rough level, that is, rougher than $\sim$ or even logarithmic asymptotics as used in large deviations theory).

Note that the argument contains some information on how $X_1 + X_2$ exceeds x: U must be approximately $u = a^2 \log x = \rho \log x$ and either V_1 or V_2 but not both large. Translated back to X_1, X_2, this means that one is larger than x and the other of order $\mathrm{e}^u = x^\rho$.

The above proposition provides an example of lognormal marginals and tail independence (through the Gaussian copula), in which the tail asymptotics of the sum are insensitive to increasing dependence. Following the discussion in Section 3.1, this raises the question whether lognormal marginals are "heavy enough" so that tail independence implies this insensitivity in general (as was the case for regularly varying marginals). The following counter-example shows that this is actually not the case and, more than that, despite tail independence the tail asymptotics of the sum may differ substantially from the independent case:

Proposition 3.4. *There exists a tail-independent exchangable random vector* (X_1, X_2) *with lognormal marginals and*

$$\liminf_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} = \infty.$$

Lemma 3.5. *There exists a tail-independent exchangable random vector* (Y_1, Y_2) *with standard normal marginals and* $\left|Y_1 - Y_2\right| = c$ *whenever* $Y_1 + Y_2 > y_0$ *for a given* $c > 0$ *and* $y_0 > 0$.

Proof. For $y_1 + y_2 < 0$ simply define the joint distribution as the restriction of the bivariate standard normal distribution with independent marginals to $\{y_1 + y_2 < 0\}$. For $y_1 + y_2 > 0$, let $f(y)$ denote the density of $Y_1 + Y_2$. The problem is to determine f such that

$$\int_0^\infty f(y)\,dy = \frac{1}{2} \tag{14}$$

and

$$\varphi(y) = \tfrac{1}{2} f(y-c) + \tfrac{1}{2} f(y+c)\,, \tag{15}$$

where $\varphi(y)$ denotes the density of the standard normal distribution. Let us rewrite

$$f(y) \;=\; \varphi(y)\mathrm{e}^{-cy} g_1(y) \tag{16}$$

for some function $g_1(y)$. Using $\varphi(y+d) = \varphi(y)\mathrm{e}^{-d^2/2-dy}$, (15) then becomes

$$1 = \tfrac{1}{2}\mathrm{e}^{-c^2/2}\left[\mathrm{e}^{cy}\mathrm{e}^{-c(y-c)} g_1(y-c) + \mathrm{e}^{-cy}\mathrm{e}^{-c(y+c)} g_1(y+c)\right],$$

that is

$$2\mathrm{e}^{-c^2/2} \;=\; g_1(y-c) \;+ \mathrm{e}^{-2c^2}\,\mathrm{e}^{-2cy} g_1(y+c)\,.$$

Trying the solution $g_1(y) = \sum_{n=0}^\infty r_n \mathrm{e}^{-2ncy}$, we obtain

$$\begin{aligned} 2\mathrm{e}^{-c^2/2} &= \sum_{n=0}^\infty r_n \mathrm{e}^{-2nc(y-c)} + \mathrm{e}^{-2c^2}\,\mathrm{e}^{-2cy}\sum_{n=0}^\infty r_n \mathrm{e}^{-2nc(y+c)} \\ &= \sum_{n=0}^\infty r_n \mathrm{e}^{2nc^2}\mathrm{e}^{-2ncy} + \mathrm{e}^{-2c^2}\sum_{n=1}^\infty r_{n-1}\mathrm{e}^{-2(n-1)c^2}\mathrm{e}^{-2ncy}. \end{aligned}$$

Identifying coefficients yields $r_0 = 2\mathrm{e}^{-c^2/2}$ and

$$r_n \;=\; -\mathrm{e}^{-4nc^2} r_{n-1}, \quad n \geq 1,$$

leading to

$$r_n = (-1)^n 2\mathrm{e}^{-2c^2(n+1)n - c^2/2}, \quad n \geq 0.$$

Hence

$$g_1(y) = 2\mathrm{e}^{-c^2/2} \sum_{n=0}^{\infty} (-1)^n \, \mathrm{e}^{-2c^2(n+1)n} \, \mathrm{e}^{-2ncy},$$

which is a convergent series for every $y \geq 0$, since it is alternating with coefficients decreasing to zero monotonically. Moreover, $\lim_{y\to\infty} g_1(y) = 2\mathrm{e}^{-c^2/2}$, so that $\int_0^\infty \varphi(y)\mathrm{e}^{-cy} g_1(y)\, dy < \infty$ and the integrand can be normalized in such a way that (14) holds. Finally, from (16) we see that $f(y+c) = o\,(f(y-c))$ as $y \to \infty$ and thus $\lambda = \lim_{y\to\infty} \mathbb{P}(Y_2 > y | Y_1 > y) = 0$. □

