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Abstract

In reduced form default models, the instantaneous default intensity is classically the modeling object. Survival probabilities are then given by the Laplace transform of the cumulative hazard defined as the integrated intensity process. Instead, recent literature has shown a tendency towards specifying the cumulative hazard process directly.

Within this framework we present a new model class where cumulative hazards are described by self-similar additive processes, also known as Sato processes. Furthermore we also analyze specifications obtained via a simple deterministic time-change of a homogeneous Lévy process. While the processes in these two classes share the same average behavior over time, the associated intensities exhibit very different properties.

Concrete specifications are calibrated to data on the single names included in the iTraxx Europe index. The performances are compared with those of a recently proposed class of intensity models based on Ornstein-Uhlenbeck type processes. It is shown how the time-inhomogeneous Lévy models achieve comparable calibration errors with fever parameters, and with more stable parameter estimates over time. However, the calibration performances of the Sato processes and the time-change specifications are practically indistinguishable.

Key words: Credit default swap, reduced form model, Sato process, time-changed Lévy process, cumulative hazard.

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1 Introduction

In recent years the market for credit derivatives has experienced an exponential growth. The estimated total value of the credit default swap market in the US has risen from $900 billion in 2000 to the breathtaking number of more than $62 trillion in 2008 - roughly twice the size of the entire US stock market. A parallel explosion in the academic literature concerning the modeling of corporate default risk has followed. The reduced form models developed by Jarrow and Turnbull (1995), Jarrow, Lando, and Turnbull (1997), Lando (1998), Duffie and Singleton (1997), Duffie and Singleton (1999), Madan and Unal (1998) and Elliott, Jeanblanc, and Yor (2000) have been particularly popular among practitioners. In this framework the time of default of a firm is modeled as the first jump time of an underlying process, typically described by a Cox process with stochastic intensity rate λ. The survival probabilities up to time t are then given as

$$\mathbb{Q}(\tau > t) = \mathbb{E}\left[e^{-A_t}\right] \tag{1}$$

with the process A defined as the integrated instantaneous intensity

$$A_t = \int_0^t \lambda_s ds \tag{2}$$

and with intensity rate λ often described by a positive and stationary affine process. This intensity based approach and its multivariate extensions have become quite popular and widely used in practice. However, problems are still present when considering parameter stability over time, leading to large variations in sensitivities and hedge parameters, which is problematic from a risk management perspective.

Recent literature, although in the context of multivariate modeling, has shown a tendency towards specifying directly the process A in (1), which we will refer to as the *cumulative hazard process*, as opposed to the instantaneous intensity rate λ. Cumulative hazard processes displaying jumps have been adopted in concrete applications e.g. by Joshi and Stacey (2006), Di Graziano and Rogers (2006) and Hull and White (2008). However, their performances on single-name derivatives have not been empirically investigated, focus being on matching market-inferred correlations.

In this work, also motivated by the ongoing financial turmoil which has almost entirely wiped out the market for multivariate credit instruments, we focus on single name default modeling. We present two new model classes where the cumulative hazard process A belongs to:

1. The class of self-similar additive processes introduced by Sato (1991) and therefore termed Sato processes. Sato processes have been employed for option pricing and interest rate modeling by e.g. Carr, Ge-

man, Madan, and Yor (2005), Carr, Geman, Madan, and Yor (2007), Eberlein and Madan (2007) and Skovmand (2008).

2. The class of time-changed Lévy processes obtained by evaluating a homogeneous Lévy process at a re-scaled point in time t^γ. While sharing with Sato processes the same average time behavior, cumulative hazards in this model class are not self-similar and exhibit rather different characteristics.

Both family of processes are extremely convenient for a number of reasons. First, they display enormous flexibility in terms of distribution modeling. They are analytically tractable and they allow for closed form expressions for default probabilities hereby enabling straightforward calibration to credit default swap prices. Finally, they allow for more flexibility and non-linearity in the long term behavior of the cumulative hazard in contrast to the more traditional framework described in (2) where, if the intensity λ is ergodic, it holds that

$$t^{-1} \int_0^t \lambda_s ds \to \overline{\lambda} \quad \text{as} \quad t \to \infty$$

with $\overline{\lambda}$ denoting the long-run average of λ.

The alternative concrete specifications here selected for calibration to market data are obtained by setting the unit time law of the cumulative hazard as either a Gamma or an Inverse Gaussian distribution. Both these distributions are well known in the financial literature and have been widely used both in the field of econometrics and derivative pricing by Madan and Seneta (1990), Rydberg (1999), Barndorff-Nielsen and Shephard (2001), Nicolato and Venardos (2003) among others. In the context of credit risk, Cariboni and Schoutens (2006) employ both Gamma and Inverse Gaussian Ornstein-Uhlenbeck type processes to describe instantaneous default intensities and demonstrate their superior performance over the classical CIR specification. On the basis of these findings, we include as comparative benchmarks the two Ornstein-Uhlenbeck type models in the empirical analysis of the Sato and time-changed Lévy cumulative hazards.

In order to assess both pricing capabilities and parameter stability over time, all the models are calibrated to weekly observations of the single names included in the iTraxx Europe Series 8 index in the period from September 17, 2007 to March 14, 2008.

The rest of the paper is structured as follows: Section 2 describes the general cumulative hazard modeling framework. Sato specifications are analyzed in Section 3 while time-changed Lévy processes are considered in Section 4. Section 5 illustrates the benchmark models and the calibration results are presented in Section 6. Last, Section 7 concludes.

2 The General Cumulative Hazard Modelling Framework

In the classical intensity based reduced-form modeling approach introduced in Lando (1998), the default time τ of a company is modeled as the first jump-time of a Cox process $\widetilde{N}$ with stochastic intensity rate $\lambda = (\lambda_t)$

$$\tau = \inf \left\{ t > 0 \mid \widetilde{N}_t > 0 \right\},$$

or equivalently as

$$\tau = \inf \left\{ t > 0 \mid \int_0^t \lambda_s ds \ \geq E_1 \right\}, \tag{3}$$

where E_1 is an exponential random variable with mean 1 independent of the process λ. The survival probabilities are then given by

$$\mathbb{Q}\left(\tau > t\right) = \mathbb{E}[e^{-\int_0^t \lambda_s ds}] \tag{4}$$

with the intensity rate λ representing the instantaneous probability of default.

More recently, a number of authors have proposed reduced form models where the default time is described, similarly to the classical framework in (3), as the first time an increasing process $A = (A_t)$ reaches or is above a level of an independent exponential random variable, with the exception that A is no longer constrained to be absolutely continuous w.r.t. the Lebesgue measure. In what follows we describe and examine in some detail this new modeling approach, also known as the Cox construction of a default time. A comprehensive and unifying theoretical analysis of the general reduced form modeling approach is given in Jeanblanc and Le Cam (2007).

We start by considering a filtered probability space $\left(\Omega, \mathcal{G}, (\mathcal{F})_{t\geq 0}, \mathbb{Q}\right)$ which is large enough to support a càdlàg, adapted process A and a mean one exponential random variable E_1 which is independent of $\mathcal{F}_\infty$. Furthermore, we assume that the process A, which we will refer to as the *cumulative hazard* (CH) process, has (strictly) increasing paths and $A_0 = 0$. The default time τ is then defined as follows

$$\tau = \inf \left\{ t > 0 \mid A_t \ \geq E_1 \right\}, \tag{5}$$

and the survival probabilities are given by (1). Notice that the default time τ may still be related to the first jump-time of a process $\widetilde{N} = \left(\widetilde{N}_t\right)$. In fact, if the probability space is large enough to support a standard unit rate Poisson process N independent of A and defining $\widetilde{N}_t = N_{A_t}$ i.e. by subordination of N to A, we see that

$$\tau = \inf\left\{t > 0 : \widetilde{N}_t > 0\right\}. \tag{6}$$

The process $\widetilde{N}$ displays jumps of size one (or zero) when the CH A has continuous trajectories but in whole generality we can only state that the jump sizes take values in the non negative integer numbers.

