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Ruin Problems and Tail Asymptotics

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Contents

Preface & acknowledgments

This thesis constitutes the result of my PhD studies at the department of mathematical Sciences, Aarhus University. My PhD studies were carried out during Feb. 2007 - Feb 2010 and have been conducted under the careful supervision of Søren Asmussen, to whom I owe many thanks for his advice, both that of mathematical nature, and regarding more general questions on scientific research. I will also thank Martin Jacobsen from the University of Copenhagen, who was my supervisor at my master's degree thesis. He has been very helpful and given many cheerful comments – especially in the work with the papers A and B.
Finally, I would like to thank all my colleagues at the Department of Mathematics, in particular my office-mates Elisabeth and Linda. Also a special thank to Sarah Bentzen, who has helped with the spell checking.

Introduction

1 Summary

This thesis contains a short introductory section followed by four papers. The introduction presents the topics and main results of the papers. The papers A, B and D concerns risk theory. In A and B premium depending risk processes are studied, while the subject in Paper D is an additive model in a Markovian environment. Paper C is about failure recovery via RESTART.

Paper A A certain class of diffusions with jumps is considered. Between jumps the process behaves like a Cox–Ingersoll–Ross process driven by a Brownian motion independent of the jump process. The jumps arrive with exponential waiting times and are allowed to be two–sided. The jumps that are mixtures of exponential distributions are assumed to form an independent, identically sequence, independent of everything else. The fact that downward jumps are always allowed makes passage of a given lower level possible both by continuity and by a jump. In case of passage by a jump the resulting undershoot is considered too. As one of the main results of the paper, the joint Laplace transform of the first passage time and the undershoot is determined. So is the probability of passage in finite time. Both the joint Laplace transform and the probability of passage in finite time are decomposed according to the type of passage: Jump or continuity.

Paper B A class of Ornstein–Uhlenbeck processes driven by compound Poisson processes is considered. The jumps arrive with exponential waiting times and are allowed to be two-sided. The jumps are assumed to form an iid sequence with distribution a mixture (not necessarily convex) of exponential distributions, independent of everything else. When the drift is negative the probability of ever crossing a given lower level is less than one and its asymptotic behaviour when the initial state of the process tends to infinity is determined explicitly. The situation where the level to cross decreases to minus infinity is more involved: The level to cross under plays a much more fundamental role in the expressions for the ruin probabilities than the initial state of the process. The asymptotics of the ruin probability in the positive drift case and the limit of the distribution of the undershoot in the negative drift case is derived in the case where the lower limit decreases to minus infinity.

Paper C A task such as the execution of a computer program or the transfer of a file on a communications link may fail and then needs to be restarted. Let the ideal task time be a constant ℓ and the actual task time X, a random variable. Tail asymptotics for $\mathbb{P}(X > x)$ is given under three different models: **1**: a time-dependent failure rate $\mu(t)$; **2**: Poisson failures and a time-dependent deterministic work rate $r(t)$; **3**: as **2**, but $r(t)$ is random and a function of a finite Markov process. Also results close to being necessary and sufficient are presented for X to be finite a.s. The results complement those of Asmussen, Fiorini, Lipsky, Rolski & Sheahan [*Math. Oper. Res.* **33**, 932–944, 2008] who took $r(t) \equiv 1$ and assumed the failure rate to be a function of the time elapsed

Paper D This is a work in progress. We consider a risk process $\{R\}_{t\geq 0}$ with the property that the rate β of the Poisson arrival process and the distribution B of the claim sizes depends on the state of an underlying Harris recurrent Markov process $\{X_t\}_{t\geq 0}$. In this setup we derive a version of Lundberg's Inequality. This involves finding eigenfunctions in the setup of a Markov–modulated random walk.

2 Risk processes and ruin probabilities

Risk theory is often associated with the mathematical problems that are faced by an insurance company that has to decide how much the premiums should cost and how big the capital reserve should be in order to minimise the probability of bankruptcy.

The capital reserve is modelled over time by the so–called *risk reserve process* $\{R_t\}_{t\geq 0}$. Let $u = R_0$ denote the initial reserve, and let $\psi(u)$ be the probability of ultimate ruin – the probability that the reserve ever drops below zero. For this define $\tau(u) = \inf\{t \geq 0 \mid R_t < 0\}$ as the time of ruin (when the initial value is u). Then

$$\psi(u) = \mathbb{P}(\tau(u) < \infty) = \mathbb{P}(\inf_{t\geq 0} R_t < 0 \mid R_0 = u)\,.$$

Often this setup is reformulated into using the *claim surplus process* $\{S_t\}_{t\geq 0}$ defined by $S_t = u - R_t$. Then instead

$$\tau(u) = \inf\{t \geq 0 \mid S_t > u\}\,,$$

and $\psi(u)$ can be expressed as

$$\psi(u) = \mathbb{P}(\tau(u) < \infty) = \mathbb{P}(M > u)\,,$$

where $M = \sup_{t\geq 0} S_t$.

The claim surplus process is very often assumed to have the following basic form

$$S_t = \sum_{i=1}^{N_t} U_i - pt\,,$$

where $\{U_i\}$ are the claims and N_t the number of claims in the time interval $[0,t]$. Furthermore p is rate at which the premiums flow in per time unit. To make the definition meaningful it is necessary that N_t is finite for all t.

A classical version of this is the Cramér–Lundberg model, where the claims $\{U_i\}_{i\in\mathbb{N}}$ are assumed to be iid, and $\{N_t\}_{t\geq 0}$ is a time homogeneous Poisson process independent of $\{U_i\}_{i\in\mathbb{N}}$. For this model several more and less specific results for $\psi(u)$ are well–known and can e.g. be seen in Chapter III of [6]. A classical result is the Cramér–Lundberg Approximation, stating

$$\psi(u) \sim C\mathrm{e}^{-\gamma u}$$

as $u \to \infty$, where γ is derived as the solution of the so–called Lundberg equation.

A meaningful generalisation of this model is Markov–modulation. Instead of having constant claim intensity and identically distributed claims, one could assume that they depend on some underlying Markov process. An example is that both claim sizes and the number of claims reasonably depend on the type of weather (sun, wind, rain, etc). Such a model in a rather general setup is studied in Paper D.

A type of model that differs from the classical setup is when the premiums depend on the current reserve. Then the risk process $\{R_t\}$ is modelled by the equation

$$R_t = u - \sum_{i=1}^{N_t} U_i + \int_0^t p(R_s)\,\mathrm{d}s\,,$$

where $r(t)$ is some function that decides how big the premiums should be depending on the current reserve. Such a model is the subject of study both in Paper A and Paper B.

Note that in the classical model only the difference between the starting point u and the level to cross under (which is zero) is of importance because of the additive structure. This is not the case in a premium depending model. Hence different levels to cross under – denoted ℓ – will be of interest in Paper A and Paper B.

3 The setup for Paper A and Paper B

In Paper A the risk process of interest, X, is given by the following stochastic differential equation

$$dX_t = \kappa X_t\,dt + dV_t + \sigma\sqrt{|X_t|}\,dB_t \tag{3.1}$$

and in Paper B the simpler process

$$\mathrm{d}X_t = \kappa X_t\,\mathrm{d}t + \mathrm{d}V_t\,, \tag{3.2}$$

is the subject. In both (3.1) and (3.2) $\{V_t\}_{t\geq 0}$ is a compound Poisson process defined by

$$V_t = \sum_{n=1}^{N_t} U_n\,.$$

Here $\{U_n\}$ are iid with distribution G, and $\{N_t\}$ is a homogeneous Poisson process with parameter λ independent of $\{U_n\}$. In (3.1) $\{B_t\}$ is a Brownian motion independent of everything else.

In both (3.1) and (3.2) some assumptions concerning the jumps are made. It is assumed that the downward part of the jump distribution G has a distribution that has a density that is a linear combination of exponential densities. We use the decomposition $G = pG_- + qG_+$, where $0 < p \leq 1$, $q = 1-p$, G_- is the restriction to $\mathbb{R}_- =]-\infty; 0[$, and G_+ is the restriction to $\mathbb{R}_+ =]0; \infty[$. Then it is assumed that

$$G_-(du) = g_-(u)\,du = \sum_{k=1}^{r} \alpha_k \mu_k \mathrm{e}^{\mu_k u} \qquad \text{for } u < 0\,. \tag{3.3}$$

In Paper B and in some situations of Paper A also the upward part of the jumps are assumed to be such linear combinations of exponential distributions

$$G_+(du) = g_+(u)\,du = \sum_{d=1}^{s} \beta_d \nu_d \mathrm{e}^{-\nu_d u} \qquad \text{for } u > 0\,. \tag{3.4}$$

The distribution parameters are supposed to secure that the definitions above actually form distributions (e.g. $\sum_i \alpha_i = 1$ and $g_+ \geq 0$). The parameters are arranged such that $0 < \mu_1 < \cdots < \mu_r$ and $0 < \nu_1 < \cdots < \nu_s$.

The process in (3.2) behaves deterministically like an exponential function

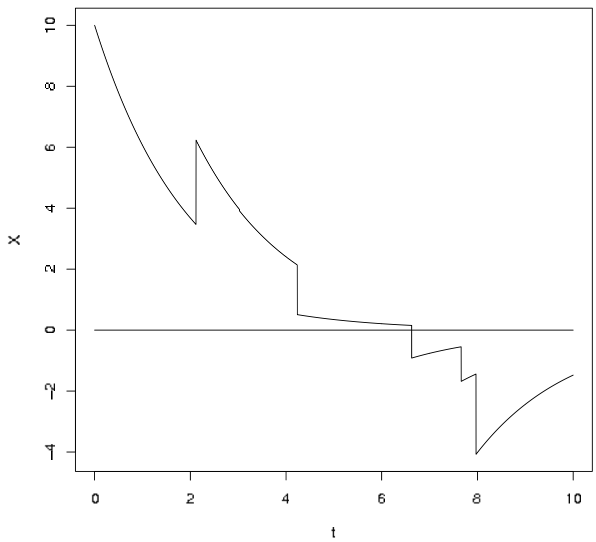

Figure .1: Illustrates a path for the process defined in (3.2)

between jumps. A path of the process can be seen in Figure .1. The process in the figure has a negative drift paramtre κ. In that case the process is recurrent. If on the other hand the drift is positive, the process will be transient.

The process given by (3.1) is not deterministic between jumps. Instead it evolves like the following Cox–Ingersoll–Ross diffusion process

$$dY_t = \kappa Y_t\,dt + \sigma\sqrt{|Y_t|}\,dB_t\,.$$

An important property of this process is that 0 is an absorbing state.
A simulated path of the process (3.1) is seen in Figure .2. Also for this

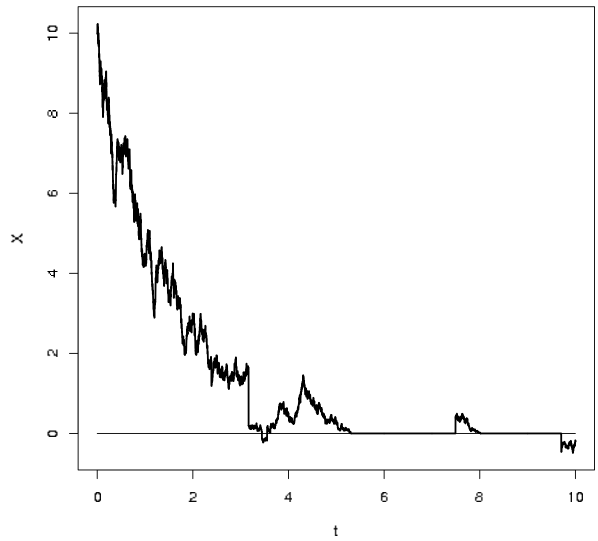

Figure .2: Illustrates a path for the process defined in (3.1)

process the sign of the drift parametre κ is substantial: If $\kappa < 0$ the process is recurrent, while it is transient when $\kappa > 0$.

Assume that $x > 0$ and write $\mathbb{P}_x$ for the probability space, where $X_0 = x$ $\mathbb{P}_x$–almost surely. Let $\mathbb{E}_x$ be the corresponding expectation. For some $\ell < x$ we shall (both in Paper A and B) be interested in the stopping time τ (the ruin time) given by

$$\tau = \tau(\ell) = \inf\{t > 0 | X_t \leq \ell\} \tag{3.5}$$

Furthermore define the *undershoot* Z

$$Z = \ell - X_\tau \,, \tag{3.6}$$

which is well–defined on the set $\{\tau < \infty\}$. It is important to notice that in most cases the level ℓ can by crossed through continuity as well as a result of a downward jump. Nevertheless the process in (3.1) might not be able to cross ℓ through continuity. That will be in one of the two situations **1**: $\kappa > 0$, $\ell > 0$ or **2**: $\kappa < 0$, $\ell < 0$.

In Paper A the joint distribution of τ and Z will be of interest. This distribution can be expressed through the joint Laplace transform for τ and Z which may be defined as

$$\mathbb{E}_x[e^{-\theta\tau - \zeta Z}; A_j] \quad \text{and} \quad \mathbb{E}_x[e^{-\theta\tau}; A_c] \,, \tag{3.7}$$

where A_j and A_c is a partition of the set $\{\tau < \infty\}$ into the jump case $A_j = \{\tau < \infty, X_\tau < \ell\}$ and the continuity case $A_c = \{\tau < \infty, X_\tau = \ell\}$.

In Paper B the focus will particularly be on the probability of ruin within finite time given by $\mathbb{P}_x(\tau < \infty)$. Note that for both of the processes (3.1) and (3.2) this probability will be 1 in the recurrent situations, where $\kappa < 0$.

4 The results in Paper A

The aim in Paper A is to describe the joint distribution of τ and Z for the model (3.1) through finding an explicit expression for the joint Laplace transform given by (3.7). The method applied in this paper for finding the joint Laplace transform has previously been used in [34], where the simpler model (3.2) was studied.

In the present paper a version of Itô's formula is established

$$e^{-\theta(\tau\wedge t)} f(X_{\tau\wedge t}) = f(X_0) + \int_0^{\tau\wedge t} e^{-\theta s}(\mathcal{A}f(X_s) - \theta f(X_s))\, ds + M_t\,, \qquad (4.1)$$

where $\mathcal{A}$ is the infinitesimal generator for X:

$$\mathcal{A}f(x) = \kappa x f'(x) + \tfrac{\sigma^2}{2}|x| f''(x) + \lambda \int_{\mathbb{R}} \Big(f(x+y) - f(x)\Big) G(\mathrm{d}y)$$

defined for $x \in [\ell, \infty[\setminus\{x\}$. here M is some mean–zero martingale. Suppose that a partial eigenfunction for $\mathcal{A}$ can be found. That is a function $f : \mathbb{R} \to \mathbb{C}$ that is bounded and continuous on $[\ell, \infty[$ and furthermore two times differentiable on $[\ell, \infty[\setminus\{x\}$ with

$$\mathcal{A}f(x) = \theta f(x) \quad \text{for all } x \in [\ell, \infty[\,.$$

Then obviously (4.1) becomes much simpler, and from using optional stopping it is obtained that

$$\mathbb{E}_x\left[\mathrm{e}^{-\theta\tau} f(X_\tau)\right] = f(x)\,.$$

If f furthermore have the form

$$f(x) = \mathrm{e}^{-\zeta(\ell - x)}$$

on the interval $]-\infty, \ell[$ then the equation becomes

$$\mathbb{E}_x[\mathrm{e}^{-\theta\tau - \zeta Z}; A_j] + f_i(\ell)\mathbb{E}_x[\mathrm{e}^{-\theta\tau}; A_c] = f(x)\,.$$

This can be solved w.r.t. $\mathbb{E}_x[\mathrm{e}^{-\theta\tau-\zeta Z}; A_j]$ and $\mathbb{E}_x[\mathrm{e}^{-\theta\tau}; A_c]$ if two partial eigenfunctions for $\mathcal{A}$ are found.

After this the main task is to construct such partial eigenfunctions. These partial eigenfunctions are defined as linear combinations of functions on the form

$$f_\Gamma(y) = \begin{cases} \int_\Gamma \psi_i(z) \mathrm{e}^{-yz}\, \mathrm{d}z & y \geq \ell \\ 0 & y < \ell \end{cases}\,, \qquad (4.2)$$

where ψ_1 and ψ_2 are some specified complex integration kernel and Γ some (as well specified) contours in the complex plane.

For these functions the generator is particularly nice (it is achieved that $(\mathcal{A}-\theta I)f$ is a simple exponential function on $[\ell,\infty[$) and hence with f an adequate combination of these f_Γ–functions makes $(\mathcal{A}-\theta I)f = 0$ on $[\ell,\infty[$.

A major part of the work is finding an adequate number of contours Γ such that the right linear combination of the f_Γ–functions is possible. The choice of these contours seems to depend on the parameters of the downward jump distribution (3.3) if $\ell > 0$, and both the downward and the upward jump part in the case, where $\ell < 0$.

5 The results in Paper B

In this paper the simpler model (3.2) is considered. When the drift parameter κ is positive we have (as mentioned above) that

$$\mathbb{P}_x(\tau(\ell) < \infty) < 1\,.$$

We are studying the behaviour of this probability when either $x \to \infty$ or $\ell \to -\infty$. The basis for this study is the results from [34], where an explicit formula for the probability was derived. It turns out that the results in [34] need to be reformulated in a way that resembles the formulation in Paper A. Also the choice of integration contours will have to be slightly different.

For the $x \to \infty$ limit it all boils down to finding the asymptotic behaviour of the f_Γ–functions which is done in two central lemmas. After this the main result is easily derived:

$$\lim_{x\to\infty} \frac{\mathbb{P}_x(\tau(\ell) < \infty)}{\mathrm{e}^{-\mu_1 x} x^{-\frac{p\alpha_1\lambda}{\kappa}-1}} = K\,,$$

where K is some specified constant, and μ_1 is the dominating part of the downward jumps.

In the $\ell \to -\infty$ case things get more complicated since ℓ is a more fundamental part of the definition of the partial eigenfunction than the initial state x. When ℓ changes, the constants in the linear combination defining the partial eigenfunction also changes. Hence finding the limit is a question about keeping track of the solution vector of a multidimensional linear equation. Here results similar to the Lemmas mentioned above are used to describe each parameter in the equation system. The following result is obtained

$$\lim_{\ell\to\infty} \mathbb{P}_x(\tau(\ell) < \infty) = -\sum_{i=1}^{r} c_i f^1_{\Gamma_{i,1}}(x)\,,$$

where the functions $f^1_{\Gamma_{i,1}}(x)$ are on the form described in (4.2) and $\{c_i\}$ are some specified constants.

Finally also the limit when $\ell \to -\infty$ of the Laplace transform for the undershoot is studied in the case, where $\kappa < 0$ and hence $\mathbb{P}_x(\tau(\ell) < \infty) = 1$. This involves arguments similar to the ones in the $\lim_{\ell\to-\infty} \mathbb{P}_x(\tau(\ell) < \infty)$ case, and it is obtained that for all $\zeta \geq 0$

$$\lim_{\ell\to-\infty} \mathbb{E}_x[\mathrm{e}^{-\zeta Z}] = \frac{\mu_1}{\mu_1 + \zeta}\,.$$

Hence the undershoot converges to a simple exponential distribution defined by the dominating exponential parameter in the downward jump distribution.

6 The setup and results of Paper D

This paper is a work in progress. It can be regarded as a first attempt to consider ruin problems in the presented rather general setup.

Setup

We assume that the arrivals are not homogeneous in time but are determined by the process X, where X is a Harris recurrent Markov process. Given X the sum $\sum_{i=1}^{N_t} U_i$ is an inhomogeneous Poisson process: Claims are independent and at time t

- The arrival intensity is $\beta(X_t)$
- A claim arriving has distribution B_{X_t}.

Among several other regularity conditions we assume that

$$x \mapsto \beta(x) \quad \text{and} \quad x \mapsto \hat{B}_x[\alpha]$$

are bounded functions (here the notation $\hat{B}_x[s] = \int_{\mathbb{R}} e^{sy} B_x(\mathrm{d}y)$ has been used). Let $\psi_x(u) := \mathbb{P}(\tau(u) < \infty)$, where $\mathbb{P}_x$ is defined such that $X_0 = x$.

Results

Apparently the direction of the drift of $\{S_t\}$ is seen from the sign of η defined below.

For each $x \in E$ let $\mu(x) := \int_0^\infty y\, B_x(\mathrm{d}y)$ and $\mu^{(2)}(x) = \int_0^\infty y^2\, B_x(\mathrm{d}y)$ denote the mean and second order moment of B_x. Define

$$\rho(x) := \beta(x)\mu(x)\,,$$

and let furthermore

$$\rho^* := \int_E \rho(x)\,\pi(\mathrm{d}x) \quad \text{and} \quad \eta := \frac{1-\rho^*}{\rho^*}\,.$$

Then it is shown that

$$S_t/t \overset{\mathbb{P}_x-a.s.}{\longrightarrow} \rho^* - 1\,,$$

and thereby that $\psi_x(u) = 1$ if $\eta \leq 0$, and that $\psi_x(u) < 1$ if $\eta > 0$.

Hence the case, where $\eta > 0$, is of interest when considering the asymptotics of $\psi_x(u)$. The main idea is changing measure to a situation, where $\eta < 0$. A fundamental part of this change of measure is the existence of an eigenfunction for the operator

$$\mathbf{P}_t^\alpha f(x) = \mathbb{E}_x[\mathrm{e}^{\alpha S_t} f(X_t)]$$

with α chosen such that the corresponding eigenvalue is 1. If h is this eigenfunction it is shown that

$$L_t^\alpha = \frac{h(X_t)}{h(x)} \mathrm{e}^{\alpha S_t - t\kappa(\alpha)}$$

is a non–negative martingale with mean 1. Then the changed measure is defined with $\{L_t^\alpha\}$ being the likelihood process. Furthermore it is shown that under the changed measure $\mathbb{P}_x^\alpha$ the process $\{S_t, X_t\}$ has a distribution similar to the original distribution under $\mathbb{P}_x$, but now with $\eta < 0$. Thereby it becomes possible to derive asymptotic results for $\psi_x(u)$.

7 The setup and results of Paper C

We consider a RESTART setting with some job that ordinarily would take the task time (ideal task time) T to be executed. At some point during the execution a failure may occur at time S_1. After that the execution will have to be restarted, and at this point the time until the task is performed is supposed to be T. However another failure may occur at time S_2 and after this the system will have to be restarted again. This procedure will go on until the task has been performed without failures, that is an interval between two failure occurrences of size at least T.

The total task time will be denoted X. Hence obviously $X \geq T$. Furthermore we shall denote the waiting times between failure times $\{U_n\}_{n\in\mathbb{N}}$ (that is $U_n = S_n - S_{n-1}$). With the definition

$$R := \inf\{n \in \mathbb{N} \mid U_n > T\}$$

we have that

$$X = S_{R-1} + T\,.$$

In this setup we are interested in the asymptotic behaviour of the probability

$$\mathbb{P}(X > x)\,,$$

when $x \to \infty$.

In Asmussen et al. [8] these tail asymptotics were found under a variety of distributions of T and the U–waiting times. It is assumed that the $\{U_n\}$–variables are iid with some common distribution. A particular important case is the one with exponential waiting times (hence $\{S_n\}$ is a Poisson process) and a fixed ideal task time $T \equiv \ell$. Then it can be shown that

$$\mathbb{P}(X > x) \sim c\mathrm{e}^{-\gamma x}\,, \tag{7.1}$$

where c is some constant, and γ is found as the root of

$$1 = \int_0^\ell \mu \mathrm{e}^{(\gamma-\mu)y}\mathrm{d}y\,, \tag{7.2}$$

with μ the parameter of the exponential waiting times.

In Paper C we consider a variety of slightly different scenarios. First we assume that the Poisson process $\{S_n\}$ is non–homogeneous with some varying (but deterministic) rate $\mu(t)$. A very similar problem is assuming that $\{S_n\}$ has constant rate but that the system works on the task with some deterministically varying rate $r(t)$. A third model, that is considered in the paper, is where the rate depends on an underlying Markov process. Altogether we have the three models:

Model 1 Failures at time t after the start of the task occur at deterministic rate $\mu(t)$.

Model 2 Failures occur according to a Poisson(μ^*) process with constant rate μ^*. At time t after the start of the task the system works on the task at rate $r(t)$.

Model 3 As Model 2, but the rate function $r(t)$ is given as $r(t) = r_{V(t)}$ where $\big\{V(t)\big\}_{t\geq 0}$ is an ergodic Markov process with $p < \infty$ states and $r_1, \dots, r_p$ are constants with $r_i > 0$ for at least one i.

In both **Model 1** and **Model 2** it is possible that $\mathbb{P}(X = \infty) > 0$, and a first result shows that

$$\begin{aligned}
\limsup_{t\to\infty} \mu(t)/\log(t) < 1/\ell \quad &\Rightarrow \quad \mathbb{P}(X=\infty) = 0\\
\liminf_{t\to\infty} \mu(t)/\log(t) > 1/\ell \quad &\Rightarrow \quad \mathbb{P}(X=\infty) > 0\,.
\end{aligned}$$

Hence only a modest increase in $\mu(t)$ may cause that the task never terminates. A similar result is reached for **Model 2** stating that with a small decrease in $r(t)$ it is obtained that $\mathbb{P}(X = \infty) > 0$. The results for **Model 2** follow from the similar results concerning **Model 1** via a rather simple time–change argument.

In the cases of a decreasing $\mu(t)$ in **Model 1** and an increasing $r(t)$ in **Model**

2, where obviously $\mathbb{P}(X = \infty) = 0$, we study the asymptotics of $\mathbb{P}(X > x)$ when $x \to \infty$. For **Model 1** we have with $\mu(t) \sim at^{-\beta}$ and $0 < \beta < 1$ that

$$\mathbb{P}(X > x) \approx_{\log} \mathrm{e}^{-c_1 x \log x} = x^{-c_1 x},$$

where $c_1 = (1-\beta)/\ell$ (with the notation $f(x) \approx_{\log} g(x)$ if $\log f(x) \sim \log g(x)$). The proof is based on an exponential change of measure to a situation, where the sequence $\{U_n\}$ is iid with distributions concentrated on $[0, \ell]$. A similar result is obtained for **Model 2**: If $r(t) \sim at^{\eta}$ with $\eta > 0$, then

$$\mathbb{P}(X > x) \approx_{\log} \mathrm{e}^{-c_2 x^{\eta+1} \log x} = x^{-c_2 x^{\eta+1}},$$

where $c_2 = a\eta/(\eta+1)\ell$. Also here the arguments are derived rather easily from **Model 1** by applying a time–change argument.

Both with respect to proof and result **Model 3** resembles the simple model in (7.1). From applying the Markov renewal theorem it is obtained that

$$\mathbb{P}(X > x) \sim c\mathrm{e}^{-\gamma x},$$

where γ is found as the solution of some multi–dimensional version of (7.2).

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PAPER A

The Ruin Time for a certain Class of Diffusion Processes with Jumps

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Abstract

A certain class of diffusions with jumps is considered. Between jumps the process behaves like a Cox–Ingersoll–Ross process driven by a Brownian Motion independent of the jump process. The jumps arrive with exponential waiting times and are allowed to be two–sided. The jumps that are mixtures of exponential distributions are assumed to form an independent, identically sequence, independent of everything else. The fact that downward jumps are always allowed makes passage of a given lower level possible both by continuity and by a jump. In case of passage by jump the resulting undershoot is considered too. As one of the main results of the thesis, the joint Laplace transform of the first passage and the undershoot is determined. So is the probability of passage in finite time. Both the joint Laplace transform and the probability of passage in finite time are decomposed according to the type of passage: Jump or continuity.

Determining the Laplace transform uses the martingales that can be derived from Itô's formula if a partial eigenfunction for the infinitesimal generator of the process can be found.

Finding partial eigenfunctions involve using complex contour integrals as the eigenfunctions are defined as linear combinations of such integrals. An important part of the search for a partial eigenfunction is finding the sufficient number of contours for integration.

1 Introduction

In the present paper the process

$$dX_t = \kappa X_t \, dt + dU_t + \sigma \sqrt{|X_t|} \, dB_t \tag{1.1}$$

is studied where (U_t) is a compound Poisson process with jumps (both the upward and downward parts) that are allowed to have densities that are linear combinations of exponential densities. Furthermore (B_t) is a Brownian motion independent of (U_t). The aim is to determine the distribution of the first passage time of a given level ℓ when the process has initial state $x > \ell$. The passage of the lower level ℓ can be a result of a downward jump as well as a continuous motion. We shall distinguish between the two types of passage when considering the passage time. In case of passage because of a jump the distribution of the so–called *undershoot* will be determined as well.

The joint distribution of the passage time and the undershoot will be determined by establishing an expression for the joint Laplace transform. This is found from Itô's formula applied to partial eigenfunctions for the infinitesimal generator $\mathcal{A}$ for X. The method has previously been used in [13]. Here the simpler model corresponding to the case $\sigma = 0$ for the process mentioned above was studied (that is an Ornstein–Uhlenbeck process driven by the compound Poisson process U). A major part of the work in [13] and the present paper is the very construction of the partial eigenfunctions. They are defined as linear combinations of functions given as contour integrals in the complex plane. It turns out to be crucial that the jump structure is given as mentioned above and that the waiting times between jumps are exponential. With these assumptions it is obtained that when the generator $\mathcal{A}$ is applied to the contour integral functions, the resulting functions are simple linear combinations of exponentials. When an adequate number of these functions are combined in the right way it can be obtained that the combination is a partial eigenfunction. Hence a crucial part in the construction of the partial eigenfunction will be finding the adequate number of integration contours in the complex plane.

As a result of the larger complexity of the model (1.1) compared to the one studied in [13] both establishing Itô's formula and finding the sufficient number of contours becomes more involved: For Itô to be true some additional constraints are imposed to the eigenfunction and now contours for two different integration kernels are requested as a result of the more complicated version of the generator $\mathcal{A}$.

The technique of using partial eigenfunctions for the infinitesimal generator has appeared before. In [17] Paulsen and Gjessing considers a model like (1.1) but in the more general (and also different) setup

$$dX_t = (p + \kappa X_t) \, dt - dU_t + \sqrt{\sigma_1^2 + \sigma_2^2 X_t^2} \, dB_t + X_t d\tilde{U}_t \, . \tag{1.2}$$

Here both U and $\tilde{U}$ are compound Poisson processes. In [17] it is shown that a partial eigenfunction for the corresponding infinitesimal generator for (1.2) will lead to the ruin probability and also the Laplace transform for the ruin time. In [8] Gaier and Grandits show – without σ_1^2 and $\tilde{U}$ in the model – the existence of this partial eigenfunction under some smoothness assumptions about the jump distributions in U. This result is extended to weaker assumptions in [9].

The fact that both the ruin distribution *and* the undershoot is of interest is reflected in the literature through the so–called Gerber–Shiu penalty function

$$\Phi_x(\alpha) = \mathbb{E}^x[g(X_{\tau-}, X_\tau)\mathrm{e}^{-\alpha\tau}] \, .$$

In [4] integro–differential equations similar to the ones involved in finding partial eigenfunctions are derived for $\Phi_x(\alpha)$. This is in the model (1.2), but with $\sigma_1^2 = 0$ and some assumptions about the jump distribution.

Another example of solving equations that is similar to finding partial eigenfunctions for the generator can be found in [6]. Here – as it is also done in the present paper – the ruin probability is decomposed into ruin as a result of a jump and ruin due to continuity and the equations that these functions satisfy are expressed.

For the OU case where $\sigma = 0$ and $\kappa > 0$ some explicit results are achieved in [17] for models with negative jumps (positive jumps are not allowed). The jumps are either a mixture of two exponential distributions or a Γ distribution. Here an explicit formula for the Laplace transform of the time to ruin is expressed. For the case of exponential negative jumps also see Asmussen [2], Chapter VII.

In Novikov et al [16] an OU–process very much like the one in [13] and [17] is studied. But here it is assumed that the drift κ is negative and only negative jumps are allowed. In both the cases exponential and uniform jumps the Laplace transform for the passage time is determined. The technique is finding a partial eigenfunction for the infinitesimal generator. In [3] this result is extended to a more general driving Lévy process instead of a compound Poisson process. Whereas the only downward jumps allowed are still exponentials. A model similar to the one in [16] is studied in Kella and Stadje [15]. But here the first *hitting* time (not passage) is considered for both an upper and a lower level. In the case of a lower level the hitting time obviously differs from the passage time. For both the upper and lower level the Laplace transform for the hitting time is expressed in terms of real valued integrals.

A technique very similar to the present paper and [13] was used in [10] and [12]. In [12] the Markov additive model given by

$$X_t = x + \int_0^t \beta_{J_s}\,\mathrm{d}s + \int_0^t \sigma_{J_{s-}}\mathrm{d}B_s - \sum_{n=1}^{N_t} U_n \tag{1.3}$$

is considered, and in [10] a similar but simpler model is the subject. Above J is a discrete spaced Markov Chain that also governs the jump times of (N_t).

Here a version of Itô's formula for the joint process (X, J) is established and partial eigenfunctions for the generator of this process are requested. The eigenfunctions are constructed as simple linear combinations of exponential functions. The much more complex partial eigenfunctions constructed in the present paper for the model (1.1) makes these phase–type waiting times from (1.3) too complicated to handle.

The paper is organised in a way that resembles [13], but all results will have to be reformulated and proved again due to the more complicated setup. First (Section 2) the setup is defined. In Section 3 some constraints are found for the functions that Itô's formula can be applied to. The additional conditions that makes these functions partial eigenfunctions are stated afterwards. In Section 4 the joint Laplace transform for the passage time and the undershoot is expressed under the assumption that it is possible to find two functions that meet the conditions from Section 3. In the following Section 5 a skeleton of how these functions look like is defined and it is proved that they actually meet the conditions from above if a sufficient number of integration contours for some complex kernel can be found. Finally in Section 6 a suggestion of how to choose these contours is made. It is shown that the proposed contours fulfil the conditions established in Section 5 and it is also argued that no more contours can be found.

2 The model

We consider a process X defined by the following stochastic differential equation:

$$dX_t = \kappa X_t \, dt + dU_t + \sigma \sqrt{|X_t|} \, dB_t \tag{2.1}$$

where (B_t) is a standard Brownian motion and (U_t) is a compound Poisson process defined by

$$U_t = \sum_{n=1}^{N_t} V_n. \tag{2.2}$$

Here (V_n) are iid with distribution G and (N_t) is a Poisson process with parameter λ.

The solution process X is a process that between jumps behaves like a continuous diffusion process (with drift part $\kappa X_t dt$ and diffusion part $\sigma \sqrt{|X_t|} \, dB_t$). The jumps arrive with exponential waiting times.

Assume that $x > 0$ and write $\mathbb{P}_x$ for the probability space where $X_0 = x$ $\mathbb{P}_x$–almost surely. Let $\mathbb{E}_x$ be the corresponding expectation. Define for $\ell < x$ the stopping time τ by

$$\tau = \tau(\ell) := \inf\{t > 0 | X_t \leq \ell\} \, . \tag{2.3}$$

For ease of notation ℓ is most often suppressed. Furthermore define the *undershoot* Z

$$Z := \ell - X_\tau \tag{2.4}$$

which is well defined on the set $\{\tau < \infty\}$. Of interest is the joint distribution of τ and Z. This distribution will be expressed through the joint Laplace transform defined by

$$\mathbb{E}_x[e^{-\theta\tau-\zeta Z}; A_j] \quad \text{and} \quad \mathbb{E}_x[e^{-\theta\tau}; A_c]\,, \tag{2.5}$$

where A_j and A_c is a partition of the set $\{\tau < \infty\}$ into the jump case $A_j = \{\tau < \infty, X_\tau < \ell\}$ and the continuity case $A_c = \{\tau < \infty, X_\tau = \ell\}$.

It can be shown that the process is transient with

$$\mathbb{P}_x(\tau(\ell) < \infty) < 1$$

if the drift is positive, and recurrent with

$$\mathbb{P}_x(\tau(\ell) < \infty) = 1$$

if the drift is negative.

3 Itô's formula

We are looking for some functions f for which the following version of Itô's formula holds

$$e^{-\theta(\tau\wedge t)} f(X_{\tau\wedge t}) = f(X_0) + \int_0^{\tau\wedge t} e^{-\theta s}(\mathcal{A}f(X_s) - \theta f(X_s))\,ds + \int_0^{\tau\wedge t} e^{-\theta s}\,dM_s\,. \tag{3.1}$$

Here $\mathcal{A}$ is the generator of X and M is some suitable martingale. By considering the diffusion part and the jump part of the process X separately we can find an expression for the generator and by that the martingale as well.

Between jumps the process behaves like the diffusion process Y given by the following differential equation

$$dY_t = \kappa Y_t\,dt + \sigma\sqrt{|Y_t|}\,dB_t\,.$$

It can be shown that 0 is an absorbing state for this process (the arguments can be carried out via the results formulated in e.g. Freedman [7]). That means that the original process X can be absorbed by 0 between jumps. When the next jump arrives the X–process will leave the state again.

Because of the special property of the state 0 some difficulties arise when applying the standard Itô formula to this process. Therefore the process $\tilde{Y}$ is considered where

$$d\tilde{Y}_t = \kappa\tilde{Y}_t\,d(t\wedge\tilde{\tau}_a) + \sigma\sqrt{|\tilde{Y}_t|}\,dB_{t\wedge\tilde{\tau}_a}$$

and

$$\tilde{\tau}_a = \inf\{t > 0 | \tilde{Y}_t \leq a\}$$

with $a > 0$. Itô gives that the following result holds

$$df(\tilde{Y}_t) = (\kappa f'(\tilde{Y}_t)Y_t + \tfrac{1}{2}f''(\tilde{Y}_t)\sigma^2|\tilde{Y}_t|)\, d(\tilde{\tau}_a \wedge t) + f'(\tilde{Y}_t)\sigma\sqrt{|\tilde{Y}_t|}\, dB_t^{\tilde{\tau}_a} \tag{3.2}$$

when f is bounded and two times differentiable (on the interval $[a, \infty[$).

Similarly we can find the generator and martingale for the jump part of X. The jump part is simply the process (U_t) that can be viewed as a marked point process. The following result is obtained by using the version of Itô's formula that concerns marked point process theory (see [11]):

$$f(U_t) = f(U_0) + \lambda \int_{]0;t]\times\mathbb{R}} (f(U_s + y) - f(U_s))\, G(dy)\, ds + \tilde{M}_t\,,$$

where $(\tilde{M}_t)$ is the martingale defined by

$$\tilde{M}_t = \int_{]0,t]\times\mathbb{R}} f(U_{s-} + y) - f(U_{s-}))\, M^\circ(ds, dy)\,.$$

Here M° is the martingale measure from the Poisson jump structure:

$$M^\circ(ds, dy) = \mu(ds, dy) - \lambda\, ds\, G(dy)$$

with μ the counting measure defined by for all $t \geq 0$ and $B \in \mathcal{B}$

$$\mu(]0,t] \times B) \;=\; \text{number of jumps in the time interval }]0,t] \text{ of a size within } B\,.$$

Combining the results for the continuous part and the jump part gives the following version of Itô on the interval $[0, \tau_a]$ when f is bounded and two times differentiable and furthermore $\mathcal{A}f$ is bounded

$$f(X_{\tau_a\wedge t}) = f(X_0) + \int_0^{\tau_a\wedge t} \mathcal{A}f(X_s)\, ds + M_{\tau_a\wedge t}\,, \tag{3.3}$$

where τ_a is the stopping time for X that corresponds to $\tilde{\tau}_a$:

$$\tau_a = \inf\{t > 0 | X_t \leq a\}\,.$$

Here $a > 0$. The generator $\mathcal{A}$ is

$$\mathcal{A}f(x) = \kappa x f'(x) + \tfrac{\sigma^2}{2}|x| f''(x) + \lambda \int_{\mathbb{R}} (f(x + y) - f(x))\, G(dy)\,, \tag{3.4}$$

and $(M_{\tau_a\wedge t})$ is a zero–mean martingale given by

$$dM_t = f'(X_{t-})\sigma\sqrt{X_{t-}}\, dB_t + \int_{\mathbb{R}} f(X_{t-} + y) - f(X_{t-})\, M^\circ(dt, dy)\,. \tag{3.5}$$

Hence we get that for all $\theta \geq 0$

$$e^{-\theta(\tau_a \wedge t)} f(X_{\tau_a \wedge t}) = f(X_0) + \int_0^{\tau_a \wedge t} e^{-\theta s} (\mathcal{A} f(X_s) - \theta f(X_s))\, ds + \int_0^{\tau_a \wedge t} e^{-\theta s}\, dM_s\,, \tag{3.6}$$

where the last part is a martingale.

The aim in the following is to find the class of functions f where the formula (3.6) holds on all the time interval $[0, \tau]$.

If $\ell > 0$ we can just choose $a = \ell$. Remembering that $\tau = \tau(\ell)$ gives the formula (3.1) when f is a bounded and two times differentiable function such that $\mathcal{A}f$ is bounded as well.

The case when $\ell \leq 0$

If $\ell \leq 0$ the problem becomes more complicated as a result of the passage of the state 0. We have conditions on the functions f such that (3.6) is fulfilled on the interval $[a, \infty[$.

Now consider a bounded, continuous function $f : \mathbb{R} \to \mathbb{C}$ that is two times differentiable on $[\ell, \infty[\setminus \{0\}$ and satisfy that $\mathcal{A}f$ is bounded on $[\ell, \infty[\setminus \{0\}$. With such a function f the formula (3.6) holds for all $a > 0$ and thereby it makes sense to consider the limit when $a \to 0$.