Proof of Proposition 3.4: Since the copula of a bivariate distribution stays invariant under strictly increasing transformations of the marginals and the tail dependence coefficient is a function of the copula only, Lemma 3.5 can be carried over to the random vector $(X_1, X_2) = (\mathrm{e}^{Y_1}, \mathrm{e}^{Y_2})$ with lognormal marginals. In particular, for large x we then either have $X_1 = X_2\, \mathrm{e}^c$ or $X_2 = X_1\, \mathrm{e}^c$. Hence

$$\begin{aligned} \mathbb{P}(X_1 + X_2 > x) &\sim \mathbb{P}(X_1 + X_2 > x, X_1 = X_2\, \mathrm{e}^c) + \mathbb{P}(X_1 + X_2 > x, X_2 = X_1\, \mathrm{e}^c) \\ &= 2\, \mathbb{P}\left(X_1 > x/(1 + e^c)\right). \end{aligned}$$

As for a lognormal random variable $X_1 = e^{Y_1}$ with $Y_1 \sim N(0,1)$, the tail is asymptotically

$$\overline{F}(x) \sim \frac{1}{\sqrt{2\pi}\log x}\, \mathrm{e}^{-\log^2 x},$$

the assertion follows. □

3.3 Archimedean copulae

Archimedean copulae are of the form

$$C(a,b) = \phi^{[-1]}(\phi(a) + \phi(b)), \quad 0 \leq a, b \leq 1, \tag{17}$$

where the generator $\phi(t)$ is a continuous, convex and strictly decreasing function from [0,1] to $[0,\infty]$ such that $\phi(1) = 0$ and $\phi^{[-1]}$ denotes the pseudo-inverse of ϕ defined by

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t), & 0 \leq t \leq \phi(0), \\ 0, & \phi(0) \leq t \leq \infty. \end{cases}$$

If $\phi(0) = \infty$, then ϕ is called a *strict* generator.

Proposition 3.6. *Let $F \in \mathcal{S}$ with X_1 and X_2 being dependent according to an Archimedean copula with generator ϕ. Then*

$$\liminf_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} \geq 2 - \frac{\lambda}{2} \tag{18}$$

Proof. Definition (17) implies

$$c_b(a,b) = \frac{\phi'(b)}{\phi'(\phi^{[-1]}(\phi(a)+\phi(b)))}.$$

From Corollary 2.4, we have the lower bound

$$\liminf_{x\to\infty} \frac{\mathbb{P}(X_1+X_2>x)}{\overline{F}(x)} \geq 2 - \min\left\{\lambda,\ \frac{\lambda}{2} + \left|\frac{\phi'(1)}{\phi'(0)}\right|\right\}$$

with tail dependence coefficient

$$\lambda = 2 - 2\lim_{u\to 1} \frac{\phi'(u)}{\phi'(\phi^{-1}(2\phi(u)))}. \tag{19}$$

Now $\lambda = 0$ unless $\phi'(1) = 0$, in which case the minimum is attained for the second term. □

Remark 3.2. The above implies that for Archimedean copulae the distribution of the sum is not determined by the distribution of the maximum, if $\lambda > 0$.

Example 3.2. Consider the generator $\phi(t) = \log(1-\theta\log t)$, where $\theta \in (0,1]$ is a dependence parameter (with the limiting case $\theta = 0$ representing independence). The copulae in this family are usually referred to as Gumbel-Barnett copulae, see for instance Nelsen [20, p.97]. From (19) we see that $\lambda = 0$ and hence $\liminf \frac{\mathbb{P}(X_1+X_2>x)}{\overline{F}(x)} \geq 2$. Condition (6) does not apply; however, a direct evaluation of (3), which in this case translates into

$$\lim \frac{\mathbb{P}(X_1+X_2>x)}{\overline{F}(x)} = 1 + \lim_{x\to\infty} \int_0^x \frac{1 - F(x-z)^{1-\theta\log F(z)}(1-\theta\log F(x-z))}{\overline{F}(x)}\, F(dz), \tag{20}$$

