WORKING PAPER F-2008-07

WP

Esben Høg, Per Frederiksen & Daniel Schiemert

On the Generalized Brownian Motion and its Applications in Finance

Finance
Research Group

On the Generalized Brownian Motion and its Applications in Finance*

Esben P. Høg[†] Per Frederiksen[‡] Daniel Schiemert[§]
Aarhus School of Business *Nordea Markets* *Universität Stuttgart*

November 2008

Abstract

This paper deals with dynamic term structure models (DTSMs) and proposes a new way to handle the limitation of the classical affine models. In particular, the paper expands the flexibility of the DTSMs by applying generalized Brownian motions with dependent increments as the governing force of the state variables instead of standard Brownian motions. This is a new direction in pricing non defaultable bonds.

By extending the theory developed by Dippon & Schiemert (2006*a*), the paper developes a bond market with memory, and proves the absence of arbitrage. The framework is readily extendable to other markets or multi factors. As a complement the paper shows an example of how to derive the implied bond pricing parameters using the ordinary Kalman filter.

JEL Classification: C22, C23, C51, G12.

Keywords: Generalized Brownian motion, Bond market with memory, Fractional bond pricing equation, fractional Ornstein-Uhlenbeck process, long memory, Kalman filter.

*We acknowledge comments received from participants at the 13th International Conference in Computing in Economics and Finance (CEF), Montreal, June 2007, and critical comments from an anonymous referee.

[†]Corresponding author. Postal address: Finance Research Group, Department of Business Studies, Aarhus School of Business, University of Aarhus, Fuglesangs Allé 4, DK-8210 Aarhus V, Denmark; phone +45 89 48 63 54; email eh@asb.dk.

[‡]Equity and Fund Linked Derivatives, Nordea Markets , Denmark; email per.frederiksen@nordea.com.

[§]Universität Stuttgart; email Daniel.schiemert@gmx.de.

1 Introduction

A stochastic integral driven by a Gaussian process with dependent increments is developed in Dippon & Schiemert (2006*b*). This class can be seen as a class of generalized Brownian motions. The stochastic integral is using the Wick product and is defined in the Hida distribution sense similar to Hu & Øksendal (2003). The calculus includes a related chain rule, which contains Itô's rule in the case of the Brownian motion, and an existence and uniqueness theorem for certain types of stochastic differential equations driven by Gaussian processes with dependent increments.

An element of the class of generalized Brownian motions is the fractional Brownian motion (fBm). This Gaussian process has dependent increments which makes it interesting for applications in network simulations and finance, see Norros (1995), Hu & Øksendal (2003) and Bender (2003*a*). In order to use this process in finance it is helpful to have a stochastic calculus wrt. fractional Brownian motion. Especially a stochastic integral driven by a fractional Brownian motion is practical. Duncan, Hu & Pasik-Duncan (2000) suggested to define this integral by the use of the Malliavin calculus and the Wick product. In their paper they derive a fractional Itô rule, which turned out to be incorrect, see Bender (2003*a*). Alternatively, Hu & Øksendal (2003) defined a stochastic integral driven by a fractional Brownian motion in the Hida distribution sense and by the use of the Wick product. Their construction of a Black-Scholes model is to use the so-called Wick portfolio, where the product is replaced by the Wick product. As the Wick product is not declared path-wise, in contrast to the ordinary product, several problems of the economic interpretation of this portfolio arise, see Björk & Hult (2005). However, Dippon & Schiemert (2006*a*) defined a Black-Scholes market with memory using a Wick portfolio similar to Hu & Øksendal (2003), and showed that there is a sufficient answer to how to treat the critics of Björk & Hult (2005) about the interpretation of the Wick portfolio. Dippon & Schiemert (2006*a*) showed that for a Wick portfolio there exists an equivalent ordinary portfolio in terms of value and vice versa.

In the present paper we apply the stochastic calculus of Dippon & Schiemert (2006*a*) and develop a bond market with memory, where the memory arises from a Gaussian process with dependent increments where the fractional Brownian motion is a special case. We use Wick portfolios and show that they have a sensible economic interpretation and extend the setup to other possible applications in finance.

The development of a bond market with memory should be seen as an unexplored path in alleviating the critics of the standard dynamic term structure models (DTSMs). Recent research has been considering models with correlated state variables (Dai & Singleton (2000)), nonlinear dynamics or more flexible forms of the market price of risk (see, for example, Ahn & Gao (1999), Duarte (2004), Leippold & Wu (2000), and Ahn, Dittmer & Gallant (2002)), quadratic models (see Leippold & Wu (2000) and Ahn et al. (2002)), bond prices with jumps (Johannes (2004), Zhou (2001), and Das (2002)), and regime shifts (Ang & Bekaert (2002)) but have not explored the possibility that the underlying state variable could be governed by a generalized Brownian motion.

Therefore, this paper takes a different path. We expand the flexibility of the model by applying a generalized Brownian motion (gBm) as the governing force of the state variable instead of the usual Brownian motion, but still embed our model in the settings of the class of affine DTSMs. Besides this theoretical contribution, we propose an estimation methodology based on the ordinary Kalman filter approach.

The remainder of the paper is organized as follows. The next section presents the concept of Gaussian processes with dependent increments. Section 3 derives the model for the short rate and Section 4 derives the arbitrage free pricing. Section 5 shows that the setup is readily extendable to a multi factor setting, and Section 6 offers an estimation methodology based on the ordinary Kalman filter. Section 7 gives a brief overview of other possible applicabilities and Section 8 offers concluding remarks.

All proofs are in the appendices.