On the set $\{X_{\tau_0} \leq 0, X_{\tau_0 -} > 0\}$ it follows easily that

$$e^{-\theta(\tau_0 \wedge t)} f(X_{\tau_0 \wedge t}) = f(X_0) + \int_0^{\tau_0 \wedge t} e^{-\theta s} (\mathcal{A} f(X_s) - \theta f(X_s))\, ds + \int_0^{\tau_0 \wedge t} e^{-\theta s}\, dM_s\,, \tag{3.7}$$

because an $a > 0$ could be found such that $\tau_a = \tau_0$.

The situation that corresponds to the set $\{X_{\tau_0} = 0, X_{\tau_0 -} = 0\}$ is more complicated. Now the process reaches the state 0 as a result of a continuous movement. As f is assumed to be continuous the left hand side of (3.6) has the limit

$$e^{-\theta(\tau_0 \wedge t)} f(0)\,.$$

When the left hand side has a limit when $a \to 0$ the right hand side must have one as well. Because both f and $\mathcal{A}f$ are assumed to be bounded each of the two integrals will even converge separately. So the desired formula is now shown until the time τ_0:

$$e^{-\theta t} f(X_t) = f(X_0) + \int_0^t e^{-\theta s} (\mathcal{A} f(X_s) - \theta f(X_s))\, ds + \int_0^t e^{-\theta s}\, dM_s \tag{3.8}$$

for $t \in [0; \tau_0]$. It should be noted that $\mathcal{A}f(X_s)$ is not defined at the time point τ_0, but this is not essential as it is the integrand in a Lebesgue integral.

The next problem is to show that the formula is still satisfied after the time τ_0

on the set $\{X_{\tau_0} = 0, X_{\tau_0-} = 0\}$. As mentioned above the state 0 is absorbing for the diffusion part of X, so on the set $\{X_{\tau_0} = 0, X_{\tau_0-} = 0\}$ the process will stay in the state until the next jump. The time of this jump is denoted as

$$\tau^* = \inf\{t > \tau_0 | X_t \neq 0\}\,,$$

and the formula (3.8) is considered on the time interval $]\tau_0, \tau^*[$. On this interval $X \equiv 0$ so the left hand side of (3.8) is simply

$$e^{-\theta t} f(0)\,, \tag{3.9}$$

and the question is whether the right hand side can be reduced to something similar. In order to make the expression meaningful an extension of the definition of $\mathcal{A}$ to cover the situation $x = 0$ seems necessary. Hence define

$$\mathcal{A}f(0) = \lambda \int_{\mathbb{R}} (f(y) - f(0))\, G(dy)\,. \tag{3.10}$$

A change in the definition of M as well will be convenient:

$$dM_t = f'(X_{t-})\sigma\sqrt{X_{t-}}1_{(X_{t-}\neq 0)}\, dB_t + \int_{\mathbb{R}} f(X_{t-} + y) - f(X_{t-})\; M^\circ(dt, dy). \tag{3.11}$$

It is important to notice that this is equivalent to the previous definition of M on the time interval $[0, \tau_0]$ so the results obtained till now will also hold with this definition. By using (3.10) and (3.11) some straightforward calculations show that the left hand side of (3.8) reduces to (3.9) .

By this it has been shown that the equation (3.8) is true for all $t \in [0, \tau^*[$. That the formula is still true at the time point τ^* can be shown rather easily: At the left hand side the increment

$$e^{-\theta\tau^*}\,(f(X_{\tau^*}) - f(0)) \tag{3.12}$$

is observed while the right hand side is affected by the increment

$$\begin{aligned} \int_{\{\tau^*\}} e^{-\theta s}\, dM_s &= \int_{\{\tau^*\}\times\mathbb{R}} e^{-\theta u}\big(f(0+y) - f(0)\big)\big(\mu(du, dy) \\ &= e^{-\theta\tau^*}\,(f(X_{\tau^*}) - f(0))\;, \end{aligned}$$

that equals (3.12).

At time τ^* the process will leave 0 by virtue of a jump. This jump can be either positive or negative. In the first case an analogous argument will explain why (3.8) is true until the next time where the state 0 is attained and passed. So two new stopping times are defined:

$$\tau_{02} = \inf\{t > \tau^* | X_t \leq 0\} \quad \text{and} \quad \tau_2^* = \inf\{t > \tau_{02} | X_t \neq 0\}\,,$$

and similar to the previous arguments it is shown that (3.8) holds on the time intervals $[\tau^*, \tau_{02}]$ and $]\tau_{02}, \tau_2^*]$.

In the case of a negative jump – where $X_{\tau^*} < 0$ – the formula is shown in the same way as above. This is a result of the symmetry around 0 of the original Itô's formula on the interval $[0, \tau_a]$: If the starting point x is negative and the stopping time

$$\tau_a' = \inf\{t \geq 0 | X_t \geq a\}$$

is defined for some $a > x$ the formula (3.8) can be shown for all $t \in [0, \tau_a' \wedge \tau]$. By extension arguments that involve stopping times similar to τ_0 and τ^* it can be concluded that (3.8) is true until the next time 0 is passed.

Overall it has been shown that in spite of the possible movements across the state 0 the formula (3.8) is true on all the interval $[0, \tau]$. Hence

$$e^{-\theta(\tau\wedge t)} f(X_t) = f(X_0) + \int_0^{\tau\wedge t} e^{-\theta s} (\mathcal{A} f(X_s) - \theta f(X_s))\, ds + \int_0^{\tau\wedge t} e^{-\theta s}\, dM_s \tag{3.13}$$

for all $t \in \mathbb{R}$ where M is defined by (3.11).

It is left to show that the last part in (3.13) is a martingale.

Given that $t \mapsto e^{-\theta t}$ is bounded it satisfies to show that $(M_{\tau\wedge t})_{t\geq 0}$ is a martingale. Using (3.13) with $\theta = 0$ yields

$$M_{\tau\wedge t} = f(X_{\tau\wedge t}) - f(X_0) - \int_0^{\tau\wedge t} \mathcal{A} f(X_s)\, ds$$

from which it can be seen that $(M_{\tau\wedge t})_{0\leq s\leq t}$ is bounded for all $t \geq 0$ since both f and $\mathcal{A}f$ are assumed to be bounded. A further result of f being bounded is that

$$\left(\int_{[0;\tau\wedge t]\times\mathbb{R}} f(X_{s-} + y) - f(X_{s-})\ M^\circ(ds, dy) \right)_{t\geq 0}$$

is a martingale. In addition to this

$$\left(\int_0^{\tau\wedge t} f'(X_{s-}) \sigma \sqrt{|X_{s-}|} 1_{(X_{s-}\neq 0)}\, dB_s \right)_{t\geq 0}$$

is a local martingale since the integrand is predictable. Furthermore $(M_{\tau\wedge t})_{t\geq 0}$ can be written as the sum of two local martingales

$$\begin{aligned} M_{\tau\wedge t} = &\int_0^{\tau\wedge t} f'(X_{s-}) \sigma \sqrt{|X_{s-}|} 1_{(X_{s-}\neq 0)}\, dB_s \\ &+ \int_{[0;\tau\wedge t]\times\mathbb{R}} f(X_{s-} + y) - f(X_{s-})\ M^\circ(ds, dy) \end{aligned}$$

and hence it is a local martingale itself. Additionally $(M_{\tau\wedge t})_{t\geq 0}$ is shown to be bounded on finite intervals so it is a true martingale.

All together it is shown that if $f:\mathbb{R}\to\mathbb{C}$ is continuous and bounded and satisfy that if

$\ell > 0$:

- f is two times differentiable on $[\ell,\infty[$ and
- $\mathcal{A}f$ defined by

  $$\mathcal{A}f(x)=\kappa x f'(x)+\tfrac{\sigma^2}{2}|x|f''(x)+\lambda\int_{\mathbb{R}}\left(f(x+y)-f(x)\right)G(dy)$$

  for $x\in[\ell,\infty[$ is bounded, and if

$\ell < 0$:

- f is two times differentiable on $[\ell,\infty[\setminus\{0\}$ and
- $\mathcal{A}f$ defined by

  $$\mathcal{A}f(x)=\begin{cases}\kappa x f'(x)+\frac{\sigma^2}{2}|x|f''(x)+\lambda\int_{\mathbb{R}}\left(f(x+y)-f(x)\right)G(dy) & x\neq 0\\ \lambda\int_{\mathbb{R}}(f(y)-f(0))\,G(dy) & x=0\end{cases}$$

  for $x\in[l,\infty[$ is bounded,

then it holds that

$$e^{-\theta(\tau\wedge t)}f(X_t)=f(X_0)+\int_0^{\tau\wedge t}\mathcal{A}f(X_s)-\theta f(X_s)\,ds+\int_0^{\tau\wedge t}e^{-\theta s}\,dM_s \tag{3.14}$$

for all $t\in\mathbb{R}$, where the last part is a zero–mean martingale.

Now assume the function $f:\mathbb{R}\to\mathbb{C}$ to be continuous and bounded, two times differentiable on $[\ell,\infty[\setminus\{0\}$ (or $[l,\infty[$ in the $l>0$ case) and is chosen such that it is a *partial eigenfunction* for $\mathcal{A}$ on $[\ell,\infty[\setminus\{0\}$:

$$\mathcal{A}f(x)=\theta f(x)\quad\text{for all } x\in[l;\infty[\setminus\{0\}\,. \tag{3.15}$$

With this property satisfied $\mathcal{A}f$ is of course bounded. If in addition to this f in the $\ell<0$ case satisfy the condition

$$f(0)=-\frac{\lambda}{\lambda+\theta}\int_{\mathbb{R}}f(y)\,G(dy)\quad\Leftrightarrow\quad\theta f(0)=\lambda\int_{\mathbb{R}}(f(y)-f(0))\,G(dy)\,,$$

the definition $\mathcal{A}f(0)=\lambda\int_{\mathbb{R}}(f(y)-f(0))\,G(dy)$ yields that $\mathcal{A}f(0)=\theta f(0)$. By this the above conditions are satisfied such that (3.13) is true. Besides the formula can be rewritten in the following much simpler way:

$$e^{-\theta(\tau\wedge t)}f(X_t)=f(X_0)+\int_0^{\tau\wedge t}e^{-\theta s}\,dM_s\,. \tag{3.16}$$

Since the last part is a zero mean martingale taking expectation on both sides gives the following

$$\mathbb{E}_x e^{-\theta(\tau\wedge t)} f(X_{\tau\wedge t}) = f(x) \quad \text{for all } t \geq 0\,. \tag{3.17}$$

All together we have obtained the following result

Lemma 3.1. *For a function $f:\mathbb{R}\to\mathbb{C}$ the equation (3.17) holds if*

$\ell > 0$:
- *f is bounded, continuous and two times differentiable on $[\ell;\infty[$*
- *$\mathcal{A}f(y) = \theta f(y)$ for $y \in [\ell;\infty[$*
- *$f(y) = Le^{-\zeta(l-y)}$ for $y < l$*

$\ell = 0$:
- *f is bounded and continuous on $[0;\infty[$, together with two times differentiable on $]0;\infty[$*
- *$\mathcal{A}f(y) = \theta f(y)$ for $y \in]0;\infty[$*
- *$f(y) = Le^{-\zeta(\ell-y)}$ for $y < \ell$*

$\ell < 0$:
- *f is continuous and bounded on $[\ell;\infty[$*
- *f is two times differentiable on $[\ell;\infty[\setminus\{0\}$*
- *$\mathcal{A}f(y) = \theta f(y)$ for $y \in [\ell;\infty[\setminus\{0\}$*
- *$f(0) = \frac{\lambda}{\lambda+\theta}\int_{\mathbb{R}} f(u)\, G(du)$*
- *$f(y) = Le^{-\zeta(\ell-y)}$ for $y < \ell$.*

4 The Laplace Transform

In this section the joint Laplace transform is derived under the assumption that a sufficient number of partial eigenfunctions f can be found satisfying the conditions in Lemma 3.1.

The $\theta > 0$ case

Since f is assumed to be bounded the use of dominated convergence yields the following in the equation (3.17) when $t \to \infty$

$$\mathbb{E}_x e^{-\theta\tau} f(X_\tau) = f(x)\,. \tag{4.1}$$

Remark that on the set $\{\tau = \infty\}$ the expression has limit 0 since f is bounded. This means that the expectation is well–defined although there is no contribution from this set.

Hence only the set $\{\tau < \infty\}$ is of interest when taking expectation. The set is divided into the two cases $A_j = \{\tau < \infty, X_\tau < \ell\}$ and $A_c = \{\tau <$

$\infty, X_\tau = \ell\}$. A partitioning that separate the case where τ is reached as a result of a jump from the continuous case. This gives

$$\mathbb{E}_x[e^{-\theta\tau} f(X_\tau); A_j] + f(\ell)\mathbb{E}_x[e^{-\theta\tau}; A_c] = f(x)\,. \tag{4.2}$$

To make the formula less complicated it is assumed that f has the following form on $]-\infty; l[$

$$f(x) = Le^{-\zeta(\ell - x)} \qquad \text{when } x < \ell\,. \tag{4.3}$$

This yields that (4.2) can be rewritten as follows

$$L\mathbb{E}_x[e^{-\theta\tau - \zeta Z}; A_j] + f(\ell)\mathbb{E}_x[e^{-\theta\tau}; A_c] = f(x)\,. \tag{4.4}$$

It is from this equation the two requested expressions for the joint Laplace transform are attained. This requires two different partial eigenfunctions f_1 and f_2 that both satisfy the conditions and have the form (4.3) on the interval $]-\infty; l[$. From these functions two versions of the equation (4.4) emerge. From this expressions for both $\mathbb{E}_x[e^{-\theta\tau-\zeta Z}; A_j]$ and $\mathbb{E}_x[e^{-\theta\tau}; A_c]$ can be found:

$$\mathbb{E}_x[e^{-\theta\tau-\zeta Z}; A_j] = \frac{f_1(\ell)f_2(x) - f_1(x)f_2(\ell)}{L_2 f_1(\ell) - L_1 f_2(\ell)} \tag{4.5}$$

$$\mathbb{E}_x[e^{-\theta\tau}; A_c] = \frac{L_2 f_1(x) - L_1 f_2(x)}{L_2 f_1(\ell) - L_1 f_2(\ell)}\,. \tag{4.6}$$

The $\theta = 0$ case

When it is assumed that $\theta = 0$ the equation (3.17) must be treated differently. Now it has the form

$$\mathbb{E}_x f(X_{\tau\wedge t}) = f(x)$$

that can be divided into

$$\mathbb{E}_x[f(X_\tau); \tau \le t] + \mathbb{E}_x[f(X_t); \tau > t] = f(x)\,.$$

If it aside from the assumption that f is bounded is assumed that either $\lim_{t\to\infty} f(X_t) = 0$ a.s. on $\{\tau = \infty\}$ or $\mathbb{P}_x(\tau < \infty) = 1$ then using dominated convergence yields that

$$\mathbb{E}_x[f(X_\tau); \tau < \infty] = f(x)\,.$$

By dividing into the jump and the conituity cases the equation becomes the following when it is assumed that f has the form (4.3) on the interval $]-\infty; l[$

$$L\mathbb{E}_x[e^{-\zeta Z}; A_j] + f(\ell)\mathbb{P}_x(A_c) = f(x)\,. \tag{4.7}$$

If two partial eigenfunctions f_1 and f_2 with the requested properties can be found the expressions for the joint Laplace transform is as follows – similar to

(4.5) and (4.6):

$$\mathbb{E}_x[e^{-\zeta Z}; A_j] = \frac{f_1(\ell)f_2(x) - f_1(x)f_2(\ell)}{L_2 f_1(\ell) - L_1 f_2(\ell)} \tag{4.8}$$

$$\mathbb{P}_x(A_c) = \frac{L_2 f_1(x) - L_1 f_2(x)}{L_2 f_1(\ell) - L_1 f_2(\ell)}\,. \tag{4.9}$$

If $\zeta = 0$ as well (4.7) becomes even more simple

$$L\mathbb{P}_x(A_j) + f(\ell)\mathbb{P}_x(A_c) = f(x)\,. \tag{4.10}$$

5 Partial Eigenfunctions

In Section 4 an expression for the joint Laplace transform was derived. This was done under the assumption that two partial eigenfunctions f_1 and f_2 could be found. That is the assumption that two functions f_1 and f_2 satisfy the conditions formulated in Lemma 3.1.

In this section a template for how these functions could look like is made. In the next section this template is exploited more concrete. The main focus will be on the partial eigenfunction part of the conditions:

$$\mathcal{A}f(y) = \theta f(y) \qquad \text{for } y \in [\ell, \infty[\backslash\{0\}\ .$$

Subsequently the other conditions will be considered. It seems practical to decompose the jump distribution into a positive and a negative part

$$G = pG_- + qG_+,$$

where $0 < p \leq 1$, $q = 1 - p$, G_+ is a distribution on $\mathbb{R}_+ =]0;\infty[$ and G_- is a distribution on $\mathbb{R}_- =]-\infty;0[$. In addition it is assumed that G_- is a linear combination of exponential distributions:

$$G_-(du) = g_-(u)\,du = \sum_{k=1}^{r} \alpha_k \mu_k e^{\mu_k u} \qquad \text{for } u < 0\,, \tag{5.1}$$

where it is assumed that $r \in \mathbb{N}$, $0 < \mu_1 < \cdots < \mu_r$ and that the α_k's fulfil that $\alpha_k \neq 0$ and g_- is a density on $\mathbb{R}_-$. The last property induce that $\sum \alpha_k = 1$ and the fact that g_- is a density induce that $\alpha_1 > 0$.

This assumption concerning the structure of the downward jumps is essential in the further calculations since it causes that $\mathcal{A}f(y)$ attains an exponential structure. This restriction is not as hard as it might seem since the class of these distributions is dense among all distributions on $\mathbb{R}$ (with respect to the topology induced by week convergence).

It seems necessary to consider the cases $l < 0$ and $l \geq 0$ separately.

The situation where l is non–negative

Now define $f_0 : \mathbb{R} \to \mathbb{C}$ by

$$f_0(y) = \begin{cases} 0 & y \geq \ell \\ Le^{-\zeta(\ell-y)} & y < \ell \end{cases} , \tag{5.2}$$

where L is some complex constant. By straightforward calculations it can be shown that

$$(\mathcal{A} - \theta I) f_0(y) = \lambda L \sum_{k=1}^{r} \tfrac{\alpha_k \mu_k}{\mu_k + \zeta} e^{\mu_k(\ell-y)} .$$

Furthermore define $f_\Gamma : \mathbb{R} \to \mathbb{C}$ by

$$f_\Gamma(y) = \begin{cases} \int_\Gamma \psi_0(z) e^{-yz}\, dz & y \geq \ell \\ 0 & y < \ell \end{cases} , \tag{5.3}$$

where ψ_0 is some suitable complex–valued kernel and Γ is a contour in the positive part of the complex plane $\mathbb{C}_+ = \{z \in \mathbb{C} : z \geq 0\}$. Γ is assumed to have the form

$$\Gamma = \{\gamma(t) : \delta_1 < t < \delta_2\}$$

with γ a continuous function that is differentiable except at a finite number of points. An expression for $f_\Gamma(y)$ can be found by using the following substitution when $y \geq \ell$

$$f_\Gamma(y) = \int_\Gamma \psi_0(z) e^{-yz}\, dz = \int_{\delta_1}^{\delta_2} \gamma'(t) \psi_0(\gamma(t)) e^{-y\gamma(t)}\, dt . \tag{5.4}$$

The integral is well–defined if

$$\int_{|\Gamma|} |\psi_0(z)| e^{-y\mathrm{Re} z}\, dz := \int_{\delta_1}^{\delta_2} |\gamma'(t)\psi_0(\gamma(t))| e^{-y\mathrm{Re}(\gamma(t))}\, dt < \infty$$

and with the same notation it holds that

$$\int_{|\Gamma|} |\psi_0(z)| e^{-y_1 \mathrm{Re} z}\, dz \geq \int_{|\Gamma|} |\psi_0(z)| e^{-y_2 \mathrm{Re} z}\, dz$$

whenever $y_1 \leq y_2$. Furthermore it holds that

$$\Big| f_\Gamma(y) \Big| \leq \int_{|\Gamma|} |\psi_0(z)| e^{-y\mathrm{Re} z}\, dz .$$

By this it is seen that f_Γ is bounded and two times differentiable on $[\ell;\infty[$ with

$$f_\Gamma'(y) = \int_\Gamma \psi_0(z)(-z) e^{-yz}\, dz \quad \text{og} \quad f_\Gamma''(y) = \int_\Gamma \psi_0(z) z^2 e^{-yz}\, dz ,$$

when $y \in [l;\infty[$ if only

- $\int_{|\Gamma|} |\psi_0(z)| e^{-\ell \text{Re} z} \, dz < \infty$
- $\int_{|\Gamma|} |\psi_0(z)| \, |z| e^{-\ell \text{Re} z} \, dz$
- $\int_{|\Gamma|} |\psi_0(z)| \, |z|^2 e^{-\ell \text{Re} z} \, dz \, .$

The strategy of the following is to choose ψ_0 and Γ in such a way that $(\mathcal{A} - \theta I) f_\Gamma(y)$ attains the same exponential structure as $(\mathcal{A} - \theta I) f_0(y)$ when $y \in [\ell; \infty[$.

We shall assume that ψ_0 has the form

$$\begin{aligned} \psi_0(z) &= \big(\tfrac{\sigma^2}{2\kappa} z - 1\big)^{\frac{\theta}{\kappa} + \sum \frac{2p\lambda\alpha_k}{2\kappa - \mu_k \sigma^2} - 1} z^{-\frac{\theta}{\kappa} - 1} \\ & \quad \times \left(\prod_{k=1}^{r} (z - \mu_k)^{-\frac{2p\lambda\alpha_k}{2\kappa - \mu_k \sigma^2}} \right) \exp\left(\frac{q\lambda}{\kappa} F(z) \right) \end{aligned} \tag{5.5}$$

with $F(z)$ some primitive of

$$\frac{h(z)}{\frac{\sigma^2}{2\kappa} z - 1} \, .$$

With these definitions we have the result:

Theorem 5.1. *Let $\Gamma = \{\gamma(t) : \delta_1 < t < \delta_2\} \subseteq \mathbb{C}_+$ be a complex curve with γ a continuous function that is differentiable except at a finite amount of points. Assume that a version of ψ_0 (given by (5.5)) exists that is holomorphic in an area containing Γ. Assume furthermore that*

(i) $\int_{|\Gamma|} |\psi_0(z)| e^{-\ell \text{Re} z} \, dz < \infty$

(ii) $\int_{|\Gamma|} |\psi_0(z)| \, |z| e^{-\ell \text{Re} z} \, dz < \infty$

(iii) $\int_{|\Gamma|} |\psi_0(z)| \, |z|^2 e^{-\ell \text{Re} z} \, dz < \infty$

(iv) $\int_{|\Gamma|} |\frac{\psi_0(z)}{z - \mu_k}| e^{-\ell \text{Re} z} \, dz < \infty$

(v) $\psi_0(\gamma(\delta_1)) \gamma(\delta_1) \big(\kappa - \frac{\sigma^2}{2} \gamma(\delta_1)\big) e^{-y\gamma(\delta_1)} = \psi_0(\gamma(\delta_2)) \gamma(\delta_2) \big(\kappa - \frac{\sigma^2}{2} \gamma(\delta_2)\big) e^{-y\gamma(\delta_2)}$ *for $y \geq \ell$.*

Then with f_Γ defined by

$$f_\Gamma(y) = \begin{cases} \int_\Gamma \psi_0(z) e^{-yz} \, dz & y \geq l \\ 0 & y < \ell \end{cases} \, ,$$

it holds for $y \geq l$ that

$$(\mathcal{A} - \theta I) f_\Gamma(y) = p\lambda \sum_{k=1}^{r} \alpha_k \mu_k \Big(\int_\Gamma \frac{\psi_0(z)}{z - \mu_k} e^{-\ell z} \, dz \Big) e^{\mu_k(\ell - y)} \, ,$$

where

$$M^k_{\Gamma_i} = \int_{\Gamma_i} \frac{\psi_0(z)}{z-\mu_k} e^{-\ell z}\, dz\,. \tag{5.6}$$

Proof. For $y \geq \ell$ it is seen that

$$\begin{aligned}
(\mathcal{A}-\theta I) f_\Gamma(y) \quad = \quad & y \int_\Gamma \psi_0(z) z(\tfrac{\sigma^2}{2} z - \kappa) e^{-yz}\, dz \\
& + p\lambda \int_{\ell - y}^0 \sum_{k=1}^r \alpha_k \mu_k e^{\mu_k u} \int_\Gamma \psi_0(z) e^{-(y+u)z}\, dz\, du \\
& + q\lambda \int_0^\infty \int_\Gamma \psi_0(z) e^{-(y+u)z}\, dz\, G_+(du) \\
& -(\lambda+\theta) \int_\Gamma \psi_0(z) e^{-yz}\, dz\,.
\end{aligned} \tag{5.7}$$

A substitution similar to (5.4) used on the first term yields that it equals

$$y \int_{\delta_1}^{\delta_2} \gamma'(t) \psi_0(\gamma(t)) \gamma(t) (\tfrac{\sigma^2}{2} \gamma(t) - \kappa) e^{-y\gamma(t)}\, dt$$

and by partial integration and another substitution this expression

$$= - \int_\Gamma \Big(\psi_0(z)(\kappa - \sigma^2 z) + \psi_0'(z) z (\kappa - \tfrac{\sigma^2}{2} z) \Big) e^{-yz}\, dz$$

under the condition that

$$\psi_0(\gamma(\delta_1))\gamma(\delta_1)\big(\kappa - \tfrac{\sigma^2}{2}\gamma(\delta_1)\big) e^{-y\gamma(\delta_1)} = \psi_0(\gamma(\delta_2))\gamma(\delta_2)\big(\kappa - \tfrac{\sigma^2}{2}\gamma(\delta_2)\big) e^{-y\gamma(\delta_2)}\,.$$

Straightforward calculations applied to the second term in (5.7) gives that it

$$\begin{aligned}
= \quad & p\lambda \int_\Gamma \psi_0(z) \Big(\sum_{k=1}^r \frac{\alpha_k \mu_k}{\mu_k - z} \Big) e^{-yz}\, dz \\
& + p\lambda \sum_{k=1}^r \alpha_k \mu_k \Big(\frac{\psi_0(z)}{z - \mu_k} e^{-\ell z}\, dz \Big) e^{\mu_k(\ell - y)}\,.
\end{aligned}$$

Now define L_+ as the generalised Laplace transform for the G_+ distribution:

$$L_+(z) = \int_0^\infty e^{-zu}\, G_+(du)\,.$$

This is well-defined for all complex numbers z with $\mathrm{Re} z \geq 0$. By this definition the third term from (5.7) can be rewritten in the following way

$$q\lambda \int_\Gamma \psi_0(z) L_+(z) e^{-yz}\, dz\,.$$

Altogether (5.7) has become

$$
\begin{aligned}
(\mathcal{A}-\theta I)f_\Gamma(y) \quad = \quad & -\int_\Gamma \Big(\psi_0(z)(\kappa-\sigma^2 z)+\psi_0'(z)z(\kappa-\tfrac{\sigma^2}{2}z)\Big)e^{-yz}\,dz \\
& +p\lambda\int_\Gamma \psi_0(z)\Big(\sum_{k=1}^{r}\frac{\alpha_k\mu_k}{\mu_k-z}\Big)e^{-yz}\,dz \\
& +p\lambda\sum_{k=1}^{r}\alpha_k\mu_k\Big(\int_\Gamma \frac{\psi_0(z)}{z-\mu_k}e^{-\ell z}\,dz\Big)e^{\mu_k(l-y)} \\
& +q\lambda\int_\Gamma \psi_0(z)L_+(z)e^{-yz}\,dz \\
& -(\lambda+\theta)\int_\Gamma \psi_0(z)e^{-yz}\,dz\,. \qquad (5.8)
\end{aligned}
$$

If all terms including an integral of the form $\int_\Gamma(\cdots)e^{-yz}\,dz$ has the sum 0 (5.8) is reduced to

$$
(\mathcal{A}-\theta I)f_\Gamma(y)=p\lambda\sum_{k=1}^{r}\alpha_k\mu_k\Big(\int_\Gamma\frac{\psi_0(z)}{z-\mu_k}e^{-\ell z}\,dz\Big)e^{\mu_k(l-y)}\,.
$$

That result is achieved if the integration kernel ψ_0 solves the differential equation

$$
\psi_0(z)(\kappa-\sigma^2 z)+\psi_0'(z)z(\kappa-\tfrac{\sigma^2}{2}z)=\psi_0(z)\Big(-(\lambda+\theta)+\sum_{k=1}^{r}p\lambda\alpha_k\frac{\mu_k}{\mu_k-z}+q\lambda L_+(z)\Big)\,,
$$

that is equivalent with the equation

$$
\psi_0'(z)=\frac{1}{\kappa z(\frac{\sigma^2}{2\kappa}z-1)}\left(\theta+\kappa-\sigma^2 z+\sum_{k=1}^{r}p\lambda\alpha_k\frac{z}{z-\mu_k}+q\lambda z h(z)\right)\psi_0(z)\,, \quad (5.9)
$$

where

$$
h(z)=\frac{1-L_+(z)}{z}\,.
$$

Since the kernel ψ_0 solves this equation the result from the Theorem has been proved. □

It has been shown that for both f_0 and f_Γ it holds that $(\mathcal{A}-\theta I)f_0$ and $(\mathcal{A}-\theta I)f_\Gamma$ respectively are linear combinations of the exponential functions $y\mapsto e^{\mu_k(\ell-y)}$. A solution strategy is to define (the potential partial eigenfunction) f as some suitable linear combination of f_0 and the f_Γ–functions. With this idea in mind we have the following result from the use of Theorem 5.1:

Theorem 5.2. *Let $\theta \geq 0$ and $\zeta \geq 0$ be given, and assume that f_0 and f_{Γ_i} for $i = 1, \ldots, m$ are defined as in Theorem 5.1 such that the conditions $(i) - (v)$ are satisfied. Define*

$$f(y) = \sum_{i=1}^{m} c_i f_{\Gamma_i}(y) + f_0(y)\,. \tag{5.10}$$

If the constants $c_1, \ldots, c_m$ and L solve the equations

$$\sum_{i=1}^{m} c_i M_{\Gamma_i}^k + L\frac{1}{\mu_k + \zeta} = 0 \tag{5.11}$$

for $k = 1, \ldots, r$, then f is a partial eigenfunction for the generator $\mathcal{A}$ on $[l, \infty[$:

$$(\mathcal{A} - \theta I)f(y) = 0 \quad \textit{for } y \in [\ell, \infty[\tag{5.12}$$

Remark 5.1. *The theorem explains what it takes to construct partial eigenfunctions: Sufficiently many f_{Γ_i}–functions should be constructed in order to be able to construct the equations (5.11). That boils down to finding sufficiently many different integration contours Γ_i for the kernel ψ_0. It is a homogeneous linear equation system consisting of r equations and the $m+1$ unknowns $c_1, \ldots, c_m$ and L. Hence $m = r$ contours are necessary to construct a partial eigenfunction.*

Since two partial eigenfunctions f_1 and f_2 are needed to solve the equations (4.4) and (4.7) at least one of the integration contours in f_2 should be different from the ones in f_1. All together that takes $m = r + 1$ contours.

Furthermore it is worth considering which amount of difference is required between the $m = r+1$ integration contours. E.g.: Obviously the same contour should not be reused several times. Here it is crucial that the equation system given by (5.11) can be solved w.r.t $c_1, \ldots, c_m$ and L. Hence the vectors of constants

$$M_{\Gamma_i} = (M_{\Gamma_i}^1, \ldots, M_{\Gamma_i}^m)$$

associated to the contours Γ_i for $i = 1, \ldots, r$ should be linearly independent. Otherwise this would correspond to a situation with less unknowns in the equation system.

When ℓ is negative

In this case it is necessary to make assumptions about both the upward and downward jumps. So assume that the positive part of the jump has the same structure as the negative:

$$G_+(\mathrm{d}u) = g_+(u)\mathrm{d}u = \sum_{d=1}^{s} \beta_d \nu_d \mathrm{e}^{-\nu_d u}\,.$$

Furthermore define the function f_0 as before, but replace the definition of f_Γ with the following

$$f^1_{\Gamma_1}(y) = \begin{cases} \int_{\Gamma_1} \psi_1(z) e^{-yz}\, dz & y \geq 0 \\ 0 & y < 0 \end{cases} \tag{5.13}$$

$$f^2_{\Gamma_2}(y) = \begin{cases} \int_{\Gamma_2} \psi_2(z) e^{-yz}\, dz & \ell \leq y \leq 0 \\ 0 & \text{otherwise} \end{cases}, \tag{5.14}$$

where $\Gamma_1 \subset \mathbb{C}_+$, $\Gamma_2 \subset \mathbb{C}$ are contours as before and ψ_1 and ψ_2 are integration kernels given by

$$\begin{aligned} \psi_1(z) &= \left(\tfrac{\sigma^2}{2\kappa} z - 1\right)^{\frac{\theta}{\kappa} + \sum^r \frac{2p\lambda\alpha_k}{2\kappa - \mu_k\sigma^2} + \sum^s \frac{2q\lambda\beta_d}{2\kappa + \nu_d\sigma^2} - 1} z^{-\frac{\theta}{\kappa} - 1} \\ &\quad \times \left(\prod_{k=1}^r (z - \mu_k)^{-\frac{2p\lambda\alpha_k}{2\kappa - \mu_k\sigma^2}} \right) \left(\prod_{d=1}^s (z + \nu_d)^{-\frac{2q\lambda\beta_d}{2\kappa + \nu_d\sigma^2}} \right) \end{aligned}$$

and

$$\begin{aligned} \psi_2(z) &= \left(\tfrac{-\sigma^2}{2\kappa} z - 1\right)^{\frac{\theta}{\kappa} + \sum^r \frac{2p\lambda\alpha_k}{2\kappa + \mu_k\sigma^2} + \sum^s \frac{2q\lambda\beta_d}{2\kappa - \nu_d\sigma^2} - 1} z^{-\frac{\theta}{\kappa} - 1} \\ &\quad \times \left(\prod_{k=1}^r (z - \mu_k)^{-\frac{2p\lambda\alpha_k}{2\kappa + \mu_k\sigma^2}} \right) \left(\prod_{d=1}^s (z + \nu_d)^{-\frac{2q\lambda\beta_d}{2\kappa - \nu_d\sigma^2}} \right). \end{aligned}$$

Definition 5.1. *For convenience we shall use the following definitions*

$$\begin{aligned} M^{1k}_{\Gamma_i} &= \int_{\Gamma_{i1}} \frac{\psi_1(z)}{z - \mu_k} dz \qquad i = 1, \ldots, m, \quad k = 1, \ldots, r \\ M^{2d}_{\Gamma_i} &= \int_{\Gamma_{i1}} \frac{\psi_1(z)}{\nu_d + z} dz \qquad i = 1, \ldots, m, \quad d = 1, \ldots, s \\ N^{1k}_{\Gamma_j} &= \int_{\Gamma_{j2}} \frac{\psi_2(z)}{\mu_k - z} dz \qquad j = 1, \ldots, n, \quad k = 1, \ldots, r \\ N^{2d}_{\Gamma_j} &= \int_{\Gamma_{j2}} \frac{\psi_2(z)}{\nu_d + z} dz \qquad j = 1, \ldots, n, \quad d = 1, \ldots, s \\ N^{3k}_{\Gamma_j} &= \int_{\Gamma_{j2}} \frac{\psi_2(z)}{z - \mu_k} e^{-lz} dz \qquad j = 1, \ldots, n, \quad k = 1, \ldots, r\,. \end{aligned}$$

Similar to the conditions $(i)-(v)$ in Theorem 5.1 we will refer to the conditions in the Notation below.

Notation 5.1. *Let $\theta \geq 0$, $\zeta \geq 0$ be given and let f_0, $f^1_{\Gamma_{i1}}$ and $f^2_{\Gamma_{j2}}$ be defined as above for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ such that for all $f^t_{\Gamma_{it}}$ it is true that $\Gamma_{i1} = \{\gamma_{i1}(u) : \delta^{i1}_1 < u < \delta^{i1}_2\} \subseteq \mathbb{C}_+$ and $\Gamma_{i2} = \{\gamma_{i2}(u) : \delta^{i2}_1 < u < \delta^{i2}_2\} \subseteq \mathbb{C}$ are complex curves with γ_{it} continuous functions that are differentiable except at finitely many points for $t = 1, 2$. Furthermore assume that there exists a* *holomorphic version of each of the ψ-kernels that contains the relevant contour Γ.*

Assume for ψ_1 and Γ_{i1}, $i = 1, \ldots, m$, that

(i) $\int_{|\Gamma_{i1}|} |\psi_1(z)|\, dz < \infty$

(ii) $\int_{|\Gamma_{i1}|} |\psi_1(z)|\, |z| e^{-y\mathrm{Re}z}\, dz < \infty$ *for all* $y > 0$

(iii) $\int_{|\Gamma_{i1}|} |\psi_1(z)|\, |z|^2 e^{-y\mathrm{Re}z}\, dz < \infty$ *for all* $y > 0$

(iv) $\int_{|\Gamma_{i1}|} |\frac{\psi_1(z)}{z-\mu_k}|\, dz < \infty$ *for* $k = 1, \ldots, r$

(v) $\int_{|\Gamma_{i1}|} |\frac{\psi_1(z)}{z+\nu_d}|\, dz < \infty$ *for* $d = 1, \ldots, s$

(vi)
$$\psi_1(\gamma_{i1}(\delta_{i1}))\gamma_{i1}(\delta^1_{i1})\big(\kappa - \tfrac{\sigma^2}{2}\gamma_{i1}(\delta^1_{i1})\big)e^{-y\gamma_{i1}(\delta^1_{i1})}$$
$$= \psi_1(\gamma_{i1}(\delta^1_{i2}))\gamma_1(\delta^1_{i2})\big(\kappa - \tfrac{\sigma^2}{2}\gamma_{i1}(\delta^1_{i2})\big)e^{-y\gamma_{i1}(\delta^1_{i2})}$$
for all $y > 0$

and similarly for ψ_2 and Γ_{j2} for $j = 1, \ldots, n$:

(i') $\int_{|\Gamma_{i2}|} |\psi_2(z)|\, dz < \infty$

(ii') $\int_{|\Gamma_{i2}|} |\psi_2(z)| e^{-\ell Rez}\, dz < \infty$

(iii') $\int_{|\Gamma_{i2}|} |\psi_2(z)|\, |z| e^{-y\mathrm{Re}z}\, dz < \infty$ *for all* $y \in [\ell; 0[$

(iv') $\int_{|\Gamma_{i2}|} |\psi_2(z)|\, |z|^2 e^{-y\mathrm{Re}z}\, dz < \infty$ *for all* $y \in [\ell; 0[$

(v') $\int_{|\Gamma_{i2}|} |\frac{\psi_2(z)}{z-\mu_k}|\, dz < \infty$ *for* $k = 1, \ldots, r$

(vi') $\int_{|\Gamma_{i2}|} |\frac{\psi_2(z)}{z-\mu_k}| e^{-\ell z}\, dz < \infty$ *for* $k = 1, \ldots, r$

(vii') $\int_{|\Gamma_{i2}|} |\frac{\psi_2(z)}{z+\nu_d}|\, dz < \infty$ *for* $d = 1, \ldots, s$

(viii')
$$\psi_2(\gamma_{j2}(\delta^2_{j1}))\gamma_{j2}(\delta^2_{j1})\big(\kappa + \tfrac{\sigma^2}{2}\gamma_{j2}(\delta^2_{j1})\big)e^{-y\gamma_{j2}(\delta^2_{j1})}$$
$$= \psi_2(\gamma_{j2}(\delta^2_{j2}))\gamma_{j2}(\delta^2_{j2})\big(\kappa + \tfrac{\sigma^2}{2}\gamma_{j2}(\delta^2_{j2})\big)e^{-y\gamma_{j2}(\delta^2_{j2})}$$
for all $y \in [\ell; 0[$.