shows that indeed the above limit is 2 for any $\theta \in (0,1]$ and any $F \in \mathcal{S}$:
For that purpose, consider the decomposition of the above integral

$$\int_0^x = \int_0^{c_0} + \int_{c_0}^{x-c_1} + \int_{x-c_1}^x = A_1(x) + A_2(x) + A_3(x)$$

for some constants $c_0, c_1 > 0$. For $z \in [0, c_0]$, we have $F(x-z)^{1-\theta\log F(z)} \sim 1 - (1-\theta\log F(z))\,\overline{F}(x-z)$ and $(1-\theta\log F(x-z)) \sim (1+\theta\overline{F}(x-z))$ as $x\to\infty$. Together with $\frac{\overline{F}(x-z)}{\overline{F}(x)} \leq \frac{\overline{F}(x-c_0)}{\overline{F}(x)}$ and $\overline{F}^{\,2}(x-z) = o(\overline{F}(x-z))$, this shows that the integrand in $A_1(x)$ is upper-bounded by $c_2(1-\theta-\theta\log F(z))$ for c_2 chosen sufficiently large. Since the latter is integrable w.r.t. $F(dz)$, dominated convergence applies to $A_1(x)$. For $z \in [c_0, x-c_1]$, one has

$$F(x-z)^{1-\theta\log F(z)} \geq \mathrm{e}^{-(1-\theta\log F(z))\overline{F}(x-z)} \geq 1 - (1-\theta\log F(z))\overline{F}(x-z)$$

and hence

$$A_2(x) \leq (1-\theta\log F(c_0))\,(1-\theta\log F(c_1)) \int_{c_0}^{x-c_1} \frac{\overline{F}(x-z)}{\overline{F}(x)}\, F(dz).$$

But the latter integral is bounded, and thus interchanging limits is justified by Pratt's Lemma. Finally,

$$A_3(x) \leq \int_{x-c_1}^{x} \frac{F(dz)}{\overline{F}(x)} = \frac{\overline{F}(x-c_1) - \overline{F}(x)}{\overline{F}(x)} \to 0.$$

Hence one can interchange limits in (20), leading to $\lim_{x\to\infty} \frac{\mathbb{P}(X_1+X_2>x)}{\overline{F}(x)} = 2$. □

Recall the definition of a survival copula

$$\widehat{C}(a,b) = a + b - 1 + C(1-a, 1-b), \quad 0 \leq a, b \leq 1,$$

corresponding to the copula $C(a,b)$ (cf. [20]). $\widehat{C}(a,b)$ is itself a copula and exchanges the role of upper and lower tails. Representation (3) can then be rewritten in the form

$$\frac{\mathbb{P}(X_1+X_2>x)}{\overline{F}(x)} = 1 + \int_0^x \frac{\widehat{c}_a(\overline{F}(z), \overline{F}(x-z))}{\overline{F}(x)} F(dz). \tag{21}$$

For survival copulae of certain Archimedean type, Alink *et al.* [2] recently derived the following remarkable explicit result:

Proposition 3.7 (Alink *et al.* 2004)**.** *Let the survival copula be Archimedean with generator $\widehat{\phi}$ regularly varying at 0^+ with index $-\alpha < 0$, and let Y_α denote a positive random variable with density $f_\alpha(y) = (1+y^\alpha)^{-1/\alpha-1}$.*

(a) If $\overline{F} \in \mathcal{R}_{-\beta}$ with $\beta > 0$, then

$$\mathbb{P}(X_1+X_2>x) \sim \left(1 + \mathbb{E}\left(1 + Y_\alpha^{-1/\beta}\right)^{\beta-1}\right) \overline{F}(x).$$

(b) If $F \in \mathcal{S}$ and for any $a \in \mathbb{R}$ the relation

$$\lim_{x\to\infty} \overline{F}(x + a\,e(x))/\overline{F}(x) = \mathrm{e}^{-a} \tag{22}$$

holds, where $e(x)$ is the mean excess function corresponding to F, then

$$\mathbb{P}(X_1+X_2>x) \sim \frac{\Gamma^2\left(1+\frac{1}{2\alpha}\right)}{\Gamma\left(1+\frac{1}{\alpha}\right)} \overline{F}\left(\frac{x}{2}\right). \tag{23}$$