2 A class of Gaussian processes with dependent increments

Let $|\cdot|_0$ denote the norm and $\langle\cdot,\cdot\rangle$ the inner product of $L^2(\mathbb{R})$. If $\eta \in C^\infty(\mathbb{R})$ and for all nonnegative integers n, k

$$|u^k \eta^{(n)}(u)| \longrightarrow 0, \qquad |u| \to \infty, \tag{1}$$

then η is called *rapidly decreasing*. The set $\mathcal{S}(\mathbb{R})$ denotes the set of rapidly decreasing functions and is called a *Schwartz space*. By the inequality

$$|\eta(u)| \leq \frac{C}{|u|},$$

for sufficient large C and sufficient large $|u|$, which is a consequence of (1), it is established that $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$. Let $\mathcal{S}'(\mathbb{R})$ denote the dual space of $\mathcal{S}(\mathbb{R})$, which is called the *space of tempered distributions*. As a result of the Hilbert space theory the inner product of $L^2(\mathbb{R})$ can be extended to the bilinear form $\langle\omega, \eta\rangle$, where $\omega \in \mathcal{S}'(\mathbb{R})$ and $\eta \in \mathcal{S}(\mathbb{R})$.

The calculus presented here is defined on the probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$ with the Gaussian measure μ, see Appendix A. Suppose $m(\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ such that for all $t \in \mathbb{R}$ the function $m(\cdot, t) \in L^2(\mathbb{R})$. Define $v(s,t) = \int_{\mathbb{R}} m(u,t)m(u,s)\,du$ and the stochastic process B_t^v by

$$B_t^v := \int_{\mathbb{R}} m(u,t)\,dB_u, \quad t \in \mathbb{R},$$

with ordinary Brownian motion B_u. We choose $m(u,t)$ such that $\frac{d}{dt}m(\cdot,t) \in \mathcal{S}'(\mathbb{R})$ for all $t \in \mathbb{R}$ i.e. it is differentiable in the sense of tempered distributions. The process B_t^v is a Gaussian process. Its covariance function is given by the Itô isometry

$$v(s,t) = \int_{\mathcal{S}'(\mathbb{R})} B_s^v B_t^v \, d\mu = \int_{\mathbb{R}} m(u,s) m(u,t) \, du.$$

Let $m(u,0) \equiv 0$, hence $B_0^v = 0$, and B_t^v is a centered Gaussian process. The properties $v(t,t) \geq 0$ and $v(s,t) = v(t,s)$ are obvious.

Example 2.1 (Ordinary Brownian motion) *The stochastic process $B_t^v = B_t$ is the ordinary Brownian motion if $m(u,t) = 1_{[0,t]}(u)$. Its covariance function is $v(s,t) = \min(t,s)$. This example is further discussed in Kuo (1996, Chapter 3.1).*

Example 2.2 (A Gaussian process with short range dependency) *Let B_t^s, $t \in \mathbb{R}$, be a centered Gaussian process with covariance function $v(s,t)$, such that $v(s,\cdot)$ has a global maximum and $\lim_{t\to\infty} v(s,t) = 0$ for all fixed $s \in \mathbb{R}$. Then we call B_t^s a short range Brownian motion. As an example consider $m(u,t) = t^2 \exp(-(u-t)^2)$, hence $v(s,t) = \sqrt{\pi} t^2 s^2 \exp(-(t-s)^2/2)$.*

Example 2.3 (Fractional Brownian motion) *We use the following operator $M_\pm^d$ to introduce the fractional Brownian motion and later on its derivative. Define $M_\pm^d$ with memory parameter $d \in (-1/2, 1/2)$ for $\eta \in \mathcal{S}(\mathbb{R})$ as*

$$(M_+^d \eta)(t) := \begin{cases} \frac{K_d}{\Gamma(d)} \int_{-\infty}^t \eta(s)(t-s)^{d-1} \, ds & \text{for } d > 0 \\ \eta(t) & \text{for } d = 0 \\ \frac{K_d(d)}{\Gamma(-d)} \lim_{\epsilon\to 0} \int_\epsilon^\infty \frac{\eta(t)-\eta(t-s)}{(s)^{1-d}} \, ds & \text{for } d < 0 \end{cases}$$

and

$$(M_-^d \eta)(t) := \begin{cases} \frac{K_d}{\Gamma(d)} \int_{-\infty}^t \eta(s)(t-s)^{d-1} \, ds & \text{for } d > 0 \\ \eta(t) & \text{for } d = 0 \\ \frac{K_d(d)}{\Gamma(-d)} \lim_{\epsilon\to 0} \int_\epsilon^\infty \frac{\eta(t)-\eta(t+s)}{(s)^{1-d}} \, ds & \text{for } d < 0 \end{cases}$$

with

$$K_d = \Gamma(d+1) \left(\frac{2d + 1\Gamma(1-d)}{\Gamma(d+1)\Gamma(1-2d)} \right)^{1/2}.$$

The operator $M_\pm^d$ is essentially the Riemann-Liouville fractional integral for $d > 0$ and the Marchaud fractional derivative for $d < 0$. For further information, see Samko, Kilbas & Marichev (1993, Chapter 6). The fractional Brownian motion B_t^d is thus a modification of $\int_{\mathbb{R}} (M_-^d 1_{[0,t]}(s)) \, dB_s$. Another representation of the fractional Brownian motion up to modification is $B_t^d = \langle \cdot, M_-^d(1_{[0,t]}) \rangle$. The covariance function is therefore $v(s,t) = 1/2(|t|^{2d+1} + |s|^{2d+1} - |s-t|^{2d+1})$.