With all these definitions we can state and prove the following:

Theorem 5.3. *Assume that the integration contours Γ_{i1}, $i = 1, \ldots, m$ and Γ_{j2}, $j = 1, \ldots, n$ satisfy the conditions in Notation 5.1. Define $f : \mathbb{R} \to \mathbb{C}$ by*

$$f(y) = \sum_{i=1}^{m} c_i f^1_{\Gamma_{i1}}(y) + \sum_{j=1}^{m} b_j f^2_{\Gamma_{j2}}(y) + f_0(y) \quad \text{for } y \in [l; \infty[\setminus\{0\}\,, \tag{5.15}$$

then f is bounded and two times differentiable on $[l, \infty[\setminus\{0\}$. If the constants $c_1, \ldots, c_m$, $b_1, \ldots, b_n$ and L solves the following equations

$$\sum_{j=1}^{n} b_j N^{3k}_{\Gamma_j} + L\frac{1}{\mu_k + \zeta} = 0 \tag{5.16}$$

and

$$\left(\sum_{i=1}^{m} c_i M^{1k}_{\Gamma_i}\right) + \left(\sum_{j=1}^{n} b_j N^{1k}_{\Gamma_j}\right) = 0 \tag{5.17}$$

for $k = 1, \ldots, r$, and furthermore

$$\left(\sum_{j=1}^{n} b_j N^{2d}_{\Gamma_j}\right) - \left(\sum_{i=1}^{m} c_i M^{2d}_{\Gamma_j}\right) = 0 \tag{5.18}$$

for $d = 1, \ldots, s$, then f is a partial eigenfunction for the generator $\mathcal{A}$ on $[\ell, \infty[\setminus\{0\}$:

$$(\mathcal{A} - \theta I)f(y) = 0 \quad \text{for all } y \in [\ell, \infty[\setminus\{0\}\,.$$

Finally f is continuous in 0 with the value $f(0) = \frac{\lambda}{\lambda+\theta}\int_{\mathbb{R}} f(u)\, G(\mathrm{d}u)$ if furthermore

$$\begin{aligned}\sum_{i=1}^{m} c_i f^1_{\Gamma_{i1}}(0) \quad &= \quad \frac{\lambda}{\lambda+\theta}\Bigg(\sum_{i=1}^{m} c_i \int_0^\infty f^1_{\Gamma_{i1}}(y)\, G(dy) \\ &\quad + \sum_{j=1}^{n} b_j \int_l^0 f^2_{\Gamma_{j1}}(y)\, G(dy) + Lp\sum_{k=1}^{r} \frac{\alpha_k\mu_k}{\eta+\mu_k} e^{l\mu_k}\Bigg)\end{aligned} \tag{5.19}$$

and

$$\begin{aligned}\sum_{j=1}^{n} b_j f^2_{\Gamma_{j2}}(0) \quad &= \quad \frac{\lambda}{\lambda+\theta}\Bigg(\sum_{i=1}^{m} c_i \int_0^\infty f^1_{\Gamma_{i1}}(y)\, G(dy) \\ &\quad + \sum_{j=1}^{n} b_j \int_l^0 f^2_{\Gamma_{j1}}(y)\, G(dy) + Lp\sum_{k=1}^{r} \frac{\alpha_k\mu_k}{\eta+\mu_k} e^{l\mu_k}\Bigg)\end{aligned} \tag{5.20}$$

Proof. Under the conditions in Notation 5.1 we have for $y > 0$

$$\begin{aligned}
(\mathcal{A}-\theta I)f^1_{\Gamma_1}(y) &= \kappa y \int_{\Gamma_1} \psi_1(z)(-z)e^{-yz}\,dz \\
&\quad + p\lambda \int_{-y}^{0} \sum_{k=1}^{r} \alpha_k \mu_k e^{\mu_k u} \int_{\Gamma_1} \psi_1(z) e^{-(y+u)z}\,dz\,du \\
&\quad + q\lambda \int_{0}^{\infty} \sum_{d=1}^{s} \beta_d \nu_d e^{\nu_d u} \int_{\Gamma_1} \psi_1(z) e^{-(y+u)z}\,dz\,du \\
&\quad - (\lambda+\theta) \int_{\Gamma_1} \psi_1(z) e^{-yz}\,dz \\
&\quad + \tfrac{\sigma^2}{2} y \int_{\Gamma_1} \psi_1(z) z^2 e^{-yz}\,dz\,. \qquad (5.21)
\end{aligned}$$

The second term can be rephrased as

$$\begin{aligned}
&= p\lambda \int_{\Gamma_1} \psi_1(z) \sum_{k=1}^{r} \int_{-y}^{0} \alpha_k\mu_k e^{\mu_k u} e^{-(y+u)z} du\,dz \\
&= p\lambda \int_{\Gamma_1} \psi_1(z) \sum_{k=1}^{r} \int_{-y}^{0} \alpha_k\mu_k e^{u(\mu_k - z)} du\, e^{-yz} dz \\
&= p\lambda \int_{\Gamma_1} \psi_1(z) \sum_{k=1}^{r} \left(\frac{\alpha_k\mu_k}{\mu_k - z} - \frac{\alpha_k\mu_k}{\mu_k - z} e^{y(\mu_k - z)} \right) e^{-yz} dz \\
&= p\lambda \int_{\Gamma_1} \psi_1(z) \left(\sum_{k=1}^{r} \frac{\alpha_k\mu_k}{\mu_k - z} \right) e^{-yz} dz \\
&\quad + p\lambda \sum_{k=1}^{r} \alpha_k\mu_k \left(\int_{\Gamma_1} \frac{\psi_1(z)}{z-\mu_k}\,dz \right) e^{-\mu_k y}\,.
\end{aligned}$$

If it as in the $\ell \geq 0$ case is assumed that

$$\psi_1(\gamma_1(\delta_1))\gamma_1(\delta_1^1)\big(\kappa - \tfrac{\sigma^2}{2}\gamma_1(\delta_1^1)\big)e^{-y\gamma_1(\delta_1^1)} = \psi_1(\gamma_1(\delta_2^1))\gamma_1(\delta_2^1)\big(\kappa - \tfrac{\sigma^2}{2}\gamma_1(\delta_2^1)\big)e^{-y\gamma_1(\delta_2^1)}\,.$$

we can obtain the following result

$$\begin{aligned}
&(\mathcal{A}-\theta I)f^1_{\Gamma_1}(y) \\
&= -\int_{\Gamma_1} \Big(\psi_1(z)(\kappa - \sigma^2 z) + \psi_1'(z) z (\kappa - \tfrac{\sigma^2}{2} z) \Big) e^{-yz}\,dz \\
&\quad + p\lambda \int_{\Gamma_1} \psi_1(z) \Big(\sum_{k=1}^{r} \frac{\alpha_k\mu_k}{\mu_k - z} \Big) e^{-yz}\,dz + p\lambda \sum_{k=1}^{r} \alpha_k\mu_k \left(\int_{\Gamma_1} \frac{\psi_1(z)}{z-\mu_k}\,dz \right) e^{-\mu_k y} \\
&\quad + q\lambda \int_{\Gamma_1} \psi_1(z) L_+(z) e^{-yz}\,dz - (\lambda+\theta) \int_{\Gamma_1} \psi_1(z) e^{-yz}\,dz
\end{aligned}$$

and since ψ_1 solves the differential equation (5.9) we have

$$(\mathcal{A}-\theta I)f^1_{\Gamma_1}(y) = p\lambda\sum_{k=1}^{r}\alpha_k\mu_k\left(\int_{\Gamma_1}\frac{\psi_1(z)}{z-\mu_k}\,dz\right)e^{-\mu_k y}\,. \qquad (5.22)$$

For $\ell \leq y < 0$ we have (recall that $f^1_{\Gamma_1}(y)$ $f'^1_{\Gamma_1}(y)$ and $f''^1_{\Gamma_1}(y)$ are 0)

$$\begin{aligned}
(\mathcal{A}-\theta I)f^1_{\Gamma_1}(y) &= q\lambda\int_{-y}^{\infty}\int_{\Gamma_1}\psi_1(z)e^{-(y+u)z}dz\,G_+(du)\\
&= q\lambda\int_{\Gamma_1}\psi_1(z)\sum_{d=1}^{s}\int_{-y}^{\infty}\beta_d\nu_d e^{-u(\nu_d+z)}du\,e^{-yz}\,dz\\
&= q\lambda\int_{\Gamma_1}\psi_1(z)\sum_{d=1}^{s}\left[\frac{\beta_d\nu_d}{\nu_d+z}e^{-u(\nu_d+z)}\right]_{-y}^{\infty}e^{-yz}\,dz\\
&= -q\lambda\sum_{d=1}^{s}\beta_d\nu_d\left(\int_{\Gamma_1}\frac{\psi_1(z)}{\nu_d+z}\,dz\right)e^{\nu_d y}\,. \qquad (5.23)
\end{aligned}$$

Considering $f^2_{\Gamma_2}$ when $y \geq 0$ yields

$$\begin{aligned}
(\mathcal{A}-\theta I)f^2_{\Gamma_2}(y) &= p\lambda\int_{l-y}^{-y}\sum_{k=1}^{r}\alpha_k\mu_k e^{\mu_k y}\int_{\Gamma_2}\psi_2(z)e^{-(y+u)z}dz\,du\\
&= p\lambda\int_{\Gamma_2}\psi_2(z)\sum_{k=1}^{r}\int_{\ell-y}^{-y}\alpha_k\mu_k e^{\mu_k u}e^{-(y+u)z}du\,dz\\
&= p\lambda\int_{\Gamma_2}\psi_2(z)\sum_{k=1}^{r}\int_{\ell-y}^{-y}\alpha_k\mu_k e^{u(\mu_k-z)}du\,e^{-yz}dz\\
&= p\lambda\int_{\Gamma_2}\psi_2(z)\sum_{k=1}^{r}\frac{\alpha_k\mu_k}{\mu_k-z}\left(e^{-y(\mu_k-z)}-e^{(\ell-y)(\mu_k-z)}\right)e^{-yz}dz\\
&= p\lambda\sum_{k=1}^{r}\alpha_k\mu_k\left(\int_{\Gamma_2}\frac{\psi_2(z)}{\mu_k-z}\,dz\right)e^{-\mu_k y}\\
&\quad +p\lambda\sum_{k=1}^{r}\alpha_k\mu_k\left(\int_{\Gamma_2}\frac{\psi_2(z)}{z-\mu_k}e^{-\ell z}\,dz\right)e^{\mu_k\ell}e^{-\mu_k y}\,, \qquad (5.24)
\end{aligned}$$

and when $\ell \leq y < 0$ we get

$$\begin{aligned}(\mathcal{A}-\theta I)f^2_{\Gamma_2}(y) = \quad & \kappa y \int_{\Gamma_2} \psi_2(z)(-z)e^{-yz}\,dz \\ & +p\lambda \int_{\ell-y}^{0} \sum_{k=1}^{r} \alpha_k \mu_k e^{\mu_k u} \int_{\Gamma_2} \psi_2(z) e^{-(y+u)z}\,dz\,du \\ & +q\lambda \int_0^{-y} \int_{\Gamma_2} \psi_2(z) e^{-(y+u)z}\,dz\,G_+(du) \\ & -(\lambda+\theta)\int_{\Gamma_2} \psi_2(z)e^{-yz}\,dz \\ & -\tfrac{\sigma^2}{2} y \int_{\Gamma_2} \psi_2(z) z^2 e^{-yz}\,dz\,, \qquad (5.25)\end{aligned}$$

where the third term can be rewritten as

$$\begin{aligned} = \quad & q\lambda \int_{\Gamma_2} \psi_2(z) L_+(z) e^{-yz}\,dz\,G_+(du) \\ & +q\lambda \sum_{d=1}^{s} \beta_d \nu_d \left(\int_{\Gamma_2} \frac{\psi_2(z)}{\nu_d + z}\,dz \right) e^{\nu_d y}\,. \end{aligned}$$

With an argument similar to the one before we have that

$$\begin{aligned}(\mathcal{A}-\theta I)f^2_{\Gamma_2}(y) \quad = \quad & p\lambda \sum_{k=1}^{r} \alpha_k \mu_k \left(\int_{\Gamma_2} \frac{\psi_2(z)}{z-\mu_k} e^{-lz}\,dz \right) e^{-\mu_k(y-l)} \\ & +q\lambda \sum_{d=1}^{s} \beta_d \nu_d \left(\int_{\Gamma_2} \frac{\psi_2(z)}{\nu_d+z}\,dz \right) e^{\nu_d y}\end{aligned}$$

if

$$\begin{aligned} & \psi_2(\gamma_2(\delta_1^2))\gamma_2(\delta_1^2)\big(\kappa + \tfrac{\sigma^2}{2}\gamma_2(\delta_1^2)\big)e^{-y\gamma_2(\delta_1^2)} \\ & = \psi_2(\gamma_2(\delta_2^2))\gamma_2(\delta_2^2)\big(\kappa + \tfrac{\sigma^2}{2}\gamma_2(\delta_2^2)\big)e^{-y\gamma_2(\delta_2^2)}\,. \end{aligned}$$

Collecting the results attained so far and using the definitions in Notation 5.1 completes the proof of the partial eigenfunction part.

The two final conditions (5.19) and (5.20) follow immediately if f is adjusted to have the same limit from left and right in 0. □

Remark 5.2. *It has been shown that with f defined as in (5.15) the equation (3.17) is true if the conditions in Notation 5.1 are fulfilled and the constants $c_1, \ldots, c_m$, $b_1, \ldots, b_n$ and L satisfy the equations (5.16)-(5.20).*

The equations (5.16)-(5.20) form together $2r+s+2$ equations in the $m+n+1$ unknowns $c_1, \ldots, c_m$, $b_1, \ldots, b_n$ and L. Hence the total number of integration contours represented by $m+n$ should satisfy $m+n = 2r+s+2$. Since two *functions f_1 and f_2 are needed when the joint Laplace transform is derived an additional contour is needed in order to construct two different eigenfunctions. Hence*

$$m + n = 2r + s + 3$$

is necessary.

Similar to Remark 5.1 it should be discussed what it takes to the contours to be counted as different. For a contours Γ_{i1} the vectors

$$M_{\Gamma_{i1}} = (M^{11}_{\Gamma_{i1}}, \ldots, M^{1r}_{\Gamma_{i1}}, M^{21}_{\Gamma_{i1}}, \ldots, M^{2s}_{\Gamma_{i1}})$$

needs to be linearly independent. Equivalent for the contours of the form Γ_{j2} the vectors

$$N_{\Gamma_{j2}} = (N^{11}_{\Gamma_{j2}}, \ldots, N^{1r}_{\Gamma_{j2}}, N^{21}_{\Gamma_{j2}}, \ldots, N^{2s}_{\Gamma_{j2}}, N^{31}_{\Gamma_{j2}}, \ldots, N^{3r}_{\Gamma_{j2}})$$

are supposed to be linearly independent.

6 Integration contours

In Section 4 it was shown that with two partial eigenfunctions satisfying the conditions in Lemma 3.1 an expression for the joint Laplace transform can be derived. In Section 5 a template for these functions were made. It was seen that if an adequate number of integration contours can be found satisfying the conditions in Notation 5.1 then the two eigenfunctions can be constructed. In this Section we shall se one way to choose these contours.

Once again we consider the situations $\ell \geq 0$ and $\ell < 0$ separately.

Non–negative ℓ

Here we will assume that the upward jumps have the same form as the downward:

$$G_+(\mathrm{d}u) = g_+(u)\mathrm{d}u = \sum_{d=1}^{s} \beta_d \nu_d \mathrm{e}^{-\nu_d u}\,.$$

Then ψ_0 becomes similar two ψ_1 in the $\ell < 0$ case:

$$\begin{aligned}\psi_0(z) \quad &= \quad \left(\tfrac{\sigma^2}{2\kappa} z - 1\right)^{\frac{\theta}{\kappa} + \sum^r \frac{2p\lambda\alpha_k}{2\kappa - \mu_k\sigma^2} + \sum^s \frac{2q\lambda\beta_d}{2\kappa + \nu_d\sigma^2} - 1} z^{-\frac{\theta}{\kappa} - 1} \\ &\quad \times \left(\prod_{k=1}^{r} (z - \mu_k)^{-\frac{2p\lambda\alpha_k}{2\kappa - \mu_k\sigma^2}}\right)\left(\prod_{d=1}^{s} (z + \nu_d)^{-\frac{2q\lambda\beta_d}{2\kappa + \nu_d\sigma^2}}\right) \qquad (6.1)\end{aligned}$$

and we are looking for $m = r + 1$ integration contours $\Gamma_1, \ldots, \Gamma_m$ for ψ_0. Note that

$$|\psi_0(z)| = O\left(|z|^{-2}\right) \qquad \text{when } |z| \to \infty\,. \qquad (6.2)$$

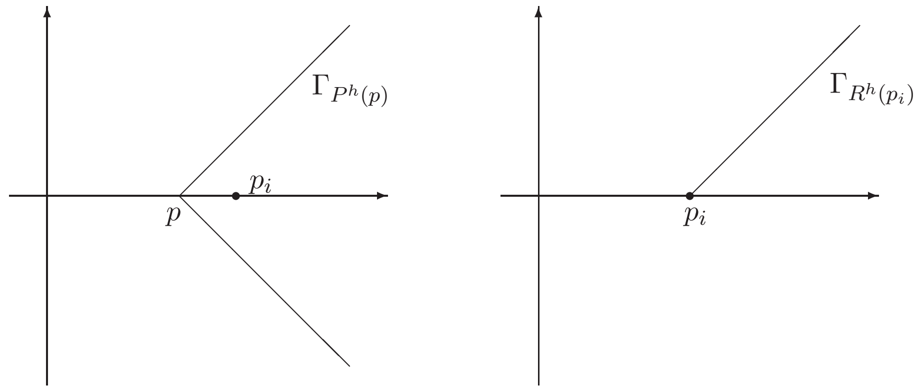

Figure A.1: Left: Shows Γ_i in case p_i is a singularity. Right: Shows Γ_i in case p_i is a zero.

Also note that each of the points $0, \frac{2\kappa}{\sigma^2}, \mu_1, \ldots, \mu_r, -\nu_1, \ldots, -\nu_s$ are either zeros or singularities for ψ_0 depending on the parameters of the model. As a result of that we have to consider the cases positive drift ($\kappa > 0$) and negative drift ($\kappa < 0$) separately. Whether $\theta > 0$ or $\theta = 0$ might also have influence on the considerations.

Positive drift, $\theta \geq 0$:

We are looking for $r + 1$ integration contours contained in $\mathbb{C}_+$ that satisfy the conditions $(i) - (v)$ in Theorem 5.1. The fifth condition demands that the function $z(\kappa - \frac{\sigma^2}{2}z)\psi_0(z)\mathrm{e}^{-yz}$ has the same value in both ends of the contours Γ. In practice this condition will be met by choosing the endpoints as zeros for the function. Here the exponential factor e^{-yz} will be useful since an endpoint could be a limit where $|z| \to \infty$.

Since it is assumed that $\kappa > 0$ we have that $\frac{2\kappa}{\sigma^2} > 0$ and that 0 is a singularity for $z(\kappa - \frac{\sigma^2}{2}z)\psi_0(z)\mathrm{e}^{-yz}$.

Now let $p_1, \ldots, p_{r+1}$ be an ordered version of $\frac{2\kappa}{\sigma^2}, \mu_1, \ldots, \mu_r$ and define $\Gamma_1, \ldots, \Gamma_{r+1}$ by the following recipe:

- If p_i is a zero for $z(\kappa - \frac{\sigma^2}{2}z)\psi_0(z)\mathrm{e}^{-yz}$ define

$$\Gamma_i = \Gamma_{Z^r(p_i)} := \{p_i + (1+i)t : 0 \leq t < \infty\}\,.$$

- If p_i is a singularity $z(\kappa - \frac{\sigma^2}{2}z)\psi_0(z)\mathrm{e}^{-yz}$ define

$$\Gamma_i = \Gamma_{S^r(p)} := \{p + (-1+i)t : -\infty < t \leq 0\} \cup \{p + (1+i)t : 0 \leq t < \infty\}$$

  for a $p \in]p_{i-1}, p_i[$ (with the convention $p_0 = 0$).

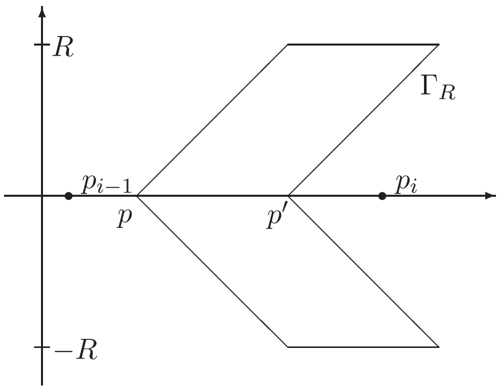

Figure A.2: Shows the closed curve Γ_R

A sketch showing the two types of contours can be seen on Figure A.1. It is important to notice that the definition of ψ_0 depends on the choice of complex logarithms in each of the power functions. These choices – that might vary with the contours – should satisfy that all argument functions are continuous along the contour (that is: ψ_0 is holomorphic in an area containing Γ).

These contours are constructed such that the fifth condition is satisfied. Left is checking $(i)-(iv)$. The boundedness condition (i) is fulfilled since

$$\int_{|\Gamma_i|} |\psi_0(z)|\,\mathrm{d}z < \infty$$

as a result of (6.2). The integration conditions (ii) and (iii) are satisfied since $\psi_0(z)$ is a power function in z with no singularities along the contours and the factor e^{-yz} decreases exponentially in the "infinite ends". That (iv) is fulfilled follows similarly but here it is important that each of the zeros for ψ_0 at most can be singularities of order > -1 for $\frac{\psi_0(z)}{z-\mu_k}$.

Remark 6.1. *Regarding the definition of* $\Gamma_{S^r(p)}$ *it is important to notice that the different choices of* $p \in]p_{i-1}, p_i[$ *give rise to the same results: If* $p_{i-1} < p < p' < p_i$ *integrating along the closed curve* Γ_R *sketched in Figure A.2 will give 0. When* $R \to \infty$ *the contributions along the horizontal parts vanish and hence the integrals along* $\Gamma_{S^r(p)}$ *and* $\Gamma_{S^r(p')}$ *equals.*

Contrary if p *and* p' *are separated by a singularity* μ *the corresponding integrals (e.g.* $f_{\Gamma_{S^r(p)}}$ *and* $f_{\Gamma_{S^r(p')}}$*) will be different. If the singularity* μ *is of order* $\rho < 0$ *with* $\rho \neq \mathbb{Z}$ *the fact that* $f_{\Gamma_{S^r(p)}} \neq f_{\Gamma_{S^r(p')}}$ *is a result of the use of different versions of the complex logarithm in the respective domains of the contours. If on the other hand* ρ *is an integer (and hence* ≤ -1*) the argument is based on Cauchy's Theorem.*

Remark 6.2. *It is worth considering if any other useful contours could be found. In the case where* p_{r+1} *is a singularity consider the contour* $\Gamma_{S^r(p)}$ *for* *a* $p > p_{r+1}$. *Since there a no singularities to the right of* p_{i+1} *it must hold that* $f_{\Gamma_R} \equiv 0$ *where* Γ_R *is the closed curve that combines the cut off version of* $\Gamma_{S^r(p)}$ *given by*

$$\{p + (-1+i)t : -\tfrac{1}{\sqrt{2}}R < t \leq 0\} \cup \{p + (1+i)t : 0 \leq t < \tfrac{1}{\sqrt{2}}R\}$$

and the circular arc C_R *with radius* R*:*

$$C_R = \{p + Re^{it} : -\tfrac{1}{4}\pi \leq t \leq \tfrac{1}{4}\pi\}\,.$$

For any $y \geq 0$ *we have* $|f_{C_R}(y)| \leq \int_{|C_R|} |\psi_0(z)| \mathrm{e}^{-py}\, \mathrm{d}z$ *and since the arc length of* C_R *is* $O(R)$ *the property (6.2) yields* $\lim_{R\to\infty} f_{C_R}(y) = 0$. *Hence* $f_{\Gamma_{S^r(p)}} \equiv 0$. *In the exact same way it is seen that the vector of constants* $M_{\Gamma_{S^r(p)}}$ *is zero. That is the* f_Γ*–function corresponding to the contour* $\Gamma_{S^r(p)}$ *makes no contribution in the construction of the partial eigenfunctions (cf Remark 5.1).*

There is after all one special situation where it is not completely obvious that an additional contour makes no contribution. Consider the case where $\frac{2\kappa}{\sigma^2}$ *is a singularity for* ψ_0 *of order* $\rho \in]-1, 0[$. *Then* $\frac{2\kappa}{\sigma^2}$ *is a both a zero of* $z(\kappa - \frac{\sigma^2}{2}z)\psi_0(z)\mathrm{e}^{-yz}$ *and a singularity for* $\psi_0(z)$. *We have already defined the contour* $\Gamma_{Z^r(\frac{2\kappa}{\sigma^2})}$*, but there is no reason why the contour*

$$\Gamma_{S^r(p)} := \{\tfrac{2\kappa}{\sigma^2} + (1+i)t : 0 \leq t < \infty\}$$

should not satisfy the conditions $(i) - (v)$ *for some* $p < \frac{2\kappa}{\sigma^2}$. *Also assume that there is some singularity* $\mu > \frac{2\kappa}{\sigma^2}$ *with corresponding contour* $\Gamma_{S^r(\tilde{p})}$ *where* $\tilde{p} \in]\frac{2\kappa}{\sigma^2}, \mu[$. *As a result of the comments in Remark 6.1 the two contours* $\Gamma_{S^r(p)}$ *and* $\Gamma_{S^r(\tilde{p})}$ *will cause different integrals but we shall see that together with* $\Gamma_{Z^r(\frac{2\kappa}{\sigma^2})}$ *the contour* $\Gamma_{S^r(p)}$ *will be redundant anyway. Now consider the factors in* ψ_0. *All factors except the complex power function* $z \mapsto (\frac{\sigma^2}{2\kappa}z - 1)^\rho$ *can be chosen holomorphic in areas containing both* $\Gamma_{S^r(p)}$ *and* $\Gamma_{S^r(\frac{2\kappa}{\sigma^2})}$. *For the contour* $\Gamma_{S^r(\frac{2\kappa}{\sigma^2})}$ *this function (denote it* ϕ*) can be chosen holomorphic on* $\mathbb{C} \setminus \{z \in \mathbb{R} | z \geq \frac{2\kappa}{\sigma^2}\}$ *and for* $\Gamma_{S^r(p)}$ *it (denote it* $\tilde{\phi}$*) can be chosen holomorphic on* $\mathbb{C} \setminus \{z \in \mathbb{R} | z \leq \frac{2\kappa}{\sigma^2}\}$. *The two versions of the power function can be chosen such that they are equal on* $\mathbb{C}_{\mathrm{Im}+} := \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$. *Consequently it holds that*

$$\phi(z) = \alpha\tilde{\phi}(z) \quad \text{when } z \in \mathbb{C}_{\mathrm{Im}-} := \{z \in \mathbb{C} \mid \mathrm{Im} z < 0\}$$

with $\alpha \in \mathbb{C}$ *so* $|\alpha| = 1$. *With* ψ_0 *and* $\tilde{\psi}_0$ *defined using* ϕ *and* $\tilde{\phi}$ *respectively we have* $\psi_0(z) = \alpha\tilde{\psi}_0(z)$ *when* $z \in \mathbb{C}_{\mathrm{Im}-}$ *and therefore*

$$\int_{-\Gamma_{Z^r(\frac{2\kappa}{\sigma^2})}} \psi_0(z)\mathrm{e}^{-yz}\, \mathrm{d}z = \alpha \int_{-\Gamma_{Z^r(\frac{2\kappa}{\sigma^2})}} \tilde{\psi}_0(z)\mathrm{e}^{-yz}\, \mathrm{d}z\,.$$

Here $-\Gamma_{Z^r(\frac{2\kappa}{\sigma^2})}$ *is the contour defined by*

$$-\Gamma_{Z^r(\frac{2\kappa}{\sigma^2})} := \{\tfrac{2\kappa}{\sigma^2} + (1-i)t : 0 \le t < \infty\}\,.$$

By letting $\tilde{p} \downarrow \frac{2\kappa}{\sigma^2}$ *and using Cauchy's Theorem one can see (remembering* $\rho > -1$*) that for* $\tilde{p} > \frac{2\kappa}{\sigma^2}$

$$\begin{aligned} f_{\Gamma_{S^r(\tilde{p})}}(y) &= \int_{\Gamma_{Z^r(\frac{2\kappa}{\sigma^2})}} \tilde{\psi}_0(z)\mathrm{e}^{-yz}\,\mathrm{d}z - \int_{-\Gamma_{Z^r(\frac{2\kappa}{\sigma^2})}} \tilde{\psi}_0(z)\mathrm{e}^{-yz}\,\mathrm{d}z \\ &= \int_{\Gamma_{Z^r(\frac{2\kappa}{\sigma^2})}} \psi_0(z)\mathrm{e}^{-yz}\,\mathrm{d}z - \alpha\int_{-\Gamma_{Z^r(\frac{2\kappa}{\sigma^2})}} \psi_0(z)\mathrm{e}^{-yz}\,\mathrm{d}z \end{aligned}$$

and similar for $p < \frac{2\kappa}{\sigma^2}$

$$f_{\Gamma_{S^r(p)}}(y) = \int_{\Gamma_{Z^r(\frac{2\kappa}{\sigma^2})}} \psi_0(z)\mathrm{e}^{-yz}\,\mathrm{d}z - \int_{-\Gamma_{Z^r(\frac{2\kappa}{\sigma^2})}} \psi_0(z)\mathrm{e}^{-yz}\,\mathrm{d}\,.z$$

It is seen from these two equations that integration along $\Gamma_{S^r(p)}$ *is linearly dependent of the integrals achieved by integrating along* $\Gamma_{S^r(\tilde{p})}$ *and* $\Gamma_{Z^r(\frac{2\kappa}{\sigma^2})}$*. Hence the contour is redundant.*

The $\theta = 0$–case:
It should be mentioned that the contours proposed above also applies when $\theta = 0$. But in this case an additional assumption (see subsection 4) was that $\lim_{x\to\infty} f(x) = 0$. This can be seen rather easily: In [18] it is even shown that functions similar to the f_Γ–functions decreases exponentially with this choice of integration contours.

Negative drift, $\theta > 0$:

When $\kappa < 0$ we have $\frac{2\kappa}{\sigma^2} > 0$ so this point cannot be used as above because of the requirement that $\Gamma_i \subseteq \mathbb{C}_+$. Instead 0 is a zero for $z(\kappa - \frac{\sigma^2}{2}z)\psi_0(z)\mathrm{e}^{-yz}$ and hence an integration contour of the form $\Gamma_{Z^r(0)}$ is useful. With $p_1, \ldots, p_{r+1}$ denoting $0, \mu_1, \ldots, \mu_r$ the definition of $\Gamma_1, \ldots, \Gamma_{r+1}$ from above can be reused.

Negative drift, $\theta = 0$:

When $\theta = 0$ then 0 is a singularity for ψ_0 of order -1 and hence 0 is no longer a zero for $z(\kappa - \frac{\sigma^2}{2}z)\psi_0(z)\mathrm{e}^{-yz}$. Thereby the integration contour corresponding to 0 will vanish.

Here it makes sense to consider the cases $\zeta = 0$ and $\zeta > 0$ separately. If $\zeta = 0$ the two probabilities $\mathbb{P}_x(A_j)$ and $\mathbb{P}_x(A_c)$ are requested and we have the equation

$$L\mathbb{P}_x(A_j) + f(l)\mathbb{P}_x(A_c) = f(x)\,.$$

This equation can be solved using only one partial eigenfunction if we use the fact that $\mathbb{P}_x(\tau < \infty) = 1$ when $\kappa < 0$. If only one partial eigenfunction is needed the r remaining integration contours are sufficient.

If on the other hand $\zeta > 0$ two partial eigenfunctions are needed. Hence it is necessary to find a function that substitutes an additional f_Γ–function. Define

$$f^* = \begin{cases} 1 & y \geq l \\ 0 & y < l \end{cases},$$

From straightforward calculation we have

$$\mathcal{A}f^*(y) = -\lambda p \sum_{k=1}^{r} \alpha_k e^{\mu_k(l-y)}$$

which is a sum of the same exponential functions that both $\mathcal{A}f_0(y)$ and $f_{\Gamma_i}(y)$ consists of. Hence f^* can be used in the construction of the partial eigenfunctions similar to the f_Γ–functions – this corresponds to an additional integration contour.

Negative ℓ

Here we are looking for integration contours Γ_{i1} and Γ_{j2} for

$$\begin{aligned}
\psi_1(z) &= \left(\tfrac{\sigma^2}{2\kappa}z - 1\right)^{\frac{\theta}{\kappa} + \sum^r \frac{2p\lambda\alpha_k}{2\kappa-\mu_k\sigma^2} + \sum^s \frac{2q\lambda\beta_d}{2\kappa+\nu_d\sigma^2} - 1} z^{-\frac{\theta}{\kappa}-1} \\
&\quad \times \left(\prod_{k=1}^{r}(z-\mu_k)^{-\frac{2p\lambda\alpha_k}{2\kappa-\mu_k\sigma^2}}\right)\left(\prod_{d=1}^{s}(z+\nu_d)^{-\frac{2q\lambda\beta_d}{2\kappa+\nu_d\sigma^2}}\right) \\
\psi_2(z) &= \left(\tfrac{-\sigma^2}{2\kappa}z - 1\right)^{\frac{\theta}{\kappa} + \sum^r \frac{2p\lambda\alpha_k}{2\kappa+\mu_k\sigma^2} + \sum^s \frac{2q\lambda\beta_d}{2\kappa-\nu_d\sigma^2} - 1} z^{-\frac{\theta}{\kappa}-1} \\
&\quad \times \left(\prod_{k=1}^{r}(z-\mu_k)^{-\frac{2p\lambda\alpha_k}{2\kappa+\mu_k\sigma^2}}\right)\left(\prod_{d=1}^{s}(z+\nu_d)^{-\frac{2q\lambda\beta_d}{2\kappa-\nu_d\sigma^2}}\right)
\end{aligned}$$

respectively. Note that ψ_1 equals ψ_0 in (6.1) and that both ψ_1 and ψ_2 has the property (as (6.2)): $|\psi(z)| = O(|z|^{-2})$ when $|z| \to \infty$.

Once again it is convenient to consider different cases of κ and θ separately.

Positive drift, $\theta \geq 0$:

With the same procedure as in the $\ell \geq 0$ case we can define $\Gamma_{11}, \ldots, \Gamma_{r+1,1}$ as integration contours for ψ_1 (with the notation $p_1, \ldots, p_{r+1}$ for the ordered version for $\frac{2\kappa}{\sigma^2}, \mu_1, \ldots, \mu_r$):

- If p_i is a zero for $z(\kappa - \frac{\sigma^2}{2}z)\psi_0(z)\mathrm{e}^{-yz}$ define

$$\Gamma_i = \Gamma_{Z^r(p_i)} := \{p_i + (1+i)t : 0 \leq t < \infty\}.$$

- If p_i is a singularity $z(\kappa - \frac{\sigma^2}{2}z)\psi_0(z)\mathrm{e}^{-yz}$ define

  $$\Gamma_i = \Gamma_{S^r(p)} := \{p + (-1+i)t : -\infty < t \leq 0\} \cup \{p + (1+i)t : 0 \leq t < \infty\}$$

  for a $p \in]p_{i-1}, p_i[$ (with the convention $p_0 = 0$).

Here the conditions $(i) - (vi)$ in Notation 5.1 have to be satisfied. The conditions $(i) - (v)$ are identical with the ones in Theorem 5.1. The remaining condition (vi) does not cause any problems since the function $\frac{\psi_1(z)}{z+\nu_d}$ has no additional singularities along the Γ_{i1} contours compared to $\psi_1(z)$.

Hence $r+1$ integration contours are found for ψ_1. Thereby it remains to construct $r+s+2$ contours for ψ_2. Since the contours for ψ_2 are allowed to be placed in the entire complex plane all the points $-\nu_s, \dots, -\nu_1, \frac{2\kappa}{-\sigma^2}, 0, \mu_1, \dots, \mu_r$ that are either zeros or singularities for ψ_2 are of interest. Let $q_1, \dots, q_{r+s+2}$ be the ordered version of these points and define $\Gamma_{12}, \dots, \Gamma_{r+s+2,2}$ by the following:

- If q_j is a zero for $z(\kappa - \frac{-\sigma^2}{2}z)\psi_0(z)\mathrm{e}^{-yz}$ define

  $$\Gamma_{j,2} = \Gamma_{Z^l(q_j)} := \{q_j + (-1+i)t : 0 \leq t < \infty\}$$

- If q_j is a singularity for $z(\kappa - \frac{-\sigma^2}{2}z)\psi_0(z)\mathrm{e}^{-yz}$ define

  $$\begin{aligned}\Gamma_{j,2} = \Gamma_{S^l(q)} :=& \{q + (1+i)t : -\infty < t \leq 0\} \\ & \cup \{q + (-1+i)t : 0 \leq t < \infty\}\end{aligned}$$

  for a $q \in]q_i, q_{i+1}[$ (with the convention $q_{r+s+3} = \infty$).

A sketch showing this type of contours can be seen in Figure A.3. Note that they are heading left (thereby the index l). These contours are supposed to fulfil the conditions $(i') - (viii')$ in Notation 5.1. The arguments are similar to the ones before. Since both the contours "infinite ends" and y have changed sign it is still true that $\mathrm{e}^{-yz} \to 0$ when z varies along the contour.

Remark 6.3. *In Remark 6.2 it was seen that a contour on the form $\Gamma_{S^r(p)}$ with a p to the right of all singularities makes no contribution. It is similar (just reversed w.r.t. the real axis) for the Γ_{j2}–contours for ψ_2: It is not possible to use a contour on the form $\Gamma_{S^l(p)}$ with p to the left of all singularities for ψ_2.*

Contrary the contour $\Gamma_{i_0,2} = \Gamma_{S^l(p)}$ with i_0 the largest singularity and $p > i_0$ does make a contribution. The argument is much alike the one in Remark 6.2. Let Γ_R be the closed curve in $\mathbb{C}$ that combines the cut off version of $\Gamma_{S^l(p)}$

$$\{q + (-1-i)t : -R < t \leq 0\} \cup \{q + (-1+i)t : 0 \leq t < R\}$$

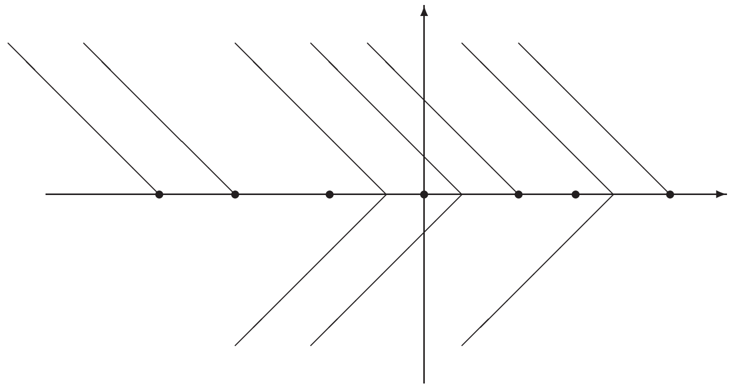

Figure A.3: The integration contours for ψ_2 .

and the circular arc

$$C_R = \{p + Re^{it} : -\tfrac{3}{4}\pi \le t \le \tfrac{3}{4}\pi\} .$$

Hence $f_{\Gamma_R}(y) = 0$ *for all* $y \in [\ell, 0[$ *and furthermore*

$$f_{C_R}(y) = \int_{-\frac{3}{4}\pi}^{\frac{3}{4}\pi} iRe^{it}\psi_0(p + Re^{it})e^{-y(p+Re^{it})}\, dt .$$

Since y *is assumed strictly negative the factor* $\mathrm{e}^{-y(p+Re^{it})}$ *is unbounded and exponentially increasing when* $R \to \infty$ *if only* $-\frac{\pi}{2} < t < \frac{\pi}{2}$*. Hence the integrand will not decrease to 0 and there is no reason why the integral should vanish. Consequently the* $f_{\Gamma_{S^l(p)}}$*–function is non–zero as well.*

With a similar argument one can see (since $\ell < 0$*) that the constants* $N^{3r}_{\Gamma_{j2}}$ *are non–zero. Thereby also the vector*

$$N_{\Gamma_{j2}} = (N^{11}_{\Gamma_{j2}}, \ldots, N^{1r}_{\Gamma_{j2}}, N^{21}_{\Gamma_{j2}}, \ldots, N^{2s}_{\Gamma_{j2}}, N^{31}_{\Gamma_{j2}}, \ldots, N^{3r}_{\Gamma_{j2}})$$

makes an useful contribution.

Negative drift, $\theta > 0$**:**

Here $\frac{2\kappa}{\sigma^2} < 0$ and 0 is a zero for $z(\kappa - \frac{\sigma^2}{2}z)\psi_0(z)\mathrm{e}^{-yz}$. With $p_1, \ldots, p_{r+1}$ and $q_1, \ldots, q_{r+s+2}$ denoting the ordered versions of $0, \mu_1, \ldots, \mu_r$ and $-\nu_s, \ldots, -\nu_1$, $0, \frac{2\kappa}{-\sigma^2}, \mu_1, \ldots, \mu_r$ the same recipe as above can be used to define the $2r+s+3$ integration contours.

Negative drift, $\theta = 0$**:**

In this case the solution strategy is the exact same as in the similar Section 6 for the positive drift case.

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PAPER B

Asymptotics for the Ruin Time of a Piecewise Exponential Markov Process with Jumps

Anders Rønn-Nielsen

Abstract

In this paper a class of Ornstein–Uhlenbeck processes driven by compound Poisson processes is considered. The jumps arrive with exponential waiting times and are allowed to be two-sided. The jumps are assumed to form an iid sequence with distribution a mixture (not necessarily convex) of exponential distributions, independent of everything else. The fact that downward jumps are allowed makes passage of a given lower level possible both by continuity and by a jump. The time of this passage and the possible undershoot (in the jump case) is considered. By finding partial eigenfunctions for the infinitesimal generator of the process an expression for the joint Laplace transform of the passage time and the undershoot can be found.

From the Laplace transform the ruin probability of ever crossing the level can be derived. When the drift is negative this probability is less than one and its asymptotic behaviour when the initial state of the process tends to infinity is determined explicitly.

The situation where the level to cross decreases to minus infinity is more involved: The level to cross under plays a much more fundamental role in the expression for the joint Laplace transform than the initial state of the process. The limit of the ruin probability in the positive drift case and the limit of the distribution of the undershoot in the negative drift case is derived.

1 Introduction

The main aim of this paper is to determine the asymptotic behaviour of the ruin probability for a certain class of time–homogeneous Markov processes with jumps. These processes, referred to as X below, can be viewed as Ornstein–Uhlenbeck processes driven by a compound Poisson process.