The directing filtration $(\mathcal{F})_{t\geq 0}$ can be seen as carrying the default-free information while the knowledge about whether default has occurred or not is contained in the enlarged filtration

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t \ ,$$

where $\mathcal{H}_t = \sigma\left\{\mathbb{1}_{\{\tau\leq s\}} : 0 \leq s \leq t\right\}$ is the filtration generated by the default process $(H_t = \mathbb{1}_{\{\tau\leq t\}})$. If the compensator $\Lambda^G = \left(\Lambda^G_t\right)$ (w.r.t the filtration $(\mathcal{G})_{t\geq 0}$) of the process H is absolutely continuous w.r.t the Lebesgue measure

$$\Lambda^G_t = \int_0^t \lambda^G_s ds,$$

the time of default is said to have *intensity rate*, or simply *intensity*, λ^G and it holds that

$$\lambda^G_t = \lim_{h\to 0} \frac{1}{h}\mathbb{Q}(t < \tau \leq t + h|\ \mathcal{G}_t)$$

(see e.g. Jeanblanc and Le Cam (2007)). The compensator Λ^G can be related to the CH process A as follows

$$\Lambda^G_t = \int_0^{t\wedge\tau} \frac{dC_s}{e^{-A_{s-}}} \ ,$$

where the process $C = (C_t)$ is the $(\mathcal{F})_{t\geq 0}$-compensator of the process $1-e^{-A}$. Hence, if C is absolutely continuous w.r.t. the Lebesgue measure, $C_t = \int_0^t c_s ds$, the time of default has intensity

$$\lambda^G_t = \mathbb{1}_{\{t<\tau\}}\lambda_t \quad \text{with} \quad \lambda_t = \frac{c_t}{e^{-A_{t-}}} \tag{7}$$

and with a slight abuse of terminology, the $(\mathcal{F})_{t\geq 0}$-adapted process λ is also referred to as the intensity process. Notice that if the CH A is specified as in the classical intensity based approach described in (3), i.e. $A_t = \int_0^t \lambda_s ds$, then the rate of growth λ is indeed the intensity process in the sense of equation (7).

For the actual computations, it is more convenient to express the intensity process λ in terms of the semimartingale characteristics of the CH process A (see Jacod and Shiryaev (1987), Chapter 2 for the definition of semimartingale characteristics).

Proposition 1 *Assume that the cumulative hazard process has absolutely continuous characteristics, i.e. relatively to the zero truncation function it has the following canonical representation*

$$A_t = \int_0^t d_s ds + x \star \mu, \tag{8}$$

where the random measure $\mu(\omega, dx, dt)$ *associated to the jumps of* A *has absolutely continuous compensator*

$$\nu(\omega, dx, dt) = K_t(\omega, dx)dt.$$

Then the time of default defined in (5) *admits intensity* λ *which is given by*

$$\lambda_t^F = d_t - \int_0^{+\infty} (e^{-x} - 1)K_t(\omega, dx). \tag{9}$$

Furthermore, if A *is a process with independent increments, i.e. an additive process, the intensity* λ *is deterministic and given by*

$$\lambda_t = -\frac{d}{dt} \log \mathbb{E}[e^{-A_t}]. \tag{10}$$

Proof. First, we compute the compensator C of the process $1 - e^{-A}$. By change of variable formula, we obtain that

$$1 - e^{-A_t} = e^{-A_{t-}} \cdot L_t$$

with

$$L_t = A_t - (e^{-x} - 1 + x) \star \mu.$$

Applying the canonical representation (8) we can rewrite L as follows

$$\begin{aligned} L_t &= x \star (\mu - \nu) + \int_0^t d_s ds - (e^{-x} - 1) \star \mu \\ &= -(e^{-x} - 1) \star (\mu - \nu) + \int_0^t d_s ds - (e^{-x} - 1) \star \nu, \end{aligned}$$

from which it follows that

$$C_t = \int_0^t e^{-A_{s-}} \left(d_s - \int_0^{+\infty} (e^{-x} - 1) K_s(\omega, dx) \right) ds$$

and the result (9) now follows from expression (7).
If furthermore A is additive, the drift process d_t and the measures $K_t(dx)$ (also known as the *Lévy system*) are deterministic and expression (10) follows

immediately by recalling the Lévy-Kintchine representation for the Laplace transform

$$\mathbb{E}\left[e^{uA_t}\right] = \exp \int_0^t \left(ud_s + \int_0^\infty (e^{ux} - 1) K_s(dx) \right) ds. \tag{11}$$

■

We conclude this section by recalling the so-called *hazard based pricing rule* (see Elliott, Jeanblanc, and Yor (2000)): given a claim $X \in \mathcal{F}_T$, it holds that

$$\mathbb{E}[X \mathbb{1}_{\{T<\tau\}} | \mathcal{G}_t] = \mathbb{1}_{\{\tau>t\}} e^{A_t} \mathbb{E}[e^{-A_T} X | \mathcal{F}_t].$$

In particular, assuming no recovery payment and that the instantaneous interest rate process $r = (r_t)$ is adapted to the default-free filtration $(\mathcal{F})_{t\geq 0}$, we obtain that the price $B(t,T)$ of a corporate bond is given by

$$B(t,T) = \mathbb{1}_{\{\tau>t\}} e^{A_t} \mathbb{E}\left[e^{-\left(A_T + \int_t^T r_s ds\right)} | \mathcal{F}_t \right].$$

By assuming independence between A and the interest rate process r, the expression further reduces to

$$\begin{aligned} B(t,T) &= p(t,T) \mathbb{1}_{\{\tau>t\}} e^{A_t} \mathbb{E}\left[e^{-A_T} | \mathcal{F}_t\right] \\ &= p(t,T) \mathbb{Q}(\tau > T | \mathcal{G}_t), \end{aligned}$$

where $p(t,T)$ is the price of a risk-free zero coupon bond with maturity T.

3 Sato Specifications

We start by recalling that a stochastic process $A = (A_t)$ is called self-similar if for any $\alpha > 0$ there exists a $\beta > 0$ such that for all t

$$A_{\alpha t} \overset{\text{law}}{=} \beta A_t, \tag{12}$$

implying that a change of time scale has an analogous effect as a change in the spatial scale. From the definition (12) it follows that there exists a $\gamma > 0$, the *exponent of self-similarity*, such that $\beta = \alpha^\gamma$. A well known class of self-similar processes is given by the strictly stable processes, i.e. processes with independent and homogeneous increments with law at unit time described by a strictly stable distribution. Sato (1991) introduced the broader class of self-similar additive processes, here termed *Sato processes*, as processes satisfying (12) and with independent but not necessarily homogeneous increments. Sato processes are intimately related with the class of self-decomposable distributions. More precisely, the unit time law of a Sato process is necessarily self-decomposable. Vice versa, for any given self-decomposable law X and an exponent $\gamma > 0$ there exists a self-similar additive process with law at unit time given by X and exponent of self-similarity given by γ. (For more information on Sato processes see e.g. Sato (1999)).

By choosing for the unit time a self-decomposable law X with support given by the positive real line and any exponent γ, we obtain that the corresponding Sato process has increasing paths and therefore is suitable to describe a CH process, here termed *Sato CH*.

Self-decomposability requires that the cumulant generating function of the unit time law X

$$\kappa_X(u) = \log \mathbb{E}[e^{uX}]$$

takes the following representation

$$\kappa_X(u) = du + \int_0^{\infty} (e^{ux} - 1)\frac{h(x)}{x}\, dx, \tag{13}$$

where $d \geq 0$ and $h(x)$ is a nonnegative decreasing function on $(0, +\infty)$ satisfying

$$\int_0^{+\infty} (1 \wedge x)\frac{h(x)}{x}\, dx < +\infty. \tag{14}$$

Self-similarity implies that

$$\mathbb{E}\left[e^{uA_t}\right] = \mathbb{E}\left[e^{ut^{\gamma}X}\right] = e^{\kappa_X(ut^{\gamma})}. \tag{15}$$

Carr, Geman, Madan, and Yor (2007) computed the precise form of the Lévy-Kintchine representation of A_t, which is given as in (11) with drift specified as

$$d_t = d\,\gamma\, t^{\gamma-1} \tag{16}$$

and Lévy system

$$K_t(dx) = -\frac{h'\left(\frac{x}{t^{\gamma}}\right)\gamma}{t^{\gamma+1}}\, dx, \quad x > 0. \tag{17}$$

Inspection of the Lévy system $K_t(dx)$ reveals that the behavior of the function $h(x)$ characterizing the unit time law determines whether the corresponding Sato CH process A displays finite or infinite activity. More precisely Carr, Geman, Madan, and Yor (2005) showed that A jumps finitely or infinitely often on finite time intervals if $h(0) < +\infty$ or $h(0) = +\infty$ respectively.