We suppose that there is a $\eta \in \mathcal{S}(\mathbb{R})$ such that for all $t \in [0,T]$, $T > 0$

$$\left\langle \frac{dm(u,t)}{dt}, \eta(u) \right\rangle = 1_{[0,T]}(t) \tag{2}$$

holds, where $1_{[0,T]}$ is the indicator function of the interval $[0,T]$. Note that in the case of the Brownian motion, it follows that equation (2) can be satisfied on $[0,T]$ with an η, which is 1 on $[0,T]$ and decays outside of this interval smoothly to 0. This is also satisfied for the fractional Brownian motion since the function $1_{[0,T]}(t)$ can be extended to the domain $t \in \mathbb{R}$ smoothly such that it has compact support, and it is therefore in the domain of the operator $M^d_{\pm}$. By using the inverse operator the existence of $\eta \in \mathcal{S}(\mathbb{R})$ is obtained, see Bender (2003*c*, corollary 4.6).

Now, since we allow for processes with memory we can in general not rely on the classical semi-martingale theory. Instead we use a different approach, which was recently put forth by Dippon & Schiemert (2006*b*). To simplify the exposition, we introduce the calculus through an example of the short interest rate.

3 The model for the short rate

As an example we generalize the Ornstein-Uhlenbeck process by replacing the ordinary Brownian motion B_t by the Gaussian process with dependent increments B^v_t. Thus,

$$dr(t) = \kappa(\theta - r(t))dt + \sigma\, dB^v_t. \tag{3}$$

To solve this stochastic differential equation we first note that if $dr_0(t) = -\kappa r_0(t)dt$, then $r_0(t) = \exp(-\kappa t)C$. Furthermore, assuming that C is a stochastic process, we can apply Dippon & Schiemert (2006*b*, proposition 9) which renders

$$dr(t) = -\kappa r(t)dt + \exp(-\kappa t)dC(t).$$

Now comparing this with equation (3) it follows that

$$\exp(-\kappa t)dC(t) = \kappa\theta dt + \sigma\, dB^v_t,$$

and thus

$$\begin{aligned} C(t) &= C(0) + \int_0^t \exp(\kappa s)\kappa\theta ds + \int_0^t \sigma \exp(\kappa s)\, dB^v_s \\ &= C(0) - \theta(1 - \exp(\kappa t)) + \int_0^t \sigma \exp(\kappa s)\, dB^v_s. \end{aligned}$$

Therefore

$$r(t) = \exp(-\kappa t)C(0) + \theta(1 - \exp(\kappa t)) + \int_0^t \sigma \exp(-\kappa(t-s))\, dB^v_s.$$

Note that if $C(0)$ is chosen to be a constant (which it usually is) and equal to the initial value $r(0)$, and that the shocks are driven by the fractional Brownian motion, the expectation and covariance function of $r(t)$ is $E(r(t)) = \exp(-\kappa t)C(0) + \theta(1 - \exp(\kappa t))$ and

$$Cov(r(t), r(s)) = E(\int_0^t \sigma \exp(-\kappa(t-u))\, dB_u^d \int_0^s \sigma \exp(-\kappa(s-u))\, dB_u^d)$$
$$= E(\sigma^2 \int_{\mathbb{R}} (M_-^d \left(1_{[0,t]}(w) \exp(-\kappa(t-w))\right)(u)) dB_u \times$$
$$\times \int_{\mathbb{R}} (M_-^d \left(1_{[0,s]}(w) \exp(-\kappa(s-w))\right)(u)) dB_u)$$

$$= \int_{\mathbb{R}} \sigma^2 (M_-^d \left(1_{[0,t]}(w) \exp(-\kappa(t-w))\right)(u))(M_-^d \left(1_{[0,s]}(w) \exp(-\kappa(s-w))\right)(u)) du,$$

respectively, see Bender (2003c) for details. By applying Bender (2003c, proposition 7.25), the variance of the process $r(t)$ is

$$\sigma^2 v(t,t) - 2\sigma^2 \int_0^t \kappa \exp(-\kappa(t-s)) v(t,s) ds$$
$$+ \sigma^2 \int_0^t \int_0^t \kappa^2 \exp(-2\kappa t + \kappa(s+u)) v(u,s) ds du,$$

where $v(t,s)$ is the covariance function of the Gaussian process B_t^v.

4 Arbitrage free pricing

Let $\|\cdot\|_0$ denote the norm of $(L^2) := L^2(\mathcal{S}'(\mathbb{R}), \mu)$. Let the wealth process of an account be described by

$$A(t) = A(0) \exp\left(\int_0^t r(s) ds\right) = A(0) \exp^\diamond \left(\int_0^t r(s) ds + \gamma(t)\right),$$

where

$$\gamma(t) := \frac{1}{2} \left\| \int_0^t \int_0^s \sigma \exp(-\kappa(s-u)) dB_u^v ds \right\|_0^2 = \frac{1}{2} \int_0^t \int_0^s \sigma^2 \exp(-2\kappa t + \kappa(u+s)) v(u,s)\, du\, ds.$$

Furthermore let $P^T(t, r(t))$ be the wealth process of a zero coupon bond with maturity T.

Definition 4.1 *Let $t \in [0,T]$ with $T > 0$. The wealth process $V^o(t)$ of an ordinary portfolio $(\varphi_A^o(t), \varphi_T^o(t))$, consisting of an account $A(t)$ and a zero coupon bond P^T with $\varphi_A^o(t), \varphi_T^o(t) \in (L^2)$, is defined by*

$$V^o(t) := \varphi_A^o(t) \cdot A(t) + \varphi_T^o(t) \cdot P^T(t, r(t))$$

such that $V^o(t) \in (L^2)$. The wealth process $V^w(t)$ of a Wick portfolio $(\varphi_A^w(t), \varphi_T^w(t))$, consisting of an account $A(t)$ and a zero coupon bond P^T with $\varphi_A^w(t), \varphi_T^w(t) \in (L^2)$, is defined by

$$V^w(t) = \varphi_A^w(t) \diamond A(t) + \varphi_T^w(t) \diamond P^T(t, r(t))$$

such that $V^w(t) \in (L^2)$.