The ruin time is defined as the time to passage below ℓ for an initial state $x > \ell$. The passage below ℓ can be a result of a downward jump, and in some cases a continuous passage through ℓ is is also possible.

The compound Poisson process (that partly is driving the process) has a special jump structure. Both the downward and upward jumps are assumed to have a density (not the same) that is a linear – not necessarily convex – combination of exponential densities

The distribution of the passage time (and by that also the ruin probability) is determined through the Laplace transform. This is found by exploiting certain stopped martingales derived from using bounded partial eigenfunctions for the infinitesimal generator for X. An explicit expression for the Laplace transform is determined in [9]. Here the partial eigenfunctions are found as linear combinations of functions given by contour integrals in the complex plane. Also the Laplace transform ends up being a linear combination of these integrals. It is the resulting Laplace transform from [9] that we shall investigate in this paper.

One should distinguish between the two very different scenarios: Whether the drift κ is positive (hence X is pushed away to $\pm\infty$, that is X is transient) or the drift is negative in which case the process X is recurrent. In the negative drift case the probability $\mathbb{P}_x(\tau(\ell) < \infty)$ (with $\tau(\ell)$ denoting the time of passage) of ever crossing below ℓ when starting at x is always 1. On the other hand when the drift κ is positive we have that $\mathbb{P}_x(\tau(\ell) < \infty) < 1$ and also that the probability decreases when either $x \to \infty$ or $\ell \to -\infty$.

In the present paper the asymptotics of $\mathbb{P}_x(\tau(\ell) < \infty) < 1$ is explored in both of the situations $x \to \infty$ and $\ell \to -\infty$. This becomes a question about finding the asymtotics for the complex contour integrals mentioned above. It turns out that the $\ell \to -\infty$ problem is the far most complicated because the dependence of ℓ in the construction of the partial eigenfunctions is much more involved. Nevertheless the need of exploring the asymptotic behaviour of the integrals is rather similar. When $x \to \infty$ we see that $\mathbb{P}_x(\tau(\ell) < \infty)$ decreases exponentially (adjusted by some specified power function) with the exponential parameter from the leading expontial part of the downward jumps.

The technique of using partial eigenfunctions for the infinitesimal generator has appeared before. In [12] Paulsen and Gjessing considers a model like the present, but in the more general (and also different) setup

$$dX_t = (p + \kappa X_t)\,dt - dU_t + \sqrt{\sigma_1^2 + \sigma_2^2 X_t^2}\,dB_t + X_t d\tilde{U}_t\,, \tag{1.1}$$

Here both U and $\tilde{U}$ are compound Poisson processes of the form $\sum_{n=1}^{N_t} S_n$. In [12] it is shown that a partial eigenfunction for the corresponding infinitesimal generator for (1.1) will lead to the ruin probability and also the Laplace transform for the ruin time. In [5] Gaier and Grandits show – without σ_1^2 and $\tilde{U}$ in the model – the existence of this partial eigenfunction under some smoothness assumptions about the jump distributions in U. This result is extended to weaker assumptions in [6].

In the case of $\sigma_1^2 = \sigma_2^2 = 0$, without $\tilde{U}$, and assuming exponential negative jump (no positive jumps are allowed) an explicit formula for the Laplace transform is determined in [12]. Furthermore the exponential decrease in $\mathbb{P}_x(\tau(\ell) < \infty)$ is derived in the $x \to \infty$ asymptotic situation for some fixed $0 < \ell < x$. For the case of exponential negative jumps also see Asmussen [1], Chapter VII.

In the present paper the jump distributions are assumed to be light tailed. The existing literature does not contain very explicit results for the asymptotic ruin probability with that kind of jump distributions. In [4] and [14] it is proved in the $\sigma_2^2 = 0$ case with $\kappa = \sup\{a \mid \mathbb{E}[e^{aU}] < \infty\}$ that for any $\epsilon > 0$

$$\lim_{x\to\infty} \mathrm{e}^{(\kappa-\epsilon)x}\mathbb{P}_x(\tau_\ell < \infty) = 0 \quad \text{and} \quad \lim_{x\to\infty} \mathrm{e}^{(\kappa+\epsilon)x}\mathbb{P}_x(\tau_\ell < \infty) = \infty\,.$$

In the case of heavy tailed jump distributions there are more explicit results for the asymptotic behaviour of the ruin probability. In [10] results are obtained for the asymptotics of the finite horizon ruin probability $\mathbb{P}_x(\tau(\ell) \le T)$ in a fairly general model with $\sigma_2^2 = 0$ and subexpontial jump distributions. Similar results are reached in [3] in the infinite horizon case. Here the jumps belong to a less general class of heavy tailed distributions.

In [7] and [8] the following model class of certain Markov modulated Lévy processes

$$X_t = x + \int_0^t \beta_{J_s}\,\mathrm{d}s + \int_0^t \sigma_{J_{s-}}\,\mathrm{d}B_s - \sum_{n=1}^{N_t} U_n$$

is studied. The same partial eigenfunction technique is applied, and it is showed that the partial eigenfunctions (and thereby also the ruin probabilities) can be expressed as a linear combination of exponential functions (evaluated in the starting point x). Hence the asymptotic behaviour of the probability when $x \to \infty$ is simply a question about finding the exponential function with the slowest decrease. Since the model is additive the level ℓ, that is to be crossed at the time of ruin, enters into the setup symmetric to x. Hence the asymptotics when $\ell \to -\infty$ are just as easy to derive. A recent paper is Novikov et. al [11] where the Laplace transform is determined for a shot–noise model with exponentially distributed downward jumps (and no positive jumps allowed) for a process with negative drift. The Laplace transform was also derived in the case of uniformly distributed downward jumps. In [2] these results are extended to a more general driving Lévy process instead of a compound Poisson process. In both [11] and [2] some asymptotic results for the distribution of $\tau(\ell)$ are carried out. Here the limit distribution of $\tau(\ell)$ is expressed when $\ell \to -\infty$ for some fixed starting point x (recall that this is for the negative drift case). This is a limit that is not considered in the present paper.

The paper will be organised in the following way. In Section 2 the setup is defined and the relevant results from [9] reproduced. The Theorem 2.1 is also reformulated in a different (and appearently more complicated) version Theorem 3.1 that turns out to fit the asymptotic considerations better. An argument for this version of the Theorem can be found in [13] (though the Theorem in that paper concerns a more complicated setup, the arguments will with some small adjustments also fit this simple scenario). In the following Section 2 the choice of some complex integration contours that are applied in the Theorems 2.1 and 3.1 is discussed. This choice differs from the proposed contours in [9] in order to suit the further calculations. In Section 4 the asymptotic behaviour of $\mathbb{P}_x(\tau(\ell) < \infty)$ is expressed when $x \to \infty$ and in Section 5 the limit when $\ell \to -\infty$ is found. Finally (also in Section 5) the limit of the distribution of the undershoot is expressed for the negative drift case when $\ell \to -\infty$.

2 The model, some definitions and previous results

Consider a process X with state space $\mathbb{R}$ defined by the following stochastic differential equation:

$$\mathrm{d}X_t = \kappa X_t \,\mathrm{d}t + \mathrm{d}U_t\,, \tag{2.1}$$

where (U_t) is a compound Poisson process defined by

$$U_t = \sum_{n=1}^{N_t} V_n. \tag{2.2}$$

Here (V_n) are iid with distribution G and (N_t) is a Poisson process with parameter λ. Both the downward and the upward part of the jump distribution G is assumed to be a linear combination of exponential distributions. We use the decomposition $G = pG_- + qG_+$ where $0 < p \le 1$, $q = 1-p$, G_- is restricted to $\mathbb{R}_- =]-\infty;0[$ and G_+ is restricted to $\mathbb{R}_+ =]0;\infty[$. That is

$$\begin{aligned} G_-(du) &= g_-(u)\,du = \sum_{k=1}^{r} \alpha_k \mu_k \mathrm{e}^{\mu_k u} \qquad \text{for } u < 0 \\ G_+(du) &= g_+(u)\,du = \sum_{d=1}^{s} \beta_d \nu_d \mathrm{e}^{-\nu_d u} \qquad \text{for } u > 0\,. \end{aligned}$$

The distribution parametres are arranged such that $0 < \mu_1 < \cdots < \mu_r$, $0 < \nu_1 < \cdots < \nu_s$ and $\alpha_i, \beta_j \neq 0$. Since g_- and g_+ need to be densities

$\sum \alpha_i = 1$ and $\sum \beta_j = 1$. Furthermore both $\alpha_1 > 0$ and $\beta_1 > 0$.

The solution process X is a process that – between jumps – behaves deterministically following an exponential function.

Assume that $x > 0$ and write $\mathbb{P}_x$ for the probability space where $X_0 = x$ $\mathbb{P}_x$–almost surely. Let $\mathbb{E}_x$ be the corresponding expectation. Define for $\ell < x$ the stopping time τ by

$$\tau = \tau(\ell) = \inf\{t > 0 | X_t \leq \ell\}\,. \tag{2.3}$$

For ease of notation ℓ is most often suppresed. Furthermore define the *undershoot* Z

$$Z = \ell - X_\tau\,, \tag{2.4}$$

which is well–defined on the set $\{\tau < \infty\}$. It is important to notice that the level ℓ can by crossed through continuity as well as a result of a downward jump. Of interest is the joint distribution of τ and Z and especially the probability $\mathbb{P}_x(\tau < \infty)$. The distribution of τ and Z is expressed through the joint Laplace transform defined by

$$\mathbb{E}_x[e^{-\theta\tau-\zeta Z}; A_j] \quad \text{and} \quad \mathbb{E}_x[e^{-\theta\tau}; A_c]\,, \tag{2.5}$$

where A_j and A_c is a partition of the set $\{\tau < \infty\}$ into the jump case $A_j = \{\tau < \infty, X_\tau < \ell\}$ and the continuity case $A_c = \{\tau < \infty, X_\tau = \ell\}$.

The expression for the joint Laplace transform in (2.5) can be found from solving two equations

$$\mathbb{E}_x[\mathrm{e}^{-\theta\tau-\zeta Z}; A_j] + f_i(\ell)\mathbb{E}_x[\mathrm{e}^{-\theta\tau}; A_c] = f_i(x) \quad i = 1, 2\,, \tag{2.6}$$

where f_1 and f_2 have to be *partial eigenfunctions* for the infinitesimal generator $\mathcal{A}$ for the process: $f_i : \mathbb{R} \to \mathbb{C}$ should be bounded and differentiable on $[\ell; \infty[$ and satisfy the condition that

$$\mathcal{A}f_i(x) = \theta f_i(x) \quad \text{for all } x \in [\ell; \infty[\,,$$

where $\mathcal{A}$ is defined by

$$\mathcal{A}f(x) = \kappa x f'(x) + \lambda \int_{\mathbb{R}} \Big(f(x+y) - f(x)\Big) G(\mathrm{d}y)\,. \tag{2.7}$$

In addition to this f_i should have the following exponential form on the interval $]-\infty; \ell[$

$$f_i(x) = \mathrm{e}^{-\zeta(\ell - x)} \quad \text{for } x < \ell\,.$$

It is important to notice that there exists some situations where only one partial eigenfunction is needed: If $\ell\kappa > 0$ the probability $\mathbb{P}_x(A_c)$ of crossing ℓ through continuity is 0 (recall that the process is deterministic and monotone between jumps). In this case finding $\mathbb{E}_x[\mathrm{e}^{-\theta\tau-\zeta Z}; A_j]$ is even simpler (from (2.6) with the A_c part equal 0):

$$\mathbb{E}_x[\mathrm{e}^{-\theta\tau-\zeta Z}; A_j] = f(x)\,, \tag{2.8}$$

where f is the single partial eigenfunction.

In the negative drift case ($\kappa < 0$) it can be shown that $\mathbb{P}_x(A_j) + \mathbb{P}_x(A_c) = \mathbb{P}_x(\tau < \infty) = 1$. If furthermore $\theta = \zeta = 0$ the Laplace transforms in (2.6) reduces to the probabilities $\mathbb{P}_x(A_j)$ and $\mathbb{P}_x(A_c)$. Hence only one partial eigenfunction is needed in order to solve the equation.

In [9] a theorem is given that schetches how to construct such partial eigenfunctions. In the following this theorem is reformulated in order to fit the further calculations. First define

$$f_0(y) = \begin{cases} 0 & y \geq \ell \\ L\mathrm{e}^{-\zeta(\ell-y)} & y < \ell \end{cases} \tag{2.9}$$

and

$$f_\Gamma(y) = \begin{cases} \int_\Gamma \psi(z)\mathrm{e}^{-yz}\,\mathrm{d}z & y \geq \ell \\ 0 & y < \ell \end{cases}\,, \tag{2.10}$$

where ψ is the complex valued kernel defined by

$$\psi(z) = z^{-\frac{\theta}{\kappa}-1}\left(\prod_{k=1}^{r}(z-\mu_k)^{-\frac{p\lambda\alpha_k}{\kappa}}\right)\left(\prod_{d=1}^{s}(z+\nu_d)^{-\frac{q\lambda\beta_d}{\kappa}}\right), \tag{2.11}$$

and Γ is some suitable curve in the complex plane of the form $\Gamma = \{\gamma(t) : \delta_1 < t < \delta_2\}$. Note that

$$|\psi(z)| = O\left(|z|^{-1-(\theta+\lambda)/\kappa}\right)\,, \tag{2.12}$$

when $|z| \to \infty$.

Theorem 2.1. *Let $\theta \geq 0$ and $\zeta \geq 0$ be given and let f_0 and f_{Γ_i} be defined as in (2.9) and (2.10) for $i = 1, \ldots, m$ such that all Γ_i are concentrated on the positive part of the complex plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}z \geq 0\}$. Assume that for each contours Γ_i a holomorfic version of ψ exists that contains the contour. Assume furthermore that for $i = 1, \ldots, m$ it holds that*

(i) $\int_{|\Gamma_i|} |\psi(z)|\mathrm{e}^{-\ell\mathrm{Re}z}\,\mathrm{d}z < \infty$

(ii) $\int_{|\Gamma_i|} |\psi(z)|\,|z|\mathrm{e}^{-\ell\mathrm{Re}z}\,\mathrm{d}z < \infty$

(iii) $\int_{|\Gamma_i|} |\frac{\psi(z)}{z-\mu_k}|\mathrm{e}^{-\ell\mathrm{Re}z}\,\mathrm{d}z < \infty$

(iv) $\psi(\gamma_i(\delta_1))\gamma_i(\delta_{i1})\mathrm{e}^{-y\gamma(\delta_{i1})} = \psi(\gamma_i(\delta_{i2}))\gamma_i(\delta_{i2})\mathrm{e}^{-y\gamma_i(\delta_{i2})}$.

Now define

$$f(y) = \sum_{i=1}^{m} c_i f_{\Gamma_i}(y) + f_0(y)\,. \tag{2.13}$$

If the constants $c_1, \ldots, c_m$ *are chosen such that*

$$\sum_{i=1}^{m} c_i M_{\Gamma_i}^k + \frac{\mu_k}{\mu_k + \zeta} = 0 \tag{2.14}$$

for $k = 1, \ldots, r$ *where* M_{ik} *is given by*

$$M_{\Gamma_i}^k = \mu_k \int_{\Gamma_i} \frac{\psi(z)}{z - \mu_k} \mathrm{e}^{-\ell z}\, \mathrm{d}z$$

for $i = 1, \ldots, m$ *and* $k = 1, \ldots, r$, *then* f *is a partial eigenfunction for the generator* $\mathcal{A}$.

The theorem shows what it takes to construct a partial eigenfunction: As many f_{Γ_i}–functions – and by that sufficiently many integration contours – such that the equation system (2.14) can be solved. To the construction of one partial eigenfunction $m = r$ integration contours are needed (note that the equation system is homogeneous and has $m + 1$ unknowns). If an additional eigenfunction is requested $m = r + 1$ different integration contours should be found.

It is worth considering which difference between the integration contours is needed. E.g. it should not be possible to repeat the same contour several times. It is essential that the equation system can be solved with respect to the unknowns $c_1, \ldots, c_m$ and L. This implies that the vectors

$$M_{\Gamma_i} = (M_{\Gamma_i}^1, \ldots, M_{\Gamma_i}^m)$$

for $i = 1, \ldots, r$ are linearly independent – otherwise the situation would correspond to one with less unknowns.

Theorem 2.1 can be used for all values of ℓ. However it might in some respect restrict the choice of integration contours. That makes the following theorem useful although it look unnecessarily complicated. See [13] for a proof of a similar theorem in a more complicated model. This theorem can be considered as a special case. First define two new versions of the f_Γ–functions:

$$\begin{aligned} f_{\Gamma_1}^1(y) &= \begin{cases} \int_{\Gamma_1} \psi(z)\mathrm{e}^{-yz}\,\mathrm{d}z & y > 0 \\ 0 & y < 0 \end{cases} \\ f_{\Gamma_2}^2(y) &= \begin{cases} \int_{\Gamma_2} \psi(z)\mathrm{e}^{-yz}\,\mathrm{d}z & \ell \le y < 0 \\ 0 & \text{otherwise} \end{cases} \,. \end{aligned} \tag{2.15}$$

For convinience we shall use the following definitions

Definition 3.

$$
\begin{aligned}
M^{1k}_{\Gamma_i} &= \int_{\Gamma_{i1}} \frac{\psi(z)}{z-\mu_k}\mathrm{d}z \qquad i=1,\ldots,m, \quad k=1,\ldots,r\\
M^{2d}_{\Gamma_i} &= \int_{\Gamma_{i1}} \frac{\psi(z)}{\nu_d+z}\mathrm{d}z \qquad i=1,\ldots,m, \quad d=1,\ldots,s\\
N^{1k}_{\Gamma_j} &= \int_{\Gamma_{j2}} \frac{\psi(z)}{\mu_k-z}\,\mathrm{d}z \qquad j=1,\ldots,n, \quad k=1,\ldots,r\\
N^{2d}_{\Gamma_j} &= \int_{\Gamma_{j2}} \frac{\psi(z)}{\nu_d+z}\mathrm{d}z \qquad j=1,\ldots,n, \quad d=1,\ldots,s\\
N^{3k}_{\Gamma_j} &= \int_{\Gamma_{j2}} \frac{\psi(z)}{z-\mu_k}\mathrm{e}^{-\ell z}\mathrm{d}z \qquad j=1,\ldots,n, \quad k=1,\ldots,r\,.
\end{aligned}
$$

Similar to the conditions $(i)-(iv)$ in Theorem 2.1 we will refer to the conditions in the Notation below.

Notation 3.1. *Let $\theta \geq 0$ and $\zeta \geq 0$ be given and let f_0, $f^1_{\Gamma_{i1}}$ and $f^2_{\Gamma_{j2}}$ be defined as in (2.15) for $i=1,\ldots,m$ and $j=1,\ldots,n$ such that all $\Gamma_{i1}\subset\mathbb{C}_+$ and $\Gamma_{j2}\subset\mathbb{C}$ are suitable complex curves (ψ should have holomorfic versions containing these curves). Assume for ψ and Γ_{i1}, $i=1,\ldots,m$, that*

(i) $\int_{|\Gamma_{i1}|} |\psi(z)|\,\mathrm{d}z < \infty$

(ii) $\int_{|\Gamma_{i1}|} |\psi(z)|\,|z|\mathrm{e}^{-y\mathrm{Re}z}\,\mathrm{d}z < \infty$ *for all* $y>0$

(iii) $\int_{|\Gamma_{i1}|} |\frac{\psi(z)}{z-\mu_k}|\,\mathrm{d}z < \infty$ *for* $k=1,\ldots,r$

(iv) $\int_{|\Gamma_{i1}|} |\frac{\psi(z)}{z+\nu_d}|\,\mathrm{d}z < \infty$ *for* $d=1,\ldots,s$

(v) $\psi(\gamma_{i1}(\delta_{i1}))\gamma_{i1}(\delta^1_{i1})\mathrm{e}^{-y\gamma_{i1}(\delta^1_{i1})} = \psi(\gamma_{i1}(\delta^1_{i2}))\gamma_{i1}(\delta^1_{i2})\mathrm{e}^{-y\gamma_{i1}(\delta^1_{i2})}$ *for all* $y>0\,$,

and similarly for ψ and Γ_{j2} that

(i') $\int_{|\Gamma_{j2}|} |\psi(z)|\,\mathrm{d}z < \infty$

(ii') $\int_{|\Gamma_{j2}|} |\psi(z)|\mathrm{e}^{-\ell Rez}\,\mathrm{d}z < \infty$

(iii') $\int_{|\Gamma_{j2}|} |\psi(z)|\,|z|\mathrm{e}^{-y\mathrm{Re}z}\,\mathrm{d}z < \infty$ *for all* $y\in[\ell;0[$

(iv') $\int_{|\Gamma_{j2}|} |\frac{\psi(z)}{z-\mu_k}|\,\mathrm{d}z < \infty$ *for* $k=1,\ldots,r$

(v') $\int_{|\Gamma_{j2}|} |\frac{\psi(z)}{z-\mu_k}|\mathrm{e}^{-\ell z}\,\mathrm{d}z < \infty$ *for* $k=1,\ldots,r$

(vi') $\int_{|\Gamma_{j2}|} |\frac{\psi(z)}{z+\nu_d}|\,\mathrm{d}z < \infty$ *for* $d=1,\ldots,s$

(vii') $\psi(\gamma_{j2}(\delta_{j1}^2))\gamma_{j2}(\delta_{j1}^2)\mathrm{e}^{-y\gamma_2(\delta_{j1}^2)} = \psi(\gamma_{j2}(\delta_{j2}^2))\gamma_{j2}(\delta_{j2}^2)\mathrm{e}^{-y\gamma_{j2}(\delta_{j2}^2)}$ *for all* $\ell \le y < 0$.

for $j = 1, \dots, n$.

With these definitions we can state

Theorem 3.1. *Assume that the integration contours* Γ_{i1}, $i = 1, \dots, m$ *and* Γ_{j2}, $j = 1, \dots, n$ *satisfy the conditions in Notation 3.1. Define* $f : \mathbb{R} \to \mathbb{C}$ *by*

$$f(y) = \sum_{i=1}^{m} c_i f_{\Gamma_{i1}}^1(y) + \sum_{j=1}^{m} b_j f_{\Gamma_{j2}}^2(y) + f_0(y) \quad \text{for } y \in [\ell; \infty[\,, \tag{3.1}$$

then f *is bounded and differentiable on* $\ell \to \infty$. *If the constants* $c_1, \dots, c_m$, $b_1, \dots, b_n$ *and* L *fulfil the equations*

$$\sum_{j=1}^{n} b_j N_{\Gamma_j}^{3k} + L \frac{1}{\mu_k + \zeta} = 0 \tag{3.2}$$

and

$$\left(\sum_{i=1}^{m} c_i M_{\Gamma_i}^{1k} \right) + \left(\sum_{j=1}^{n} b_j N_{\Gamma_j}^{1k} \right) = 0 \tag{3.3}$$

for $k = 1, \dots, r$ *together with*

$$\left(\sum_{j=1}^{n} b_j N_{\Gamma_j}^{2d} \right) - \left(\sum_{i=1}^{m} c_i M_{\Gamma_j}^{2d} \right) = 0 \tag{3.4}$$

for $d = 1, \dots, s$, *then* f *is a partial eigenfunction for* $\mathcal{A}$.

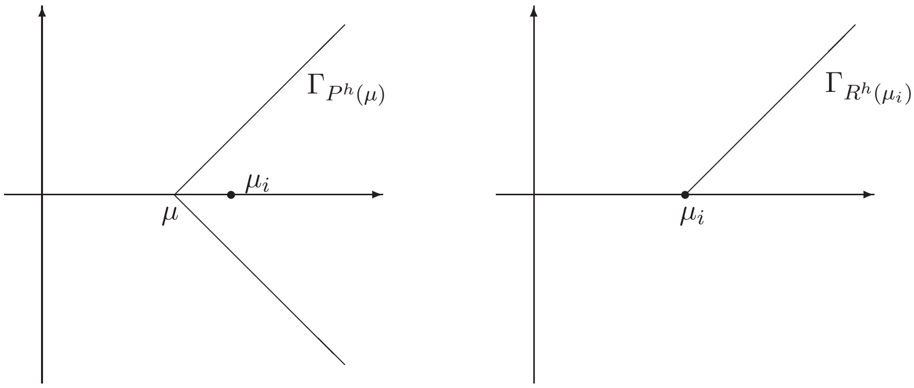
The choice of integration contours

In the following a short description of a choice for the integration contours will be given. As mentioned in [9] there are several possible choices. The one described here applies to cases with positive drift κ (the situation with $\sigma = 0$ and $\kappa < 0$ is studied in Section 5) and will differ from the ones defined in [9].

First assume that ℓ is positive. In this case only one partial eigenfunction is needed and we shall use Theorem 2.1 so the contours $\Gamma_1, \dots, \Gamma_r$ fulfilling $(i) - (iv)$ are requested. The definition of the contours has its starting point in the zeros and singularities of the kernel ψ. The real–valued points

$$-\nu_s, \dots, -\nu_1, 0, \mu_1, \dots, \mu_r$$

are all such zeros or singularities. In the case where $\ell > 0$ the contours $\Gamma_1, \dots, \Gamma_r$ are chosen as follows



μ_i is a singularity for ψ. μ_i is a zero for ψ.

Figure B.1: Shows the contour Γ_i in the two cases: μ_i is a singularity (left) for ψ and μ_i is a zero (right)

- If μ_i is a zero for ψ define

$$\Gamma_i = \Gamma_{Z^r(\mu_i)} := \{\mu_i + (1+i)t : 0 \leq t < \infty\}.$$

- If μ_i is a singularity for $\psi(z)$ define

$$\Gamma_i = \Gamma_{S^r(\mu)} := \{\mu + (-1+i)t : -\infty < t \leq 0\} \cup \{\mu + (1+i)t : 0 \leq t < \infty\}$$

for a $\mu \in]\mu_{i-1}, \mu_i[$ (with the convention $\mu_0 = 0$).

A schetch of the chosen contours can be seen in Figure B.1. Now consider the $\ell < 0$ case. Here Theorem 3.1 is used. For the contours $\Gamma_{11}, \ldots, \Gamma_{r1}$ one can use $\Gamma_1, \ldots, \Gamma_r$ from above. It remains to find $n = r + s + 1$ contours $\Gamma_{12}, \ldots, \Gamma_{r+s+1,2}$ in order to construct two eigenfunctions. For convenience let $p_1, \ldots, p_{r+s+1}$ denote the points $-\nu_s, \ldots, -\nu_1, 0, \mu_1, \ldots, \mu_r$ and use the following recipe:

- If p_i is a zero for ψ define

$$\Gamma_i = \Gamma_{Z^l(p_i)} \stackrel{def}{=} \{p_i + (-1+i)t : 0 \leq t < \infty\}$$

- If p_i is a singularity for $\psi(z)$ define

$$\Gamma_i = \Gamma_{S^l(p)} \stackrel{def}{=} \{p + (1+i)t : -\infty < t \leq 0\} \cup \{p + (-1+i)t : 0 \leq t < \infty\}$$

for a $p \in]p_i; p_{i+1}[$ (with the convention $p_{r+s+2} = \infty$).

Remark 3.1. *It is important to notice that for the contours of the form $S^r(\mu)$ the specific choice of μ in $]\mu_{i-1}, \mu_i[$ is without influence as a result of Cauchy's Theorem. In fact μ can be chosen freely in $]\mu_l, \mu_i[$ where μ_l is the biggest singularity for ψ less than μ_i (remember that 0 is a singularity so that $\mu_l \geq 0$).*

Another important fact is that it can never happen that $f_{\Gamma_i} = f_{\Gamma_{i+1}}$ in the case where both μ_i and μ_{i+1} are singularities. If μ_i – the singularity that separates the two contours – is of order $\rho < 0$ with $\rho \notin \mathbb{Z}$ the fact is secured from the use of different versions of the complex logarithm in the respective domains of the contours. If – on the other hand – the singularity μ_i is an integer the argument that $f_{\Gamma_i} \neq f_{\Gamma_{i+1}}$ is based on Cauchy's Theorem.

4 Asymptotics with increasing x

When the drift κ is positive the probability $\mathbb{P}_x(\tau < \infty)$ of ever crossing the level ℓ is less than 1. Furthermore the probability decreases when the starting point x increases. We have that (solving the equation system (2.6) w.r.t. $\mathbb{P}_x(\tau < \infty) = \mathbb{P}_x(A_c) + \mathbb{P}_x(A_j)$)

$$\mathbb{P}_x(\tau < \infty) = f_1(x)\frac{L_2 - f_2(\ell)}{L_2 f_1(\ell) - L_1 f_2(\ell)} + f_2(x)\frac{f_1(\ell) - L_1}{L_2 f_1(\ell) - L_1 f_2(\ell)} \tag{4.1}$$

if $\ell < 0$ (where f_1 and f_2 are the two partial eigenfunctions constructed in Theorem 3.1. In the $\ell > 0$ case we have

$$\mathbb{E}_x[\mathrm{e}^{-\theta\tau - \zeta Z}; A_j] = f(x)\,.$$

It is essential that the construction of the partial eigenfunctions f_1 and f_2 (or f in the $\ell > 0$ case) does not depend on the starting point x. The behaviour of the probability $\mathbb{P}_x(\tau < \infty)$ to be studied is therefore only determined by the behaviour of the two partial eigenfunctions f_1 and f_2 when $x \to \infty$. We have the following result:

Theorem 4.1. *There exists a constant K such that*

$$\lim_{x \to \infty} \frac{\mathbb{P}_x(\tau < \infty)}{\mathrm{e}^{-\mu_1 x} x^{-\frac{p\alpha_1\lambda}{\kappa} - 1}} = K\,.$$

The constant K is expressed explicitly in (4.9) below when $\ell < 0$ and in (4.10) for the $\ell > 0$ case.

For the later use of the results it is convenient to formulate part of the proof of Theorem 4.1 as self–contained lemmas:

Lemma 4.1. *Assume that $\alpha_j < 0$, that is μ_j is a zero for ψ. Hence $\Gamma_j = \{\mu_j + (1 + 2i)t : 0 \leq t < \infty\}$ and f_{Γ_j} can be written as*

$$f_{\Gamma_j}(x) = \int_\Gamma (z - \mu_j)^\rho \psi_{\setminus\{\mu_j\}}(z)\mathrm{e}^{-xz}\,\mathrm{d}z$$

with the notation

$$\psi_{\setminus\{\mu_j\}}(z) = z^{-1}\left(\prod_{k=1,k\neq j}^{r}(z-\mu_k)^{-\frac{p\alpha_k\lambda}{\kappa}}\right)\left(\prod_{d=1,}^{s}(z+\nu_d)^{-\frac{q\beta_d\lambda}{\kappa}}\right).$$

and $\rho = -p\alpha_1\lambda/\kappa > 0$. *Then it holds that*

$$\lim_{x\to\infty}\frac{f_{\Gamma_j}(x)}{\mathrm{e}^{-\mu_j x}x^{\rho-1}} = \psi_{\setminus\{\mu_j\}}(\mu_j)\int_{\Gamma_0} z^{\rho}\mathrm{e}^{-z}\,\mathrm{d}z\,, \tag{4.2}$$

where Γ_0 *is the integration contour*

$$\Gamma_0 = \{(1+i)t : 0\leq t<\infty\}\,. \tag{4.3}$$

Proof. The expression of $f_{\Gamma_j}(x)$ can be rewritten in the following way

$$\begin{aligned}
& f_{\Gamma_j}(x) \qquad (4.4)\\
= \;& \int_{\Gamma_j}(z-\mu_j)^{\rho}\psi_{\setminus\{\mu_j\}}(z)\mathrm{e}^{-xz}\,\mathrm{d}z\\
= \;& \int_0^{\infty}(1+i)\big((1+i)t\big)^{\rho}\psi_{\setminus\{\mu_j\}}\big(\mu_j+(1+i)t\big)\mathrm{e}^{-x(\mu_0+(1+i)t)}\,\mathrm{d}t\\
= \;& e^{-\mu_j x}\int_0^{\infty}(1+i)\big((1+i)t\big)^{\rho}\psi_{\setminus\{\mu_j\}}\big(\mu_j+(1+i)t\big)\mathrm{e}^{-xt(1+i)}\,\mathrm{d}t\\
= \;& \mathrm{e}^{-\mu_j x}\int_0^{\infty}\tfrac{1}{x}(1+i)\big((1+i)\tfrac{s}{x}\big)^{\rho}\psi_{\setminus\{\mu_j\}}\big(\mu_j+(1+i)\tfrac{s}{x}\big)\mathrm{e}^{-s(1+i)}\,\mathrm{d}s\\
= \;& x^{-\rho-1}\mathrm{e}^{-\mu_j x}\int_0^{\infty}(1+i)\big((1+i)s\big)^{\rho}\psi_{\setminus\{\mu_j\}}\big(\mu_j+(1+i)\tfrac{s}{x}\big)\mathrm{e}^{-s(1+i)}\,\mathrm{d}s\,,
\end{aligned}$$

where the substitution $s = tx$ has been used.

Consider the function

$$t\mapsto|\psi_{\setminus\{\mu_j\}}\big(\mu_j+(1+i)t\big)|\,,$$

which – apart from being continuous – is strictly positive. Furthermore it is $O(|\mu_j+(1+2i)t|^{-1-\lambda/\kappa-\rho})$ when $t\to\infty$. This gives the existence of a constant $C<\infty$ such that

$$|\psi_{\setminus\{\mu_j\}}\big(\mu_j+(1+i)t\big)|\leq C \quad \text{for all } t\geq 0\,.$$

Especially this holds when $t = s/x$ for all $s\geq 0$ and $x>0$. Thus the function

$$s\mapsto C|(1+i)((1+2i)s)^{\rho}|\mathrm{e}^{-s}$$

is an integrable upper bound for the integrand in the last line of (4.4) when $x \geq x_0$. By dominated convergence we get that

$$\begin{aligned}
& \lim_{x\to\infty} \int_0^\infty (1+i)\big((1+i)s\big)^\rho \psi_{\setminus\{\mu_j\}}\big(\mu_j + (1+i)\tfrac{s}{x}\big)\mathrm{e}^{-s(1+i)}\,\mathrm{d}s \\
= & \int_0^\infty (1+i) \lim_{x\to\infty} \big((1+i)s\big)^\rho \psi_{\setminus\{\mu_j\}}\big(\mu_j + (1+i)\tfrac{s}{x}\big)\mathrm{e}^{-s(1+i)}\,\mathrm{d}s \\
= & \int_0^\infty (1+i)\big((1+i)s\big)^\rho \psi_{\setminus\{\mu_j\}}\mu_j)\mathrm{e}^{-s(1+i))}\,\mathrm{d}s \\
= & \psi_{\setminus\{\mu_j\}}(\mu_j)\int_{\Gamma_0} z^\rho \mathrm{e}^{-z}\,\mathrm{d}z\,.
\end{aligned}$$

Hence the result is shown. □

Lemma 4.2. *Assume that $\alpha_j > 0$, that is μ_j is a singularity for ψ. Hence the contour Γ_j is defined as*

$$\Gamma_j = \Gamma_\mu = \{\mu + (-1+i)t : -\infty < t \leq 0\} \cup \{\mu + (1+i)t : 0 < t < \infty\}\,,$$

with some $\mu \in]\mu_{j-1}, \mu_j[$ and with the notation from Lemma 4.1 f_{Γ_j} is written as

$$f_{\Gamma_j}(x) = \int_{\Gamma_\mu} (z-\mu_j)^\rho \psi_{\setminus\{\mu_j\}}(z)\mathrm{e}^{-xz}\,\mathrm{d}z\,.$$

Then

$$\lim_{x\to\infty} \frac{f_\Gamma(x)}{x^{\rho-1}e^{-\mu_j x}} = \psi_{\setminus\{\mu_j\}}(\mu_j)\int_{\Gamma_{-a}} z^\rho \mathrm{e}^{-z}\,\mathrm{d}z\,, \tag{4.5}$$

where

$$\Gamma_{-a} = \{-a + (-1+i)t : -\infty < t \leq 0\} \cup \{-a + (1+i)t : 0 < t < \infty\}$$

and $a > 0$ is some positive real number.

Proof. In Remark 3.1 it was argued that

$$f_{\Gamma_j}(x) = f_{\Gamma_{\mu'}}(x)$$

for all μ' in $]\mu_l; \mu_j[$. Since all $\mu' \in]\mu_l; \mu_j[$ will imply the same result one could choose $\mu' = \mu_u - \frac{a}{x}$ for some suitable $a > 0$. Hence

$$\begin{aligned}
& f_{\Gamma_j}(x) \\
& = f_{\Gamma_{\mu_j - a/x}}(x) \\
& = \int_0^\infty (1+i)\left(-\tfrac{a}{x} + (1+i)t\right)^\rho \psi_{\setminus\{\mu_j\}}\big(\mu_j - \tfrac{a}{x} + (1+i)t\big)\mathrm{e}^{-x\mu_j + a - x(1+i)t}\,\mathrm{d}t \\
& + \int_{-\infty}^0 (-1+i)\left(-\tfrac{a}{x} + (-1+i)t\right)^\rho \psi_{\setminus\{\mu_j\}}\big(\mu_j - \tfrac{a}{x} + (-1+i)t\big)\mathrm{e}^{-x\mu_j + a - x(-1+i)t}\,\mathrm{d}t\,.
\end{aligned}$$

Using the substitution $s = tx$ yields that the first integral equals

$$x^{\rho-1}\mathrm{e}^{-\mu_j x}\int_0^\infty (1+2i)\big((1+i)s-a\big)^\rho \psi_{\setminus\{\mu_j\}}\big(\mu_j-\tfrac{a}{x}+(1+i)\tfrac{s}{x}\big)\mathrm{e}^{a-(1+i)s}\,\mathrm{d}s\,. \tag{4.6}$$

From dominated convergence similar to Lemma 4.1 it is seen that the limit of the integral in (4.6) as $x \to \infty$ is:

$$\psi_{\setminus\{\mu_j\}}(\mu_j)\int_0^\infty \big((1+i)s-a\big)^\rho \mathrm{e}^{-(1+i)s}\,\mathrm{d}s\,.$$

Something similar is seen for the second integral. Hence it has been shown that

$$\begin{aligned}
&\lim_{x\to\infty}\frac{f_{\Gamma_j}(x)}{x^{\rho-1}e^{-\mu_j x}} \\
&= \psi_{\setminus\{\mu_j\}}(\mu_j)\int_0^\infty (1+i)\big(-a+(1+i)s\big)^\rho \mathrm{e}^{-(-a+(1+i)s)}\,\mathrm{d}s \\
&\quad + \psi_{\setminus\{\mu_j\}}(\mu_j)\int_{-\infty}^0 (-1+i)\big(-a+(-1+i)s\big)^\rho \mathrm{e}^{-(-a+(-1+i)s)}\,\mathrm{d}s \\
&= \psi_{\setminus\{\mu_j\}}(\mu_j)\int_{\Gamma_{-a}} z^\rho \mathrm{e}^{-z}\,\mathrm{d}z\,.
\end{aligned} \tag{4.7}$$

□

Remark 4.1. *A possible lack of consistency in the argument above might be the following: The starting point of the contour, μ', was set to move right towards μ_j. Another solution could be letting it move left towards μ_l (the biggest singularity less than μ_j) with the definition $\mu' = \mu_{j-1} + \frac{a}{x}$. From redoing all the arguments the following result would be reached:*

$$\lim_{x\to\infty}\frac{f_\Gamma(x)}{x^{\rho'-1}\mathrm{e}^{-\mu_l x}} = \phi(\mu_l)\pi(\mu_l)\int_{\Gamma_a} z^{\rho'}\mathrm{e}^{-z}\,\mathrm{d}z$$

what appears to be a slower decrease towards 0. What rescues the consistency is that only one of the integrals is different from 0:

$$\int_{\Gamma_a} z^{-\rho'}\mathrm{e}^{-z}\,\mathrm{d}z = 0 \quad \textit{and} \quad \int_{\Gamma_{-a}} z^{-\rho}\mathrm{e}^{-z}\,\mathrm{d}z \neq 0\,.$$

Proof of Theorem 4.1. Consider the $\ell < 0$ case (the $\ell > 0$ will be the same just more simple). Both f_1 and f_2 are linear combinations of the f_Γ functions. Since x is assumed to be positive all $f_{\Gamma_{j2}}(x) = 0$. Then both $f_1(x)$ and $f_2(x)$ are linear combinations of

$$f^1_{\Gamma_{11}}(x), \ldots, f^1_{\Gamma_{m1}}(x)\,.$$

So in order to study $\mathbb{P}_x(\tau < \infty)$ it is sufficient to determine the behaviour of the functions $f^1_{\Gamma_{i1}}(x)$ when $x \to \infty$.

For each each $i = 1, \ldots, r$ there are two possible situations to consider: $\alpha_i < 0$ and $\alpha_i > 0$. This is explored in the Lemmas 4.1 and 4.2 respectively and it was shown that either way

$$\lim_{x\to\infty} \frac{f_{\Gamma_{i1}}(x)}{x^{\rho-1}e^{-\mu_j x}} = K$$

for some constant K_j.

Since the ruin probability $\mathbb{P}_x(\tau < \infty)$ can be written as a linear combination of these functions the study of the asymptotics is simply a question about finding the function with the slowest decrease. The function with the slowest decrease is then f_{Γ_1} and since μ_1 is always a singularity for ψ the exact asymptotic behaviour of f_{Γ_1} can be found in Lemma 4.1.