Notice also that, by Proposition 1, a Sato CH process allows for the existence of the intensity rate λ which is deterministic and given as in expression (10) which in virtue of (15) becomes

$$\lambda_t = \gamma t^{\gamma-1}\frac{d}{dt}\kappa'_X(-t^{\gamma}). \tag{18}$$

Notice that if the unit time law X has finite first moment, the corresponding Sato CH process A has average behavior given by

$$\mathbb{E}[A_t] = \mathbb{E}[X]t^\gamma \tag{19}$$

allowing for an increasing, constant or decreasing average growth rate corresponding to $\gamma > 1$, $\gamma = 1$ or $\gamma < 1$ respectively. However, the effect of the exponent of self-similarity γ on the long term behavior of the intensity λ is not completely analogous. In particular, in the case of a driftless Sato CH process with finite activity, the intensity vanishes over time for any choice of γ. A more complete description is given in the following Proposition.

Proposition 2 *Let A be a Sato CH process with exponent of self-similarity γ and unit time law X specified as in* (13)*, and let λ denote the associated intensity. Then the following holds:*

i) *If $0 < \gamma < 1$ then $\lambda_t \to 0$ as $t \to +\infty$.*

ii) *If $\gamma \geq 1$, $d = 0$ and $h(0) < +\infty$ then $\lambda_t \to 0$ as $t \to +\infty$.*

iii) *Finally if $d > 0$ then*

$$\lambda_t \to \begin{cases} \textit{constant} \geq d & \textit{if } \gamma = 1 \\ +\infty & \textit{if } \gamma > 1 \end{cases}$$

as $t \to +\infty$.

Proof. All the statements above follow immediately from the observation that by combining (13) with (18) the intensity can be expressed as follows

$$\lambda_t = \gamma t^{\gamma-1} \left(d + \int_0^{+\infty} e^{-t^\gamma x} h(x) dx \right) \tag{20}$$

and from the integrability property (14) of the function $h(x)$. ■

When dealing with a driftless Sato CH process of infinite activity and exponent of self-similarity $\gamma \geq 1$, the long term behavior of the intensity depends on the particular shape of the function $h(x)$ and it is not possible to provide a general statement (see e.g. the case of an IG-Sato process discussed below).

From (15) it follows that the unconditional survival probabilities are given by the following simple expression

$$\mathbb{Q}\left(\tau > t\right) = \mathbb{E}\left[e^{-A_t}\right] = e^{\kappa_X(-t^\gamma)}. \tag{21}$$

Therefore, if the cumulant generating function κ_X is known explicitly, the survival probabilities can be computed in closed-form enabling straightforward calibration of the model to credit default swaps. This is the case for the two concrete specifications we consider in this work, namely the *Sato-Gamma* and the *Sato-IG* cumulative hazards.

The *Sato-Gamma CH* is obtained by specifying the unit time distribution X as a Gamma law $\Gamma(a, b)$ with density function given by

$$f_X(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-b x} \quad x > 0 \; . \tag{22}$$

The cumulant generating function is given by

$$\kappa_X(u) = \log\left(\frac{1}{(1 - u\, b^{-1})^a}\right) = \int_0^\infty (e^{ux} - 1) \frac{a\, e^{-bx}}{x}\, dx \tag{23}$$

and in view of (13) we see that a Gamma law is self-decomposable. Moreover, the associated Sato-Gamma CH displays finite activity and has intensity given by

$$\lambda_t = a\gamma \frac{t^{\gamma-1}}{b + t^\gamma}.$$

The *Sato-IG CH* is obtained by setting the unit time distribution X as an Inverse Gaussian law $IG(a, b)$ with density function given by

$$f_X(x) = \frac{a}{\sqrt{2\pi}} e^{a\,b} x^{-\frac{3}{2}} e^{-(b^2 x + \frac{a^2}{x})/2} \quad x > 0 \; . \tag{24}$$

Also the $IG(a, b)$ distribution is self-decomposable[1] as it can be noticed from the cumulant generating function which has the following expression

$$\kappa_X(u) = ab - a\sqrt{b^2 - 2u} = \int_0^\infty (e^{ux} - 1) \frac{a}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2} b^2 x}}{x^{\frac{3}{2}}}\, dx. \tag{25}$$

We see that a Sato-IG CH is a process of infinity activity and the corresponding intensity is given by

$$\lambda_t = a\gamma \frac{t^{\gamma-1}}{\sqrt{b^2 + 2t^\gamma}}.$$

Notice that as $t \to +\infty$ the intensity converges to 0, a positive constant or $+\infty$ if the exponent of self-similarity γ is chosen smaller, equal or greater than 2 respectively.

[1] The self-decomposability of Inverse Gaussian distributions was first proved by Halgreen (1979).

4 Time-Changed Lévy Specifications

A simple class of CH processes sharing with Sato processes the same average time behavior (19) but otherwise displaying different characteristics can easily be constructed via a simple deterministic time-change of a Lévy process. More precisely let $L = (L_t)$ be an increasing Lévy process, i.e. a process with positive, time-homogeneous and independent increments. Then by changing the time scale of L in the following manner

$$A_t = L_{t^\gamma} \tag{26}$$

with $\gamma > 0$ we obtain an increasing additive CH process, which we will refer to as a *TC Lévy* CH process. Once again, let X denote the positive unit time law of A and $\kappa_X(u)$ its cumulant generating function. Only infinite divisibility of X and not necessarily self-decomposability is required in the TC Lévy construction (26). Therefore, $\kappa_X(u)$ takes the general form

$$\kappa_X(u) = du + \int_0^\infty (e^{ux} - 1)\xi(dx), \tag{27}$$

where $d \geq 0$ and the Lévy measure $\xi(dx)$ satisfies $\int_0^{+\infty}(1 \wedge x)\xi(dx) < +\infty$. Clearly the TC Lévy process A displays the same activity level of the subordinand process L, i.e. it is a process of finite activity if L is a compound Poisson process while it exhibits infinite activity if $\int_0^{+\infty} \xi(dx) = +\infty$. It is also immediate to verify that

$$\mathbb{E}\left[e^{uA_t}\right] = e^{t^\gamma \kappa_X(u)}$$

and that the Lévy-Kintchine representation is given as in (11) with drift specified as

$$d_t = d\,\gamma\,t^{\gamma-1}$$

and Lévy system

$$K_t(dx) = \gamma\,t^{\gamma-1}\xi(dx).$$

Hence, by Proposition 1 the default time associated with a TC Lévy CH admits intensity λ which is given by the power function

$$\lambda_t = -\kappa_X(-1)\,\gamma\,t^{\gamma-1}\ . \tag{28}$$

Notice that unlike the case of Sato specifications, the average growth rate $d\mathbb{E}[A_t]/dt$ (when defined) and the intensity λ of a TC Lévy CH display the same behavior over time, which in particular is independent of the choice of the unit time law X, albeit for a scaling constant.

The survival probabilities are given by

$$\mathbb{Q}\left(\tau > t\right) = \mathbb{E}\left[e^{-A_t}\right] = e^{t^\gamma \kappa_X(-1)} \tag{29}$$

and can again be computed in closed form whenever κ_X is known. In consistence with the concrete models considered in Section 3, we calibrate to market data the *TC Lévy-Gamma* and the *TC Lévy-IG* processes obtained by selecting respectively the $\Gamma(a,b)$ law (22) and the $IG(a,b)$ law (24) for the unit time distribution X. From expressions (23) and (25) one can see that both the TC Lévy-Gamma and TC Lévy-IG processes display infinite activity.