Using Definition 4.1, it is possible to show the following theorem

Theorem 4.2 *For an ordinary portfolio* $(\varphi_A^o(t), \varphi_T^o(t))$*, there exists a Wick portfolio* $(\varphi_A^w(t), \varphi_T^w(t))$ *such that*

$$V^w(t) = V^o(t)$$

a.s. for all $t \in [0, T]$ *and vice versa.*

Proof. See Appendix B. ■

Theorem 4.2 states that Wick portfolios can be interpreted as ordinary portfolios and thus that they provide meaningful economic interpretations, which contrast the findings of Björk & Hult (2005).

Now if we assume, without loss of generality, that $P^T(t, r(t))$ is exponentially affine in the short rate $r(t)$

$$P^T(t, r(t)) = \exp(a^o(t) + b^o(t)r(t)),$$

where a^o and b^o are defined later, we can write the price using the Wick exponential by

$$\begin{aligned} P^T(t, r(t)) &= \exp^\diamond\left(a^o(t) + b^o(t)r(t) + \frac{(b^o(t))^2}{2}\left\|\int_0^t \sigma \exp(-\kappa(t-s))dB_s^v\right\|_0^2\right) \\ &= \exp^\diamond(a(t) + b(t)r(t)), \end{aligned} \tag{4}$$

where $a(t) := a^o(t) + \frac{(b^o(t))^2}{2}\left\|\int_0^t \sigma \exp(-\kappa(t-s))dB_s^v\right\|_0^2$ and $b(t) := b^o(t)$. By this definition it is possible to derive the Ricatti equations

$$\frac{da(t)}{dt} = (\lambda\sigma - \kappa\theta)b(t) + \frac{d\gamma(t)}{dt}$$

and

$$\frac{db(t)}{dt} = 1 + \kappa b(t),$$

subject to $a(T), b(T) = 0$, such that

$$b(t) = \frac{\exp(-k(T-t)) - 1}{k}, \tag{5}$$

and

$$\begin{aligned} a(t) &= (\lambda\sigma - \kappa\theta)\frac{(\exp(-k(T-t)) - 1)/k + T - t}{k} + \gamma(t) - \gamma(T) \\ &= \left(\frac{\lambda\sigma}{\kappa} - \theta\right)(b(t) + T - t) + \gamma(t) - \gamma(T), \end{aligned} \tag{6}$$

where λ is the market price of risk. Note that the market price of risk is in general defined by

$$\lambda(t) \diamond \sigma^T(t, r(t))) := \alpha^T(t, r(t)) - \left(r(t) + \frac{d}{2dt} \left\| \int_0^t \int_0^s \sigma \exp(-\kappa(s-u))\, dB_u^v\, ds \right\|_0^2 \right),$$

and derived from the chain rule for $P^T(t, r(t))$

$$\begin{aligned} dP^T &= \left(P_t^T + P_r^T \diamond (\kappa(\theta - r(t))) \right) dt + P_r^T \sigma\, dB_t^v \\ &= \alpha^T \diamond P^T dt + \sigma^T \diamond P^T dB_t^v, \end{aligned}$$

where

$$\alpha^T(t, r(t)) := \left(P_t^T + P_r^T \diamond (\kappa(\theta - r(t))) \right) \diamond (P^T)^{-\diamond}$$

and

$$\sigma^T(t, r(t)) := \sigma P_r^T \diamond (P^T)^{-\diamond}.$$

Here $^{-\diamond}$ denotes the Wick inverse, which exists as $P^T > 0$ a.s. Note that from the absence of arbitrage, it follows that the market price of risk λ is independent of the maturity of the zero coupon bond P^T, see Appendix B for the proof.

Now, recall that a portfolio (φ_A, φ_T) is self-financing if

$$V(t) - V(0) = \int_0^t \varphi_A(t) dA(t) + \int_0^t \varphi_T(t) dP^T$$

holds a.s. for all $t \in [0, T]$. Also recall that the self-financing portfolio is called an arbitrage if $V(0) = 0$, $V(T) \geq 0$ a.s., and $E(V(T)) > 0$. Applying this definition, and assuming that $P^T(t, r(t))$ follows (4), we can state the following theorem

Theorem 4.3 *Assume that equation (2) holds and that $P^T(t, r(t))$ is given by (4) with $a(t)$ and $b(t)$ given by (6) and (5), respectively, then there is no arbitrage in the class of self-financing portfolios.*

Proof. See Appendix B. ■

Applying Theorem 4.3, we can state the following bond price equation

Theorem 4.4 *The price of a zero coupon bond P^T with maturity T satisfies the equation*

$$\left(\lambda \sigma^T + r(t) + \psi(t) \right) \diamond P^T = P_t^T + P_r^T \diamond (\kappa(\theta - r(t)))$$

with $P^T(T, r(T)) = 1$. Thus

$$\begin{aligned} P^T(t, r(t)) &= \exp^\diamond(a(t) + b(t) r(t)) \\ &= \exp\left(a(t) + b(t) r(t) - \frac{1}{2} b^2(t) \left\| \int_0^t \sigma \exp(-\kappa(t-s)) dB_s^v \right\|_0^2 \right). \end{aligned}$$

Proof. See Appendix B. ■

5 Multi-factor Ornstein-Uhlenbeck model with memory

To show the flexibility of the setup derived in Section 4, this section extends the results to a multi-factor setting with correlated state variables.