Let the two partial eigenfunctions f_1 and f_2 be the linear combinations

$$f_1(x) = \sum_{i=1}^{r} c_i^1 f_{\Gamma_{1i}}(x) \quad \text{resp.} \quad f_2(x) = \sum_{i=1}^{r} c_i^2 f_{\Gamma_{1i}}(x) \tag{4.8}$$

for $x > 0$. Then

$$\begin{aligned} & \lim_{x\to\infty} \frac{\mathbb{P}_x(\tau < \infty)}{\mathrm{e}^{-\mu_1 x} x^{-\frac{p\alpha_l\lambda}{\kappa}-1}} \\ = & \lim_{x\to\infty} \frac{f_{\Gamma_{\mu_1}}(x)}{\mathrm{e}^{-p\mu_1 x} x^{-\frac{p\alpha_l\lambda}{\kappa}-1}} \left(c_1^1 \frac{L_2 - f_2(\ell)}{L_2 f_1(\ell) - L_1 f_2(\ell)} + c_1^2 \frac{f_1(\ell) - L_1}{L_2 f_1(\ell) - L_1 f_2(\ell)} \right) \\ = & K\,, \end{aligned}$$

where K is given by

$$\begin{aligned} K = & \left(\psi_{\setminus\{\mu_1\}}(\mu_1) \int_{\Gamma_{-a}} z^{\frac{p\alpha_l\lambda}{\kappa}} \mathrm{e}^{-z}\,\mathrm{d}z \right) \\ & \left(c_1^1 \frac{L_2 - f_2(\ell)}{L_2 f_1(\ell) - L_1 f_2(\ell)} + c_1^2 \frac{f_1(\ell) - L_1}{L_2 f_1(\ell) - L_1 f_2(\ell)} \right) . \end{aligned} \tag{4.9}$$

Hence the theorem has been proved in the $\ell < 0$ case. With exactly the same arguments when $\ell > 0$ we derive

$$K = c_1 \left(\psi_{\setminus\{\mu_1\}}(\mu_1) \int_{\Gamma_{-a}} z^{\frac{p\alpha_l\lambda}{\kappa}} \mathrm{e}^{-z}\,\mathrm{d}z \right) . \tag{4.10}$$

□

5 Asymptotics as $\ell \to -\infty$

Asymptotics of the ruin probability, positive drift

When considering the situation where $\ell \to -\infty$ the setup becomes more complicated: The constants $c_1, \ldots, c_m$ and $b_1, \ldots, b_n$ in the construction of the partial eigenfunctions change as ℓ decreases.

In the study of the behaviour of $\mathbb{P}_x(\tau(\ell) < \infty)$ given by (4.1) both $f_i(x)$ and $f_i(\ell)$ are of interest, $i = 1, 2$. With $x > 0$ and $\ell < 0$ the expressions are the following for the first of the two eigenfunctions

$$\begin{aligned} f_1(\ell) &= \sum_{j=-s}^{r-1} b_j(\ell) f^2_{\Gamma_{j,2}}(\ell) \\ f_1(x) &= \sum_{i=1}^{r} c_i(\ell) f^1_{\Gamma_{i,1}}(x)\,. \end{aligned}$$

This definition excludes the last of the integration contours $\Gamma_{1,-s}, \ldots, \Gamma_{r,2}$. Similarly $f_2(\ell)$ and $f_2(x)$ are defined by

$$\begin{aligned} f_2(\ell) &= \sum_{j=-s+1}^{r} \tilde{b}_j(\ell) f^2_{\Gamma_{j,2}}(\ell) \\ f_2(x) &= \sum_{i=1}^{r} \tilde{c}_i(\ell) f^1_{\Gamma_{i,1}}(x)\,. \end{aligned}$$

The constants $c_1(\ell), \ldots, c_r(\ell)$ and $b_{-s}(\ell), \ldots, b_{r-1}(\ell)$ are found as the solution in a linear equation:

$$\underbrace{\begin{bmatrix} 0 & \ldots & 0 & N^{31}_{\Gamma_{-s}}(\ell) & \ldots & N^{31}_{\Gamma_{r-1}}(\ell) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & N^{3r}_{\Gamma_{-s}}(\ell) & \ldots & N^{3r}_{\Gamma_{r-1}}(\ell) \\ M^{11}_{\Gamma_1} & \ldots & M^{11}_{\Gamma_r} & N^{11}_{\Gamma_{-s}} & \ldots & N^{11}_{\Gamma_{r-1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M^{1r}_{\Gamma_1} & \ldots & M^{1r}_{\Gamma_r} & N^{1r}_{\Gamma_{-s}} & \ldots & N^{1r}_{\Gamma_{r-1}} \\ -M^{21}_{\Gamma_1} & \ldots & -M^{21}_{\Gamma_r} & N^{21}_{\Gamma_{-s}} & \ldots & N^{21}_{\Gamma_{r-1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -M^{2s}_{\Gamma_1} & \ldots & -M^{2s}_{\Gamma_r} & N^{2s}_{\Gamma_{-s}} & \ldots & N^{2s}_{\Gamma_{r-1}} \end{bmatrix}}_{=A(\ell)} \begin{bmatrix} c_1(\ell) \\ \vdots \\ c_r(\ell) \\ b_{-s}(\ell) \\ \vdots \\ b_{r-1}(\ell) \end{bmatrix} = \begin{bmatrix} \frac{1}{\mu_1} \\ \vdots \\ \frac{1}{\mu_r} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{5.1}$$

With these definitions a limit of the probability $\mathbb{P}_x(\tau(\ell) < \infty)$ when $\ell \to -\infty$ can be derived:

Theorem 5.1. *The limits $c_i = \lim_{\ell \to -\infty} c_i(\ell)$ are well defined and non–zero for $i = 1, \ldots, r$, and*

$$\lim_{\ell \to \infty} \mathbb{P}_x(\tau(\ell) < \infty) = -\sum_{i=1}^{r} c_i f^1_{\Gamma_{i,1}}(x)\,.$$

Expressions of the c_i constants are found in the Corollary 5.1 below.

Proof. It is important to note that in the matrix $A(\ell)$ only $N^{3k}_{\Gamma_j}(\ell)$ (for $k = 1,\dots,r$ and $j = -s,\dots,r-1$) depends on ℓ. Exploring this dependence by applying the same technique as in the $x \to \infty$ case yields for $k = 1,\dots,r$ and $i = -s,\dots,-1$ that

$$\begin{aligned}\lim_{\ell\to-\infty} \frac{N^{3k}_{\Gamma_i}(\ell)}{\mathrm{e}^{\ell\nu_{-i}}(-\ell)^{\frac{q\beta_{-i}\lambda}{\kappa}-1}} &= \lim_{\ell\to-\infty} \frac{1}{\mathrm{e}^{\ell\nu_{-i}}(-\ell)^{\frac{q\beta_{-i}\lambda}{\kappa}-1}} \int_{\Gamma_{i,2}} \frac{\psi(z)}{z-\mu_k}\mathrm{e}^{-\ell z}\,\mathrm{d}z \\ &= \frac{\psi_{\setminus\{-\nu_{-i}\}}(-\nu_{-i})}{-\nu_{-i}-\mu_k} \int_{\tilde\Gamma} z^{-\frac{q\beta_{-i}\lambda}{\kappa}} \mathrm{e}^z\,\mathrm{d}z \qquad (5.2)\end{aligned}$$

if $-\nu_{-i}$ is a root for ψ_0. Here

$$\tilde\Gamma = \{(-1+i)t : 0 \le t < \infty\}$$

and

$$\psi_{\setminus\{-\nu_{-i}\}} = z^{-1}\left(\prod_{k=1}^{r}(z-\mu_k)^{-\frac{p\alpha_k\lambda}{\kappa}}\right)\left(\prod_{d=1,d\neq i}^{s}(z+\nu_d)^{-\frac{q\beta_d\lambda}{\kappa}}\right).$$

If instead $-\nu_{-i}$ is a singularity the result is

$$\lim_{\ell\to-\infty} \frac{N^{3k}_{\Gamma_i}(\ell)}{\mathrm{e}^{\ell\nu_{-i}}(-\ell)^{\frac{q\beta_{-i}\lambda}{\kappa}-1}} = \frac{\psi_{\setminus\{-\nu_{-i}\}}(-\nu_{-i})}{-\nu_{-i}-\mu_k} \int_{\tilde\Gamma} z^{-\frac{q\beta_{-i}\lambda}{\kappa}} \mathrm{e}^z\,\mathrm{d}z\,, \qquad (5.3)$$

where

$$\tilde\Gamma_a = \{a + (1+i)t : -\infty < t \le 0\} + \{a + (-1+i)t : 0 \le t < \infty\}.$$

Furthermore

$$\lim_{\ell\to-\infty} N^{3k}_{\Gamma_0}(\ell) = \frac{\psi_{\setminus\{0\}}(0)}{-\mu_k} \int_{\tilde\Gamma_a} z^{-1}\mathrm{e}^z\,\mathrm{d}z\,. \qquad (5.4)$$

Finally the constants that relate to $\mu_1,\dots,\mu_r$ satisfy the following if μ_i is a root

$$\begin{aligned}\lim_{\ell\to-\infty} \frac{N^{3i}_{\Gamma_i}}{\mathrm{e}^{-\ell\mu_i}(-\ell)^{-\frac{p\alpha_i\lambda}{\kappa}}} &= \psi_{\setminus\{\mu_i\}}(\mu_i) \int_{\tilde\Gamma} z^{-\frac{p\alpha_i\lambda}{\kappa}-1}\mathrm{e}^z\,\mathrm{d}z && (5.5)\\ \lim_{\ell\to-\infty} \frac{N^{3k}_{\Gamma_i}}{\mathrm{e}^{-\ell\mu_i}(-\ell)^{-\frac{p\alpha_i\lambda}{\kappa}}-1} &= \frac{\psi_{\setminus\{\mu_i\}}(\mu_i)}{\mu_i-\mu_k} \int_{\tilde\Gamma} z^{-\frac{p\alpha_i\lambda}{\kappa}}\mathrm{e}^z\,\mathrm{d}z \quad \text{if } k\neq i && (5.6)\end{aligned}$$

and if it is a singularity

$$\begin{aligned}\lim_{\ell\to-\infty} \frac{N^{3i}_{\Gamma_i}}{\mathrm{e}^{-\ell\mu_i}(-\ell)^{-\frac{p\alpha_i\lambda}{\kappa}}} &= \psi_{\setminus\{\mu_i\}}(\mu_i) \int_{\tilde\Gamma_a} z^{-\frac{p\alpha_i\lambda}{\kappa}-1}\mathrm{e}^z\,\mathrm{d}z && (5.7)\\ \lim_{\ell\to-\infty} \frac{N^{3k}_{\Gamma_i}}{\mathrm{e}^{-\ell\mu_i}(-\ell)^{-\frac{p\alpha_i\lambda}{\kappa}}-1} &= \frac{\psi_{\setminus\{\mu_i\}}(\mu_i)}{\mu_i-\mu_k} \int_{\tilde\Gamma_a} z^{-\frac{p\alpha_i\lambda}{\kappa}}\mathrm{e}^z\,\mathrm{d}z \quad \text{if } k\neq i && (5.8)\end{aligned}$$

When the determinant of $A(\ell)$ is considered it is crucial that $N^{3k}_{\Gamma_i}(\ell)$ has the largest rate of growth when $k = i$. It is also important to note that if μ_i is a singularity of an order in $]0;1[$ and $k \neq i$ then the limit integral for $N^{3k}_{\Gamma_i}(\ell)$ is zero while the integral in the limit of $N^{3i}_{\Gamma_i}(\ell)$ is not.

Now define the matrices

$$M = \begin{bmatrix} M^{11}_{\Gamma_1} & \dots & M^{11}_{\Gamma_r} & N^{11}_{\Gamma_{-s}} & \dots & N^{11}_{\Gamma_{-1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M^{1r}_{\Gamma_1} & \dots & M^{1r}_{\Gamma_r} & N^{1r}_{\Gamma_{-s}} & \dots & N^{1r}_{\Gamma_{-1}} \\ -M^{21}_{\Gamma_1} & \dots & -M^{21}_{\Gamma_r} & N^{21}_{\Gamma_{-s}} & \dots & N^{21}_{\Gamma_{-1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -M^{2s}_{\Gamma_1} & \dots & -M^{2s}_{\Gamma_r} & N^{2s}_{\Gamma_{-s}} & \dots & N^{2s}_{\Gamma_{-1}} \end{bmatrix}$$

and

$$N(\ell) = \begin{bmatrix} N^{31}_{\Gamma_0}(\ell) & \dots & N^{31}_{\Gamma_{r-1}}(\ell) \\ \vdots & \ddots & \vdots \\ N^{3r}_{\Gamma_0}(\ell) & \dots & N^{3r}_{\Gamma_{r-1}}(\ell) \end{bmatrix}.$$

The formulas (5.2) – (5.8) yield that

$$\det(A(\ell)) \sim \big(\det(N(\ell))(-1)^{r+s+1}\det(M)\big)$$

and by using that $N^{3i}_{\Gamma_i}(\ell)$ has the most rapid growth compared to $N^{3k}_{\Gamma_i}(\ell)$ when $k \neq i$ it is seen that

$$\det(N(\ell)) \sim \left(N^{3r}_{\Gamma_0}\prod_{i=1}^{r-1} N^{3i}_{\Gamma_i}(\ell)\right)$$

which means that

$$\det(N(\ell)) = O\left(\mathrm{e}^{\ell\sum_{j=1}^{r-1}\mu_j}(-\ell)^{\sum_{j=1}^{r-1}\frac{p\alpha_j\lambda}{\kappa}}\right).$$

By Cramers Rule it is possible to find an expression for the constants $c_1(\ell),\dots,c_r(\ell)$

and $b_{-s}(\ell), \ldots, b_r(\ell)$ in the equation system (5.1):

$$c_1(\ell) = \frac{1}{\det(A(\ell))} \det \left(\underbrace{\begin{bmatrix} \frac{1}{\mu_1} & 0 & \ldots & 0 & N^{31}_{\Gamma_{-s}}(\ell) & \ldots & N^{31}_{\Gamma_{r-1}}(\ell) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\mu_r} & 0 & \ldots & 0 & N^{3r}_{\Gamma_{-s}}(\ell) & \ldots & N^{3r}_{\Gamma_{r-1}}(\ell) \\ 0 & M^{11}_{\Gamma_2} & \ldots & M^{11}_{\Gamma_r} & N^{11}_{\Gamma_{-s}} & \ldots & N^{11}_{\Gamma_{r-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & M^{1r}_{\Gamma_2} & \ldots & M^{1r}_{\Gamma_r} & N^{1r}_{\Gamma_{-s}} & \ldots & N^{1r}_{\Gamma_{r-1}} \\ 0 & -M^{21}_{\Gamma_2} & \ldots & -M^{21}_{\Gamma_r} & N^{21}_{\Gamma_{-s}} & \ldots & N^{21}_{\Gamma_{r-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & -M^{2s}_{\Gamma_2} & \ldots & -M^{2s}_{\Gamma_r} & N^{2s}_{\Gamma_{-s}} & \ldots & N^{2s}_{\Gamma_{r-1}} \end{bmatrix}}_{=A_1(\ell)} \right)$$

and similarly for the remaining constants. It is seen that

$$\det(A_i(\ell)) = O\left(\mathrm{e}^{\ell \sum_{j=1}^{r-1} \mu_j} (-\ell)^{\sum_{j=1}^{r-1} \frac{p\alpha_j \lambda}{\kappa}} \right)$$

for $i = 1, \ldots, r+s$ and therefore

$$\begin{aligned} c_i(\ell) &= \frac{\det(A_i(\ell))}{\det(A(\ell))} = O(1) \qquad i = 1, \ldots, r \\ b_j(\ell) &= \frac{\det(A_{r+s+1+j}(\ell))}{\det(A(\ell))} = O(1) \qquad j = -s, \ldots, -1 \,. \end{aligned}$$

Furthermore

$$\begin{aligned} \det(A_{r+s+1}(\ell)) &\sim \left(\det(M) \times \frac{1}{\mu_r} \prod_{i=1}^{r-1} N^{3i}_{\Gamma_i}(\ell) \right) \\ \det(A_{r+s+1+j}(\ell)) &\sim \left(\det(M) \times \frac{1}{\mu_j} N^{3r}_{\Gamma_0}(\ell) \prod_{i=1, i\neq j}^{r-1} N^{3i}_{\Gamma_i}(\ell) \right) \qquad j = 1, \ldots, r-1 \end{aligned}$$

such that

$$\begin{aligned} b_0(\ell) &= \frac{\det(A_{r+s+1}(\ell))}{\det(A(\ell))} \sim \left(\frac{1}{\mu_r} \frac{1}{N^{3r}_{\Gamma_0}(\ell)} \right) \\ b_j(\ell) &= \frac{\det(A_{r+s+1+j}(\ell))}{\det(A(\ell))} \sim \left(\frac{1}{\mu_j} \frac{1}{N^{3j}_{\Gamma_j}(\ell)} \right) \\ &= O\left(\mathrm{e}^{\ell \mu_j} (-\ell)^{\frac{p\alpha_j \lambda}{\kappa}} \right) \qquad j = 1, \ldots, r-1 \,. \end{aligned}$$

The equivalent constants $\tilde{c}_1(\ell),\dots,\tilde{c}_r(\ell)$ and $\tilde{b}_{-s+1}(\ell),\dots,\tilde{b}_r(\ell)$ that belongs to the second partial eigenfunction solve an equation system similar to (5.1):

$$\underbrace{\begin{bmatrix} 0 & \dots & 0 & N^{31}_{\Gamma_{-s+1}}(\ell) & \dots & N^{31}_{\Gamma_r}(\ell) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & N^{3r}_{\Gamma_{-s+1}}(\ell) & \dots & N^{3r}_{\Gamma_r}(\ell) \\ M^{11}_{\Gamma_1} & \dots & M^{11}_{\Gamma_r} & N^{11}_{\Gamma_{-s+1}} & \dots & N^{11}_{\Gamma_r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M^{1r}_{\Gamma_1} & \dots & M^{1r}_{\Gamma_r} & N^{1r}_{\Gamma_{-s+1}} & \dots & N^{1r}_{\Gamma_r} \\ -M^{21}_{\Gamma_1} & \dots & -M^{21}_{\Gamma_r} & N^{21}_{\Gamma_{-s+1}} & \dots & N^{21}_{\Gamma_r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -M^{2s}_{\Gamma_1} & \dots & -M^{2s}_{\Gamma_r} & N^{2s}_{\Gamma_{-s+1}} & \dots & N^{2s}_{\Gamma_r} \end{bmatrix}}_{=\tilde{A}(\ell)} \begin{bmatrix} \tilde{c}_1(\ell) \\ \vdots \\ \tilde{c}_r(\ell) \\ \tilde{b}_{-s+1}(\ell) \\ \vdots \\ \tilde{b}_r(\ell) \end{bmatrix} = \begin{bmatrix} \frac{1}{\mu_1} \\ \vdots \\ \frac{1}{\mu_r} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{5.9}$$

where the integration contour Γ_{-s} is replaced by Γ_r in order to obtain a new and independent partial eigenfunction. It can be shown that the constants have the following asymptotics as functions of ℓ

$$\begin{aligned} \tilde{c}_i(\ell) &= O\left(\frac{1}{\mu_1}\frac{1}{N^{31}_{\Gamma_1}(\ell)}\right) = O\left(\mathrm{e}^{-\ell\mu_1}(-\ell)^{\frac{p\alpha_1\lambda}{\kappa}}\right) \qquad i=-s,\dots,-1 \\ \tilde{b}_j(\ell) &= O\left(\frac{1}{\mu_1}\frac{1}{N^{31}_{\Gamma_1}}(\ell)\right) = O\left(\mathrm{e}^{-\ell\mu_1}(-\ell)^{\frac{p\alpha_1\lambda}{\kappa}}\right) \qquad j=-s+1,\dots,0 \\ \tilde{b}_j(\ell) &\sim \left(\frac{1}{\mu_j}\frac{1}{N^{3j}_{\Gamma_j}}(\ell)\right) = O\left(\mathrm{e}^{-\ell\mu_j}(-\ell)^{\frac{p\alpha_j\lambda}{\kappa}}\right) \qquad j=1,\dots,r\,. \end{aligned}$$

The asymptotic behaviour of the $f^2_{\Gamma_{j,2}}$ functions is of interest as well. Similar to the previous analysis it is seen that

For $j=-s,\dots,-1$:

$$\begin{aligned} \lim_{\ell\to-\infty}\frac{f^2_{\Gamma_{j,2}}(\ell)}{\mathrm{e}^{\ell\nu_{-j}}(-\ell)^{\frac{q\beta_j\lambda}{\kappa}-1}} &= \psi_{\setminus\{-\nu_{-j}\}}(-\nu_{-j})\int_{\tilde{\Gamma}_a} z^{-\frac{q\beta_j\lambda}{\kappa}}\mathrm{e}^z\,\mathrm{d}z\,,\ \text{if } \nu_{-j} \text{ is a singularity} \\ \lim_{\ell\to-\infty}\frac{f^2_{\Gamma_{j,2}}(\ell)}{\mathrm{e}^{\ell\nu_{-j}}(-\ell)^{\frac{q\beta_j\lambda}{\kappa}-1}} &= \psi_{\setminus\{-\nu_{-j}\}}(-\nu_{-j})\int_{\tilde{\Gamma}} z^{-\frac{q\beta_j\lambda}{\kappa}}\mathrm{e}^z\,\mathrm{d}z\,,\ \text{if } \nu_{-j} \text{ is a root}\,. \end{aligned}$$

For $j=0$:

$$\lim_{j\to-\infty} f^2_{\Gamma_{0,2}}(\ell) = \psi_{\setminus\{0\}}(0)\int_{\tilde{\Gamma}_a} z^{-1}\mathrm{e}^z\,\mathrm{d}z\,.$$

For $j = 1, \dots, r$:

$$
\begin{aligned}
\lim_{\ell \to -\infty} \frac{f^2_{\Gamma_{j,2}}(\ell)}{\mathrm{e}^{-\ell\mu_j}(-\ell)^{\frac{p\alpha_j\lambda}{\kappa}-1}} &= \psi_{\setminus\{\mu_j\}}(\mu_j) \int_{\tilde{\Gamma}_a} z^{-\frac{p\alpha_j\lambda}{\kappa}} \mathrm{e}^z \, \mathrm{d}z\,, \quad \text{if } \mu_j \text{ is a singularity} \\
\lim_{\ell \to -\infty} \frac{f^2_{\Gamma_{j,2}}(\ell)}{\mathrm{e}^{-\ell\mu_j}(-\ell)^{\frac{p\alpha_j\lambda}{\kappa}-1}} &= \psi_{\setminus\{\mu_j\}}(\mu_j) \int_{\tilde{\Gamma}} z^{-\frac{q\alpha_j\lambda}{\kappa}} \mathrm{e}^z \, \mathrm{d}z\,, \quad \text{if } \mu_j \text{ is a root}\,.
\end{aligned}
$$

By comparing these results with the asymptotics for the constants $c_i(\ell)$, $\tilde{c}_i(\ell)$, $b_j(\ell)$ and $\tilde{b}_j(\ell)$ it is seen that

- $b_j(\ell) f^2_{\Gamma_{j,2}}(\ell)$ tends to zero exponentially fast as $\ell \to -\infty$ for $j = -s, \dots, -1$
- $\tilde{b}_j(\ell) f^2_{\Gamma_{j,2}}(\ell)$ tends to zero exponentially fast as $\ell \to -\infty$ for $j = -s + 1, \dots, 0$
- $b_j(\ell) f^2_{\Gamma_{j,2}}(\ell) = O\left(\frac{1}{-\ell}\right)$ for $\ell \to -\infty$ when $j = 1, \dots, r-1$
- $\tilde{b}_j(\ell) f^2_{\Gamma_{j,2}}(\ell) = O\left(\frac{1}{-\ell}\right)$ for $\ell \to -\infty$ when $j = 1, \dots, r$.

Left is finding the non–zero limit of $b_0(\ell) f^2_{\Gamma_{0,2}}(\ell)$ when $\ell \to -\infty$:

$$
\begin{aligned}
\lim_{\ell \to -\infty} b_0(\ell) f^2_{\Gamma_{0,2}}(\ell) &= \lim_{\ell \to -\infty} \frac{1}{\mu_r} \frac{1}{N^{3r}_{\Gamma_0}(\ell)} f^2_{\Gamma_{0,2}}(\ell) \\
&= \frac{1}{\mu_r} \frac{\psi_{\setminus\{0\}}(0) \int_{\tilde{\Gamma}_a} z^{-1} \mathrm{e}^z \, \mathrm{d}z}{\frac{\psi_{\setminus\{0\}}(0)}{-\mu_r} \int_{\tilde{\Gamma}_a} z^{-1} \mathrm{e}^z \, \mathrm{d}z} \\
&= -1\,.
\end{aligned}
$$

Hence it has been shown that

$$
\begin{aligned}
\lim_{\ell \to -\infty} f_1(\ell) &= \lim_{\ell \to -\infty} \sum_{j=-s}^{r-1} b_j(\ell) f^2_{\Gamma_{j,2}}(\ell) = -1 \\
\lim_{\ell \to -\infty} f_2(\ell) &= \lim_{\ell \to -\infty} \sum_{j=-s+1}^{r} \tilde{b}_j(\ell) f^2_{\Gamma_{j,2}}(\ell) = 0\,.
\end{aligned}
$$

Furthermore it has been shown that all $\tilde{c}_i(\ell)$ decreases to zero so

$$
\lim_{\ell \to -\infty} f_2(x) = \lim_{t \to -\infty} \sum_{i=1}^{r} \tilde{c}_i(\ell) f^1_{\Gamma_{i,1}}(x) = 0
$$

and since all c_i has a non–zero limit the limit $\lim_{\ell\to-\infty} f_1(x)$ is well–defined and non–zero. Then

$$\begin{aligned}\lim_{\ell\to-\infty}\mathbb{P}_x(\tau<\infty) &= \lim_{\ell\to-\infty} f_1(x)\frac{1-f_2(\ell)}{f_1(\ell)-f_2(\ell)}+f_2(x)\frac{f_1(\ell)-1}{f_1(\ell)-f_2(\ell)}\\ &= \lim_{\ell\to-\infty} f_1(x)\frac{1-0}{-1-0}+0\frac{-1-1}{-1-0}\\ &= -\lim_{\ell\to-\infty} f_1(x)\,.\end{aligned}$$

□

The asymptotic expression for $c_i(\ell)$ can be rewritten as

$$c_i(\ell)\sim\left(\frac{|M_i|}{|M|}\frac{1}{\mu_r N^{3r}_{\Gamma_0}(\ell)}\right),$$

where

$$M_i=\begin{bmatrix} M^{11}_{\Gamma_1} & \dots & M^{11}_{\Gamma_{i-1}} & M^{11}_{\Gamma_{i+1}} & \dots & M^{11}_{\Gamma_r} & N^{11}_{\Gamma_{-s}} & \dots & N^{11}_{\Gamma_{-1}}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ M^{1r}_{\Gamma_1} & \dots & M^{1r}_{\Gamma_{i-1}} & M^{1r}_{\Gamma_{i+1}} & \dots & M^{1r}_{\Gamma_r} & N^{1r}_{\Gamma_{-s}} & \dots & N^{1r}_{\Gamma_{-1}}\\ -M^{21}_{\Gamma_1} & \dots & -M^{21}_{\Gamma_{i-1}} & -M^{21}_{\Gamma_{i+1}} & \dots & -M^{21}_{\Gamma_r} & N^{21}_{\Gamma_{-s}} & \dots & N^{21}_{\Gamma_{-1}}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ -M^{2s}_{\Gamma_1} & \dots & -M^{2s}_{\Gamma_{i-1}} & -M^{2s}_{\Gamma_{i+1}} & \dots & -M^{2s}_{\Gamma_r} & N^{2s}_{\Gamma_{-s}} & \dots & N^{2s}_{\Gamma_{-1}}\end{bmatrix}.$$

Hence we have this result:

Corollary 5.1. *For $i=1,\dots,r$ it holds that*

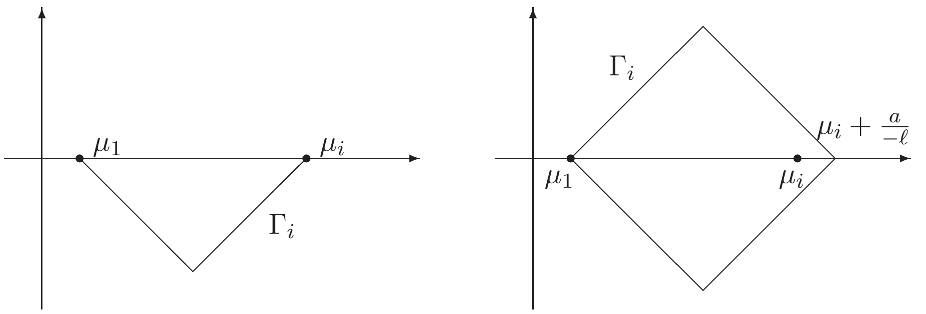
$$\begin{aligned}\lim_{\ell\to-\infty} c_i(\ell) &= \frac{\det(M_i)}{\det(M)}\frac{1}{\mu_r}\left(\frac{\psi_{\setminus\{0\}}(0)}{-\mu_k}\int_{\tilde{\Gamma}_a} z^{-1}\mathrm{e}^z\,\mathrm{d}z\right)^{-1}\\ &= -\frac{\det(M_i)}{\det(M)}\left(\psi_{\setminus\{0\}}(0)\int_{\tilde{\Gamma}_a} z^{-1}\mathrm{e}^z\,\mathrm{d}z\right)^{-1}.\end{aligned}$$

The undershoot when the drift is negative

Now consider the negative drift case, $\kappa<0$. This situation is particularly simple because only one partial eigenfunction, f, is needed since crossing ℓ through continuity is not possible. The Laplace transform of the undershoot is therefore expressed by the rather simple formula

$$\mathbb{E}_x[\mathrm{e}^{-\zeta Z}]=f(x)\,.$$

Since ψ satisfies that $|\psi(z)|=O(|z|^{-1-\frac{\lambda}{\kappa}})$ the negative κ makes infinite integration contours impossible. Instead the following finite contours are chosen



p_i is a zero for ψ. μ_i is a singularity for ψ.

Figure B.2: The choice of contours in the negative drift case.

using that μ_1 is always a zero for ψ. For each $i = 2, \ldots, r$ there are two possible definitions:
If μ_i is a zero define Γ_i as

$$\{\mu_i + (-1-i)t \ : \ 0 \leq t \leq \frac{\mu_i - \mu_1}{2}\}$$
$$\cup \{\mu_i - i(\mu_i - \mu_1) + (-1+i)t \ : \ \frac{\mu_i - \mu_1}{2} \leq t \leq \mu_i - \mu_1\}\,.$$

If μ_i is a singularity define Γ_i as

$$\{\mu_i + \tfrac{a}{-\ell} + i(\mu_i + \tfrac{a}{-\ell} - \mu_1) + (1+i)t : -(\mu_i + \tfrac{a}{-\ell} - \mu_1) \leq t \leq -\frac{\mu_i + \frac{a}{-\ell} - \mu_1}{2}\}$$
$$\cup \{\mu_i + \tfrac{a}{-\ell} + (1-i)t : -\frac{\mu_i + \frac{a}{-\ell} - \mu_1}{2} \leq t \leq 0\}$$
$$\cup \{\mu_i + \tfrac{a}{-\ell} + (-1-i)t : \frac{\mu_i + \frac{a}{-\ell} - \mu_1}{2} \leq t \leq 0\}$$
$$\cup \{\mu_i - \tfrac{a}{-\ell} + i(\mu_i + \tfrac{a}{-\ell} - \mu_1) + (-1+i)t : \frac{\mu_i + \frac{a}{-\ell} - \mu_1}{2} \leq t \leq \mu_i + \tfrac{a}{-\ell} - \mu_1\}\,.$$

A rough sketch of the two contours can be seen on Figure B.2

Remark 5.1. *In [9] these contours are suggested to be half–circles and circles but that choice makes the further calculations too complicated.*

The partial eigenfunction f is defined by

$$f(y) = \sum_{i=2}^{r} c_i f_{\Gamma_i}(y) + U f^*(y) + f_0(y)\,, \tag{5.10}$$

where

$$f^*(y) = 1_{[\ell;\infty[}(y)$$

and the parameters $c_2, \dots, c_r$ and U are the solutions of the equation (when putting $L = 1$)

$$\underbrace{\begin{bmatrix} -\frac{1}{\mu_1}(\ell) & M^1_{\Gamma_2}(\ell) & \cdots & M^1_{\Gamma_r}(\ell) \\ -\frac{1}{\mu_2}(\ell) & M^2_{\Gamma_2}(\ell) & \cdots & M^2_{\Gamma_r}(\ell) \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\mu_r}(\ell) & M^r_{\Gamma_2}(\ell) & \cdots & M^r_{\Gamma_r}(\ell) \end{bmatrix}}_{B(\ell)} \begin{bmatrix} U \\ c_2 \\ \vdots \\ c_r \end{bmatrix} = \begin{bmatrix} -\frac{1}{\mu_1+\zeta} \\ \vdots \\ -\frac{1}{\mu_r+\zeta} \end{bmatrix} \tag{5.11}$$

where – as usual – the constants $M^k_{\Gamma_i}(\ell)$ are given as

$$M^k_{\Gamma_i}(\ell) = \int_{\Gamma_i} \frac{\psi(z)}{z-\mu_k} \mathrm{e}^{-\ell z}\, \mathrm{d}z$$

for $i = 2, \dots, r$ and $k = 1, \dots, r$. In order to explore the asymptotic behaviour of $U, c_2, \dots, c_r$ and through that the behaviour of f it is necessary to study these constants.

From the use of these definitions the following result, which states that the limit of the undershoot is a simple expontential distribution with the parameter from the dominating part of the downward jumps, can be reached.

Theorem 5.2. *For all $\zeta \geq 0$ it holds that*

$$\lim_{\ell \to -\infty} \mathbb{E}_x[\mathrm{e}^{-\zeta Z}] = \frac{\mu_1}{\mu_1 + \zeta}\,.$$

Proof. First the behaviour of the constants $M^k_{\Gamma_i}(\ell)$ when $\ell \to -\infty$ is explored.

In the case where μ_i is a zero (for some $i = 2, \dots, r$) and $i \neq k$ the corresponding constant can be written as

$$M^k_{\Gamma_i}(\ell) = \int_0^{\frac{\mu_i-\mu_1}{2}} (-1-i) \frac{\psi(\mu_i + (-1-i)t)}{\mu_i + (-1-i)t - \mu_k} \mathrm{e}^{-\ell(\mu_i+(-1-i)t)}\, \mathrm{d}t \tag{5.12}$$
$$+ \int_{\frac{\mu_i-\mu_1}{2}}^{\mu_i-\mu_1} (-1+i) \frac{\psi(\mu_i - i(\mu_i-\mu_1) + (-1+i)t)}{\mu_i - i(\mu_i - \mu_1) + (-1+i)t - \mu_k} \mathrm{e}^{-\ell(\mu_i - i(\mu_i-\mu_1)+(-1+i)t)}\, \mathrm{d}t\,.$$

Rewriting the expression and applying the usual substitution $s = (-\ell)t$ to the first part in (5.12) yields that it

$$= \mathrm{e}^{-\ell\mu_i} \int_0^{\frac{\mu_i-\mu_1}{2}} (-1-i) \frac{\psi_{\setminus\{\mu_i\}}(\mu_i + (-1-i)t)}{\mu_i + (-1-i)t - \mu_k} ((-1-i)t)^{-\frac{p\lambda\alpha_i}{\kappa}} \mathrm{e}^{-\ell t(-1-i)}\, \mathrm{d}t$$
$$\stackrel{s=-\ell t}{=} \mathrm{e}^{-\ell\mu_i} (-\ell)^{\frac{p\lambda\alpha_i}{\kappa}-1} \int_0^{(-\ell)\frac{\mu_i-\mu_1}{2}} (-1-i) \frac{\psi_{\setminus\{\mu_i\}}(\mu_i + (-1-i)\frac{s}{-\ell})}{\mu_i + (-1-i)\frac{s}{-\ell} - \mu_k}$$
$$((-1-i)s)^{-\frac{p\lambda\alpha_i}{\kappa}} \mathrm{e}^{s(-1-i)}\, \mathrm{d}s\,.$$

Hence by dominated convergence it is seen that the integral in the last line has the limit

$$\begin{aligned} &\frac{\psi_{\setminus\{\mu_i\}}(\mu_i)}{\mu_i - \mu_k} \int_0^\infty (-1-i)((-1-i)s)^{-\frac{p\lambda\alpha_i}{\kappa}} \mathrm{e}^{s(-1-i)}\,\mathrm{d}s \\ = &\frac{\psi_{\setminus\{\mu_i\}}(\mu_i)}{\mu_i - \mu_k} \int_{-\Gamma} z^{-\frac{p\lambda\alpha_i}{\kappa}} \mathrm{e}^z\,\mathrm{d}z\,, \end{aligned}$$

where

$$-\Gamma = \{(-1-i)t : 0 \le t < \infty\}\,.$$

Now remains to discuss the asymptotics of the second part in (5.12). Here the substitution $s = (-\ell)(t - (\mu_i - \mu_1))$ is used and by that the expression equals

$$\begin{aligned} &\mathrm{e}^{-\ell(\frac{\mu_1+\mu_i}{2} - i\frac{\mu_i-\mu_1}{2})}(-\ell)^{-1} \\ \times &\textstyle\int_0^{(-\ell)\frac{\mu_i-\mu_1}{2}} (-1+i)\psi\left(\frac{\mu_1+\mu_i}{2} - i\frac{\mu_i-\mu_1}{2} + (-1+i)\frac{s}{-\ell}\right) \mathrm{e}^{s(-1+i)}\,\mathrm{d}s\,. \end{aligned}$$

The integral has the following limit for $\ell \to -\infty$

$$\psi\left(\frac{\mu_1+\mu_i}{2} - i\frac{\mu_i-\mu_1}{2}\right) \int_{\tilde{\Gamma}} \mathrm{e}^z\,\mathrm{d}z$$

when using dominated convergence. The definition $\tilde{\Gamma} = \{(-1+i)t : 0 \le t < \infty\}$ has been used. The study of the asymptotics in (5.12) concludes that the first part grows with a bigger rate than the last part. Therefore

$$\lim_{\ell\to-\infty} \frac{M^k_{\Gamma_i}(\ell)}{\mathrm{e}^{-\ell\mu_i}(-\ell)^{-\frac{p\lambda\alpha_i}{\kappa}-1}} = \frac{\psi_{\setminus\{\mu_i\}}(\mu_i)}{\mu_i - \mu_k} \int_{-\Gamma} z^{-\frac{p\lambda\alpha_i}{\kappa}} \mathrm{e}^z\,\mathrm{d}z\,. \tag{5.13}$$

A result completely similar is found in the case where $i = k$:

$$\lim_{\ell\to-\infty} \frac{M^k_{\Gamma_i}(\ell)}{\mathrm{e}^{-\ell\mu_i}(-\ell)^{-\frac{p\lambda\alpha_i}{\kappa}}} = \psi_{\setminus\{\mu_i\}}(\mu_i) \int_{-\Gamma} z^{-\frac{p\lambda\alpha_i}{\kappa}-1} \mathrm{e}^z\,\mathrm{d}z\,. \tag{5.14}$$

The same substitution technique yields results in the cases where μ_i are singularities for ψ. That gives

$$\lim_{\ell\to-\infty} \frac{M^k_{\Gamma_i}(\ell)}{\mathrm{e}^{-\ell\mu_i}(-\ell)^{-\frac{p\lambda\alpha_i}{\kappa}-1}} = \frac{\psi_{\setminus\{\mu_i\}}(\mu_i)}{\mu_i - \mu_k} \int_{-\Gamma_a} z^{-\frac{p\lambda\alpha_i}{\kappa}} \mathrm{e}^z\,\mathrm{d}z \tag{5.15}$$

if $i \neq k$ and

$$\lim_{\ell\to-\infty} \frac{M^k_{\Gamma_i}(\ell)}{\mathrm{e}^{-\ell\mu_i}(-\ell)^{-\frac{p\lambda\alpha_i}{\kappa}}} = \psi_{\setminus\{\mu_i\}}(\mu_i) \int_{-\Gamma_a} z^{-\frac{p\lambda\alpha_i}{\kappa}-1} \mathrm{e}^z\,\mathrm{d}z \tag{5.16}$$

when $i = k$. Here

$$-\Gamma_a = \{a + (1-i)t : -\infty < t \leq 0\} \cup \{a + (-1-i)t : 0 \leq t < \infty\}.$$

With the results (5.13)-(5.16) the behaviour of the solutions $U, c_2, \ldots, c_r$ in (5.11) can be expressed. The following asymptotic behaviour of the determinant of the matrix $B(\ell)$ is observed

$$\det(B(\ell)) \sim \left(-\frac{1}{\mu_1}\prod_{i=2}^{r} M^i_{\Gamma_i}(\ell)\right). \tag{5.17}$$

And with B_i denoting B where the ith column is replaced by the vector $[-\frac{1}{\mu_1+\zeta}, \ldots, -\frac{1}{\mu_r+\zeta}]^T$ it is seen that

$$\begin{aligned}\det(B_1(\ell)) &\sim \left(-\frac{1}{\mu_1+\zeta}\prod_{i=2}^{r} M^i_{\Gamma_i}(\ell)\right) &(5.18)\\ \det(B_i(\ell)) &\sim \left((\frac{-1}{\mu_1}\frac{1}{\mu_i+\zeta} - \frac{-1}{\mu_i}\frac{1}{\mu_1+\zeta})\prod_{j\in\{2,\ldots,r\},j=i} M^j_{\Gamma_j}(\ell)\right). &(5.19)\end{aligned}$$

The solutions of the equation (5.11) are obtained from the use of Cramer's rule. In addition the asymptotic behaviour is determined from the results (5.17)-(5.19). Together that is:

$$\begin{aligned} U(\ell) &= \frac{\det(B_1(\ell))}{\det(B(\ell))} \sim \left(\frac{\frac{-1}{\mu_1+\zeta}}{\frac{-1}{\mu_1}}\right) = \frac{\mu_1}{\mu_1+\zeta}\\ c_i(\ell) &= \frac{\det(B_i(\ell))}{\det(B(\ell))} \sim \left(\frac{\frac{-1}{\mu_1}\frac{1}{\mu_i+\zeta} - \frac{-1}{\mu_i}\frac{1}{\mu_1+\zeta}}{\frac{-1}{\mu_1}}\frac{1}{M^i_{\Gamma_i}(\ell)}\right)\end{aligned}$$

with $i = 2, \ldots, r$. Since all $M^i_{\Gamma_i}(\ell)$ are growing exponentially fast the asymptotics for f defined in (5.10) are easily determined. So is the limit of the Laplace transform for the undershoot:

$$\begin{aligned}\lim_{\ell\to-\infty} \mathbb{E}_x[\mathrm{e}^{-\zeta Z}] &= \lim_{\ell\to-\infty}\left(\sum_{i=2}^{r} c_i(\ell) f_{\Gamma_i}(x) + U(\ell) f^*(x)\right)\\ &= \lim_{\ell\to-\infty} U(\ell)\cdot 1\\ &= \frac{\mu_1}{\mu_1+\zeta}.\end{aligned}$$

□

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PAPER C

Failure Recovery via RESTART: Wallclock Models

Anders Rønn-Nielsen and Søren Asmussen

Abstract

A task such as the execution of a computer program or the transfer of a file on a communications link may fail and then needs to be restarted. Let the ideal task time be a constant ℓ and the actual task time X, a random variable. Tail asymptotics for $\mathbb{P}(X > x)$ is given under three different models: **1**: a time-dependent failure rate $\mu(t)$; **2**: Poisson failures and a time-dependent deterministic work rate $r(t)$; **3**: as **2**, but $r(t)$ is random and a function of a finite Markov process. Also results close to being necessary and sufficient are presented for X to be finite a.s. The results complement those of Asmussen, Fiorini, Lipsky, Rolski & Sheahan [*Math. Oper. Res.* **33**, 932–944, 2008] who took $r(t) \equiv 1$ and assumed the failure rate to be a function of the time elapsed since the last restart rather than wallclock time

Keywords change of measure, computer reliability, fluid model, inhomogeneous Poisson process, Markov-modulation, Markov renewal theorem, tail asymptotics, time transformation

1 Introduction and Statement of Results

Tasks such as the execution of a computer program or the transfer of a file on a communications link may fail. There is a considerable literature on protocols for handling such failures. We mention in particular RESUME where the task is resumed after repair, REPLACE where the task is abandoned and a new one taken from the pile of waiting tasks, RESTART where the task needs to be restarted from scratch, and CHECKPOINTING where the task contains checkpoints such that performed work is saved at checkpoint times and that upon a failure, the task only needs to be restarted from the last checkpoint.