5 The Ornstein-Uhlenbeck Type Intensity Rates Models

In this section we introduce and briefly illustrate the models which will be used as benchmark in order to assess the performances of the CH models described above.

Cariboni and Schoutens (2006) have recently proposed a new default model where the intensity rate is governed by a positive process of the Ornstein-Uhlenbeck type (OU type process henceforth). Recall that a positive OU process $\{\lambda_t : t > 0\}$ is defined as the solution of the stochastic differential equation

$$d\lambda_t = -\theta\lambda_t dt + dZ_{\theta t} \qquad \lambda_0 > 0, \tag{30}$$

where θ is a positive real number and $Z = (Z_t)$ is a subordinator, i.e. a purely jumping process with independent, positive and stationary increments, which is often termed the *Background Driving Lévy Process* (BDLP henceforth). The process λ is mean-reverting, it increases only by jumps and between jumps it decays exponentially. This might be considered a natural property in the credit market if we think of the jumps as the arrival of bad information, which drives the intensity process up, while during periods with no arrival of bad information the intensity decays. Under (mild) integrability conditions on the BDLP Z, the OU process λ described in (30) is also stationary with invariant distribution X which is self-decomposable and, due to the unusual timing of $Z_{\theta t}$, does not depend on the mean reversion parameter θ. In fact the converse statement is also true in the sense that for any given self-decomposable distribution X, there exists a Lévy process Z such that for any $\theta > 0$ the corresponding OU type process driven by Z as in (30) has invariant law given by X.

Moreover the cumulant function $\kappa_X(u)$ of X and the cumulant function $\kappa_Z(u)$ of Z_1 are related by the formula

$$\kappa_Z(u) = u\,\frac{d\kappa_X}{du}(u). \tag{31}$$

For further details and a thorough analysis of OU processes see Sato (1999).

Cariboni and Schoutens (2006) examine the two concrete stationary OU processes, denoted by IG(a,b)-OU and Gamma(a,b)-OU, characterized by having invariant (self-decomposable) law given by the $\Gamma(a,b)$ and the $IG(a,b)$ distributions respectively.

From expressions (23) and (31) it follows that the Gamma(a,b)-OU specification is obtained by choosing the BDLP Z as a compound Poisson process with Lévy density

$$w(x) = a\,b\exp(-bx)\ , \tag{32}$$

i.e. Z is a process with jumps arriving at the same jump times of a Poisson process with intensity a and having jump-size distributed according to an exponential law with parameter b.

Analogous derivations imply that the IG(a,b)-OU process is obtained by specifying the Lévy density of Z as

$$w(x) = \frac{a}{2\sqrt{2\pi}} x^{-\frac{3}{2}}(1+b^2x)e^{-\frac{1}{2}b^2x}. \tag{33}$$

In this case Z is not a compound Poisson process, since the density in (33) is not integrable. In other words, Z (and therefore λ) jumps infinitely often in finite time intervals. For a complete derivation of the results above see Barndorff-Nielsen and Shephard (2001).

Both in the Gamma(a,b)-OU and the IG(a,b)-OU case, the Laplace transform of the CH $A_t = \int_0^t \lambda_s ds$ is available in terms of elementary functions and thereby provides explicit expressions for the survival probabilities in (4). Having specified λ as a Gamma(a,b)-OU process with dynamics as in (30), one obtains

$$\mathbb{E}[e^{u\,A_t}] = \exp\left(\frac{u\lambda_0}{\theta}(1-e^{-\theta t}) + \frac{\theta a}{u-\theta b}\left(b\log\left(\frac{b}{b-u\theta^{-1}(1-e^{-\theta t})}\right) - ut\right)\right) \tag{34}$$

while for the IG(a,b)-OU specification the Laplace transform takes the form

$$\mathbb{E}[e^{u\,A_t}] = \left(\frac{\lambda_0 u}{\theta}\left(1-e^{-\theta t}\right) + \frac{2au}{b\theta}B\left(u,t\right)\right), \tag{35}$$

where

$$\begin{aligned} B\left(u,t\right) =& \frac{1-\sqrt{1+v\left(1-e^{-\theta t}\right)}}{v} \qquad (36)\\ &+ \frac{1}{\sqrt{1+v}}\left(\operatorname{arctanh}\left(\frac{\sqrt{1+v\left(1-e^{-\theta t}\right)}}{\sqrt{1+v}}\right) - \operatorname{arctanh}\left(\frac{1}{\sqrt{1+v}}\right)\right)\\ v =& \frac{-2u}{b^2\theta}\ . \end{aligned}$$

For the derivation, see Nicolato and Venardos (2003) or Cariboni and Schoutens

(2006). However, Cariboni and Schoutens (2006) find that while both the Gamma(a, b)-OU and IG(a, b)-OU processes provide accurate fits to the inferred survival probabilities, the results obtained in the Gamma(a, b)-OU specification are more stable. Moreover, the Gamma(a, b)-OU process allows for exact and fast simulation, due to the fact that it is driven by a compound Poisson process Z. This is a very convenient feature if portfolios of credit derivatives has to be priced.

Finally, Cariboni and Schoutens (2006) compare the performance of OU type intensity processes to other intensity models and find that Ornstein-Uhlenbeck type processes exhibit the most stable parameters and are able to fit market prices as good as the classical CIR process. These findings motivate our choice to include only Ornstein-Uhlenbeck type processes in the empirical analysis below.

6 Calibration of the Models

A total of six CH models have been introduced and discussed: Sato-Gamma, Sato-IG, TC Lévy-Gamma, TC Lévy-IG, Gamma-OU and IG-OU.

The models are calibrated by matching the market inferred credit default swap (CDS) spreads on each name to the model inferred spreads.

Recall that a CDS insures the protection buyer against default of an underlying company in exchange for a stream of payments to the protection seller. The payments continue until the maturity of the contract or the underlying company defaults. In case of default, the contract is terminated prematurely and the protection seller pays the face value of the corporate bond minus a possible recovery on the bond to the protection buyer.

Assuming no counter party risk and independence between the CH and interest rates, the price of a CDS with maturity T is given by the difference between the discounted protection payment and the discounted continuously paid CDS spread c

$$CDS = (1 - R) \int_0^T p(0, s)\, d\mathbb{Q}(s) - c \int_0^T p(0, s)\, \mathbb{Q}(\tau > s)\, ds,$$

where $\mathbb{Q}(s) = \mathbb{Q}(\tau \leq s)$ is the default probability up to time s, R is the recovery on the bond and $\{p(0, s), s \in [0, T]\}$ the observed default-free zero coupon bond prices. Consequently, the par spread equals

$$c^* = \frac{(1 - R) \int_0^T p(0, s)\, d\mathbb{Q}(s)}{\int_0^T p(0, s)\, \mathbb{Q}(\tau > s)\, ds}. \tag{37}$$

Since survival probabilities are given by analytically closed expressions in all the described models, the parameters can be directly calibrated by matching the model spreads in (37) to those observed in the market.

We calibrate by minimizing the root mean square error (RMSE) given by

$$\text{RMSE} = \sqrt{\sum_{\text{CDSs}} \frac{\left(c_{Market} - c_{Model}\right)^2}{\#\left\{\text{CDS prices}\right\}}}$$

using the Nelder-Mead simplex algorithm.

Standardly, the recovery is set to 0.4 and the interest rate r is chosen constant equal to 4 percent.

The data consists of weekly observations on the 125 single names in the iTraxx Europe Series 8 index during the period from September 17th, 2007 to March 14th, 2008 (a total of 26 weeks) in order to check both the pricing capabilities of the models and the stability of the parameters over time.[2] The models are fitted to spreads for the five days in the week for maturities 1,3,5,7 and 10 years, which gives a total of 25 market prices for each company per observation. For each week the calibration is initialized with the estimated parameters from the previous week. Moreover, one of the parameters in each of the models has been fixed as otherwise the numerical routine seems to encounter some identification issues. In the Sato and TC Lévy specifications, we set a to 1, while in the OU-case the parameter a is set to 100. Without this fixation parameter the pricing errors are reduced slightly, but at the cost of a severe reduction in parameter stability.