Consider the short rate constructed by multiple factors

$$r_n(t) = \alpha_0 + \sum_{l=1}^{n} \alpha_l p_l(t),$$

where α_l, $l \in \{0, .., n\}$, are constant, and p_l satisfies

$$dp_l(t) = k_l(\theta_l - p_l(t))dt + \sigma_l \, dB_t^{v_l} \tag{7}$$

for all $l \in \{1, .., n\}$, $\sigma_l, \theta_l, k_l > 0$.

Here $B_t^{v_l}$ are Gaussian processes with dependent increments. Since the solution of equation (7) is given by

$$p_l(t) = c_l \exp(-k_l t) + \theta_l(1 - \exp(-k_l t)) + \int_0^t \sigma_l \exp(-k_l(t-s)dB_t^{v_l}$$

with constants c_l, the short rate can be written as

$$r_n(t) = \alpha_0 + \sum_{l=1}^{n} \alpha_l(c_l \exp(-k_l t) + \theta_l(1 - \exp(-k_l t))) + \sum_{l=1}^{n} \int_0^t \alpha_l \exp(-k_l(t-s))\sigma_l dB_s^{v_l}$$

with the differential form

$$dr_n(t) = \sum_{l=1}^{n} \alpha_l k_l(\theta_l - p_l(t))dt + \sigma_l \alpha_l dB_t^{v_l}.$$

To derive the multi-factor bond price $P^{T,n}$, let

$$\psi_n(t) := \frac{d}{2dt} \left\| \int_0^t \int_0^s \sum_{l=1}^{n} \alpha_l \sigma_l \exp(-k_l(s-u)) \, dB_u^{v_l} \, ds \right\|_0^2, \tag{8}$$

and assume that there is an $\eta \in \mathcal{S}(\mathbb{R})$, $\eta = \sum_{l=1}^{n} \eta_l$, such that for all $l \in \{1, .., n\}$ and for all $t \in [0, T]$ it holds that

$$\left\langle \frac{dm_l(u,t)}{dt}, \eta_l(u) \right\rangle = 1_{[0,T](t)}, \tag{9}$$

and

$$\left\langle \frac{dm_l(u,t)}{dt}, \eta_j(u) \right\rangle = 0, \text{ for } j \neq l. \tag{10}$$

Furthermore, assume that the bond price is exponentially affine

$$P^{T,n}(t, r_n(t)) = \exp^{\diamond}\left(a(t) + \sum_{l=1}^{n} b(t)p_l(t)\right), \tag{11}$$

where the deterministic functions a, b satisfy the differential equations on $[0, T]$

$$-\alpha_l + b_l'(t) - k_l b_l(t) = 0 \tag{12}$$

for all $l \in \{1, .., n\}$ and $b_l(T) = 0$, and

$$a'(t) - \psi_n(t) - \alpha_0 = \sum_{l=1}^{n}(\zeta_l - k_l\theta_l)b_l(t), \tag{13}$$

with $a(T) = 0$. ζ_l are unspecified constants.

Now, since it can be shown that all definitions and theories of section 4 holds with the above assumptions, interchanging $r(t)$ and $r_n(t)$, we can state the following

Theorem 5.1 *The price of the zero coupon bond $P^{T,n}$ with maturity T satisfies in $[0, T]$ the equation*

$$P^{T,n}(t, r_n(t)) = \exp^{\diamond}\left(a(t) + \sum_{l=1}^{n} b_l(t)p_l(t)\right),$$

where $a(t), b_l(t)$ satisfy the equations (13) and (12), such that

$$b_l(t) = \alpha_l \frac{\exp(-k_l(T-t)) - 1}{k_l}$$

and

$$a(t) = a(0) + \int_0^t \psi_n(t) + \alpha_0 - \sum_{l=1}^{n}(\zeta_l - k_l\theta_l)b_l(t)dt.$$

Proof. See Appendix B ■

Note that nowhere in the derivation of Theorem (5.1) have we assumed that $B_t^{v_i}$ and $B_t^{v_j}$ are uncorrelated. Thus, it is possible to derive a bond price governed by intercorrelated factors with (long-range) dependent increments.

6 An Econometric Methodology

To give an example of how to materialize the proposed setup, we apply a state space formulation.

If the short rate $r(t)$ for example follows an Ornstein-Uhlenbeck process, it is known that such a process can be written as the following transition

$$r(t) = \mathrm{e}^{-\kappa(t-s)}r(s) + \theta(1 - \mathrm{e}^{-\kappa(t-s)}) + \sigma \int_s^t \mathrm{e}^{-\kappa(t-u)}\mathrm{d}B_u,$$

where B_u is a Brownian motion.

The exact state space formulation is based on the transition density, i.e. the conditional density of $r(t)$ given $r(s)$. It is known from Vasicek (1977) that the exact transition density is the product of normal densities, i.e. the equidistant state process $\{r(t + \Delta\, k)\}_{k\in Z}$ is a Markov process for any $\Delta > 0$.

In case $r(t)$ e.g. is a fractional Ornstein-Uhlenbeck process, then the dth difference of $r(t)$, $r^{(d)}(t)$, may be written as

$$r^{(d)}(t) = \mathrm{e}^{-\kappa\Delta}\, r^{(d)}(s) + \theta(1 - \mathrm{e}^{-\kappa\Delta}) + \sigma \int_s^t \mathrm{e}^{-\kappa\Delta} \mathrm{d}B_u, \tag{14}$$

where $t-s = \Delta$. In other words, $r(t)$ sampled at equidistant time points is an ARFIMA$(1, d, 0)$ process. In this case $\{r(t + \Delta\, k)\}_{k\in Z}$ is not Markov, and to apply the Kalman filter, we have to define the state variable (or vector) differently than in the ordinary case.

If we let $\mathbf{y}(t) = (y(1, t), \ldots, y(n, t))'$ be an n-dimensional vector of yields observed at time t, then the approach can be described as follows.