The protocols RESUME and REPLACE are fairly easy to analyze, see e.g. Kulkarni *et al.* [14], [15] and Bobbio & Trivedi [8]. In contrast, RESTART (Castillo [9], Chimento & Trivedi [10]) resisted analysis for a long time until the recent work of Sheahan *et al.* [17] and Asmussen *et al.* [5] (see also Jelenkovic & Tan [12, 13] for in part parallel work). Recent results for CHECKPOINTING as well as references to earlier work can be found in Asmussen & Lipsky [7].

The model of Asmussen *et al.* [5] assumes that failures occur at a time after each restart with the same distribution G for each restart (a particular important case is of course the exponential distribution). However, it is easy to imagine situations where the model behaviour is determined by the time of the day (the clock on the wall) rather than the time elapsed since the last restart. Think, e.g., of a time-varying load in the system which may influence the failure rate and/or the speed at which the task is performed. For example, the load could be identified with the number of busy tellers in a call centre or the number of users in a LAN (local area network) currently using the central server. The purpose of the present paper is to provide some first insight in the behaviour of such models.

We denote by X the total task time, including failures (a precise definition is given below). One of our goals is to describe the asymptotics of the tail $\mathbb{P}(X > x)$ as $x \to \infty$. For simple restart with constant task time and Poisson failures, this is easy via a renewal argument. In fact, the details as given in [5] lead to:

Proposition 1.1. *Consider simple RESTART with ideal task time ℓ and Poisson(μ^*) failures. Let $\gamma_0 = \gamma_0(\ell, \mu^*) > 0$ denote the root of*

$$1 \; = \; \int_0^\ell \mu^* \mathrm{e}^{(\gamma_0 - \mu^*)y} \, \mathrm{d}y \, . \tag{1.1}$$

Then $\mathbb{P}(X > x) \sim c_0 \mathrm{e}^{-\gamma_0 x}$ as $x \to \infty$ for some $0 < c_0 < \infty$

Here and in the following $f(x) \sim g(x)$ means $f(x)/g(x) \to 1$. Similarly, we will write $f(x) \approx_{\log} g(x)$ if $\log f(x) \sim \log g(x)$. This is the logarithmic asymptotics familiar from large deviations theory (though we will not use results or tools from that area!). It summarizes the main asymptotical features, but does not allow to capture constants like c_0, prefactors of smaller magnitude etc.

It should be noted that c_0 is explicit given γ, but the value needs not concern us here.

The emphasis in [5] is on the more difficult case of a random rather than a constant ideal task time. However as a first attempt, we shall in the present paper throughout assume a constant ideal task time of length ℓ. We will consider three models:

Model 1 Failures at time t after the start of the task occur at deterministic rate $\mu(t)$.

Model 2 Failures occur according to a Poisson(μ^*) process with constant rate μ^*. At time t after the start of the task, the system works on the task at rate $r(t)$.

Model 3 As Model 2, but the rate function $r(t)$ is given as $r(t) = r_{V(t)}$ where $\{V(t)\}_{t\geq 0}$ is an ergodic Markov process with $p < \infty$ states and $r_1, \ldots, r_p$ are constants with $r_i > 0$ for at least one i.

Models 1, 2 are self-explanatory. Model 3 could for example describe a LAN with p users, where $V(t)$ is the number of users currently using the central server and $r_0 = 0$, $r_i = r_1/i$ for $i > 1$.

Models 1 and 2 exhibit a feature not found in simple RESTART: it is possible that $\mathbb{P}(X = \infty) > 0$. This would occur in Model 1 if $\mu(t) \to \infty$ fast enough, and in Model 2 if $r(t) \to 0$ fast enough. Our first main result gives the critical rates:

Theorem 1.1. *(1) Consider Model 1. If* $\limsup_{t\to\infty} \mu(t)/\log t < 1/\ell$, *then* $X < \infty$ *a.s., whereas* $\mathbb{P}(X = \infty) > 0$ *if* $\liminf_{t\to\infty} \mu(t)/\log t > 1/\ell$.
(2) Consider Model 2 and assume that $\int_0^\infty r(s)\,\mathrm{d}s = \infty$ *and* $R(t) = \int_0^t r(s)\mathrm{d}s < \infty$ *for all* $t \geq 0$. *If* $\liminf_{t\to\infty} r(R(t)) \log t/\mu^* > \ell$, *then* $X < \infty$ *a.s., whereas* $\mathbb{P}(X = \infty) > 0$ *if* $\limsup_{t\to\infty} r(R(t)) \log t/\mu^* < \ell$.

The result shows that in Model 1 only a very modest rate of increase to ∞ of $\mu(t)$ may cause the task never to terminate, and that the same is the case for Model 2 with only a very modest rate of decrease to 0 of $r(t)$. In view of this, it seems reasonable to concentrate on decreasing $\mu(t)$ in Model 1 and increasing $r(t)$ in Model 2. The simplest case is of course the power case, and our second main result gives the asymptotics of $\mathbb{P}(X > x)$ in this case:

Theorem 1.2. *(1) Consider Model 1 and assume that* $\mu(t)$ *is strictly positive with* $\mu(t) \sim at^{-\beta}$ *with* $0 < \beta < 1$. *Then*

$$\mathbb{P}(X > x) \approx_{log} \mathrm{e}^{-c_1 x \log x} = x^{-c_1 x},$$

where $c_1 = (1-\beta)/\ell$.
(2) Consider Model 2 and assume that $r(t) \sim at^\eta$ *with* $\eta > 0$. *Then*

$$\mathbb{P}(X > x) \approx_{log} \mathrm{e}^{-c_2 x^{\eta+1} \log x} = x^{-c_2 x^{\eta+1}},$$

where $c_2 = a\eta/(\eta+1)\ell$.

Note that $\beta = 0$ in (1) or $\eta = 0$ in (2) corresponds to the standard RESTART setting, which is why we exclude these cases. Note also that in both Model 1 and Model 2 the decay rate is faster than any exponential. In Model 1 this is intuitive by comparing with Proposition 1.1 since $\gamma \to \infty$ as $\mu \to 0$ with ℓ fixed. This is also the intuitive explanation in Model 2, but to see this, one needs an intermediate step of time reversal given below.

For Model 3 it is trivial that $X < \infty$ a.s. because there is an infinity of sojourn periods in the state with $r_i > 0$ and the probability of task completion in such a period is > 0. For the asymptotics, we need properties of the fluid model

$$F(t) = \int_0^t r_{V(s)} \, \mathrm{d}s .$$

More precisely:

Theorem 1.3. *In Model 3, let* $\kappa(s)$ *denote the largest real value of the* $p \times p$ *matrix* $K[s]$ *with* ij*th element*

$$\int_0^\infty \mu^* \mathrm{e}^{(s-\mu^*)t} \mathbb{P}_i\big(F(t) < \ell, V(t) = j\big) \, \mathrm{d}t .$$

Then $\kappa(s)$ *increases monotonically from* $\kappa(0) < 1$ *to* ∞ *in the interval* $s \in [0, \infty)$. *If* γ_3 *denotes the unique value with* $\kappa(\gamma_3) = 1$, *then* $\mathbb{P}_i(X > x) \sim d_i \mathrm{e}^{-\gamma_3 x}$ *for suitable constants* $d_1, \ldots, d_p$.

Here $d_1, \ldots, d_p$ are again explicit, see Section 4, and as usual, $\mathbb{P}_i$ refers to the case $V(0) = i$.

The outline of proofs is that first Model 1 is considered (Section 2). The results for Model 2 then follow by exploiting the time-transformation connection between homogeneous and inhomogeneous Poisson processes (Section 3). Theorem 1.3 for Model 3 is an easy consequence of the Markov renewal theorem, once it has been recognized how to write up an appropriate Markov renewal equation.

Finally, Section 3 also contains a numerical example.

Notation For the Poisson process with constant rate μ^*, we write $S_1^*, S_2^*, \ldots$ for the event times and $U_n^* = S_n^* - S_{n-1}^*$ for the interevent times ($S_0^* = 0$ is not considered an event time). Similarly, the notation $S_1, S_2, \ldots$ and $U_n = S_n - S_{n-1}$ is used for the inhomogeneous Poisson process of failures in Model 1, and $S_1', S_2', \ldots$ and $U_n' = S_n' - S_{n-1}'$ for a certain auxiliary inhomogeneous Poisson process with rate function $\mu'(s)$ in Section 2. The corresponding counting processes are denoted by $N^*(t), N(t), N'(t)$.

2 Proofs: Model 1

Let $N(t)$ denote the number of failures before t. Then the counting process $\{N(t)\}_{t\geq 0}$ given by $N(t) = \sup\{n : S_n < t\}$ is a time-inhomogeneous Poisson process with rate function $\{\mu(t)\}_{t\geq 0}$.

Define the stopping time $\tau = \inf\{n \in \mathbb{N} \,|\, U_n > \ell\}$. Then the total task time is the r.v. $X = S_{\tau-1} + \ell$ if $\tau < \infty$ and $X = \infty$ otherwise.

In the proof of Theorem 1.1(1) and in the following, define the integrated intensity as $M(t) = \int_0^t \mu(s)\,\mathrm{d}s$. It is then standard that $\{N(t)\}_{t\geq 0}$ can be represented by taking the event times as $S_n = M^{-1}(S_n^*)$.

Proof of Theorem 1.1. Let ℓ' be fixed, let $\mu'(s) = (\log s)^+/\ell'$ and define $X', M', U'n, S_n'$ etc. the obvious way (the ideal task time remains ℓ, not ℓ'!). Then for $s > 1$,

$$\begin{aligned} M'(s) &= s\log s/\ell' + \mathrm{O}(1)\,, \\ M'(s+\ell) &= (s+\ell)\big[\log s + \mathrm{O}(1/s)\big]/\ell' + \mathrm{O}(1) \;=\; (s+\ell)\log s/\ell' + \mathrm{O}(1)\,, \end{aligned}$$

and hence

$$\int_1^\infty \mu'(s)\exp\big\{-M'(s+\ell)+M'(s)\big\}\,\mathrm{d}s \tag{2.1}$$

$$= \int_1^\infty \mathrm{O}(1)\log s\cdot s^{-\ell/\ell'}\,\mathrm{d}s \begin{cases} <\infty & \text{if } \ell' < \ell \\ =\infty & \text{if } \ell' > \ell \end{cases}. \tag{2.2}$$

For the intuition, note that (2.1) equals $\mathbb{E}\sum_1^\infty\{n : U_n' > \ell\}$, the expected number of interevent intervals that would have completed the task, had the task not been completed by the start of the interval.

Assume first $\ell' < \ell$ and let $A'(s)$ be event that $U_n' \leq \ell$ for all n with $S_{n-1}' \leq s$. Clearly, $\mathbb{P}\big(A'(s)\big) > 0$. Defining $K'(s)$ as the number of n with $S_{n-1}' > s$, $U_n' > \ell$ and letting $\mathcal{F}'(s) = \sigma\big(N'(v) : v \leq s\big)$, we have

$$\mathbb{E}\big[K'(s)\,\big|\,\mathcal{F}'(s)\big] \;\leq\; \int_s^\infty \mu'(v)\exp\big\{-M'(v+\ell)+M'(v)\big\}\,\mathrm{d}v\,.$$

By (2.2), we can choose s so large that this integral is (say) $< 1/2$ and get $\mathbb{P}\big(K'(s) \geq 1\,\big|\,\mathcal{F}'(s)\big) \leq 1/2$ such that

$$\begin{aligned} \mathbb{P}(X' = \infty) &= \mathbb{P}\big(A'(s) \cap \{K'(s) = 0\}\big) \\ &= \mathbb{E}\Big[I\big(A'(s)\big)\cdot\mathbb{P}\big(K'(s) = 0\,\big|\,\mathcal{F}'(s)\big)\Big] \;\geq\; \mathbb{P}\big(A'(s)\big)/2 > 0\,. \end{aligned}$$

Let next $\ell' > \ell$. The above estimates for M' imply that $M'^{-1}(s) = s\ell'/\log s\big(1+\mathrm{o}(1)\big)$ as $s\to\infty$, and hence that

$$S_{n-1}' \;=\; \frac{S_{n-1}\ell'}{\log S_{n-1}}\big(1+\mathrm{o}(1)\big) \;=\; \frac{n\ell'}{\log n}\big(1+\mathrm{o}(1)\big) \quad \text{a.s.}$$

Thus

$$\sum_{n=1}^{\infty} \mathbb{P}\bigl(U'_n > \ell \,\big|\, \mathcal{F}'(S'_{n-1})\bigr) \;=\; \sum_{n=1}^{\infty} \exp\bigl\{-M'(S'_{n-1}+\ell) + M'(S'_{n-1})\bigr\}$$

$$= \sum_{n=1}^{\infty} \exp\bigl\{-\ell \log S'_{n-1}/\ell' + \mathrm{O}(1)\bigr\} \;=\; \infty \text{ a.s.}$$

The conditional Borel-Cantelli lemma therefore implies that $U'_n > \ell$ for infinitely many n.

Now consider a general $\mu(s)$. If $\limsup_{s\to\infty} \mu(s)/\log s < 1/\ell$, then for some s_0 and some $\ell' > \ell$ we have $\mu(s) < \mu'(s)$ for all $s > s_0$. Then, realizing N on (s_0, ∞) as the independent sum of N' and an inhomogeneous Poisson process with rate $\mu'(s) - \mu(s)$, we may assume

$$\bigl\{S_{n-1} : S_{n-1} > s_0\bigr\} \;\subseteq \bigl\{S'_{n-1} : S'_{n-1} > s_0\bigr\}\,.$$

Since $U'_n > \ell$ for infinitely many n with $S'_{n-1} > s_0$, this implies $U_n > \ell$ for infinitely many n with $S_{n-1} > s_0$ and $X < \infty$. Similarly, if $\liminf_{s\to\infty} \mu(s)/\log s > 1/\ell$, then for some s_0 and some $\ell' > \ell$ we have $\mu(s) > \mu'(s)$ for all $s > s_0$, and $U_n > \ell$ for some n with $S_{n-1} > s_0$ implies $S'_n > \ell$ for some n with $S'_{n-1} > s_0$. Therefore the event that $U_n > \ell$ for some n with $S_{n-1} > s_0$ cannot have probability one, which as above implies $\mathbb{P}(X = \infty) > 0$. □

We next consider the proof of Theorem 1.2 (1), describing the tail of X in the most standard case, a Weibull type rate function $\mu(t) =\sim at^{-\beta}$ with $0 < \beta < 1$. Note that $\beta = 0$ corresponds to the simple RESTART setting with Poisson failures with $\mu^* = a$. $\beta < 0$ is excluded because then $\mathbb{P}(X = \infty) > 0$, and $\beta > 1$ is excluded because then $M(\infty) < \infty$, a case that appears somewhat pathological and that we do not study.

Before turning to the setup of Theorem 1.2 (1) we shall prove some less clear results for a general $\mu(t)$. Assume that $\mu(t)$ is decreasing with limit 0.

The probability $\mathbb{P}(X > x)$ can be written as $\mathbb{P}(X > x) = \mathbb{P}\bigl(B(x)\bigr)$ where

$$B(x) = \;\bigl\{U_1 \le \ell, \ldots, U_{\tau(x)-1} \le \ell, x - S_{\tau(x)-1} \le \ell\bigr\}\,,\;\; \tau(x) \;=\; \inf\{n : S_n > x\}\,.$$

Obviously, we must have $B(x) \subseteq C(x - \ell)$ where

$$C(x) = \{U_1 \le \ell, \ldots, U_{\tau(x)} \le \ell\}\,.$$

But in fact $x \ge S_{\tau(x-\ell)} \ge x - \ell$ implies $U_{\tau(x-\ell)+1} \le \ell, \ldots, U_{\tau(x)-1} \le \ell$ and $x - S_{\tau(x)-1} \le \ell$ (see Figure C.1). That is,

$$B(x) \;=\; C(x - \ell)\,, \tag{2.3}$$

so deriving the asymptotics for $\mathbb{P}\bigl(C(x - t)\bigr)$ will solve the problem.

Figure C.1: It holds that $x - S_{\tau(x)-1} \le \ell$ when $U_{\tau(x-\ell)} \le \ell$.

Choose $\gamma = \gamma(\ell)$ such that

$$1 \;=\; \int_0^\ell \mathrm{e}^{\gamma x}\,\mathrm{d}x \;=\; \frac{1}{\gamma}\Big[\mathrm{e}^{\gamma\ell} - 1\Big]\,. \tag{2.4}$$

Let $\mathbb{Q}$ be the probability measure where $(U_n)_{n\in\mathbb{N}}$ are i.i.d. each with density $u \to \mathrm{e}^{\gamma u}$ on $(0,\ell)$. Note that $\mathbb{Q}\big(C(x)\big) = 1$ since $\mathbb{Q}(U_1 \le \ell) = 1$.

Proposition 2.1. *For any $\mu(t)$ it holds that*

$$\mathbb{P}\big(C(x)\big) \;=\; \mathbb{E}_{\mathbb{Q}}\Big[\big(\prod_{k=1}^{\tau(x)} \mu(S_n)\big)\exp\Big(-M(S_{\tau(x)}) - \gamma S_{\tau(x)}\Big)\Big]\,.$$

Proof. Define $(\mathcal{F}_n)_{n\in\mathbb{N}}$ as the natural filtration for $(U_n)_{n\in\mathbb{N}}$:

$$\mathcal{F}_n = \sigma(U_1,\dots,U_n) \qquad (n\in\mathbb{N})\,.$$

Note that $\tau(x)$ is a stopping time with $\mathbb{P}(\tau(x)<\infty) = 1$ (such that $C(x)$ is well-defined) and that $C(x) \in \mathcal{F}_{\tau(x)}$.

Given S_{n-1} , the conditional distribution of U_n is determined by the density

$$f_{|S_{n-1}}(s) = \mu(s)\exp\Big(-M(s) + M(S_{n-1}))\Big) \qquad (s > S_{n-1})$$

(on $\{S_{n-1}<\infty\}$ in the case of $M(\infty)<\infty$). If $M(\infty)<\infty$, this distribution is defective with

$$\mathbb{P}_{|S_{n-1}}(U_n = \infty) = \exp\Big(-\int_{S_{n-1}}^\infty \mu(u)\,\mathrm{d}u\Big) = \exp\Big(-M(\infty) + M(S_{n-1}))\Big)\,.$$

Therefore the joint density of $(U_1,\dots,U_n)$ (w.r.t. the Lebesgue measure on $(0,\infty)^n$) is

$$\begin{aligned} g_n(u_1,\dots,u_n) \;&=\; \prod_{k=1}^n g_{|s_{k-1}}(u_k) \;=\; \prod_{k=1}^n \mu(s_k)\exp\Big(-M(s_k) + M(s_{k-1})\Big) \\ &=\; \Big(\prod_{k=1}^n \mu(s_k)\Big)\exp\{-M(s_n)\} \end{aligned}$$

for $(u_1,\dots,u_n) \in (0;\infty)^n$. Here the notation $s_k = u_1 + \cdots + u_k$ has been used.

Under $\mathbb{Q}$ the vector $(U_1, \dots, U_n)$ has density

$$h_n(u_1, \dots, u_n) = \mathrm{e}^{\gamma s_n}\,.$$

Define

$$F_n = \{U_1 \le \ell, \dots, U_n \le \ell\}\,.$$

Then $F_n \in \mathcal{F}_n$. If $D_n \subseteq F_n$ and $D_n \in \mathcal{F}_n$ (that is $D_n = \{(U_1, \dots, U_n) \in C_n\}$, where $C_n \subseteq [0, \ell]^n$ is Borel measurable), we have

$$\mathbb{P}(D_n) \quad = \quad \mathbb{E}_{\mathbb{Q}}\Big[\frac{g_n(U_1, \dots, U_n)}{h_n(U_1, \dots, U_n)}; D_n\Big]\,.$$

Thus by a standard extension to stopping times (e.g. [4] pp. 131–132)

$$\mathbb{P}\big(C(x)\big) \quad = \quad \mathbb{E}_{\mathbb{Q}}\Big[\frac{g_{\tau(x)}(U_1, \dots, U_{\tau(x)})}{h_{\tau(x)}(U_1, \dots, U_{\tau(x)})}\Big]\,,$$

where we have used that $\mathbb{Q}\big(C(x)\big) = 1$ and $\mathbb{P}(\tau(x) < \infty) = \mathbb{Q}(\tau(x) < \infty) = 1$. When the expressions for g_n and h_n are inserted, this becomes the requested result. □

□

Proposition 2.2. *If $\mu(t)$ is decreasing with limit 0 then*
(i)

$$\mathbb{P}\big(C(x)\big) \;\le\; c_3\Big(\prod_{k=0}^{[x/\ell]} \mu(x - k\ell)\Big) \exp\big(-M(x) - \gamma x\big)$$

when $x \to \infty$, for some constant c_3.
(ii)

$$\begin{aligned}\mathbb{P}\big(C(x)\big) \;&\ge c_4 \exp\Big(\frac{1}{\ell} \log\big(\mu(x+\ell)\big)(1 + \mathrm{o}(1))(x+\ell)\Big)\\ &\times\; \exp\Big(-M(x+\ell) - (1 - 1/\ell)\gamma x\Big)\end{aligned}$$

when $x \to \infty$, for some constant c_4.

Proof. For (i), recall that $x < S_{\tau(x)} \le x + \ell$. If either $\gamma > 0$ or $\gamma < 0$, the expression for $\mathbb{P}\big(C(x)\big)$ in Proposition 2.1 is bounded up by a constant times

$$\mathbb{E}_{\mathbb{Q}}\Big[\prod_{k=1}^{\tau(x)} \mu(S_n)\Big] \exp\big(-M(x) - \gamma x\big) \tag{2.5}$$

Left is exploring the behaviour of the expectation in (2.5). First rewrite it as

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\Big[\prod_{k=1}^{\tau(x)}\mu(S_k)\Big] &= \mathbb{E}_{\mathbb{Q}}\Big[\Big(\prod_{k=1}^{\tau(x)-\lfloor x/\ell\rfloor-1}\mu(S_k)\Big)\Big(\prod_{k=\tau(x)-\lfloor x/\ell\rfloor}^{\tau(x)}\mu(S_k)\Big)\Big]\\ &\le \mathbb{E}_{\mathbb{Q}}\Big[\prod_{k=1}^{\tau(x)-\lfloor x/\ell\rfloor-1}\mu(S_k)\Big]\Big(\prod_{k=0}^{\lfloor x/t\rfloor}\mu(x-k\ell)\Big).\end{aligned}$$

In the inequality we have used that μ is decreasing and the fact that

$$S_{\tau(x)} > x,\ S_{\tau(x)-1} > x-\ell\ ,\ S_{\tau(x)-2} > x-2\ell\ ,\ldots,\ S_{\tau(x)-\lfloor x/\ell\rfloor} > x-\lfloor x/\ell\rfloor\ell\,.$$

Since $\lim_{t\to\infty}\mu(t)=0$ the second factor above obviously decreases – very fast – to 0. We show that the first factor – the expectation – decreases to 0 and thereby is bounded such that we have the result from the theorem.

We have

$$\frac{\tau(x)}{x/\mathbb{E}_{\mathbb{Q}}[U_1]} \overset{x\to\infty}{\longrightarrow} 1 \quad \mathbb{Q}\text{–a.s.}$$

Since $\mathbb{E}_{\mathbb{Q}}[U_1] < \ell$ and therefore $x/\mathbb{E}_{\mathbb{Q}}[U_1] - \lfloor x/\ell\lfloor \to \infty$, this yields that

$$\tau(x) - \Big\lfloor\frac{x}{\ell}\Big\rfloor \overset{x\to\infty}{\longrightarrow} \infty \quad \mathbb{Q}\text{–a.s.}$$

Together with the fact that $S_n\to\infty$ $\mathbb{Q}$–a.s., this leads to

$$\prod_{k=1}^{\tau(x)-\lfloor x/\ell\rfloor-1}\mu(S_k) \overset{x\to\infty}{\longrightarrow} 0 \quad \mathbb{Q}\text{–a.s.}$$

because the factors in the product decrease to 0 and $\tau(x)-\lfloor x/\ell\rfloor-1\to\infty$. Now let $a>0$ be a constant such that $\mu(t)<1$ for $t>a$ and define the stopping time

$$\sigma = \inf\{n\in\mathbb{N}\,|\,S_n > a\}\,.$$

Then we have the following upper bound for the integrand in the expectation:

$$\begin{aligned}\prod_{k=1}^{\tau(x)-\lfloor x/\ell\rfloor-1}\mu(S_k) &= \Big(\prod_{k=1}^{(\tau(x)-[x/\ell]-1)\wedge\sigma}\mu(S_k)\Big)\Big(\prod_{k=(\tau(x)-\lfloor x/\ell\rfloor-1)\wedge\sigma}^{\tau(x)-\lfloor x/\ell\rfloor-1}\mu(S_k)\Big)\\ &\le \prod_{k=1}^{(\tau(x)-\lfloor x/\ell\rfloor-1)\wedge\sigma}\lambda(S_k) \ \le \prod_{k=1}^{(\tau(x)-\lfloor x/\ell\rfloor-1)\wedge\sigma}\mu(0) \ \le\ \mu(0)^{\sigma}\,.\end{aligned}$$

From Lemma 5.1 in the Appendix we have that this upper bound has finite expectation. Hence by dominated convergence we can conclude that

$$\mathbb{E}_{\mathbb{Q}}\Big[\prod_{k=1}^{\tau(x)-\lfloor x/\ell\rfloor-1}\mu(S_k)\Big] \overset{x\to\infty}{\longrightarrow} 0\,.$$

(ii): As in (i) we have

$$\mathbb{P}\big(C(x)\big) \;=\; \mathbb{E}_{\mathbb{Q}}\Big[\Big(\prod_{k=1}^{\tau(x)} \mu(S_n)\Big) \exp\big(-M(S_{\tau(x)}) - \gamma S_{\tau(x)}\big)\Big]$$

and similarly the r.h.s. is bounded below by a constant times

$$\mathbb{E}_{\mathbb{Q}}\Big[\prod_{k=1}^{\tau(x)} \mu(S_k)\Big] \exp\Big(-M(x+\ell) - \gamma x\Big)\,. \tag{2.6}$$

Recall that $S_k \le x + \ell$ on $\{\tau(x) \ge k\}$ so that a lower bound for (2.6) is

$$\mathbb{E}_{\mathbb{Q}}\Big[(\mu(x+\ell)^{\tau(x)}\Big] \exp\big(-\Lambda(x+\ell) - \gamma x\big)\,.$$

From Proposition 5.1 in the Appendix we have that this is bounded below by

$$\exp\big(-\varphi^{-1}(\mu(x+\ell))(x+\ell)\big)\exp\Big(-M(x+\ell) - \gamma x\Big)$$

with

$$\varphi(z) \;=\; \frac{z+\gamma-1}{\mathrm{e}^{\ell(z+\gamma-1)}-1}\,.$$

Combined with the result from Proposition 5.2 in the Appendix this gives

$$\mathbb{P}\big(C(x)\big) \;\ge\; \exp\Big(\frac{1}{\ell}\log(\mu(x+\ell))(1+\mathrm{o}(1))(x+\ell)\Big)\exp\big(-M(x+\ell) - \gamma x\big)$$

when $x \to \infty$ (remember that $\mu(x) \to 0$). □

□

As a result of Proposition 2.2 and (2.3), we immediately get

Corollary 2.1. *In the setup from above we have that*

(i) $\mathbb{P}\big(A(x)\big) \;\le\; c_5 \left(\prod_{k=1}^{\lfloor x/\ell \rfloor} \mu(x-(k+1)\ell)\right) \exp\big(-M(x-\ell) - \gamma x\big)$ *when* $x \to \infty$*, for some constant* c_5*.*

(ii)

$$\mathbb{P}\big(A(x)\big) \;\ge\; c_6 \exp\Big(\frac{1}{\ell}\log\big(\mu(x)\big)\big(1+\mathrm{o}(1)\big)x\Big)\exp\big(-M(x) - (1-1/\ell)\gamma x\big)$$

when $x \to \infty$*, for some constant* c_6*.*

In the case where $M(\infty) < \infty$ the result becomes simpler:

Corollary 2.2. *If furthermore $M(\infty) < \infty$ it holds that*
(i) $\mathbb{P}(X > x) \ \leq \ c_7 \left(\prod_{k=1}^{\lfloor x/\ell \rfloor} \mu(x-(k+1)\ell) \right) \exp\big(-\gamma x\big)$ *when* $x \to \infty$*, for some constant* c_7*.*
(ii)

$$\mathbb{P}(X > x) \ \geq \ c_8 \exp\Big(\frac{1}{\ell}\log\big(\mu(x)\big)(1+o(1))x\Big)\exp\big(-(1-1/\ell)\gamma x\big)$$

when $x \to \infty$*, for some constant* c_8*.*

Proof of Theorem 1.1(i). If the intensity process is strictly positive and satisfies $\mu(s) \sim as^{-\beta}$ with $0 < \beta < 1$, then $\underline{\mu}$, $\overline{\mu}$ exists on the form $cs^{-\beta}$ with $\underline{\mu} \leq \mu(s) \leq \overline{\mu}(s)$. With $\underline{M}$, $\overline{M}$ the corresponding integrated intensity processes we have e.g. $\underline{M}(s) = cs^{1-\beta}/(1-\beta)$. Let furthermore $\underline{X}$, $\overline{X}$ denote total task times corresponding to $\underline{\mu}$, $\overline{\mu}$ respectively. Then $\mathbb{P}(\underline{X} > x) \leq \mathbb{P}(X > x) \leq \mathbb{P}(\overline{X} > x)$ and

$$\begin{aligned}
&\prod_{k=1}^{\lfloor x/\ell \rfloor} \overline{\mu}(x-k\ell) \\
&= \prod_{k=1}^{\lfloor x/\ell \rfloor} \frac{a}{(b+x-k\ell)^{1-\beta}} \\
&= \prod_{k=1}^{\lfloor x/\ell \rfloor} \frac{a}{\ell^{1-\beta}(\frac{b+x}{\ell}-k)^{1-\beta}} \\
&\leq C\frac{1}{\left(\frac{b+x}{\ell}-\lfloor\frac{x}{\ell}\rfloor\right)^{1-\beta}} \prod_{k=1}^{\lfloor x/\ell \rfloor -1} \frac{1}{(\lfloor\frac{b+x}{\ell}\rfloor-k)^{1-\beta}} \\
&= C\frac{1}{\left(\frac{b+x}{\ell}-\lfloor\frac{x}{\ell}\rfloor\right)^{1-\beta}} \left(\frac{(\lfloor\frac{b+x}{\ell}\rfloor-\lfloor\frac{x}{\ell}\rfloor)!}{(\lfloor\frac{b+x}{\ell}\rfloor-1)!}\right)^{1-\beta} \\
&\leq C\frac{1}{\left(\frac{b+x}{\ell}-\lfloor\frac{x}{\ell}\rfloor\right)^{1-\beta}} \left(\frac{(\lfloor\frac{b}{\ell}\rfloor+1)!}{(\lfloor\frac{b+x}{\ell}\rfloor-1)!}\right)^{1-\beta} \\
&\sim \tilde{C}\frac{1}{\left(x\sqrt{(\lfloor\frac{b+x}{\ell}\rfloor-1)}\,(\lfloor\frac{b+x}{\ell}\rfloor-1)^{\lfloor\frac{b+x}{\ell}\rfloor-1}\exp\big(-(\lfloor\frac{b+x}{\ell}\rfloor-1)\big)\right)^{1-\beta}}.
\end{aligned}$$

From Corollary 2.1 (i) we have that

$$\begin{aligned}
\mathbb{P}(X > x) \leq & C\left(\left\lfloor\frac{b+x}{\ell}\right\rfloor-1\right)^{-(1-\beta)(\lfloor\frac{b+x}{\ell}\rfloor-1)} \\
& \times \ \exp\Big(-\frac{a}{\beta}(b+x)^{\beta}+(\frac{1-\beta}{\ell}-\gamma)x\Big)x^{-\frac{3(1-\beta)}{2}}
\end{aligned}$$

when $x \to \infty$ for some constant C. The expression on the r.h.s. above is

$$\begin{aligned} &\approx_{\log} \quad \left(\left\lfloor \frac{b+x}{\ell} \right\rfloor - 1 \right)^{-(1-\beta)\left(\left\lfloor \frac{b+x}{\ell} \right\rfloor - 1\right)} \\ &\approx_{\log} \quad x^{\frac{1-\beta}{\ell} x} . \end{aligned}$$

From (ii) we get

$$\begin{aligned} \mathbb{P}(X > x) \quad &\geq \quad \tilde{C} \exp\Big(-\frac{1}{\ell}(1-\beta) \log\big(b+x\big)(1+o(1))x \Big) \\ &\qquad \times \exp\Big(-\frac{a}{\beta}(b+x)^{\beta} - (1-1/\ell)\gamma x \Big) . \end{aligned}$$

Here the first factor decreases faster than the second so that this

$$\begin{aligned} &\approx_{\log} \quad \exp\Big(-\frac{1}{\ell}(1-\beta) \log\big(b+x\big)(1+o(1))x \Big) \\ &\approx_{\log} \quad \exp\Big(-\frac{1}{\ell}(1-\beta) \log\big(b+x\big)x \Big) \\ &\approx_{\log} \quad x^{-\frac{1-\beta}{\ell} x} . \end{aligned}$$

All together we have shown that

$$\mathbb{P}(X > x) \approx_{\log} x^{\frac{1-\beta}{\ell} x}$$

when $x \to \infty$. □

3 Proofs: Model 2

Recall that in Model 2 the failure times $(N_t^*)_{t \geq 0}$ form a homogeneous Poisson process with intensity parameter μ^*, event times $(S_n^*)_{n \in \mathbb{N}}$, and interevent times $(U_n^*)_{n \in \mathbb{N}}$. Again, ℓ is the ideal task time, and it is assumed that at time t the system works on the task at rate $r(t)$, where r is a nonnegative measurable function.

Define the continuous and increasing function R as

$$R(t) \;=\; \int_0^t r(s)\,\mathrm{d}s\,.$$

It is obvious that $\mathbb{P}(X = \infty) > 0$ if $R(\infty) < \infty$, so assume that $R(\infty) = \infty$. Also assume that $R(t) < \infty$ for all $t \geq 0$.

A straightforward calculation shows that the inverse R^{-1} of R is continuous and increasing function and given by

$$R^{-1}(y) \;=\; \int_0^y \frac{1}{r\big(R(s)\big)}\,\mathrm{d}s\,.$$

Since $R(t)$ is the amount of work that has been spent on the task up to time t provided the task has not been completed, the total task time in absence of failures is given by $R(X) = \ell$, i.e. $X = R^{-1}(\ell)$. More generally, if the task is not completed at the time S^*_{n-1} of the $(n-1)$th failure, then the task is still uncompleted at the time S^*_n of the nth failure if and only if $R^{-1}(S^*_n) - R^{-1}(S^*_{n-1}) < \ell$ [these observations are close to some standard facts in storage processes, see [3] p. 381].

It follows that the total task time X can be calculated as follows. First define the time ω as

$$\omega = \inf\{n \in \mathbb{N} \mid \int_{S^*_{n-1}}^{S^*_n} r(t)\,\mathrm{d}t > \ell\}\,,$$

and let ℓ^* satisfy

$$\int_{S^*_{\omega-1}}^{\ell^*} r(t)\,\mathrm{d}t = \ell\,.$$

Then the total task time X is

$$X = S^*_{\omega-1} + \ell^*\,,$$

if $\omega < \infty$ and $X = \infty$ when $\omega = \infty$.

Proof of Theorem 1.1(ii). From the definition of N we can construct another point process: Let S'_n, $n \in \mathbb{N}$ be defined by $S'_n = R(S^*_n)$ for all $n \geq 0$. Since $S'_n = R^{-1}(S^*_n)$ it is well–known that $(S_n)_{n\in\mathbb{N}}$ are the event times of an inhomogeneous Poisson process with rate function $\mu(t) = \mu^*/r\big(R(t)\big)$.