For the two integrated OU-intensity models the calibration for all the companies and for a given point in time (i.e. one week) takes less than one minute. The procedure is even faster for the Sato and TC Lévy specifications, approximately 30 seconds on a 2 GB ram, 2 GHz dual core pc using MatLab.

The performances of the models are illustrated by focusing on three representative companies : Banco Espirito Santo SA, Endesa SA and Deutsche Post AG. However, similar results are obtained for the other names in the index. In Figure 1 the spreads for the three companies during the analyzed period are illustrated, where the increasing behavior over time clearly depicts the credit crisis which erupted in August 2007 and gained strength in early 2008.

Figure 2 shows the term structures obtained for the different models and companies after calibration on a single day data. The corresponding RMSE performances are reported in Table 1 where, for completeness, we also compute the resulting average absolute error as a percentage of the

[2]September 17th was the first Monday after the roll over of the index and March 14th the last Friday before the next roll over.

mean price (APE) and the average relative percentage error (ARPE)

$$\text{APE} = \frac{1}{\text{mean CDS spread}} \sum_{\text{CDSs}} \frac{|c_{Market} - c_{Model}|}{\#\{\text{CDS prices}\}}$$

$$\text{ARPE} = \frac{1}{\#\{\text{CDS prices}\}} \sum_{\text{CDSs}} \frac{|c_{Market} - c_{Model}|}{c_{Market}} .$$

All the models perform satisfactorily with ARPE and APE errors below 3% with the two OU models producing slightly smaller errors. This is not surprising since in the OU models 3 parameters are calibrated as opposite to the 2 parameters involved in the Sato and TC Lévy models. It should also be noticed that the two TC Lévy models achieve spreads almost identical to the Sato-IG model.

Analogous results are obtained when looking at the entire sample of weekly data as illustrated by the average errors RMSE, APE and ARPE across all firms and weeks reported in Table 2. The slightly better fit produced by the OU-models is confirmed by Figures 3 and 4 where all the 3250 $(26 \cdot 125)$ errors, APE and ARPE, are collected in histograms for the different models. The overall quality of fit is satisfactory with the two OU models producing error distributions shifted a little more to the left in comparison to the other four models.

The time series of the model parameters resulting from the calibration over the 26 weeks in the sample are depicted in Figures 5 to 7. The graphical illustrations suggest that the Sato and TC Lévy models exhibit a more stable parameter behavior over time. This intuition is supported when considering the lag-i autocorrelation for each of the parameter time series as also suggested by Cariboni and Schoutens (2006). Recall that the lag-i autocorrelation for a time serie $X = \{X_t\}_{t=1,\dots,N}$ is given by

$$\rho_i = \frac{\mathbb{E}\left[\left(X_t - \mathbb{E}\left[X_t\right]\right)\left(X_{t+i} - \mathbb{E}\left[X_t\right]\right)\right]}{\sqrt{\mathbb{E}\left[\left(X_t - \mathbb{E}\left[X_t\right]\right)^2\right]\mathbb{E}\left[\left(X_{t+i} - \mathbb{E}\left[X_t\right]\right)^2\right]}} .$$

yielding a suitable measure of stability. For a given parameter in a given model, the lag-1 autocorrelation coefficients obtained for the 125 names in the index are collected in histograms in Figure 8. Notice that the Sato and the TC Lévy models display autocorrelation distributions shifted to the right in comparison to the two OU models, and thus more stable parameters in time. Finally, Figure 9 shows the average autocorrelation for each model parameter across firms for the first five lags. Again, as far as stability is concerned, the Sato and the TC Lévy models outperform the two OU models. However, when comparing Sato with TC Lévy models it is not possible to assess which model class is favorable. The four models produce almost identical errors and are equally stable in their optimal parameters. In fact, when sharing the same unit time distribution, the corresponding Sato and TC Lévy models produce almost indistinguishable calibrated parameters, in spite of the different characteristics of the employed CH processes.

7 Conclusion

In this work we have proposed and analyzed two new reduced form models where the cumulative hazard belongs to the class of Sato and deterministic TC Lévy processes respectively. Both model classes are tractable, as they allow for survival probabilities in closed form. They are very parsimonious as only 2-3 parameters are employed and display cumulative hazards with the same flexible and non-linear average long run behavior. However, the two model classes exhibit very different properties e.g. in the activity level and in the behavior of the associated intensities. Moreover, TC Lévy processes do not possess the self-similarity property characterizing the Sato specifications.

From each class two concrete specifications have been empirically investigated and compared to two OU type intensity models. All the models are calibrated to weekly observations of credit default swap spreads on the names constituting the iTraxx Europe series 8 index. The models produce comparable and satisfactory calibration errors, with the Sato and the TC Lévy models requiring one parameter less and displaying more stable parameter estimates over time.

However, on the basis of our calibration results, it is not possible to discern whether Sato models are preferable to TC Lévy models. Thus, our analysis indicates that the flexible average behavior shared by the two model classes is the critical factor in their satisfactory empirical performances.

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A Tables

Company	Model	1YR	3YR	5YR	7YR	10YR	RMSE	ARPE(%)	APE(%)
Banco Espirito	Market	37.90	45.00	51.50	54.20	55.00			
Santo SA	Sato-Gamma	37.68	46.28	50.51	53.23	55.92	0.9434	1.76	1.80
	Sato-IG	37.95	46.05	50.24	53.13	56.25	1.0387	1.83	1.92
	TC Lévy-Gamma	37.95	46.04	50.24	53.13	56.25	1.0394	1.83	1.92
	TC Lévy-IG	37.95	46.04	50.24	53.13	56.25	1.0394	1.83	1.92
	Gamma-OU	37.65	46.08	50.71	53.42	55.75	0.7769	1.48	1.50
	IG-OU	37.65	46.08	50.70	53.42	55.75	0.7768	1.48	1.50
Endesa SA	Market	44.40	63.70	74.50	81.30	88.30			
	Sato-Gamma	44.52	63.71	74.22	81.30	88.47	0.1587	0.18	0.17
	Sato-IG	45.28	62.23	73.53	80.96	89.30	0.7806	1.11	1.04
	TC Lévy-Gamma	45.28	62.23	73.52	80.96	89.31	0.7855	1.12	1.04
	TC Lévy-IG	45.28	63.23	73.52	80.96	89.30	0.7855	1.12	1.04
	Gamma-OU	44.50	63.35	74.63	81.69	88.02	0.2747	0.35	0.35
	IG-OU	44.50	63.35	74.63	81.69	88.02	0.2749	0.35	0.35
Deutsche	Market	12.50	24.80	37.20	45.30	55.00			
Post AG	Sato-Gamma	12.18	25.93	36.36	44.94	55.39	0.6897	2.18	1.74
	Sato-IG	12.51	25.93	36.13	44.71	55.64	0.7984	2.00	1.97
	TC Lévy-Gamma	12.51	25.93	36.13	44.71	55.64	0.7987	2.00	1.97
	TC Lévy-IG	12.51	25.93	36.13	44.71	55.64	0.7987	2.00	1.97
	Gamma-OU	12.20	25.71	36.52	45.20	55.18	0.5324	1.69	1.24
	IG-OU	12.20	25.71	36.52	45.20	55.18	0.5323	1.69	1.24

Table 1: The spreads from the models calibrated on Thursday, January 3rd, 2008 and the corresponding errors RMSE are reported. The resulting average errors ARPE and APE are also listed.

Model	Sato-Gamma	Sato-IG	TC Lévy-Gamma	TC Lévy-IG	Gamma-OU	IG-OU
RMSE	4.07	4.37	4.37	4.37	3.61	3.64
ARPE (%)	6.70	7.11	7.12	7.12	5.53	5.56
APE (%)	5.63	5.93	5.93	5.93	4.94	4.97

Table 2: The calibration errors RMSE, ARPE and APE on weekly data averaged across all firms and the entire sample for the different models.