Remark 3.3. The assumptions on the generator in the above proposition enforce a strictly positive tail dependence coefficient. More explicitly,

$$\lambda = \lim_{u\to 1} \frac{\widehat{C}(1-u,1-u)}{1-u} = 2\lim_{u\to 0} \widehat{c}_a(u,u) = 2\lim_{u\to 0} \frac{\widehat{\phi}'(u)}{\widehat{\phi}'(\widehat{\phi}^{-1}(2\widehat{\phi}(u)))} = 2^{-1/\alpha}. \tag{24}$$

Remark 3.4. Assumption (22) is equivalent to $F \in$ MDA(Gumbel) (i.e. F is in the maximum domain of attraction of the Gumbel distribution, cf. Embrechts *et al.* [9] and also Section 2.2). Hence the conditions in assertion (b) are in particular fulfilled for the lognormal and the Weibull distribution with parameter $\tau < 1$. The constant from (23) (which increases to 1 as $\alpha \to \infty$) can be compared with the trivial upper bound from (2) in view of (24), cf. Figure 2. In particular, it becomes visible that, roughly, for large α (e.g. strong dependence among X_1 and X_2), the dominating contribution for the sum to be large comes from both variables being large.

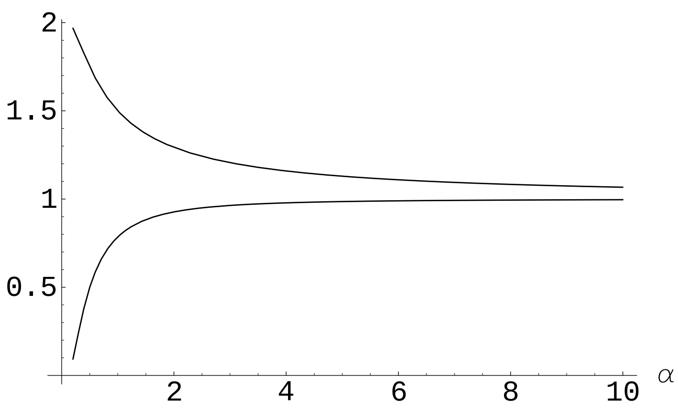

Figure 2: $F \in \mathrm{MDA(Gumbel)}$: exact value from (23) vs. trivial upper bound

Remark 3.5. Assertion (a) above can be rewritten as

$$\mathbb{P}(X_1 + X_2 > x) \sim 2^{-\beta} \left(1 + \int_0^\infty \left(1 + y^{-1/\beta}\right)^{\beta - 1} (1 + y^\alpha)^{-1-1/\alpha} \, dy \right) \overline{F} \left(\frac{x}{2} \right),$$

which is monotonically decreasing in β and converges to (23) for $\beta \to \infty$ (note that $\lim_{\beta \to \infty} 2^{-\beta} \left(1 + y^{-1/\beta}\right)^{\beta-1} = \frac{1}{2\sqrt{y}}$). Figure 3 illustrates that already for values of β around 10, the asymptotic behavior of the regularly varying case and the one of the Gumbel case are almost indistinguishable.

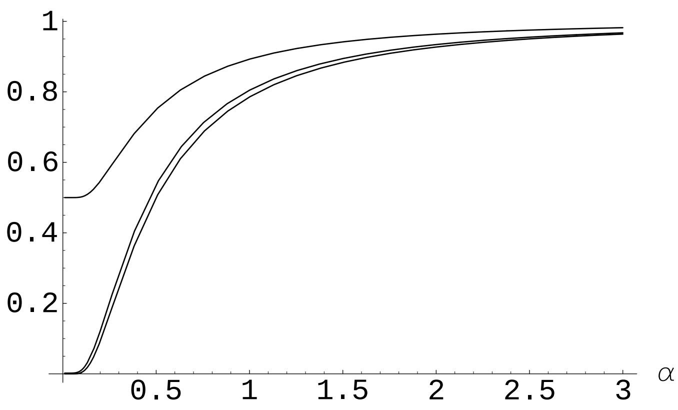

Figure 3: Comparison of constants: $\overline{F} \in \mathcal{R}_{-\alpha}$ with $\alpha = 2$ and $\alpha = 10$ and $F \in$ MDA(Gumbel) (from top to bottom)

3.4 Farlie-Gumbel-Morgenstern copula

This family of copulae is defined by

$$C(a, b) = a\, b\, (1 + 3\rho_S (1 - a)(1 - b)), \quad -1/3 \le \rho_S \le 1/3,$$

where ρ_S denotes Spearman's rank correlation coefficient. Here $c_{ab}(a, b) = 1 + 3\rho_S\,(1 - 2a)\,(1 - 2b)$ and Proposition 2.5 applies for $\rho_S < 1/3$ giving $\mathbb{P}(X_1 + X_2 >$

$x) \sim 2\overline{F}(x)$. In this case, a direct proof for this fact can also be given which additionally covers the extremal case $\rho_S = 1/3$.

