

DEGREE OF A ISOLATED REAL POINT OR A SINGULAR COMPLEX POINT ON A PLANE CURVE DEFINED OVER \mathbb{Q}

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ABSTRACT. Let X be a curve in the affine plane defined by a reduced polynomial of degree d with rational coefficients. Assume that P is an isolated real point or a singular complex point on the curve X . The coordinates of P are algebraic numbers over the rationals of degree at most d^2 .

The result in the complex case is obtained by elimination theory and the use of resultants. The result in the real case follows from the complex case using analytic uniformization.

This result in the real case has applications in the theory of games, see [2].

Theorem 1. *Let X be an affine algebraic set in the affine plane defined by a reduced polynomial $f(X, Y) \in \mathbb{Q}[X, Y]$ of degree d . Assume that $P \in X(\mathbb{R}) \subset \mathbb{A}^2(\mathbb{R})$ is an isolated real point. The coordinates of P are algebraic over \mathbb{Q} of degree at most d^2 .*

In [2], Theorem 21 shows that $(2d + 1)^2$ is an upper bound for the algebraic degree of the coordinates. This result is obtained by different methods.

The proof is presented at the end of this paper.

Theorem 2. *Let X be an affine algebraic set in the affine plane defined by a reduced polynomial $f(X, Y) \in \mathbb{Q}[X, Y]$ of degree d . Assume that $P \in X(\mathbb{C}) \subset \mathbb{A}(\mathbb{C})$ is a singular complex point. The coordinates of P are algebraic over \mathbb{Q} of degree at most d^2 .*

Recall that a polynomial f is said to be reduced (or square-free) if it can be factored in a product of distinct irreducible polynomials:

$$f = g_1 \cdots g_r, \quad g_i \neq g_j \text{ for } i \neq j$$

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FiXme Note: Kan anvendelser i game-theory uddybes, evt. med referencer? Peter Bro Miltersen

FiXme Note: Uddyb begreberne inden præsentation af sætningerne?

where the empty spaces are filled by zeros. The resultant has the following properties:

- i) $\text{Res}(f, g, Y)(X) \in k[X]$ is of degree at most $l \cdot m$, see [1], Chap. 8, §7, Lemma 5.
- ii) $\text{Res}(f, g, Y)(X) = 0$ if and only if f and g have a common factor in $k[X, Y]$ which has positive degree in Y , see [1], Chap. 6, §6, Proposition 1.
- iii) There are polynomials $a, b \in k[X, Y]$ such that

$$\text{Res}(f, g, Y)(X) = a(X, Y)f(X, Y) + b(X, Y)g(X, Y),$$

see [1], Chap. 3, §5, Proposition 9.

Proof. (Theorem 2) We prove the claim for the coordinate x , the proof for y is entirely similar. Assume Y is an irreducible factor in f :

$$f = g_1 \cdots g_{r-1} \cdot Y.$$

As for the singular loci we have

$$\text{Sing}(V(f)) = \text{Sing}(V(g_1 \cdots g_{r-1})) \cup (V(g_1 \cdots g_{r-1}) \cap V(Y)).$$

A point $P \in V(g_1 \cdots g_{r-1}) \cap V(Y)$ has coordinates $(x, 0)$, where x is a root in the univariate polynomial $g_1(X) \cdots g_{r-1}(X) \in \mathbb{Q}[X]$ of degree at most $d - 1$, so the claim is proved for such points.

We are therefore reduced to prove the theorem in the case where Y is not a factor in f . In that case X appears in all the irreducible factors g_1, \dots, g_r .

By Lemma 3 the polynomials f and $\frac{\partial f}{\partial X} \in k[X, Y]$ have no non-trivial common factor. Consider the resultant

$$R(X) = \text{Res}(f(X, Y), \frac{\partial f}{\partial X}(X, Y), Y)(X) \in \mathbb{Q}[X]$$

By Remark 5 we have that $R(X) \neq 0$, $\deg R(X) \leq d^2$ and there exist polynomials $a, b \in \mathbb{Q}[X, Y]$ such that

$$(1) \quad R(X) = a(X, Y)f(X, Y) + b(X, Y)\frac{\partial f}{\partial X}(X, Y)$$

Substituting the coordinates (x, y) in (1) above, we obtain

$$R(x) = a(x, y)f(x, y) + b(x, y)\frac{\partial f}{\partial X}(x, y) = 0,$$

as (x, y) is a singular point on $V(f)$ and we conclude that x is algebraic over \mathbb{Q} of degree at most d^2 . □

Proposition 6. *Let $X \subset \mathbb{A}^n$ be an affine algebraic set in the affine n -space defined by an ideal $I \subset \mathbb{Q}[X_1, \dots, X_n]$ generated by polynomials with coefficients in \mathbb{Q} . Let $P \in X(\mathbb{R})$ be a real point in $X(\mathbb{R})$, that is isolated in the classical topology. Then P is a singular point on $X(\mathbb{C})$.*

Proof. Assume on the contrary that $P = (a_1, \dots, a_n)$ is a smooth point on $X(\mathbb{C})$ of dimension r . We apply *analytic uniformization*, see [4], (1.26) to obtain a contradiction.

Let x_{i_1}, \dots, x_{i_r} be any sets of coordinates such that $dx_{i_1}, \dots, dx_{i_r}$ are independent linear functions on the tangent space T_P . Then there is an $\varepsilon > 0$ and uniquely determined convergent power series $g_j(x_{i_1}, \dots, x_{i_r})$, for all $j \in \{1, \dots, n\} - \{i_1, \dots, i_r\}$, such that if $|t_1|, \dots, |t_n| < \varepsilon$.

$$(a_1 + t_1, \dots, a_n + t_n) \in X \Leftrightarrow t_j = g_j(t_{i_1}, \dots, t_{i_r})$$

As X is defined over \mathbb{Q} and in particular over \mathbb{R} , the powerseries g_j have (by uniqueness) real coefficients. In conclusion, $X(\mathbb{R})$ can locally around P be parametrized by real convergent power series. This contradicts that P is isolated in $X(\mathbb{R})$. \square

Proof. (Theorem 1). The claim follows directly from Proposition 6 and Theorem 2. \square

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