B Figures

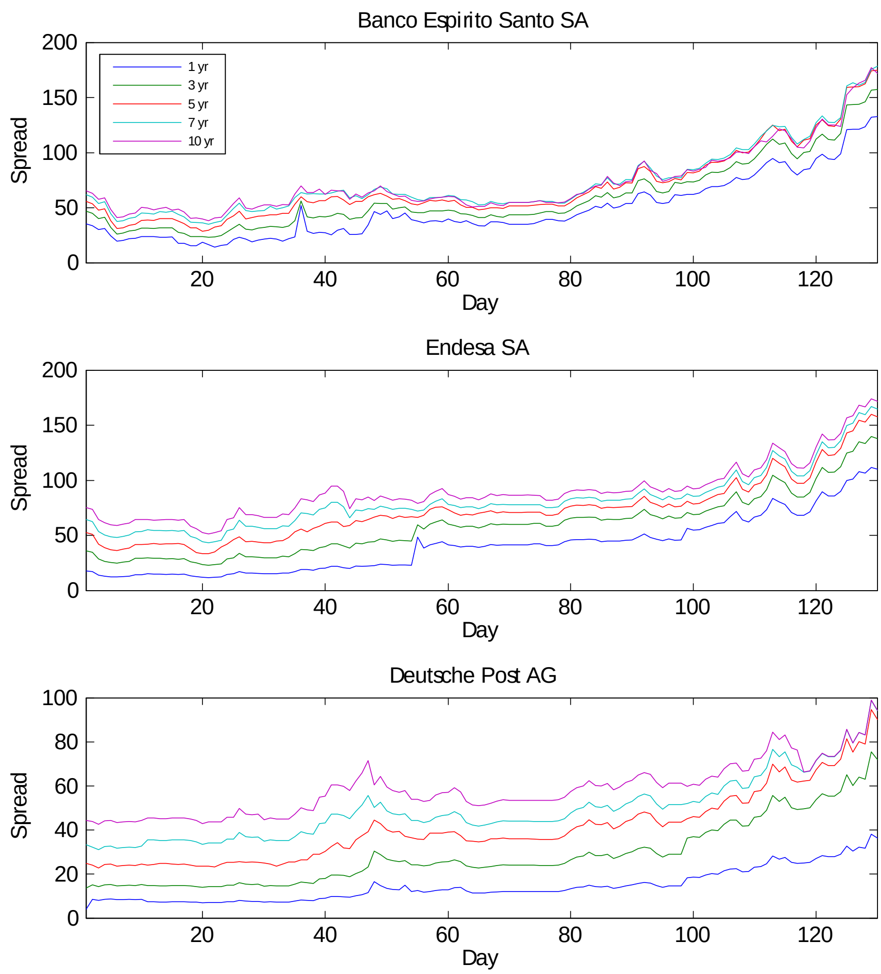

Figure 1: Depicted time series of the spreads for the 3 illustrating companies covering the period from September 17th, 2007 to March 14th, 2008.

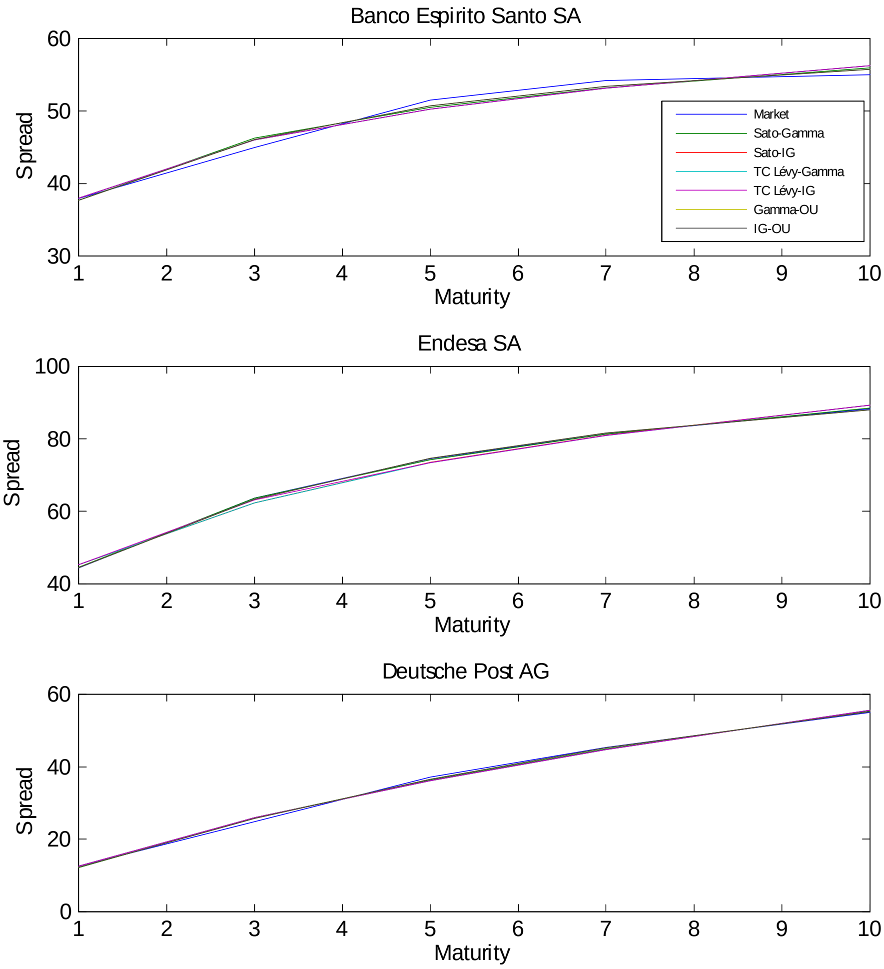

Figure 2: Calibrated spreads for the various models to the market on Thursday, January 3rd, 2008.

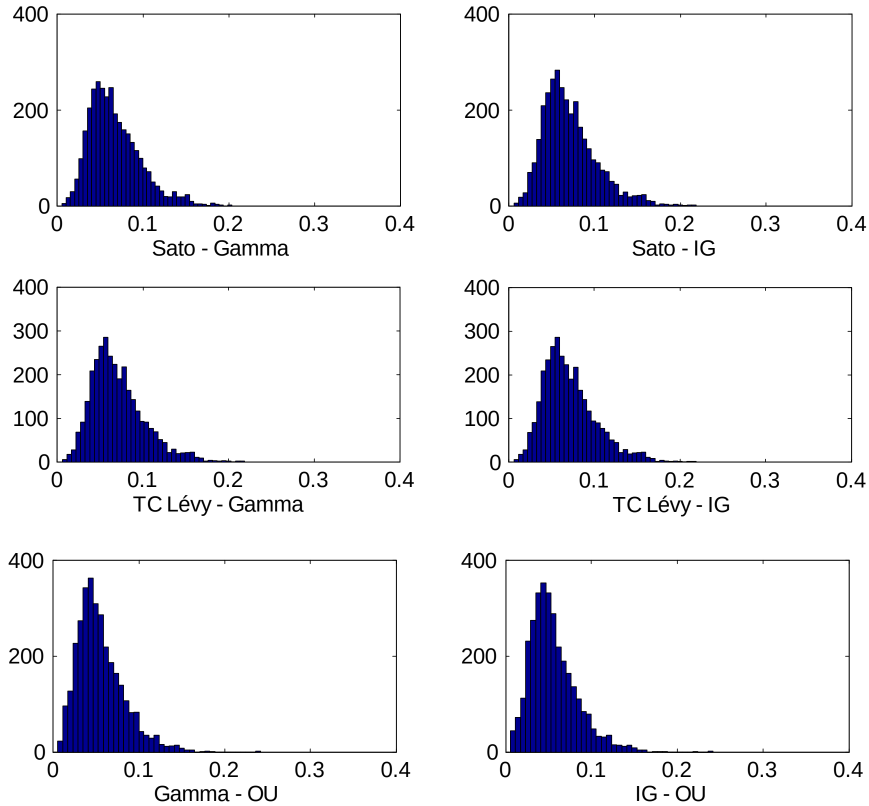

Figure 3: Distributions of ARPE in percent across companies and weeks for the different models.