Proposition 3.8. *Let subexponential random variables X_1 and X_2 be dependent according to a Farlie-Gumbel-Morgenstern copula. Then*

$$\mathbb{P}(X_1 + X_2 > x) \sim 2\overline{F}(x).$$

Proof. From $c_a(a,b) = b + 3\rho_S\, b\,(1-b)(1-2a)$, one obtains from Proposition 2.2

$$\frac{\mathbb{P}(X_1 + X_2 > x)}{\overline{F}(x)} = 1 + B_1 + B_2,$$

with

$$B_1 = \int_0^x \frac{\overline{F}(x-z)}{\overline{F}(x)}\, F(dz)$$

and

$$B_2 = -3\rho_S \int_0^x \frac{F(x-z)\,\overline{F}(x-z)\,(1-2F(z))}{\overline{F}(x)}\, F(dz).$$

Now, due to subexponentiality,

$$\mathbb{P}(X_1 + X_2 > x, X_1 \le A) = \int_0^A \overline{F}(x-z)F(dz) \sim \overline{F}(x)\, F(A)$$

for any fixed $A > 0$, and hence

$$\lim_{A\to\infty} \limsup_{x\to\infty} \frac{1}{\overline{F}(x)} \int_A^x \overline{F}(x-z)F(dz) = 0. \tag{25}$$

Since the remaining factors in the integral are bounded, we have

$$\begin{aligned} B_2 &= -3\rho_S \lim_{A\to\infty} \lim_{x\to\infty} \int_0^A \frac{F(x-z)\overline{F}(x-z)(1-2F(z))\, F(dz)}{\overline{F}(x)} \\ &\sim -3\rho_S \int_0^\infty (1-2F(z))\, F(dz) = -3\rho_S \left(2\int_0^\infty \overline{F}(z)\, F(dz) - 1\right) = 0. \end{aligned}$$

At the same time, it also follows from (25) and dominated convergence that for subexponential marginals we have $B_1 \sim 1$. □

Remark 3.6. Note that the Farlie-Gumbel-Morgenstern copula is tail-independent, providing another example of a dependence structure, for which the first order tail asymptotics of the sum is insensitive to the degree of dependence irrespective of the heaviness of the marginal tails, as long as $F \in \mathcal{S}$.

3.5 Linear Spearman copula

Finally, we briefly mention the simple case of convex combinations of independence and comonotone dependence, which admits an explicit solution as well. The positive linear Spearman copula is defined by

$$C(a,b) = \lambda \min(a,b) + (1-\lambda)\, a\, b, \quad 0 \le a, b \le 1,$$

where $\lambda \in [0, 1]$. It is easy to see that λ is indeed the tail dependence coefficient. Assuming $F \in \mathcal{S}$, we get

$$\begin{aligned}\mathbb{P}(X_1 + X_2 > x) &= \lambda \mathbb{P}(X_1 + X_2 > x | C_M) + (1 - \lambda)\mathbb{P}(X_1 + X_2 > x | C_I) \\ &\sim \lambda \overline{F}(x/2) + 2(1 - \lambda)\overline{F}(x).\end{aligned}$$

For $\overline{F} \in \mathcal{R}_{-\alpha}$ with $\alpha > 0$, we obtain $\mathbb{P}(X_1 + X_2 > x) \sim (2 - 2\lambda + \lambda\, 2^\alpha)\overline{F}(x)$. In particular, for $\alpha = 1$ the tail of the sum is asymptotically equivalent to the independent sum for all $\lambda \in [0, 1]$ (a comparison with Proposition 3.1 shows that in this example the upper bound (10) is quite rough for larger values of α). On the other hand, for distributions with $\overline{F}(x) = o(\overline{F}(x/2))$, $\mathbb{P}(X_1 + X_2 > x) \sim \lambda \overline{F}(x/2)$ scales with the tail dependence coefficient λ.

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