It is directly seen that also

$$\omega \;=\; \inf\{n \in \mathbb{N} \,|\, S'_n > \ell\}\,.$$

Applying this yields the following definition of the total task time X' corresponding to S'_n, $n \in \mathbb{N}$

$$X' = \begin{cases} S'_{\omega-1} + \ell & ,\ \omega < \infty \\ \infty & ,\ \omega = \infty \end{cases}$$

Especially we have $\{X = \infty\} = \{X' = \infty\}$ and hence the theorem follows from Theorem 1.1(i). $\square$

Define

$$f(x) := ax^\eta, \qquad F(x) := \int_0^x f(y)\,\mathrm{d}y = \frac{a}{\eta+1}x^{\eta+1}\,.$$

Lemma 3.1. *If $r(x) \sim f(x)$ then*

$$R(x) \sim F(x) = \frac{a}{\eta+1}x^{\eta+1} \quad \textit{and} \quad r(R(x)) \sim f(F(x)) = a\left(\frac{a}{\eta+1}\right)^\eta x^{\eta(\eta+1)}\,.$$

Proof. Given $\epsilon > 0$, there exits a x_0 exists such that

$$(1-\epsilon)f(x) \leq r(x) \leq (1+\epsilon)f(x) \quad \text{for } x > x_0\,.$$

Hence

$$(1-\epsilon)\int_{x_0}^{x} f(y)\,\mathrm{d}y \leq \int_{x_0}^{x} r(y)\,\mathrm{d}y \leq (1+\epsilon)\int_{x_0}^{x} f(y)\,\mathrm{d}y \quad \text{for } x > x_0$$

Thereby it is seen - since $\int_0^x f(x)\,\mathrm{d}x \to \infty$ - that choosing $x > x_0$ large enough gives

$$\frac{\int_0^x r(y)\,\mathrm{d}y}{\int_0^x f(y)\,\mathrm{d}y} = \frac{\int_0^{x_0} r(y)\,\mathrm{d}y + \int_{x_0}^x r(y)\,\mathrm{d}y}{\int_0^{x_0} f(y)\,\mathrm{d}y + \int_{x_0}^x f(y)\,\mathrm{d}y} \in (1-2\epsilon, 1+2\epsilon)\,.$$

For the second result write

$$\frac{r(R(x))}{f(F(x))} = \frac{r(R(x))}{f(R(x))}\frac{f(R(x))}{f(F(x))}\,,$$

where the first factor obviously has limit 1. For the second factor $x_0 > 0$ can be found given ϵ such that for $x > x_0$

$$(1-\epsilon)F(x) < R(x) < (1+\epsilon)F(x)$$

and hence

$$f((1-\epsilon)F(x)) < f(R(x)) < f((1+\epsilon)F(x))$$

for $x > x_0$. Furthermore

$$f((1-\epsilon)F(x)) < f(F(x)) < f((1+\epsilon)F(x))$$

so it is obtained

$$\frac{f((1-\epsilon)F(x))}{f((1+\epsilon)F(x))} < \frac{f(R(x))}{f(F(x))} < \frac{f((1+\epsilon)F(x))}{f((1-\epsilon)F(x))}\,.$$

Since

$$\frac{f((1+\epsilon)F(x))}{f((1-\epsilon)F(x))} = \left(\frac{1+\epsilon}{1-\epsilon}\right)^{\eta}$$

has limit 1 as $\epsilon \to 0$ the proof is complete. □

□

Proof of Theorem 1.2(i). Note that $\mathbb{P}(X = \infty) = 0$. With X' defined as in the proof of Theorem 1.1 it holds on $\{X < \infty\}$ that

$$\begin{aligned}\{X > x\} &= \{R(X) > R(x)\} \\ &= \{\int_0^{S'_{\omega-1}+\ell^*} r(t)\,\mathrm{d}t > R(x)\} \\ &= \{R(S'_{\omega-1}) + \ell > R(x)\} \\ &= \{X' > R(x)\}\,. \end{aligned} \tag{3.1}$$

Recall that X' is the total task time for a nonhomogeneous Poisson process with intensity process $(\mu(t))_{t\geq 0}$ where $\mu(t) = \frac{\mu^*}{r(R(t))}$. From Lemma 3.1 we have that

$$r(R(t)) \sim f(F(t)) = a\left(\frac{a}{\eta+1}\right)^{\eta} t^{\eta(\eta+1)},$$

and hence $(\mu(t))$ has a form that suits the theorem for **Model 1**. Since also $R(t) \sim F(t)$ applying the result for **Model 1** to the relation (3.1) yields

$$P(X > x) = P(\tilde{X} > R(x)) \approx_{\log} R(x)^{-\frac{\eta(\eta+1)}{\ell}R(x)} \approx_{\log} F(x)^{-\frac{\eta(\eta+1)}{\ell}F(x)}.$$

□

4 Proofs: Model 3

The renewal argument in [5] leading to Proposition 1.1 for simple RESTART uses a geometric sum representation of $D = X - \ell$. It is instructive for the following to give a direct variant. Define $Z(x) = \mathbb{P}(D > x)$ and let $z(x)$, $Z_0(x)$ be the contributions to $Z(x)$ from the events $U > x$ that the first failure time exceeds x, resp. $U \leq x$. A failure at time $t \leq x$ will contribute to $Z_0(x)$ if and only if $t \leq \ell$, which readily leads to

$$Z_0(x) \;=\; \int_0^x Z(x-t)\mu\mathrm{e}^{-\mu t} I(t \leq \ell)\,\mathrm{d}t\,.$$

Similarly but easier, $z(x) = \int_x^\infty \mu\mathrm{e}^{-\mu t} I(t \leq \ell)\,\mathrm{d}t$, and altogether,

$$Z(x) \;=\; z(x) + Z_0(x) \;=\; z(x) + \int_0^x Z(x-t)g(t)\,\mathrm{d}t\,,$$

where $g(t)$ is the defective density $\mu\mathrm{e}^{-\mu t} I(t \leq \ell)$. The rest is then standard renewal theory (e.g. [3] V.7).

Now consider Model 3 and write again $D = X - \ell$. Define $Z_i(x) = \mathbb{P}_i(D > x)$. We then get the following Markov renewal equation:

Proposition 4.1.

$$Z_i(x) \;=\; z_i(x) \;+\; \sum_{j=1}^{p} \int_0^x F_{ij}(\mathrm{d}t) Z_j(x-t)\,\mathrm{d}t\,, \tag{4.1}$$

where F_{ij} has density $\mu\mathrm{e}^{-\mu t}\mathbb{P}_i\big(F(t) \leq \ell, V(t) = j\big)$ and

$$z_i(x) \;=\; \int_x^\infty \mu\mathrm{e}^{-\mu t}\mathbb{P}\big(F(t) \leq \ell\big)\,.$$

Proof. We condition again on the time $U = t$ of the first failure. Then for $D > 0$ it is necessary that $F(t) \leq \ell$, and therefore $z_i(x)$ is the contribution to $Z_i(x)$ from the event $U > x$. Similarly, conditioning in addition on $V(t)$ shows that the second term in (4.1) is the contribution from the event $U \leq x$. □

The proof of Theorem 1.3 is now a straightforward adaptation of the defective Markov key renewal theorem, [3] pp. 209–210. To give the value of D_i is also straightforward from the formulas there, but the formulas are tedious and therefore omitted.

For computational purposes, one therefore needs to evaluate $\mathbb{P}\big(F(t) \leq \ell, V(t) = j\big)$. The four most common approaches are:

a) to let $g(t, f; ij) = (\mathrm{d}/\mathrm{d}f)\mathbb{P}\big(F(t) \leq f, V(t) = j\big)$ and derive a set of PDE's for the $g(t, f; ij)$;

b) the transform inversion method of Ahn & Ramaswami [1];

c) the series expansion of Sericola [16];

d) simulation of $\mathbb{P}\big(F(t) \leq \ell, V(t) = j\big)$.

Example 4.1. Consider a LAN with N users. Each sends a task of an exponential(ν) duration to the central unit at rate λ (no more tasks are sent before completion), the central unit works at rate 1 and uses standard processor sharing (works simultaneous on all tasks at the same rate). Thus, it seems reasonable to take $V(t) \in \{0, \ldots, N\}$ as the number of tasks currently with the server, let

$$q_{i(i+1)} = (N - i)\lambda\,, \quad q_{i(i-1)} = \nu$$

and all other off-diagonal q_{ij} equal to zero, and take $r_i = 1/i$ for $i > 0$, $r_0 = 0$. The model for $V(t)$ is an example of the so–called *Palm's Machine Repair Problem* described in [3] III.3 with only a single repairman.

With $\pi_j = \lim_{t\to\infty} \mathbb{P}_i\big(V(t) = j\big)$, the average service rate is $r^* = \sum_1^p \pi_i/i$ where π is the stationary distribution ov V. If failures occur at rate μ and a user sends a task of length ℓ to the central unit, a reasonable question is then how the exponential decay rate $\gamma(\ell)$ of this total task duration compares to that γ^* of simple RESTART with service rate r^* (that is, ideal task duration ℓ/r^*). To illustrate this, we took $N = 10$, $\lambda = 1$, $\ell = 1$ and considered 3×3 combinations of ν, μ: ν chosen such that $\mathbb{E}_\pi V(0) = 2\,,5\,,8$ (low, moderate and heavy load) and $\mu = 1/5\,,1\,,5$ (low, moderate and high failure rate). We used method d) and obtained the following table over γ and γ^* (the vaules of γ^* are in $(\cdot)$):

$\mathbb{E}_\pi V(0)$ μ	2	5	8
1/5	0.683 (0.744)	0.259 (0.304)	0.079 (0.079)
1	0.134 (0.121)	0.040 (0.021)	0.011 (0.004)
5	0.144 (0.030)	0.212 (0.050)	0.235 (0.094)

□

5 Appendix

Lemma 5.1. *Assume that $(U_n)_{n\in\mathbb{N}}$ are iid variables with $U_n > 0$. Let $(S_n)_{n\in\mathbb{N}}$ be the corresponding random walk, that is $S_n = \sum_{k=1}^n U_k$ for $n \in \mathbb{N}$. Define*

$$\tau = \inf\{n \in \mathbb{N} \,|\, S_n > a\}$$

for some $a > 0$. Then

$$\mathbb{E}[t^\tau] < \infty \qquad \textit{for all } t > 0\,.$$

Proof. The result is obvious for $t \leq 1$ so assume $t > 1$. Since $U_1 > 0$ there exist a constant $b > 0$ such that $p := \mathbb{P}(U_1 \leq b) < \frac{1}{t}$. Choose $M \in \mathbb{N}$ with $bM \geq a$.

Let $\tau_0 \equiv 0$ and define the stopping times $(\tau_m)_{m\in\mathbb{N}}$ recursively by

$$\tau_m = \inf\{n > \tau_{m-1} \,|\, U_n > b\}\,.$$

Then it holds that

$$\tau_M \geq \tau$$

and $(\sigma_m)_{m\in\mathbb{N}}$ are iid where

$$\sigma_m = \tau_m - \tau_{m-1}\,.$$

Note that

$$\sigma_1 = \tau_1 = \inf\{n \in \mathbb{N} \,|\, S_n > b\}$$

and furthermore that

$$\mathbb{P}(\sigma_1 = n) = \mathbb{P}(U_1 \leq b, \ldots, U_{n-1} \leq b, U_n > b) = p^{n-1}(1-p)\,.$$

Hence

$$\begin{aligned} \mathbb{E}[t^{\sigma_1}] &= \sum_{n=1}^\infty t^n p^{n-1}(1-p) \\ &= \frac{1}{p}(1-p)\sum_{n=1}^\infty (tp)^n \\ &< \infty\,, \end{aligned}$$

where it has been used that $p < \frac{1}{t}$. Thereby we obtain

$$\mathbb{E}[t^\tau] \le \mathbb{E}\left[t^{\sum_{k=1}^M \sigma_k}\right] = \mathbb{E}\left[\prod_{k=1}^M t^{\sigma_k}\right] = \prod_{k=1}^M \mathbb{E}[t^{\sigma_k}] = \mathbb{E}[t^{\sigma_1}]^M < \infty .$$

□

Proposition 5.1. *Assume that $(U_n)_{n\in\mathbb{N}}$ are iid variables each with density $t \mapsto \mathrm{e}^{\gamma t}$ on $[0,t]$. Let $(S_n)_{n\in\mathbb{N}}$ be the corresponding random walk, that is $S_n = \sum_{k=1}^n U_k$ for $n \in \mathbb{N}$. Define*

$$\tau(x) = \inf\{n \in \mathbb{N} \mid S_n > x\} = \inf\{n \in \mathbb{N} \mid S_n - x > 0\}$$

for some $x > 0$. Then

$$\exp\big(-\varphi^{-1}(z)(x+t)\big) \le \mathbb{E}[z^{\tau(x)}] \le \exp\big(-\varphi^{-1}(z)x\big)$$

for all $0 < z < 1$, where

$$\varphi(\theta) = \frac{\theta+\gamma}{\mathrm{e}^{t(\theta+\gamma)} - 1} .$$

Proof. Because the U_k–variables are bounded we have for all $\theta > 0$ that

$$h(\theta) := \mathbb{E}\left[\mathrm{e}^{\theta U_1}\right] < \infty .$$

Consequently

$$M_n(\theta) = \frac{\mathrm{e}^{\theta S_n}}{(h(\theta))^n} \qquad (n \in \mathbb{N})$$

is a martingale with mean 1. Define

$$\tau(x) = \inf\{n \in \mathbb{N} \mid S_n > x\} .$$

Then by optional stopping we have

$$1 = \mathbb{E}\left[\frac{\mathrm{e}^{\theta S_{\tau(x)}}}{h(\theta)^{\tau(x)}}; \tau(x) \le n\right] + \mathbb{E}\left[\frac{\mathrm{e}^{\theta S_n}}{h(\theta)^n}; \tau(x) > n\right] .$$

Let $n \to \infty$ and note that $\mathrm{e}^{\theta S_n} \le \mathrm{e}^{\theta x}$ on $\{\tau(x) > n\}$. From dominated convergence we have

$$1 = \mathbb{E}\left[\frac{\mathrm{e}^{\theta S_{\tau(x)}}}{h(\theta)^{\tau(x)}}; \tau(x) < \infty\right] = \mathbb{E}\left[\frac{\mathrm{e}^{\theta S_{\tau(x)}}}{h(\theta)^{\tau(x)}}\right] .$$

Since $x < S_{\tau(x)} \le x + t$ this yields

$$\mathbb{E}\left[\frac{1}{h(\theta)^{\tau(x)}}\right] \mathrm{e}^{\theta x} < 1 \le \mathbb{E}\left[\frac{1}{h(\theta)^{\tau(x)}}\right] \mathrm{e}^{\theta(x+t)}$$

and thereby

$$\mathrm{e}^{-\theta(x+t)} \leq \mathbb{E}\left[\frac{1}{h(\theta)^{\tau(x)}}\right] < \mathrm{e}^{-\theta x}\,.$$

Now consider the function $\theta \mapsto h(\theta) = \mathbb{E}[\exp(\theta U_1)]$. It is strictly increasing with $h(0) = 1$ and $\lim_{\theta\to\infty} h(\theta) = \infty$. Hence $\varphi(\theta) = 1/h(\theta)$ is strictly decreasing so that the inverse φ^{-1} is well–defined (on $]0,1]$). Furthermore

$$\begin{aligned} h(\theta) &= \mathbb{E}\left[\mathrm{e}^{\theta U_1}\right] \\ &= \int_0^t \mathrm{e}^{\theta y}\mathrm{e}^{\gamma y}\,\mathrm{d}y \\ &= \frac{1}{\theta+\gamma}\left[\mathrm{e}^{(\theta+\gamma)t} - 1\right]. \end{aligned}$$

This concludes the proof of the proposition. □

Proposition 5.2. *For the function $z \mapsto \varphi^{-1}(z)$ studied in Proposition 5.1 it holds that*

$$\varphi^{-1}(z) = -\frac{1}{t}\log(z)(1 + o(1))\,,$$

when $z \downarrow 0$.

Proof. We have that $\theta(z) := \varphi^{-1}(z)$ can be found as the solution w.r.t. θ of the equation

$$z = \frac{\theta+\gamma}{\mathrm{e}^{t(\theta+\gamma)} - 1}$$

which can be rewritten as

$$\theta + \gamma = z\left(\mathrm{e}^{t(\theta+\gamma)} - 1\right). \tag{5.1}$$

Now let $\delta > 0$ and define $\theta_\delta(z)$ by

$$\theta_\delta(z) = -\frac{\delta}{t}\log(z) - \gamma\,.$$

With $\theta = \theta_\delta$ the r.h.s. of (5.1) becomes $z^{1-\delta} - z$ and the l.h.s. is of order $\log z$ when $z \downarrow 0$. If $\delta > 1$ the r.h.s. increases faster than the l.h.s. as $z \downarrow 0$. With z small enough we thereby have

$$\theta_\delta(z) + \gamma \leq z\left(\mathrm{e}^{t(\theta_\delta(z)+\gamma)} - 1\right).$$

Note that the r.h.s. in (5.1) is an increasing and convex function of θ while the l.h.s. is affine. From that we can deduce that $\theta(z) < \theta_\delta(z)$. Similarly in the $\delta \leq 1$ case we can see that $\theta(z) > \theta_\delta(z)$. Hence

$$\theta(z) = -\frac{1}{t}\log(z)(1 + o(1))$$

as wanted. □

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PAPER D

Ruin Theory in a Markovian Environment

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Abstract

We consider a risk process $\{R\}_{t\geq 0}$ with the property that the rate β of the Poisson arrival process and the distribution B of the claim sizes depend on the state of an underlying Harris recurrent Markov process $\{X_t\}_{t\geq 0}$. In this setup we derive a version of Lundberg's Inequality. This involves finding eigenfunctions in the setup of a Markov–modulated random walk. This is work in progress!

1 Introduction

The aim of this paper is to determine the asymptotic behaviour of the ruin probability $\psi(u)$ for a Markov–additive risk process where the governing Markov process is rather general. The idea is to see how the generalisation of the results of Chapter VI in [2], where the Markov process is discrete–valued, carries through.

First recall the classical Cramér–Lundberg model for the capital growth of an insurance company

$$Y_t = u + t - \sum_{i=1}^{N_t} U_i \,,$$

where u is the initial capital of the company, $\{U_i\}_{i\in\mathbb{N}}$ are iid claim losses, and $\{N_t\}_{t\geq 0}$ is a Poisson process, independent of $\{U_i\}_{i\in\mathbb{N}}$, which describes the occurrence times of the claims. Of interest is the time of ruin $\tau(u) = \inf\{t \geq 0 \mid Y_t < 0\}$ and the probability of ruin within finite time $\psi(u) = \mathbb{P}(\tau(u) < \infty)$. Usually the setup is reformulated into

$$S_t = \sum_{i=1}^{N_t} U_i - t$$

leading to $\tau(u) = \inf\{t \geq 0 \mid S_t > u\}$. For this model several more and less specific results for $\psi(u)$ are well–known and can e.g. be seen in Chapter III of [2]. A classical result is the Cramér–Lundberg Approximation, stating

$$\psi(u) \sim C\mathrm{e}^{-\gamma u}$$

as $u \to \infty$, where γ is derived as the solution of the so–called Lundberg equation.

A meaningful generalisation of this model is Markov–modulation. Instead of having constant claim intensity and identically distributed claims, one could assume that they depended on some underlying Markov process. An example is that both claim sizes and the number of claims reasonably depend on the type of weather (sun, wind, rain, etc).

Such a model is studied in [2] (and originally in [1]). Here it is assumed that the underlying Markov process is ergodic and has a finite state–space. One of the main results is a version of the Cramér–Lundberg Approximation.

In [12] a rather similar (but in some aspects more complicated) Markov additive model given by

$$S_t = \sum_{i=1}^{N_t} U_i - \int_0^t \beta_{J_s}\,\mathrm{d}s - \int_0^t \sigma_{J_{s-}}\mathrm{d}B_s \tag{1.1}$$

is considered, and in [11] a simpler but similar model is the subject. Above J is a discrete spaced Markov Chain that also governs the jump times of $\{N_t\}$.

Note that here the premium income is dependent of the Markov process, and that a diffusion part as well is permitted to contribute to the risk process. Another difference is that in this model the jump distributions are supposed to be identical.

In this setup joint Laplace transforms of the time to ruin and the overshoot at ruin is determined explicitly. This is done from establishing a version of Itô's formula for the joint process $\{X, J\}$ and finding partial eigenfunctions for the generator of this process. The eigenfunctions are constructed as simple linear combinations of exponential functions. From these eigenfunctions the asymptotic exponential behaviour of the ruin probability is easily derived.

Another subject related to the present is formed by the Markov random walks. This can be considered as a discrete time version of the setup in this paper. Ruin times for such processes are e.g. studied in the papers [8], [9] and [10]. Another exposition that contains a rather clear asymptotic result similar to the Cramér–Lundberg approximation is seen in [4].

The paper is organised such that it follows the exposition of the relevant parts of Chapter VI in [2] rather closely. First the setup and the basic assumptions is introduced. In Section 3 a series of preliminary results are presented and proved. Section 4 and 5 contain several technical results necessary for the more concrete results in the final Section 6. In Section 4 – as a result of a yet unsolved problem – another very important assumption is made.

2 Setup and basic assumptions

Consider a Markov additive process $\{Y_t\}_{t\geq 0} = \{S_t, X_t\}_{t\geq 0}$. That is Y is a bivariate Markov process and X is a Markov process with state space E (a complete and separable metric space) that governs the increments of S such that

$$\mathbb{E}[f(S_{t+s} - S_t)g(X_{s+t})|\mathcal{F}_t] = \mathbb{E}_{X_t}[f(S_s)g(X_s)]\,, \tag{2.1}$$

where $\mathcal{F}_t = \sigma(\{Y_s \mid 0 \leq s \leq t\})$. Furthermore we have used the notation $\mathbb{E}_x$ for the expectation w.r.t. $\mathbb{P}_x$ under which $X_0 = x$.
It is assumed that S has the form

$$S_t = \sum_{i=1}^{N_t} U_i - t\,.$$

The arrivals are not homogeneous in time but are determined by the process X. Given X the sum $\sum_{i=1}^{N_t} U_i$ is an inhomogeneous Poisson process: Claims are independent and at time t

- The arrival intensity is $\beta(X_t)$
- A claim arriving has distribution B_{X_t}.

Here $\{B_x\}_{x\in E}$ is a family of distributions on $\mathbb{R}$. We shall use the notation

$$\hat{B}_x[s] = \int_{\mathbb{R}} e^{sy} B_x(\mathrm{d}y)$$

for the moment generating function of the distribution B_x.

Of interest is the ruin time

$$\tau(u) = \inf\{t \geq 0 \mid S_t > u\}$$

and (with the definition $M := \sup_{t\geq 0} S_t$) the ruin probabilities

$$\psi_x(u) := \mathbb{P}_x(\tau(u) < \infty) = \mathbb{P}_x(M > u), \qquad \psi_x(u,T) := \mathbb{P}_x(\tau(u) \leq T)\,.$$

We assume that X is time–homogeneous, has paths in $D = D([0,\infty), E)$, and that for any bounded continuous $f : E \to \mathbb{R}$ and any s it holds that $\mathbb{E}_x f(X_s)$ is a continuous function of x.

Let $\mathbf{A}$ denote the infinitesimal generator for X. We assume that X is Harris ergodic. As described in Chapt. VII, section 3 of [3] X then has a regeneration set R: Letting $\tau(R) = \inf\{t \geq 0 \mid X_t \in R\}$ we have $\mathbb{P}_x(\tau(R) < \infty) = 1$, and furthermore for some $r > 0$ a probability measure λ on E exists such that for some $\epsilon > 0$

$$\mathbf{P}_r(x, B) \geq \epsilon\lambda(B)\,, \qquad x \in R\,,$$

for all $B \in \mathcal{B}(E)$. From this regeneration set a renewal process $\{Z_n\}$ can be constructed w.r.t. which X is regenerative. Hence X can be divided into one–dependent identically distributed cycles of the form $\{X_t\}_{Z_n\leq t<S_{Z+1}}$. In fact $\{X_t\}_{Z_{n+1}\leq t<Z_{n+2}}$ only depends of $\{X_t\}_{Z_n\leq t<Z_{n+1}}$ through $\{X_t\}_{Z_{n+1}-r\leq t<Z_{n+1}}$. Let $\{Y_n\}$ denote the interarrival times for $\{Z_n\}$, that is $Z_n = Y_1 + \cdots + Y_n$.

Also assume that

$$t \mapsto X_t$$

is continuous in probability. Let $(\mathbf{P}_t)_{t\geq 0}$ denote the transition kernel for X, and define

$$(\mathbf{P}_t f)(x) = \mathbb{E}_x[f(X_t)] = \int_E f(y)\mathbf{P}_t(x, dy)\,.$$

Assume that

$$x \mapsto \beta(x) \quad \text{and} \quad x \mapsto \hat{B}_x[\alpha] \tag{2.2}$$

are continuous and bounded functions on E. The ladder for all α in a suitably large interval containing 0.

From this we obtain (by considering the compound Poisson process with the maximal intensity and jumps) that for all $t \geq 0$ and all $n \in \mathbb{N}$

$$\sup_{x\in E}\sup_{||h||\leq 1} \mathbb{E}_x[\mathrm{e}^{\alpha S_t}|h(X_t)|] < \infty \text{ for all } \alpha \text{ from (2.2)}\,. \tag{2.3}$$

$$\sup_{x\in E} \mathbb{E}_x[|S_t|^n] < \infty \tag{2.4}$$

3 Preliminaries

For each $x \in E$ let $\mu(x) := \int_0^\infty y\, B_x(\mathrm{d}y)$ and $\mu^{(2)}(x) := \int_0^\infty y^2\, B_x(\mathrm{d}y)$ denote the mean and second order moment of B_x. Define

$$\rho(x) := \beta(x)\mu(x)\,,$$

and let furthermore

$$\rho^* := \int_E \rho(x)\,\pi(\mathrm{d}x), \quad \beta^* := \int_E \beta(x)\pi(\mathrm{d}x), \quad \eta := \frac{1-\rho^*}{\rho^*}\,.$$

Let $A \in \mathcal{B}(E)$ and let $N_t(A)$ denote the number of claim arrivals up to time t where X is in A:

$$N_t(A) = \int_0^t 1_{\{X_s \in A\}}\, N(\mathrm{d}s)\,.$$

Also define $(\Lambda_t(A))_{t\geq 0}$ by

$$\Lambda_t(A) := \int_0^t \beta(X_s) 1_{\{X_s \in A\}}\,\mathrm{d}s\,.$$

We have

Proposition 3.1. *As $t \to \infty$,*

$$\frac{N_t(A)}{t} \stackrel{\mathbb{P}_x-a.s.}{\longrightarrow} \int_A \beta(x)\,\pi(\mathrm{d}x)$$

for $A \in \mathcal{B}(E)$. Especially it holds that $\frac{N_t}{t} \stackrel{\mathbb{P}_x-a.s.}{\longrightarrow} \beta^*$.

Proof. Simply write

$$\frac{N_t(A)}{t} = \frac{N_t(A)}{\Lambda_t(A)} \cdot \frac{\Lambda_t(A)}{t}\,.$$

Given (X_t) we have that $(N_t(A))$ is a inhomogeneous Poisson process with intensity $\beta(X_t)1_{\{X_t \in A\}}$. Hence (see e.g. Chapt. 4.5 in [15])

$$\frac{N_t(A)}{\Lambda_t(A)} \stackrel{\mathbb{P}_x-a.s.}{\longrightarrow} 1\,.$$

That the second factor has limit $\int_A \beta(x)\,\pi(\mathrm{d}x)$ follows directly from the Ergodic Theorem (see e.g. [3] Chapt. VII, prop. 3.7). □

Proposition 3.2. *As $t \to \infty$:* (a) $\mathbb{E}_x S_t/t \to \rho^* - 1$; (b) $S_t/t \stackrel{\mathbb{P}_x-a.s.}{\longrightarrow} \rho^* - 1$.

For the proof of (b) we will need the following lemma:

Lemma 3.1. *For $f : E \to \mathbb{R}$ bounded, positive and measurable we have*

$$\frac{1}{t}\int_0^t f(X_s)\,N(\mathrm{d}s) \stackrel{\mathbb{P}_x-a.s.}{\longrightarrow} \int_E f(x)\beta(x)\,\pi(\mathrm{d}x)\,.$$

Proof. If $f = 1_A$ the result equals Proposition 3.1. That the lemma also holds for simple functions is obvious. Since f is bounded we can choose simple functions s_n, t_n for each $n \in \mathbb{N}$ such that

$$s_n \le f \le t_n \quad \text{and} \quad |t_n(x) - s_n(x)| \le 1/n \text{ for all } x \in E\,.$$

Then

$$\begin{aligned}
&\limsup_{t\to\infty} \Big|\frac{1}{t}\int_0^t f(X_s)\,N(\mathrm{d}s) - \int_E f(x)\beta(x)\,\pi(\mathrm{d}x)\Big| \\
&\quad\le \limsup_{n\to\infty}\limsup_{t\to\infty} \Big|\frac{1}{t}\int_0^t t_n(X_s)\,N(\mathrm{d}s) - \int_E s_n(x)\beta(x)\,\pi(\mathrm{d}x)\Big| \\
&\qquad + \limsup_{n\to\infty}\limsup_{t\to\infty} \Big|\frac{1}{t}\int_0^t s_n(X_s)\,N(\mathrm{d}s) - \int_E t_n(x)\beta(x)\,\pi(\mathrm{d}x)\Big| \\
&\quad\le \limsup_{n\to\infty} 2\int_E \frac{1}{n}\beta(x)\pi(\mathrm{d}x) \\
&\quad= 0\,.
\end{aligned}$$

$\square$

Proof of Proposition 3.2. For the proof of (a) first notice that

$$S_t + t = \int_0^t U_{N_s}\,N(\mathrm{d}s) \tag{3.1}$$

Given X the counting process N is a non–homogeneous Poisson process with intensity $(\beta(X_t))_{t\ge0}$. Similarly we have that for $A \in \mathcal{B}(\mathbb{R})$ the number of claims of a size within A up to time t given X is a Poisson process with intensity $(\beta(X_t)B_{X_t}(A))_{t\ge0}$. That is

$$\Big(\int_0^t 1_{\{U_{N_s}\in A\}}\,N(\mathrm{d}s)\Big)\ \big|\, X \sim po((\beta(X_t)B_{X_t}(A))_{t\ge0})\,. \tag{3.2}$$

Hence

$$\mathbb{E}\Big[\int_0^t 1_{\{U_{N_s}\in A\}}\,N(\mathrm{d}s)\big|X\Big] = \int_0^t \beta(X_s)B_{X_s}(A)\,\mathrm{d}s\,,$$

and by a classical extension argument via simple functions we obtain

$$\mathbb{E}\Big[\int_0^t U_{N_s}\,N(\mathrm{d}s)\big|X\Big] = \int_0^t \beta(X_s)\mu(X_s)\,\mathrm{d}s\,.$$

When dividing by t, the right hand side has limit ρ^* $\mathbb{P}_x$–a.s. because of the Ergodic Theorem. Since β and μ are bounded functions $\Big(1/t\int_0^t \beta(X_s)\mu(X_s)\,\mathrm{d}s\Big)_{t\ge0}$ must be a bounded sequence. Taking expectations and using Dominated Convergence yields that

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_x\Big[\int_0^t U_{N_s}\,N(\mathrm{d}s)\Big] = \mathbb{E}_x\Big[\lim_{t\to\infty}\frac{1}{t}\int_0^t \beta(X_s)\mu(X_s)\,\mathrm{d}s\Big] = \rho^*\,.$$

Together with (3.1) this concludes the proof of (a).

Given X and N we have that (U_n) are independent where U_n has distribution $B_{X_{\tau_n}}$. Then $\mathbb{E}[U_n|X,N] = \mu(X_{\tau_n})$ and we can write

$$\sum_{n=1}^{N_t} \mathbb{E}[U_n|X,N] = \int_0^t \mu(X_s)\, N(\mathrm{d}s)\,.$$

Now for (b) write

$$\frac{S_t + t}{t} = \frac{1}{t}\left(\sum_{n=1}^{N_t} U_n - \sum_{n=1}^{N_t} \mathbb{E}[U_n|X,N]\right) + \frac{1}{t}\int_0^t \mu(X_s)\, N(\mathrm{d}s)\,. \qquad (3.3)$$

Note that $x \mapsto \mu(x)$ is a bounded measurable function because of (2.2). Then Lemma 3.1 yields that the second term has the $\mathbb{P}_x$–$a.s.$ limit $\int_E \mu(x)\beta(x)\,\pi(\mathrm{d}x) = \rho^*$ as $t \to \infty$.

Also because of the assumption in (2.2) we must have that $x \to \mu^{(2)}(x)$ is bounded. Then obviously

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \mathbb{V}\mathrm{ar}[U_n|X,N] < \infty\,,$$

and from a classical version of the Law of Large Numbers it follows that given X and N

$$\frac{1}{M}\left(\sum_{n=1}^{M} U_n - \sum_{n=1}^{M} \mathbb{E}[U_n|X,N]\right) \xrightarrow{a.s.} 0\,.$$

This must be true without the conditioning as well. Replace M by N_t and recall that $N_t/t \to \beta^*$ $\mathbb{P}_x$–a.s.. Then the first term in (3.3) has limit 0, and the proof of (b) is complete. □

Corollary 3.1. *If $\eta \le 0$, then $\psi_x(u) = 1$ for all x and u. If $\eta > 0$, then $\psi_x(u) < 1$ for all x and u.*

Proof. The case $\eta < 0$ is trivial since the a.s. limit $\rho^* - 1$ of S_t/t is > 0, and hence $M = \infty$. The case $\eta > 0$ is similarly easy.

Now let $\eta = 0$. Because of the existence of a regeneration set, a renewal process $\{Y_n\}_{n\in\mathbb{N}}$ exists such that $\{X_{Y_n}\}_{n\in\mathbb{N}}$ are equally distributed and the processes $\{X_t\}_{Y_{n+1}\le t\le Y_n}$ for $n \ge 1$ are one–dependent. Define

$$Z_1 = S_{Y_1}, \qquad Z_n = S_{Y_{n+1}} - S_{Y_n}\,.$$

Then $\{Z_n\}_{n\in\mathbb{N}}$ are one–dependent, and $\{Z_n\}_{n\ge 2}$ are equally distributed (if X_0 has distribution λ also $Z_1 \sim Z_2$) Since X is assumed to be positive recurrent (from the ergodicity) it holds that $\mathbb{E}_\lambda S_1 < \infty$ and furthermore from [3], Theorem 3.2, we have

$$\mathbb{E}_\lambda\Big[\int_0^{Y_1} \beta(X_t)\mu(X_t)\,\mathrm{d}t\Big] = \frac{\int_E \beta(x)\mu(x)\,\pi(\mathrm{d}x)}{\mathbb{E}_\lambda Y_1} = \frac{\rho^*}{\mathbb{E}_\lambda Y_1}\,. \tag{3.4}$$

But conditionally upon X the regeneration point Y_1 is fixed. Thus by the same extension argument as in Proposition 3.2

$$\mathbb{E}[S_{Y_1}|X] = \int_0^{Y_1} \beta(X_t)\mu(X_t)\,\mathrm{d}t - Y_1\,. \tag{3.5}$$

Combining (3.4) and (3.5) yields $\mathbb{E}_\lambda S_{Y_1} = (\rho^* - 1)/\mathbb{E}_\lambda Y_1 = 0$. Now

$$S_{Y_n} = Z_1 + \ldots + Z_n\,,$$

where $(Z_n)_{n\geq 2}$ are one dependent and equally distributed with mean 0. Then due to the argument below $\limsup_{n\to\infty} S_{Y_n} = \infty$ $\mathbb{P}_x$–a.s. and thereby $\psi_x(u) = 1$ for all u.

First note that Kolmogorov's 0–1–law is true for the tail–σ–algebra $\mathcal{F}^\infty$ of $\{Z_n\}$. We obviously have that $\limsup_{n\to\infty} \frac{1}{\sqrt{n}} S_{Y_n}$ is $\mathcal{F}^\infty$–measurable. Then a constant $c \in [-\infty, \infty]$ exists such that

$$\limsup_{n\to\infty} \tfrac{1}{\sqrt{n}} S_{Y_n} = c \quad \mathbb{P}_x\text{–a.s.}\,.$$

Now assume that $c < \infty$. Then

$$\mathbb{P}_x(\tfrac{1}{\sqrt{n}} S_{Y_n} > c + 1) \to 0\,. \tag{3.6}$$

But from a one–dependent version of the Central Limit Theorem we have that

$$\tfrac{1}{\sqrt{n}} S_{Y_n} \overset{\sim}{\to} \mathcal{N}(0, V)$$

for some variance $V > 0$, which contradicts (3.6). Hence

$$\limsup \tfrac{1}{\sqrt{n}} S_{Y_n} = \infty \quad \mathbb{P}_x\text{–a.s.}\,.$$

□

Another result that will be useful is the following. First define f_α by

$$f_\alpha(x) = \beta(x)[\hat{B}_x[\alpha] - 1] - \alpha \tag{3.7}$$

and note that f_α is bounded for $\alpha \in \Theta$ because of (2.2).

Proposition 3.3. *For α, where (2.2) is fulfilled, it holds that*

$$\mathbb{E}[\mathrm{e}^{\alpha S_t} \mid X] = \exp\Big(\int_0^t f_\alpha(X_s)\,\mathrm{d}s\Big)\,.$$

Proof. We have the equality

$$\mathrm{e}^{\alpha S_t} = \mathrm{e}^{\alpha \int_0^t U_{N_s}\, N(\mathrm{d}s) - \alpha t}\,.$$

From (3.2) it is obtained

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(\alpha \int_0^t 1_{\{U_{N_s}\in A\}} N(\mathrm{d}s)\Big)|X\Big] \\
&\quad = \exp\Big(\big(\int_0^t \beta(X_s) B_{X_s}(A)\,\mathrm{d}s\big)(\mathrm{e}^{\alpha}-1)\Big) \\
&\quad = \exp\Big(\int_0^t \beta(X_s) \int \big(\mathrm{e}^{\alpha 1_A(u)} - 1\big) B_{X_s}(\mathrm{d}u)\,\mathrm{d}s\Big)\,.
\end{aligned}$$

Note that given X and with A and B disjoint the integrals $\int_0^t 1_{\{U_{N_s}\in A\}} N(\mathrm{d}s)$ and $\int_0^t 1_{\{U_{N_s}\in B\}} N(\mathrm{d}s)$ are independent and furthermore $\mathrm{e}^{\alpha a 1_A} - 1 + \mathrm{e}^{\alpha b 1_B} - 1 = \mathrm{e}^{\alpha(a 1_A + b 1_B)}$. This leads to

$$\mathbb{E}\Big[\exp\Big(\alpha \int_0^t f(U_{N_s}) N(\mathrm{d}s)\Big)|X\Big] = \exp\Big(\int_0^t \beta(X_s) \int \big(\mathrm{e}^{\alpha f(u)} - 1\big) B_{X_s}(\mathrm{d}u)\,\mathrm{d}s\Big)$$

for simple functions f. Now the result follows from approximation. □

4 Fundamental operator and eigenfunctions

For the rest of the paper we shall consider the situation, where $\eta > 0$ and hence $\psi_x(u) < 1$ for all x and u.

For some α, where $x \mapsto \hat{B}_x[\alpha]$ is bounded, define the operator $\mathbf{P}_t^{\alpha}$ on $b(E)$ by

$$\mathbf{P}_t^{\alpha} f(x) = \mathbb{E}_x[\mathrm{e}^{\alpha S_t} f(X_t)]$$

for $f \in b(E)$ and $\alpha > 0$ (note that $\mathbf{P}_t^0 \equiv \mathbf{P}_t$). From the Markov additive property (2.1) it is seen that $(\mathbf{P}_t^{\alpha})_{t\geq 0}$ forms a semigroup:

$$\begin{aligned}
\mathbf{P}_t^{\alpha}\mathbf{P}_s^{\alpha} f(x) &= \mathbb{E}_x\Big[\mathrm{e}^{\alpha S_t}\mathbb{E}_{X_t}\big[\mathrm{e}^{\alpha S_s} f(X_s)\big]\Big] \\
&= \mathbb{E}_x\Big[\mathrm{e}^{\alpha S_t}\mathbb{E}\big[\mathrm{e}^{\alpha(S_{t+s}-S_t)} f(X_{t+s})|\mathcal{F}_t\big]\Big] \\
&= \mathbf{P}_{t+s}^{\alpha} f(x)\,,
\end{aligned}$$

And as a result of the continuity in probability we have with $f \in bC(E)$

$$\lim_{h\to 0} \mathbb{E}_x[f(X_h)] = f(x) \tag{4.1}$$

and

$$\lim_{h\to 0} \mathbf{P}_h^{\alpha} f(x) = f(x) \tag{4.2}$$

for all $x \in E$. Hence $(\mathbf{P}_t^{\alpha})_{t\geq 0}$ is a strongly continuous semigroup (see e.g. [7], Chapt. 1)

Proposition 4.1. *The semigroup* $(\mathbf{P}_t^\alpha)$ *is generated by the operator* $\mathbf{C}^\alpha$ *that for* $g \in bC(E) \cap \mathcal{D}om\mathbf{A}$ *is given by*

$$(\mathbf{C}^\alpha g)(x) = g(x)(-\alpha + \beta(x)(\hat{B}_x[\alpha] - 1)) + \mathbf{A}g(x)\,. \tag{4.3}$$

Proof. Let $z = \alpha \in \mathbb{R}$ and denote $\mathcal{G}_h = \sigma(\{X_s \mid 0 \le s \le h\}$. Then for $f \in bC(E)$ and $h \ge 0$ we have the following up to $o(h)$–terms:

$$\begin{aligned}
\mathbf{P}_h^\alpha f(x) &= \mathbb{E}_x\Big[\mathbb{E}\big[\mathrm{e}^{\alpha S_h} f(X_h)|\mathcal{G}_h\big]\Big] \\
&= \mathbb{E}_x\Big[\mathbb{E}\big[\mathrm{e}^{\alpha S_h}|\mathcal{G}_h\big] f(X_h)\Big] \\
&= \mathbb{E}_x\Big[\mathbb{E}\big[\mathrm{e}^{\alpha S_h}|\mathcal{G}_h;\ \text{no claims in } [0,t]\big] f(X_h)\Big] \\
&\quad + \mathbb{E}_x\Big[\Big(\int_0^h \mathbb{E}\big[\mathrm{e}^{\alpha S_h}|\mathcal{G}_h;\ \text{claim at time } s\big]\beta(X_s)\mathrm{e}^{-\int_0^s \beta(X_u)\mathrm{d}u}\mathrm{d}s\Big) f(X_h)\Big] \\
&= \mathbb{E}_x\big[\mathrm{e}^{-\alpha h}\big(1 - \int_0^h \beta(X_s)\mathrm{e}^{-\int_0^s \beta(X_u)\mathrm{d}u}\mathrm{d}s\big) f(X_h)\big] \\
&\quad + \mathbb{E}_x\big[\mathrm{e}^{-\alpha h}\big(\int_0^h \hat{B}_{X_s}[\alpha]\beta(X_s)\mathrm{e}^{-\int_0^s \beta(X_u)\mathrm{d}u}\mathrm{d}s\big) f(X_h)\big] \\
&= \mathrm{e}^{-\alpha h}(1 - h\beta(x))\mathbb{E}_x[f(X_h)] \\
&\quad + e^{-\alpha h} h\hat{B}_x[\alpha]\beta(x)\mathbb{E}_x[f(X_h)] \\
&= (1-\alpha h)\mathbb{E}_x[f(X_h)] + h\beta(x)(\hat{B}_x[\alpha] - 1)\mathbb{E}_x[f(X_h)] \\
&= \big[h(-\alpha + \beta(x)(\hat{B}_x[\alpha] - 1)) + 1\big]\mathbb{E}_x[f(X_h)] \\
&= hf(x)(-\alpha + \beta(x)(\hat{B}_x[\alpha] - 1)) + (\mathbf{P}_h f)(x)\,.
\end{aligned}$$

In the last equality we have used (4.1). Then for $f \in \mathcal{D}om\mathbf{A}$

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{h}\big((\mathbf{P}_h^\alpha f)(x) - f(x)\big) \\
&\qquad = f(x)(-\alpha + \beta(x)(\hat{B}_x[\alpha] - 1)) + \lim_{h\to 0}\frac{1}{h}\big(\mathbf{P}_h f(x) - f(x)\big) \\
&\qquad = f(x)(-\alpha + \beta(x)(\hat{B}_x[\alpha] - 1)) + (\mathbf{A}f)(x)\,,
\end{aligned}$$

which concludes the proof. □

For the rest of the paper we shall assume

Assumption 4.1. *There exists* $\alpha_0 > 0$ *such that*

$$\lim_{t\to\infty} \mathbb{E}_x\left[\mathrm{e}^{\alpha_0 S_t}\right] = \infty\,, \tag{4.4}$$

and such that for $\alpha \in \Theta := [0, \alpha_0]$ *a constant* $\lambda(\alpha) > 0$ *and a function* $h^\alpha \in bC(E)$ *exists with* $0 < h_{\min}^\alpha \le h^\alpha(x) \le h_{\max}^\alpha$ *for all* $x \in E$ *and*

$$\mathbf{P}_1^\alpha h^\alpha = \lambda(\alpha) h^\alpha\,. \tag{4.5}$$

Furthermore $\lambda(\alpha)$ is the spectral radius if $\mathbf{P}_1^\alpha$ and a simple eigenvalue. Assume that both $\lambda(\alpha)$ and h^α are differentiable w.r.t. α. Finally assume that (2.2) is fulfilled for all $\alpha \in \Theta$.