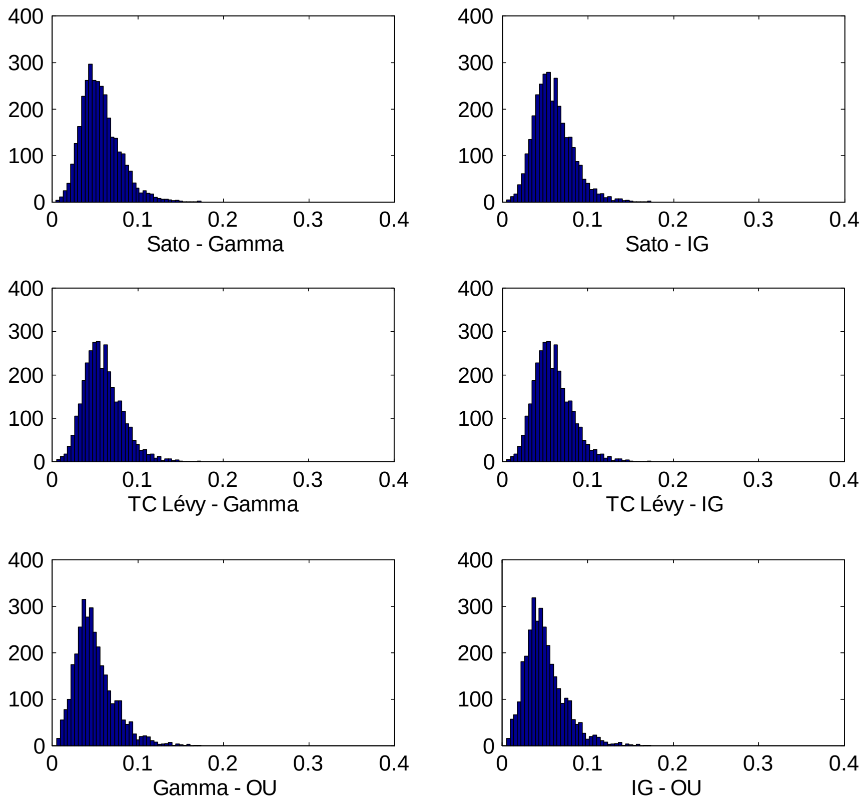

Figure 4: Distributions of APE in percent across companies and weeks for the different models.

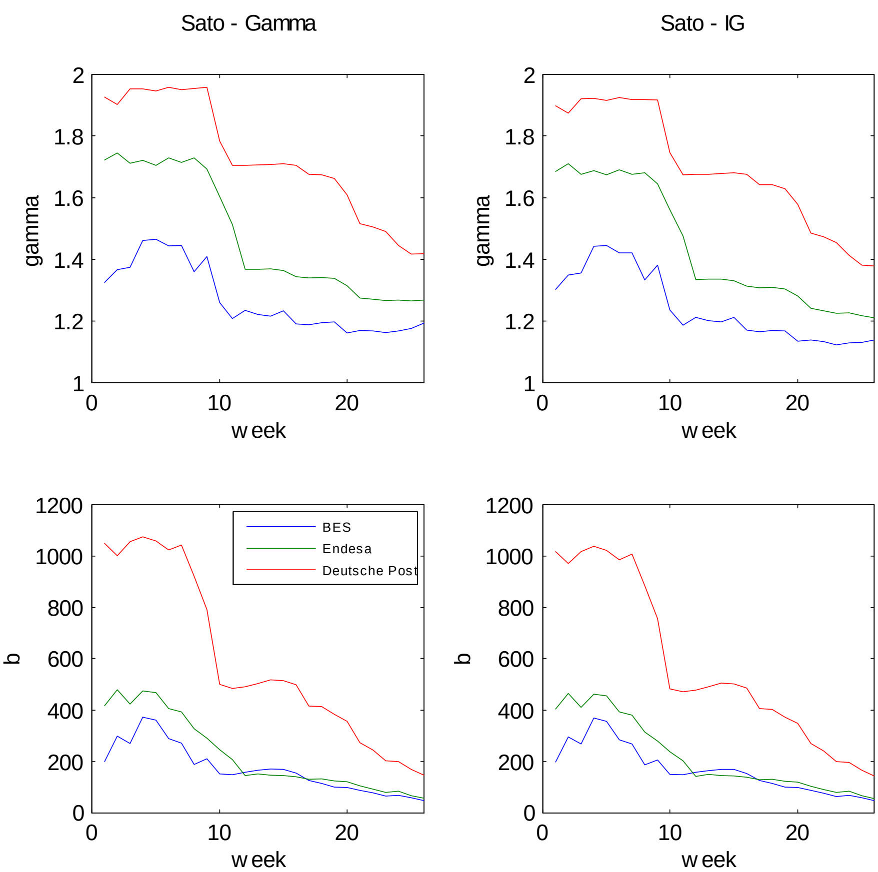

Figure 5: The calibrated γ and b parameters in the two Sato models for the 3 representative companies over the 26 weeks. The parameter a is fixed to 1 in both specifications.

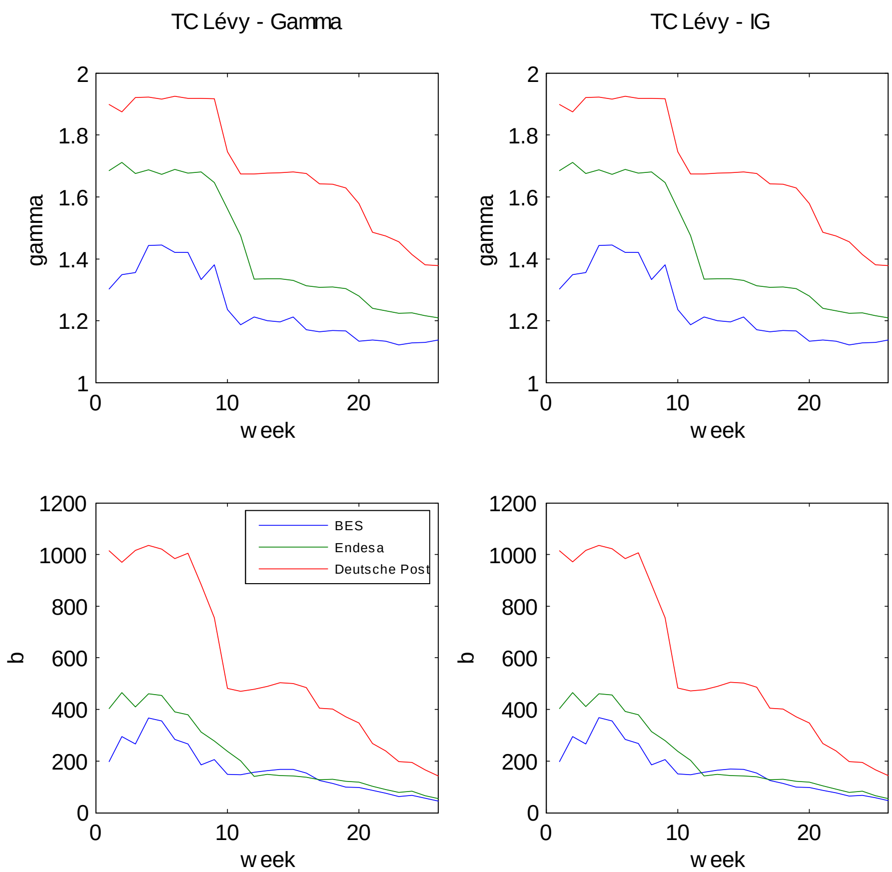

Figure 6: The calibrated γ and b parameters in the two time-changed Lévy models for the 3 representative companies over the 26 weeks. The parameter a is fixed to 1 in both specifications.