From the exponential affine relation, $P^T(t, r(t)) = \exp(a^o(t) + b^o(t)r(t))$, the yields can be expressed as

$$\mathbf{y}(t) = -\,\frac{1}{\tau} \log P^T(t, r(t)) = \mathbf{d}(t) + \tilde{\mathbf{z}}(t)\, r(t), \tag{15}$$

where $\mathbf{d}(t)$ and $\tilde{\mathbf{z}}(t)$ are the n-dimensional vectors

$$\mathbf{d}(t) = \begin{pmatrix} -a^o(t_1)/\tau_1 \\ \vdots \\ -a^o(t_n)/\tau_n \end{pmatrix}, \quad \tilde{\mathbf{z}}(t) = \begin{pmatrix} -b^o(t_1)/\tau_1 \\ \vdots \\ -b^o(t_n)/\tau_n \end{pmatrix},$$

respectively, where $\tau = T-t$. Adding an error term to (15), we obtain a measurement equation for the observed yields, where $r(t)$ is an unobserved variable.

6.1 An approximation to the Kalman Filter for the ARFIMA$(1, d, 0)$

Now let $r(t)$ be an ARFIMA model according to (14). Then

$$r(t) = \mu + \varphi\, r(t-1) + \zeta^{(d)}(t), \tag{16}$$

where $\mu = \theta(1 - \mathrm{e}^{-\kappa\Delta})$, $\varphi = \mathrm{e}^{-\kappa\Delta}$, and $\zeta^{(d)}(t)$ is integrated fractional white noise[1] with parameters d and σ^2. Then $\tilde{x}(t) = (1-L)^d r(t)$ is the AR(1) process with mean zero[2]

$$\tilde{x}(t) = \varphi\, \tilde{x}(t-1) + \tilde{\zeta}(t),$$

[1]Fractional white noise in the statistical sense is defined as a stationary ARFIMA$(0, d, 0)$ process. Note that stationarity requires $-\frac{1}{2} < d < \frac{1}{2}$.

[2]Note that if $0 < d \le 1$, then the mean $(1-L)^d\, \mu = 0$ because $\sum_{k=0}^{\infty} c_k^{(d)} = 0$.

where $\tilde{\zeta}(t)$ is a white noise process with variance $\sigma^2(1-\mathrm{e}^{-2\kappa\Delta})/2\kappa$.

To obtain a feasible version, we truncate the infinite polynomial

$$(1-L)^d = \sum_{k=0}^{\infty} \left[\frac{\Gamma(k-d)}{\Gamma(k+1)\Gamma(-d)}\right] L^k$$

at the $(m-1)$th power, thereby defining the following finite filtered variable

$$x(t) = \sum_{k=0}^{m-1} c_k^{(d)}\, r(t-k), \tag{17}$$

where

$$c_k^{(d)} = \frac{\Gamma(k-d)}{\Gamma(k+1)\Gamma(-d)}, \qquad k = 0, 1, 2, \cdots, m-1.$$

Then, according to (16)

$$x(t) = \mu \sum_{k=0}^{m-1} c_k^{(d)} + \varphi x(t-1) + \zeta(t). \tag{18}$$

Here $\zeta(t) = \sum_{k=0}^{m-1} c_k^{(d)} \zeta^{(d)}(t-k)$, which converges to a white noise process for $m \to \infty$.

Consider the m-dimensional vector $\mathbf{x}(t)$ defined from equation (18) by

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ x(t-1) \\ x(t-2) \\ \vdots \\ x(t-m+1) \end{bmatrix}.$$

Then the following transition holds

$$\mathbf{x}(t) \;\; = \;\; \boldsymbol{\mu} + \mathbf{T}\,\mathbf{x}(t-1) + \boldsymbol{\zeta}(t),$$

where $\boldsymbol{\mu}$, $\boldsymbol{\zeta}(t)$ and $\mathbf{T}$ are the two $(m \times 1)$ vectors and partitioned $(m \times m)$ matrix given as

$$\boldsymbol{\mu} = \begin{bmatrix} \mu \sum_{k=0}^{m-1} c_k^{(d)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \boldsymbol{\zeta}(t) = \begin{bmatrix} \zeta(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ and } \mathbf{T} = \begin{bmatrix} \varphi & \mathbf{0}'_{m-1} \\ \mathbf{I}_{m-1} & \mathbf{0}_{m-1}. \end{bmatrix}.$$

To be able to express the vector of observed yields as a function of the state vector $\mathbf{x}(t)$, we derive from (17)

$$\mathbf{x}(t) = \mathbf{A}_1\,\mathbf{r}(t) + \mathbf{A}_2\,\mathbf{r}(t-m),$$

where

$$\mathbf{r}(t) = \begin{bmatrix} r(t) \\ r(t-1) \\ \vdots \\ r(t-m+1) \end{bmatrix}, \qquad \text{and}$$

$$\mathbf{A}_1 = \begin{bmatrix} 1 & c_1^{(d)} & c_2^{(d)} & \cdots & c_{m-1}^{(d)} \\ 0 & 1 & c_1^{(d)} & \cdots & c_{m-2}^{(d)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ c_{m-1}^{(d)} & 0 & 0 & \cdots & 0 \\ c_{m-2}^{(d)} & c_{m-1}^{(d)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1^{(d)} & c_2^{(d)} & \cdots & c_{m-1}^{(d)} & 0 \\ \end{bmatrix}.$$