Remark 4.1. *In Appendix A a scenario is shown under which Assumption 4.1 is fulfilled without (4.4). There α_0 is simply chosen very small.*

Consider the function $\mathbf{P}_{1/n}^\alpha h^\alpha$ for $\alpha \in \Theta$ and $n \in \mathbb{N}$. We have

$$\mathbf{P}_1^\alpha \mathbf{P}_{1/n}^\alpha h^\alpha = \mathbf{P}_{1/n}^\alpha \mathbf{P}_1^\alpha h^\alpha = \lambda(\alpha)\mathbf{P}_{1/n}^\alpha h^\alpha \,.$$

Since the eigenspace for $\mathbf{P}_1^\alpha$ corresponding to $\lambda(\alpha)$ is one–dimensional we have $\mathbf{P}_{1/n}^\alpha h^\alpha = ch^\alpha$. Using that $(\mathbf{P}_{1/n}^\alpha)^n = \mathbf{P}_1^\alpha$ yields $c = \lambda(\alpha)^{1/n}$ and then

$$\mathbf{P}_q^\alpha h^\alpha = \lambda(\alpha)^q h^\alpha$$

for all rational $q \geq 0$. Since $(\mathbf{P}_t^\alpha)_{t\geq 0}$ is a continuous semigroup we have the result

$$\mathbf{P}_t^\alpha h^\alpha = \mathrm{e}^{t\log(\lambda(\alpha))} h^\alpha = \mathrm{e}^{t\kappa(\alpha)} h^\alpha \,,$$

where $\kappa(\alpha) := \log(\lambda(\alpha))$.

Proposition 4.2. *For each $\alpha \in \Theta$ we have that*

$$\kappa(\alpha) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_x[\mathrm{e}^{\alpha S_n}] \,.$$

Furthermore $\alpha \mapsto \kappa(\alpha)$ is convex.

Note that apparently the function $\kappa(\alpha)$ resembles the so–called Gärtner–Ellis limit known from large deviation theory – see e.g. Chapter 2 in Dembo and Zeitouni, [6], or the discussion in Section 7 of de Acosta, [5].

Proof. We have constants $0 < c_1 < c_2 < \infty$ such that

$$c_1 < h^\alpha(x) < c_2 \qquad \text{for all } x \in E \,.$$

Then

$$\mathbb{E}_x[\mathrm{e}^{\alpha S_n} c_1] \leq \mathbb{E}_x[\mathrm{e}^{\alpha S_n} h^\alpha(X_n)] \leq \mathbb{E}_x[\mathrm{e}^{\alpha S_n} c_2] \,,$$

and since $\mathbb{E}_x[\mathrm{e}^{\alpha S_n} h^\alpha(X_n)] = (\mathbf{P}_n^\alpha h^\alpha)(x) = \lambda(\alpha)^n h^\alpha(x)$, we have

$$c_1 \mathbb{E}_x[\mathrm{e}^{\alpha S_n}] \leq \lambda(\alpha)^n h^\alpha(x) \leq c_2 \mathbb{E}_x[\mathrm{e}^{\alpha S_n}]$$

and thereby

$$\frac{1}{n}\log(c_1) + \frac{1}{n}\log \mathbb{E}_x[\mathrm{e}^{\alpha S_n}] \leq \log\lambda(\alpha) + \frac{1}{n}\log(h^\alpha(x)) \leq \frac{1}{n}\log(c_2) + \frac{1}{n}\log \mathbb{E}_x[\mathrm{e}^{\alpha S_n}] \,.$$

Letting $n \to \infty$ yields

$$\kappa(\alpha) = \log(\lambda(\alpha)) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_x[\mathrm{e}^{\alpha S_n}] \,. \tag{4.6}$$

We will say that a positive function f is *superconvex* if $\log f$ is convex. From the remark after Corollary 1 in Kingman [14] we have that $\alpha \mapsto \mathbb{E}_x[\mathrm{e}^{\alpha S_n}]$ is superconvex for each $n \in \mathbb{N}$. Then the Lemma in [14] yields that also

$$\lambda(\alpha) = \limsup_{n\to\infty} \left(\mathbb{E}_x[\mathrm{e}^{\alpha S_n}]\right)^{1/n}$$

is superconvex. Hence κ is a convex function. $\square$

Let k^α denote the derivative of h^α w.r.t. α.

Lemma 4.1.

$$\mathbb{E}_x[S_t] = t\kappa'(0) + k^0(x) - \mathbb{E}_x[k^0(X_t)]\,.$$

Proof. From Proposition 5.1 we have $\mathbb{E}_x[\mathrm{e}^{\alpha S_t} h^\alpha(X_t)] = \mathrm{e}^{t\kappa(\alpha)} h^\alpha(x)$. By differentiation

$$\mathbb{E}_x[S_t \mathrm{e}^{\alpha S_t} h^\alpha(X_t) + \mathrm{e}^{\alpha S_t} k^\alpha(X_t)] = \mathrm{e}^{t\kappa(\alpha)} \Big(k^\alpha(X_t) + t\kappa'(\alpha) h^\alpha(x)\Big)\,.$$

Let $\alpha = 0$ and recall that $h^0(x) = 1$ for all x and $\kappa(0) = 0$. $\square$

Proposition 4.3.

$$\lim_{t\to\infty} \frac{\mathbb{E}_x[S_t]}{t} = \kappa'(0)\,.$$

Proof. Simply divide by t in Lemma 4.1 and let $t \to \infty$. $\square$

5 Martingale and exponential change of measure

Let $\theta \in \Theta$ and define the process $\{L_t\}_{t\geq 0}$ by

$$L_t^\theta = \frac{h^\theta(X_t)}{h^\theta(x)} \mathrm{e}^{\theta S_t - t\kappa(\theta)}\,.$$

Then we have the following useful result

Proposition 5.1. *$(L_t^\theta)_{t\geq 0}$ is a $\mathbb{P}_x$-martingale with mean 1. Furthermore $(L_t^\theta)_{t\geq 0}$ is a multiplicative functional.*

Proof. First we see that

$$\mathbb{E}_x[e^{\theta S_t} h^\theta(X_t)] = (\mathbf{P}_t^\theta h^\theta)(x) = \mathrm{e}^{t\kappa(\theta)} h^\theta(x)$$

and thereby also

$$\mathbb{E}_x[L_t^\theta] = \frac{\mathrm{e}^{-t\kappa(\theta)}}{h^\theta(x)}\, \mathbb{E}_x[e^{\theta S_t} h^\theta(X_t)] = 1\,.$$

Then

$$
\begin{aligned}
\mathbb{E}[L^\theta_{t+h}|\mathcal{F}_t] &= \mathbb{E}\Big[\frac{h^\theta(X_{t+h})}{h^\theta(x)}\ \mathrm{e}^{\theta S_{t+h}-(t+h)\kappa(\theta)}|\mathcal{F}_t\Big] \\
&= \frac{\mathrm{e}^{\theta S_t-(t+h)\kappa(\theta)}}{h^\theta(x)}\ \mathbb{E}[\mathrm{e}^{\theta(S_{t+h}-S_t)}h^\theta(X_{t+h})|\mathcal{F}_t] \\
&= \frac{\mathrm{e}^{\theta S_t-(t+h)\kappa(\theta)}}{h^\theta(x)}\ \mathbb{E}_{X_t}[\mathrm{e}^{\theta S_h}h^\theta(X_h)] \\
&= \frac{\mathrm{e}^{\theta S_t-(t+h)\kappa(\theta)}}{h^\theta(x)}\ \mathrm{e}^{h\kappa(\theta)}h^\theta(X_t) \\
&= L^\theta_t\,.
\end{aligned}
$$

That $(L_t)_{t\geq 0}$ is a multiplicative functional follows from

$$L^\theta_s \circ \theta_t = \frac{X_{t+s}}{X_t}\ \mathrm{e}^{\theta(S_{t+s}-S_t)-s\kappa(\theta)}\,,$$

where θ_t is the shift operator. Then obviously

$$L^\theta_{t+s} = L^\theta_t \cdot (L^\theta_s \circ \theta_t)\,.$$

$\square$

Now define the probability measure $\mathbb{P}^\theta_x$ by

$$\mathbb{P}^\theta_x(A) = \mathbb{E}_x[L^\theta_t; A] \qquad \text{for } A \in \mathcal{F}_t\,.$$

Let $\mathbb{E}^\theta_x$ be the expectation under $\mathbb{P}^\theta_x$.

Definition 5.1. *Define the following operators on $bC(E)$ (the first resembles $\mathbf{P}^\alpha_t$ under $\mathbb{P}^\theta_x$) by*

$$
\begin{aligned}
(\mathbf{P}^{\theta,\alpha}_t f)(x) &= \mathbb{E}^\theta_x[\mathrm{e}^{\alpha S_t}f(X_t)] \\
(\boldsymbol{\Delta}_{h^\theta} f)(x) &= h^\theta(x)f(x) \\
(\boldsymbol{\Delta}^{-1}_{h^\theta} f)(x) &= \frac{f(x)}{h^\theta(x)}
\end{aligned}
$$

for $f \in bC(E)$. Let furthermore

$$\tfrac{1}{h^\theta}\mathcal{D}om\mathbf{A} := \{\tfrac{f}{h^\theta} \mid f \in \mathcal{D}om\mathbf{A}\}\,.$$

Note that since $\mathcal{D}om\mathbf{A}$ is dense in $b(E)$ (see e.g. Cor. 1.6, Chapt. 1 in [7]) then so is $\frac{1}{h^\theta}\mathcal{D}om\mathbf{A}$.

Now we can state and prove

Theorem 5.1. *Under $\mathbb{P}^\theta_x$ the process $\{S_t, X_t\}_{t\geq 0}$ is a MAP. Here the distribution of $\{S, X\}$ is parametrised as follows:*

(*i*) *X has infinitesimal generator* $\mathbf{A}^\theta$ *that on* $\frac{1}{h^\theta}\mathcal{D}om\mathbf{A}$ *is given by* $\mathbf{A}^\theta = \mathbf{\Delta}_{h^\theta}^{-1}\mathbf{C}^\theta\mathbf{\Delta}_{h^\theta} - \kappa(\theta)\mathbf{I}$

(*ii*) *The arrival intensity is given by* $\beta^\theta(x) = \beta(x)\hat{B}_x[\theta]$

(*iii*) *The claims are given by the distributions* $B_x^\theta(\mathrm{d}u) = \frac{\mathrm{e}^{\theta u}}{\hat{B}_x[\theta]}B_x(\mathrm{d}u)$ *(equivalently:* $\hat{B}_x^\theta[\alpha] = \hat{B}_x[\alpha+\theta]/\hat{B}_x[\theta]$*)*

Proof. It is immediately seen from Chapt. II, Theorem 2.5 (Asmussen, 2000) that $\{S_t, X_t\}$ is a time homogeneous Markov process under $\mathbb{P}_x^\theta$ (using the fact that $\{L_t^\theta\}_{t\geq 0}$ is a multiplicative functional).

We will now show from direct calculation that $\{S, X\}$ fulfil the MAP–condition (2.1) under the changed measure $\mathbb{P}_x^\theta$. First we see that with $A \in \mathcal{F}_t$ it holds that

$$\begin{aligned}
\int_A \mathbb{E}^\theta[f(S_{t+s}-S_t)g(X_{t+s})|\mathcal{F}_t]\,\mathrm{d}\mathbb{P}_x^\theta &= \int_A f(S_{t+s}-S_t)g(X_{t+s})\,\mathrm{d}\mathbb{P}_x^\theta \\
&= \int_A f(S_{t+s}-S_t)g(X_{t+s})L_t\cdot L_s\circ\theta_t\,\mathrm{d}\mathbb{P}_x \\
&= \int_A \mathbb{E}[f(S_{t+s}-S_t)g(X_{t+s})L_t\cdot L_s\circ\theta_t|\mathcal{F}_t]\,\mathrm{d}\mathbb{P}_x \\
&= \int_A \mathbb{E}[f(S_{t+s}-S_t)g(X_{t+s})L_s\circ\theta_t|\mathcal{F}_t]L_t\,\mathrm{d}\mathbb{P}_x \\
&= \int_A \mathbb{E}[f(S_{t+s}-S_t)g(X_{t+s})L_s\circ\theta_t|\mathcal{F}_t]\,\mathrm{d}\mathbb{P}_x^\theta\,.
\end{aligned}$$

So $\mathbb{P}_x^\theta$–a.s. we have

$$\begin{aligned}
&\mathbb{E}^\theta[f(S_{t+s}-S_t)g(X_{t+s})|\mathcal{F}_t] \\
&\quad= \mathbb{E}[f(S_{t+s}-S_t)g(X_{t+s})L_s\circ\theta_t|\mathcal{F}_t] \\
&\quad= \mathbb{E}\Big[f(S_{t+s}-S_t)g(X_{t+s})\frac{h^\theta(X_{t+s})}{h^\theta(X_t)}\mathrm{e}^{\theta(S_{t+s}-S_t)-s\kappa(\theta)}|\mathcal{F}_t\Big] \\
&\quad= \frac{1}{h^\theta(X_t)}\mathbb{E}\Big[f(S_{t+s}-S_t)\mathrm{e}^{\theta(S_{t+s}-S_t)-s\kappa(\theta)}h^\theta(X_{t+s})g(X_{t+s})|\mathcal{F}_t\Big] \\
&\quad= \frac{1}{h^\theta(X_t)}\mathbb{E}_{X_t}\Big[f(S_s)\mathrm{e}^{\theta S_s-s\kappa(\theta)}h^\theta(X_s)g(X_s)\Big] \\
&\quad= \mathbb{E}_{X_t}^\theta[f(S_s)g(X_s)]\,,
\end{aligned}$$

and hence for $f \in bC(E)$ we have

$$\begin{aligned}
(\mathbf{P}_t^{\theta,\alpha} f)(x) &= \mathbb{E}_x^\theta[\mathrm{e}^{\alpha S_t} f(X_t)] \\
&= \mathbb{E}_x[L_t^\theta \mathrm{e}^{\alpha S_t} f(X_t)] \\
&= \mathbb{E}_x\Big[\frac{h^\theta(X_t)}{h^\theta(x)}\, \mathrm{e}^{\theta S_t - t\kappa(\theta)} \mathrm{e}^{\alpha S_t} f(X_t)\Big] \\
&= \frac{\mathrm{e}^{-t\kappa(\theta)}}{h^\theta(x)} \mathbb{E}_x[\mathrm{e}^{(\theta+\alpha)S_t} h^\theta(X_t) f(X_t)] \\
&= \mathrm{e}^{-t\kappa(\theta)} (\mathbf{\Delta}_{h^\theta}^{-1} \mathbf{P}_t^{\theta+\alpha} \mathbf{\Delta}_{h^\theta} f)(x)\,. \qquad (5.1)
\end{aligned}$$

Since $(\mathbf{P}_t^\alpha)_{t\geq 0}$ forms a strongly continuous semigroup it is immediately seen that so does $(\mathbf{P}_t^{\theta,\alpha})_{t\geq 0}$. Recall that for $f \in \mathcal{D}om\mathbf{A}$

$$\lim_{t\to 0} \tfrac{1}{t} \left(\mathbf{P}_t^{\theta+\alpha} f - f\right) = \mathbf{C}^{\theta+\alpha}\,.$$

From applying the exact same arguments as in the proof of Proposition 4.1 one would obtain

$$\lim_{t\to 0} \tfrac{1}{t} \left(\mathrm{e}^{-t\kappa(\theta)} \mathbf{P}_t^{\theta+\alpha} f - f\right) = \mathbf{C}^{\theta+\alpha} - \kappa(\theta)\mathbf{I}\,.$$

Then with $\mathbf{\Delta}_{h^\theta} f \in \mathcal{D}om\mathbf{A}$ ($\Leftrightarrow f \in \frac{1}{h^\theta}\mathcal{D}om\mathbf{A}$)

$$\begin{aligned}
&\lim_{t\to 0} \tfrac{1}{t} \left(\mathrm{e}^{-t\kappa(\theta)} \mathbf{\Delta}_{h^\theta}^{-1} \mathbf{P}_t^{\theta+\alpha} \mathbf{\Delta}_{h^\theta} f - f\right) \\
=&\mathbf{\Delta}_{h^\theta}^{-1} \lim_{t\to 0} \tfrac{1}{t} \left(\mathrm{e}^{-t\kappa(\theta)} \mathbf{P}_t^{\theta+\alpha} \mathbf{\Delta}_{h^\theta} f - \mathbf{\Delta}_{h^\theta} f\right) = \mathbf{\Delta}_{h^\theta}^{-1} \mathbf{C}^{\theta+\alpha} \mathbf{\Delta}_{h^\theta} - \kappa(\theta)\mathbf{I}\,.
\end{aligned}$$

Hence the semigroup $(\mathbf{P}_t^{\theta,\alpha})_{t\geq 0}$ is generated by $\mathbf{C}^{\theta,\alpha}$, where $\mathbf{C}^{\theta,\alpha} = \mathbf{\Delta}_{h^\theta}^{-1} \mathbf{C}^{\theta+\alpha} \mathbf{\Delta}_{h^\theta} - \kappa(\theta)\mathbf{I}$.

Letting $\alpha = 0$ above yields

$$\lim_{t\to 0} \tfrac{1}{t} \left(\mathbb{E}_x^\theta[f(X_t)] - f(x)\right) = \mathbf{\Delta}_{h^\theta}^{-1} \mathbf{C}^\theta \mathbf{\Delta}_{h^\theta} - \kappa(\theta)\mathbf{I}\,.$$

It is seen that X is a Markov process with infinitesimal generator $\mathbf{A}^\theta = \mathbf{\Delta}_{h^\theta}^{-1} \mathbf{C}^\theta \mathbf{\Delta}_{h^\theta} - \kappa(\theta)\mathbf{I}$ under $\mathbb{P}^\theta$. Here $\mathbf{A}^\theta$ has domain $\frac{1}{h^\theta}\mathcal{D}om\mathbf{A}$.

We have for $f \in \frac{1}{h^\theta}\mathcal{D}om\mathbf{A}$ that (recall the definition of $\mathbf{C}^\alpha$ in (4.3)!)

$$\begin{aligned}
&(\mathbf{C}^{\theta,\alpha} f)(x)\\
&= (\mathbf{\Delta}_{h^\theta}^{-1}\mathbf{C}^{\theta+\alpha}\mathbf{\Delta}_{h^\theta} f)(x) - \kappa(\theta) f(x)\\
&= \big(-(\theta+\alpha) + \beta(x)(\hat{B}_x[\theta+\alpha]-1)\big) f(x) + \big(\mathbf{\Delta}_{h^\theta}^{-1}\mathbf{A}\mathbf{\Delta}_{h^\theta} f\big)(x) - \kappa(\theta) f(x)\\
&= \big(-\alpha + \beta(x)(\hat{B}_x[\theta+\alpha]-\hat{B}_x[\theta])\big) f(x) + \big(-\theta + \beta(x)(\hat{B}_x[\theta]-1)\big) f(x)\\
&\quad + \big(\mathbf{\Delta}_{h^\theta}^{-1}\mathbf{A}\mathbf{\Delta}_{h^\theta} f\big)(x) - \kappa(\theta) f(x)\\
&= \Big(-\alpha + \frac{\beta(x)}{\hat{B}_x[\theta]}\big(\frac{\hat{B}_x[\theta+\alpha]}{\hat{B}_x[\theta]} - 1\big)\Big) f(x) + (\mathbf{\Delta}_{h^\theta}^{-1}\mathbf{C}^\theta\mathbf{\Delta}_{h^\theta} - \kappa(\theta)\mathbf{I}) f(x)\\
&= \big(-\alpha + \beta^\theta(\hat{B}_x^\theta[\alpha]-1)\big) f(x) + (\mathbf{A}^\theta f)(x)\,,
\end{aligned}$$

where $\mathbf{A}^\theta$, β^θ and $\hat{B}_x^\theta$ are as in the theorem. This resembles the expression for $\mathbf{C}^\alpha$ in (4.3) with the parameters stated in the theorem. For fixed $t \geq 0$ we have that the expression $(\mathbf{P}_t^{\theta,\alpha} f)(x) = \mathbb{E}_x^\theta[\mathrm{e}^{\alpha S_t} f(X_t)]$ determines the distribution of $\{S_t, X_t\}$ uniquely when $\alpha \in (-\delta, \delta)$ (for some arbitrary $\delta > 0$) and $f \in bC(E)$ varies. Since a strongly continuous semigroup is uniquely determined by it's generator (see. Prop. 2.9, Chapt. 1 of [7]) we can conclude that $\{S_t, X_t\}$ has distribution as a MAP with the stated parameters. □

Remark 5.1. *It is immediately seen that under $\mathbb{P}^\theta$ for $\theta < \alpha_0$ we have properties similar to (2.2):*

$$x \mapsto \beta^\theta(x) \quad \textit{and} \quad x \mapsto \hat{B}_x^\theta[\alpha]$$

are continuous and bounded functions. The ladder for $\theta + \alpha \in \Theta$.

From [2], Theorem 2.3, Chapt. II, we have

Proposition 5.2. *Let τ be any stopping time and let $G \in \mathcal{F}_\tau$ with $G \subseteq \{\tau < \infty\}$. Then*

$$\mathbb{E}_x(G) = h^\theta(x)\mathbb{E}_x^\theta\left[\tfrac{1}{h^\theta(X_\tau)}\exp\big(-\theta S_\tau + \tau\kappa(\theta)\big); G\right]. \tag{5.2}$$

Recall from (5.1) that under $\mathbb{P}_x^\theta$ the process $\{S, X\}$ is Markov–additive with fundamental operator $\mathbf{P}_t^{\theta,\alpha}$ given by

$$(\mathbf{P}_t^{\theta,\alpha} f)(x) = \mathrm{e}^{-t\kappa(\theta)}(\mathbf{\Delta}_{h^\theta}^{-1}\mathbf{P}_t^{\theta+\alpha}\mathbf{\Delta}_{h^\theta} f)(x)\,.$$

Since $(\mathbf{P}_t^{\theta+\alpha} h^{\theta+\alpha})(x) = \mathrm{e}^{t\kappa(\theta+\alpha)} h^{\theta+\alpha}(x)$ we obviously have that $h^{\theta+\alpha}/h^\theta$ is an eigenfunction for $\mathbf{P}_t^{\theta,\alpha}$ satisfying

$$(\mathbf{P}_t^{\theta,\alpha} h^{\theta+\alpha}/h^\theta)(x) = \mathrm{e}^{t(\kappa(\theta+\alpha)-\kappa(\theta))} h^{\theta+\alpha}(x)/h^\theta(x)\,.$$

Hence with κ^θ and ρ_θ^* being defined under $\mathbb{P}_x^\theta$ as κ and ρ^* are defined under $\mathbb{P}_x$ we have

Lemma 5.1. $\kappa^\theta(\alpha) = \kappa(\theta+\alpha) - \kappa(\theta)$, *and $\rho_\theta^* > 1$ whenever $\kappa'(\theta) > 0$*

Proof. The ladder comes from recalling that $\rho_\theta^* = \kappa_\theta'(0)$. □

6 Ruin probabilities

From combining (4.4) with Proposition 4.2 and Proposition 3.2 we obtain that κ is differentiable and convex on Θ with $\kappa(0) = 0$, $\kappa'(0) < 0$ and $\kappa(\alpha_0) > 0$. Hence a solution $\gamma > 0$ of the Lundberg equation $\kappa(\gamma) = 0$ exists.

Proposition 6.1. *The Markov process X is Harris recurrent under $\mathbb{P}^\gamma$ with R as regeneration set and λ as the distribution at the regeneration epochs.*

Proof. Recall
$$\tau(R) = \inf\{t > 0 \mid X_t \in R\}\,.$$

Since X is assumed to be Harris recurrent under $\mathbb{P}_x$ we have that $\mathbb{P}_x(\tau(R) < \infty) = 1$ for all $x \in E$. For convenience let $\tau := \tau(R) \wedge \tau(u)$. Since $\tau \wedge t$ is bounded Optional Stopping yields

$$\begin{aligned} 1 = \mathbb{E}_x[L^\gamma_{\tau\wedge t}] &= \mathbb{E}_x\Big[\frac{1}{h^\gamma(x)} \mathrm{e}^{\gamma S_{\tau\wedge t}} h_\gamma(X_{\tau\wedge t})\Big] \\ &= \mathbb{E}_x\Big[\frac{1}{h^\gamma(x)} \mathrm{e}^{\gamma S_\tau} h_\gamma(X_\tau); \tau \le t\Big] \\ &\quad + \mathbb{E}_x\Big[\frac{1}{h^\gamma(x)} \mathrm{e}^{\gamma S_t} h_\gamma(X_t); \tau > t\Big]\,. \end{aligned}$$

Define $h^\gamma_{\max} = \max_x h^\gamma(x)$ and $h^\gamma_{\min} = \min_x h^\gamma(x)$. Note that $\mathrm{e}^{\gamma S_t} h_\gamma(X_t)$ is bounded by $\mathrm{e}^{\gamma u} h_{\gamma,\max}$ on $\{\tau > t\}$ so by Dominated convergence the second term has limit 0 as $t \to \infty$. Since L^γ is a positive supermartingale it holds that $\mathbb{E}_x[L^\gamma_\tau] \le \mathbb{E}_x[L^\gamma_0] = 1$ and hence from applying Dominated convergence to the first term we obtain

$$\mathbb{E}_x\Big[\frac{1}{h^\gamma(x)} \mathrm{e}^{\gamma S_{\tau(R)\wedge\tau(u)}} h_\gamma(X_{\tau(R)\wedge\tau(u)})\Big] = 1\,.$$

From [2], Theorem 2.3, Chapt. II, we have

$$\begin{aligned} \mathbb{P}^\gamma_x[\tau(R)\wedge\tau(u) < \infty] &= \mathbb{E}_x\Big[\frac{1}{h^\gamma(x)} \mathrm{e}^{\gamma S_{\tau(R)\wedge\tau(u)}} h_\gamma(X_{\tau(R)\wedge\tau(u)}); \tau(R)\wedge\tau(u) < \infty\Big] \\ &= \mathbb{E}_x\Big[\frac{1}{h^\gamma(x)} \mathrm{e}^{\gamma S_{\tau(R)\wedge\tau(u)}} h_\gamma(X_{\tau(R)\wedge\tau(u)})\Big] = 1 \end{aligned}$$

using the assumption that $\mathbb{P}_x[\tau(R) < \infty] = 1$. When $u \to \infty$ we must have that $\tau(u) \to \infty$ (explosions are impossible). Then

$$\{\tau(R)\wedge\tau(u) < \infty\} \downarrow \{\tau(R) < \infty\}$$

as $u \to \infty$. Hence
$$\mathbb{P}^\gamma_x(\tau(R) < \infty) = 1\,.$$

Furthermore we have for $f \in bC(E)^+$ that

$$\begin{aligned}\mathbb{E}_x^\gamma[f(X_r)] &= \mathbb{E}_x\left[\frac{1}{h^\gamma(x)}\mathrm{e}^{\gamma S_r}h^\gamma(X_r)f(X_r)\right] \\ &\geq \frac{h^\gamma_{\text{min}}}{h^\gamma_{\text{max}}}\mathrm{e}^{-\gamma r}\mathbb{E}_x[f(X_r)] \geq \frac{h^\gamma_{\text{min}}}{h^\gamma_{\text{max}}}\mathrm{e}^{-\gamma r}\epsilon\int f\mathrm{d}\lambda\,.\end{aligned}$$

□

Proposition 6.2. *Assume that* $\sup_{x\in E}\mathbb{E}_x\left[\mathrm{e}^{rc_\alpha\tau(R)}\right] < \infty$ *for some* $r > 1$. *Then* X *is Harris ergodic under* $\mathbb{P}^\gamma$.

Proof. From a standard extension of Proposition 5.2 to stopping times (see e.g. Thm. XIII.3.2 in [3]) we obtain

$$\mathbb{E}_x[\tau(R)] = \mathbb{E}_x^\gamma\left[\frac{h^\gamma(X_{\tau(R)})}{h^\gamma(x)}\mathrm{e}^{\gamma S_{\tau(R)}}\tau(R)\right] = \mathbb{E}_x^\gamma\left[\frac{h^\gamma(X_{\tau(R)})}{h^\gamma(x)}\tau(R)\mathbb{E}^\gamma\left[\mathrm{e}^{\gamma S_{\tau(R)}}|X\right]\right]\,.$$

Using (3.3) and furthermore letting $1 = \frac{1}{r} + \frac{1}{p}$ yields

$$\mathbb{E}_x[\tau(R)] \leq \frac{h^\gamma_{\text{max}}}{h^\gamma_{\text{min}}}\mathbb{E}_x\left[\mathrm{e}^{c_\alpha\tau(R)}\tau(R)\right] \leq \frac{h^\gamma_{\text{max}}}{h^\gamma_{\text{min}}}\left(\mathbb{E}_x\left[\mathrm{e}^{rc_\alpha\tau(R)}\right]\right)^{1/r}(\mathbb{E}_x[\tau(R)^p])^{1/p}\,.$$

Hence from the assumption of the proposition we obtain $\sup_{x\in E}\mathbb{E}_x^\gamma[\tau(R)] < \infty$. Recall that the first regeneration epoch Y of X under $\mathbb{P}_\lambda^\gamma$ can be produced as follows: Let $\{\xi_n\}_{n\in\mathbb{N}}$ be iid 0–1–variables with probability ϵ' of being 1 and let $N = \inf\{n \in \mathbb{N} \mid \xi_n = 1\}$. Then

$$Y = \tau(R)_1 + r + \tau(R)_2 + r + \ldots + \tau(R)_N + r$$

where e.g. $\tau(R)_2$ is the waiting time after $\tau(R)_1 + r$ until R is hit the next time. Note that the $\tau(R)$ variables are independent of $\{\xi_n\}_{n\in\mathbb{N}}$. Then obviously

$$\mathbb{E}_\lambda^\gamma[Y|N] \leq N\sup_{x\in E}\mathbb{E}_x^\gamma[\tau(R)]$$

and since N is geometrically distributed it has finite expectation. Hence $\mathbb{E}_\lambda[Y] < \infty$. □

Under the assumption of Proposition 6.2 the results from Section 3 can be applied to the process under $\mathbb{P}^\gamma$. Note that $\kappa'(\gamma) > 0$ such that $\rho_\gamma^* > 0$ and hence by Corollary 3.1 we have $\mathbb{P}_x^\gamma(\tau(u) < \infty) = 1$.

Let $\xi(u) := S_{\tau(u)} - u$ denote the overshoot. Then

Corollary 6.1.

$$\begin{aligned}\psi_x(u,T) &= h^\gamma(x)\mathrm{e}^{-\gamma u}\mathbb{E}_x^\gamma\left[\frac{\mathrm{e}^{-\gamma\xi(u)}}{h^\gamma(X_{\tau(u)})};\tau(u)\leq T\right] \\ \psi_x(u) &= h^\gamma(x)\mathrm{e}^{-\gamma u}\mathbb{E}_x^\gamma\left[\frac{\mathrm{e}^{-\gamma\xi(u)}}{h^\gamma(X_{\tau(u)})}\right]\,.\end{aligned}$$

Noting that $\xi(u) \geq 0$ yields

Corollary 6.2.

$$\psi_x(u) \leq \frac{h^\gamma(x)}{\min_{y\in E} h^\gamma(y)} \mathrm{e}^{-\gamma u}\,.$$

7 Appendix

We will in this section use the following assumption

Assumption 7.1. *Assume that the Markov process $(X_t)_{t\geq 0}$ is irreducible (with respect to a maximal irreducible measure ϕ), aperiodic and uniformly ergodic. Let $\mathbf{Q}$ be the operator corresponding to the stationary distribution π. That is*

$$\mathbf{Q}f(x) = \int_E f(y)\,.$$

Assume furthermore that with some $\nu > 0$

$$||\mathbf{P}_t - \mathbf{Q}|| = O(\mathrm{e}^{-\nu t})\,. \tag{7.1}$$

Recall that for some bounded operator $\mathbf{T} : bC(E) \to bC(E)$ the resolvent $\rho(\mathbf{T})$ is defined as

$$\rho(\mathbf{T}) := \{y \in \mathbb{C} \mid (\mathbf{T} - y\mathbf{I})^{-1} \text{ exists}\}$$

and the spectrum $\sigma(\mathbf{T})$ is the compliment of the resolvent,

$$\sigma(\mathbf{T}) = \mathbb{C} \setminus \rho(\mathbf{T})\,.$$

For $y \in \rho(\mathbf{T})$ the operator $(\mathbf{T}-y\mathbf{I})^{-1}$ is called the resolvent at y and is denoted $\mathbf{R}(y)$.

The following decomposition (see Riesz, Chap. XI, [16]) will be very important. Assume that $\sigma(\mathbf{T}) = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are disjoint and isolated. Let I_i for $i = 1, 2$ be a closed rectifiable curve in $\rho(\mathbf{T})$ which is the boundary of an open bounded region D_i such that $\sigma_i = \sigma(\mathbf{T}) \cap D_i$. We have the result:

Theorem 7.1. *The space $bC(E)$ may be decomposed into the vector sum of two linearly independent subspaces $\mathcal{N}_1$ and $\mathcal{N}_2$, where*

$$\mathbf{T}(\mathcal{N}_i) \subseteq \mathcal{N}_i \quad \textit{and} \quad \sigma(\mathbf{T}|_{\mathcal{N}_i}) = \sigma_i\,.$$

The parallel projection of $bC(E)$ onto $\mathcal{N}_i$ is given by

$$\mathbf{P}_{\sigma_i} = -\frac{1}{2\pi i}\int_{\sigma_i} \mathbf{R}(y)\,\mathrm{d}y\,.$$

Furthermore $\mathbf{P}_{\sigma_1} = \mathbf{I}$ and $\mathbf{P}_{\sigma_2} = \mathbf{0}$ if and only if σ_1 coincides with $\sigma(\mathbf{T})$ and σ_2 is empty.

Let $\mathbf{Q}$ be the operator corresponding to the stationary distribution π:

$$\mathbf{Q}f(x) = \int f(y)\pi(\mathrm{d}y) \qquad \text{for all } x \in E\,.$$

From (7.1) we have

$$||\mathbf{P}_n - \mathbf{Q}|| = \sup_{||h|\leq 1} ||\mathbf{P}_n h - \mathbf{Q}h|| \leq \gamma\rho^n\,, \tag{7.2}$$

and because of (7.2) and direct calculation one can see that

$$\mathbf{R}(y) := \frac{1}{y-1}\mathbf{Q} + \sum_{n=0}^{\infty} \frac{1}{y^{n+1}}(\mathbf{P}_n - \mathbf{Q})$$

is well–defined and $-\mathbf{R}(y)$ is the resolvent for $\mathbf{P}_1$ at y when $y \neq 1$ and $|y| > \rho$.

With a proof similar to the one of Jensen [13], Lemma 2.2 (see also the appendices of [8], [9] and [10]) and using the formula (2.4) we have

Lemma 7.1. *There exist $K > 0$ and $\eta > 0$ such that for $|\alpha| \leq \eta$, $|y-1| > (1-\rho)/6$ and $|y| > \rho + (1-\rho)/6$ it holds that*

$$||P_1^\alpha - P_1|| \leq K|\alpha|$$

and thereby

$$\mathbf{R}^\alpha(y) := \sum_{n=0}^{\infty} \mathbf{R}(y)[(\mathbf{P}_1^z - \mathbf{P}_1)\mathbf{R}(y)]^n$$

is well–defined with $-\mathbf{R}^\alpha(y)$ the resolvent for $\mathbf{P}_1^\alpha$.

From this result it is seen that for $|\alpha| \leq \eta$ the spectrum of P_1^α lies inside the two circles $C_1 = \{y : |y-1| = (1-\rho)/3\}$ and $C_2 = \{y : |y| = \rho + (1-\rho)/3\}$. Hence the spectrum can be decomposed into two disjoint parts. From Theorem 7.1 we have the decomposition $bC(E) = \mathcal{N}_1(\alpha) \oplus \mathcal{N}_2(\alpha)$ such that

$$\mathbf{Q}^\alpha := \frac{1}{2\pi i}\int_{C_1} \mathbf{R}^\alpha(y)\mathrm{d}y, \qquad \mathbf{I} - \mathbf{Q}^\alpha := \frac{1}{2\pi i}\int_{C_2} \mathbf{R}^\alpha(y)\mathrm{d}y$$

are parallel projections of $bC(E)$ onto $\mathcal{N}_1(\alpha)$ and $\mathcal{N}_2(\alpha)$ respectively. With a proof similar to the one of Lemma 2.3 in [13] we have the following Lemma:

Lemma 7.2. *There exists $0 \leq \delta \leq \eta$ such that $\mathcal{N}_1(z)$ is one–dimensional for $|\alpha| \leq \delta$. Furthermore $\sup_{|\alpha|\leq\delta} ||\mathbf{Q}^\alpha - \mathbf{Q}|| < 1$.*

Together with the results from Theorem 7.1 we have some eigenvalue $\lambda(z)$ when $|z| \leq \delta$ such that $\mathbf{P}_1^\alpha f = \lambda(\alpha)f$ for all $f \in \mathcal{N}_1(\alpha)$. Now let $z = \alpha \in \mathbb{R}$ with $|\alpha| \leq \delta$. Define $\psi \in bC(E)$ by $\psi \equiv 1$. Then $\mathbf{Q}^\alpha\psi \in \mathcal{N}_1(\alpha)$ and obviously $\mathbf{Q}\psi = \psi$ and $\mathbf{Q}^\alpha\mathbf{Q}^\alpha\psi = \mathbf{Q}^\alpha\psi$. Then from Lemma 7.2 we have that

$$||\mathbf{Q}^\alpha\psi - \psi|| < 1\,. \tag{7.3}$$

Hence $(\mathbf{Q}^\alpha\psi)(x) \neq 0$ for all $x \in E$. Now consider

$$\mathbf{Q}^\alpha\psi = \frac{1}{2\pi i}\int_{C_1} \mathbf{R}^\alpha(y)\psi \,\mathrm{d}y\,.$$

Each term $(\mathbf{R}(y)[(\mathbf{P}_1^\alpha - \mathbf{P}_1)\mathbf{R}(y)]^n)\psi$ in $\mathbf{R}^\alpha(y)\psi$ can be written on the form

$$\sum_{n,m=0}^{\infty} \frac{1}{(y-1)^m y^n} f_{m,n}\,,$$

where all $f_{m,n}$ are real–valued functions, since $\mathbf{P}^\alpha$ is a real operator. Hence also $\mathbf{R}^\alpha(y)\psi$ has this form. Since

$$\frac{1}{2\pi i}\int_{C_1} \frac{1}{(y-1)^m y^n}\,\mathrm{d}y \in \mathbb{R}$$

for all n, m we conclude that $\mathbf{Q}^\alpha\psi$ is a real–valued function. From (7.3) it is furthermore seen to have positive values, bounded away from 0. All together we have that for $|\alpha| \leq \delta$

$$\mathbf{P}_1^\alpha(\mathbf{Q}^\alpha\psi) = \lambda(\alpha)(\mathbf{Q}^\alpha\psi)\,,$$

where $\mathbf{Q}^\alpha\psi$ is real with positive values bounded away from 0. Hence also $\lambda(\alpha)$ is real–valued.

As a result of (2.4) also Lemma 2.5 and 2.6 from Jensen, [13], can be reproved. Hence for α in a sufficiently small interval of α's around 0 both $\alpha \mapsto \lambda(\alpha)$ and $\alpha \mapsto \mathbf{Q}^\alpha\psi$ are differentiable.

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