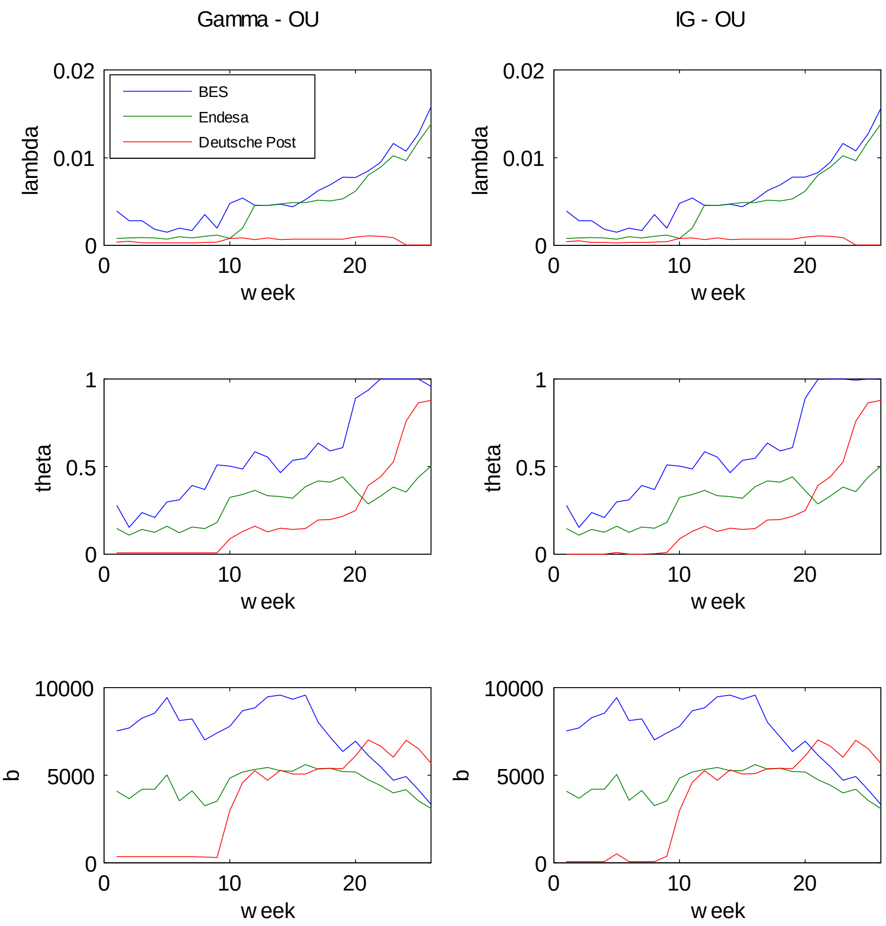

Figure 7: The calibrated λ_0, θ and $b$parameters in the two OU models for the 3 representative companies over the 26 weeks. The parameter a is fixed to 100 in both specifications.

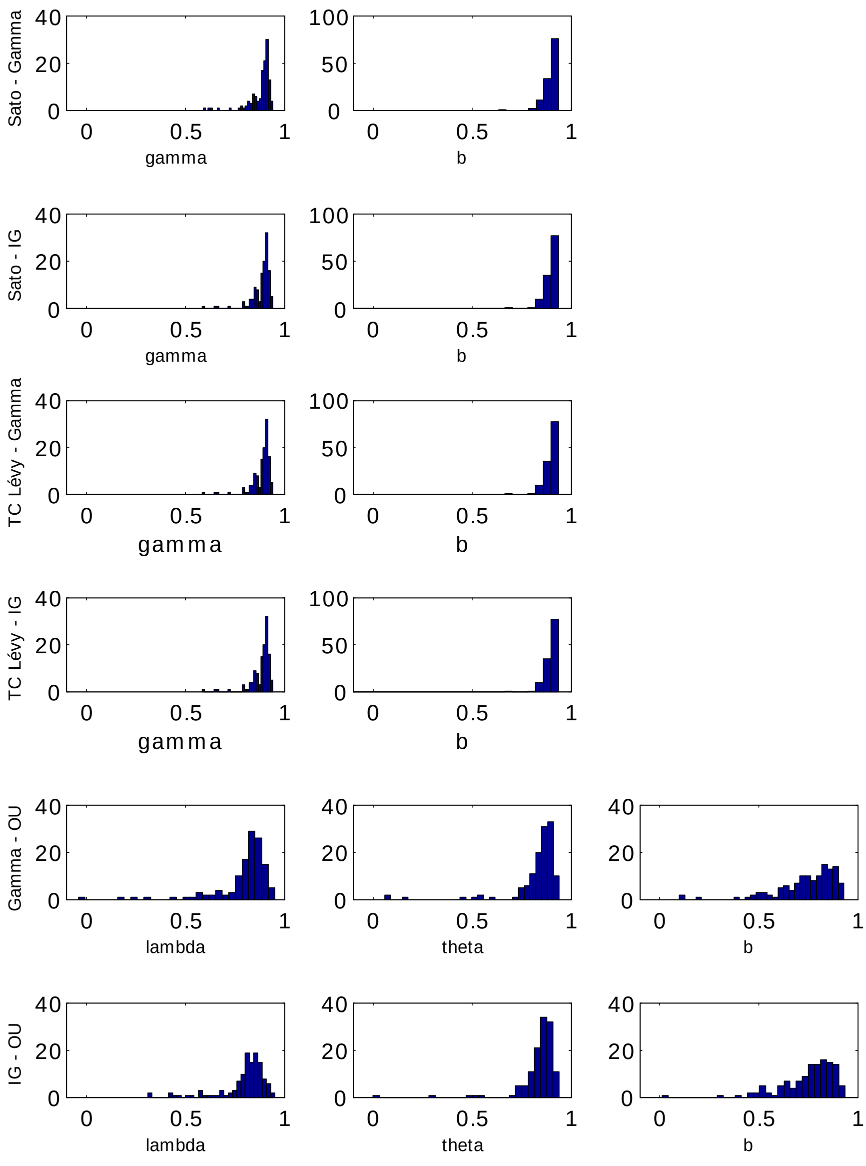

Figure 8: The 1-lag autocorrelation distributions for the parameters in the different models. Each row corresponds to a specific model.

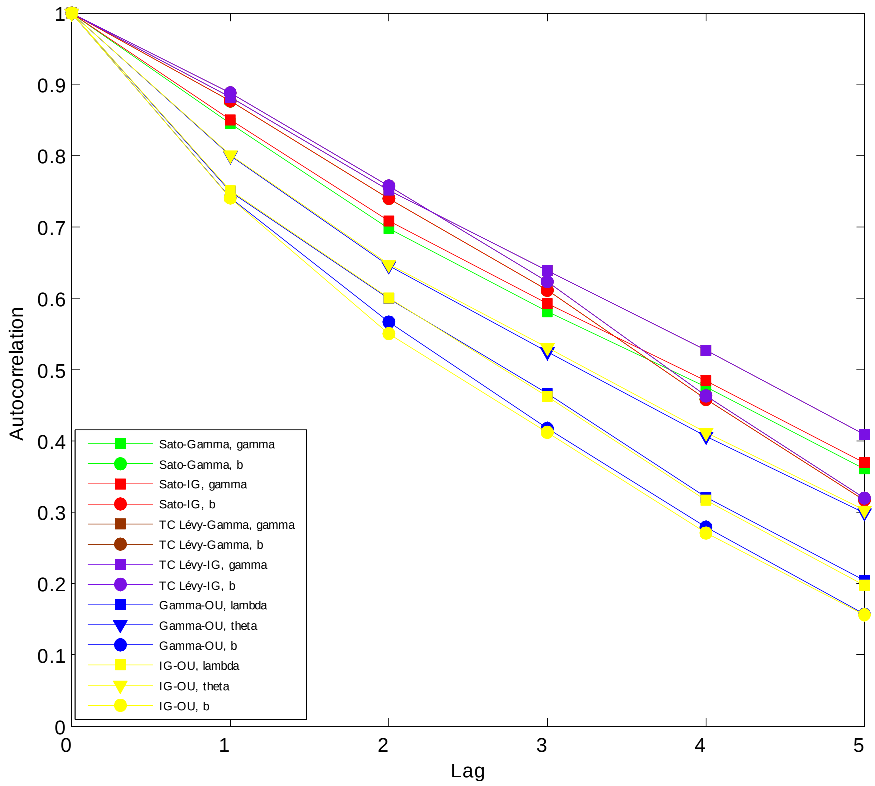

Figure 9: The average of the autocorrelation across firms for each parameter in each model as a function of the lag.

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