Since $\mathbf{A}_1$ is non singular with inverse

$$\mathbf{A}_1^{-1} = \begin{bmatrix} 1 & c_1^{(-d)} & c_2^{(-d)} & \cdots & c_{m-1}^{(-d)} \\ 0 & 1 & c_1^{(-d)} & \cdots & c_{m-2}^{(-d)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \end{bmatrix},$$

where

$$c_k^{(-d)} = \frac{\Gamma(k+d)}{\Gamma(k+1)\Gamma(d)}, \qquad k = 0, 1, 2, \cdots, m-1,$$

we obtain that

$$\mathbf{r}(t) = \mathbf{A}_1^{-1}\mathbf{x}(t) - \mathbf{A}_1^{-1}\mathbf{A}_2\mathbf{r}(t-m),$$

and the first row in this equation is

$$\begin{aligned} r(t) &= \sum_{k=0}^{m-1} c_k^{(-d)}\,x(t-k) - \sum_{k=1}^{m-1}\left(\sum_{j=k}^{m-1} c_j^{(-d)} c_{m+k-j-1}^{(d)}\right) r(t-m-k) \\ &\approx \sum_{k=0}^{m-1} c_k^{(-d)}\,x(t-k). \end{aligned} \tag{19}$$

The second term in (19) is an artifact of the truncation of $\tilde{x}(t)$ induced by the finite filter defining $x(t)$ in (17), and we omit it since if $m \to \infty$ this term vanishes.

Returning to the relation in (15), we obtain the following

$$\begin{aligned}\mathbf{y}(t) &= \mathbf{d}(t) + \tilde{\mathbf{z}}(t)\, r(t) \\ &\approx \mathbf{d}(t) + \tilde{\mathbf{z}}(t) \left(\sum_{k=0}^{m-1} c_k^{(-d)}\, x(t-k) \right) \\ &= \mathbf{Z}(t)\, \mathbf{x}(t) + \mathbf{d}(t), \end{aligned} \tag{20}$$

where[3]

$$\mathbf{Z}(t) = \tilde{\mathbf{z}}(t) \begin{bmatrix} 1 & c_1^{(-d)} & \cdots & c_{m-1}^{(-d)} \end{bmatrix}.$$

Having defined the state space equations above, one can use the ordinary Kalman filter techniques available in order to derive the implied parameters.

7 Other Applications of the Generalized Brownian Motion in Finance

Dippon & Schiemert (2006*a*) presented the Black-Scholes market with memory. They applied the Gaussian process with dependent increments B_t^v to drive the stock price $S(t)$ with $\mu \in \mathbb{R}$ and $\sigma > 0$ in

$$dS(t) = \mu S(t)dt + \sigma S(t) dB_t^v.$$

Using the unique solution

$$S(t) = S(0) \exp\left(\mu t + \sigma B_t^v - \frac{\sigma^2 v(t,t)}{2} \right),$$

Dippon & Schiemert (2006*a*) priced path-independent options. Here the wealth process, $v(t, S(t)) = V(t, \theta)$ of the self-financing portfolio θ replicating the contingent claim X, satisfied the following stochastic partial differential equation

$$r \frac{\partial v(t, S(t))}{\partial s} \diamond S(t) - r v(t, S(t)) + \frac{\partial v(t, S(t))}{\partial t} = 0$$

with boundary condition $v(T, S(T)) = X$. Using Wick calculus the equation could be written as

$$\frac{\partial v^o(t,s)}{\partial t} + rs \frac{\partial v^o(t,s)}{\partial s} + \frac{1}{2}\sigma^2 \frac{d|m(\cdot,t)|_0^2}{dt} s^2 \frac{\partial^2 v^o(t,s)}{\partial s^2} = r v^o(t,s),$$

where $v^o(T, s) = X$ and $v^o(t, S(t))$ is the representation of the wealth process with the ordinary product.

[3]Note that $\tilde{\mathbf{z}}(t)$ is $(n \times 1)$, whereas $\mathbf{Z}(t)$ is $(n \times m)$.

If $|m(\cdot,t)|_0^2$ is continuously differentiable, strictly increasing in t with $t > 0$ and satisfying

$$\lim_{t\to T} \frac{T-t}{\sqrt{|m(\cdot,T)|_0^2 - |m(\cdot,t)|_0^2}} < \infty,$$

then Dippon & Schiemert (2006a) showed that the price of the European call option in the Black-Scholes market with memory is

$$v^o(t,s) = sN(d_1(t,s)) - Ke^{-r(T-t)}N(d_2(t,s)),$$

where

$$d_1(t,s) := \frac{\ln(s/K) + r(T-t) + \frac{1}{2}\sigma^2\left(|m(\cdot,T)|_0^2 - |m(\cdot,t)|_0^2\right)}{\sigma\sqrt{|m(\cdot,T)|_0^2 - |m(\cdot,t)|_0^2}},$$

and

$$d_2(t,s) := \frac{\ln(s/K) + r(T-t) - \frac{1}{2}\sigma^2\left(|m(\cdot,T)|_0^2 - |m(\cdot,t)|_0^2\right)}{\sigma\sqrt{|m(\cdot,T)|_0^2 - |m(\cdot,t)|_0^2}}.$$

In the same manner, it is possible to consider weather derivatives.

Suppose the temperature $\theta(t)$ is given by

$$d\theta(t) = \nu(\kappa(t) - \theta(t))\,dt + \sigma\,dB_t^v \tag{21}$$

with two constants $\nu \in \mathbb{R}$, $\sigma > 0$ and a deterministic function $\kappa(t)$. Let $P^T(t,\theta(t))$ be the wealth process of a derivative on the temperature $\theta(t)$. Then it can be shown that Wick portfolios and ordinary portfolios are equal in a similar way as in section 4. By assuming that equation (2) holds, the absence of arbitrage follows. By letting the corresponding market price of risk being constant, one obtains the stochastic partial differential equation

$$\frac{\partial P^T}{\partial t} + \frac{\partial P^T}{\partial \theta} \diamond (\nu(\kappa(t) - \theta(t)) - \lambda\sigma) = rP^T$$

with the boundary condition $P^T(T,\theta(T)) = 1$. Benth (2003) developed an approach to pricing weather derivatives, where the temperature is governed by (21). However, compared to our setup, he applied the fractional Itô formula of Duncan et al. (2000), which is not correct, see Bender (2003a).

8 Conclusion

As a new way of handling the imitations of the standard affine dynamic term structure models, this paper has proven the existence of a bond market where the short rate is governed by a generalized Brownian motion with dependent increments.

We have shown that there is still absence of arbitrage and that the setup is readily extendable to other markets or multi factors. As a complement we showed an example of how to derive the implied parameters using the ordinary Kalman filter.

A bond market where the short rate is governed by e.g. a fractional Brownian motion has the advantage that the model implies that the realized bond return or yield variation is long range dependent, which is a well known feature of interest rate series. This is not investigated here, but left for future research. Furthermore, our setup captures the properties that interest rates might be integrated of order d, see Shea (1991), Backus & Zin (1993), Crato & Rothman (1994), Høg (1997), Tkacz (2001), Iglesias & Phillips (2005), Nielsen (2006).

References

Ahn, D., Dittmer, R. F. & Gallant, A. R. (2002), 'Quadratic term structure models: Theory and evidence', *Review of Financial Studies* **15**, 243–288.

Ahn, D. & Gao, B. (1999), 'A parametric nonlinear model of term structure dynamics', *Review of Financial Studies* **12**, 721–762.

Ang, A. & Bekaert, G. (2002), 'Regime switches in interest rates', *Journal of Business and Economic Statistics* **20**(2), 163–182.

Backus, D. K. & Zin, S. E. (1993), 'Long-memory inflation uncertainty: Evidence from the term structure of interest rates', *Journal of Money, Credit and Banking* **25**, 687–700.

Bender, C. (2003*a*), 'Integration with respect to a fractional brownian motion and related market models', *Dissertation, August 2003, University of Konstanz* .

Bender, C. (2003*b*), 'An ito formula for generalized functionals of a fractional brownian motion with arbitrary hurst parameter', *Stochastic Processes and their Applications* **104**, 81–106.

Bender, C. (2003*c*), 'An s-transform approach to integration with respect to a fractional brownian motion', *Bernoulli* **9**(6), 955–983.

Benth, F. E. (2003), 'On arbitrage-free pricing of weather derivatives based on fractional brownian motion', *Applied Mathematical Finance* **10**(4), 303–324.

Björk, T. & Hult, H. (2005), 'A note on wick products and the fractional black-scholes model', *Finance and Stochastics* **9**, 197–209.

Crato, N. & Rothman, P. (1994), 'Fractional integration analysis of long-run behavior for US macroeconomic time series', *Economics Letters* **45**, 287–291.

Dai, Q. & Singleton, K. J. (2000), 'Specification analysis of affine term structure models', *Journal of Finance* **55**, 1943–1978.

Das, S. (2002), 'The surprise element: Jumps in interest rates', *Journal of Finance* **106**, 27–65.

Dippon, J. & Schiemert, D. (2006*a*), 'Option pricing in a black-scholes market with memory', *University of Stuttgart, Institute for Stochastic and Applications Preprint 2006* .

Dippon, J. & Schiemert, D. (2006*b*), 'Stochastic differential equations driven by gaussian processes with dependent increments', *University of Stuttgart, Institute for Stochastic and Applications Preprint 2006* .

Duarte, J. (2004), 'Evaluating an alternative risk preference in affine term structure models', *The Review of Financial Studies* **17**(2), 370–404.

Duncan, T., Hu, Y. & Pasik-Duncan, B. (2000), 'Stochastic calculus for fractional brownian motion i. theory', *SIAM J. Control Optim.* **38**(2), 582–612.

Høg, E. (1997), 'Analyzing continuous-time long-memory models with wavelets', *Computing Science and Statistics* **29**(2), 600–606.

Hu, Y. & Øksendal, B. (2003), 'Fractional white noise calculus and applications to finance', *IDAQP* **6**, 1–32.

Iglesias, E. M. & Phillips, G. D. A. (2005), 'Analysing one-month Euro-market interest rates by fractionally integtrated models', *Applied Financial Economics* **15**, 95–106.

Johannes, M. (2004), 'The statistical and economic role of jumps in continuous-time interest rate models', *Journal of Finance* **59**(1), 227–260.

Kuo, H.-H. (1996), 'White noise distribution theory', *CRC Press, 1996, Boca Raton* .

Leippold, M. & Wu, L. (2000), 'Quadratic term structure models', *Working Paper* (University of St. Gallen).

Nielsen, M. Ø. (2006), 'Nonparametric cointegration analysis of fractional systems with unknown integration orders', *Working paper, Cornell University* .

Norros, I. (1995), 'On the use of the fractional brownian motion in the theory of connectionless networks', *IEEE J. Sel. Areas Commun.* **13**, 953–962.

Nualart, D. (1995), 'The malliavin calculus and related topics', *Springer Verlag, Heidelberg* .

Samko, S. G., Kilbas, A. A. & Marichev, O. I. (1993), 'Fractional integrals and derivatives: Theory and applications', *Gordon and Breach* .

Shea, G. S. (1991), 'Uncertainty and implied variance bounds in long memory models of the interst rate term structure', *Empirical Economics* **16**, 287–312.

Tkacz, G. (2001), 'Estimating the fractional order of interest rates using a wavelet estimator', *Studies in Nonlinear Dynamics and Econometrics* **5**(1). Article 2.

Vasicek, O. (1977), 'An equilibrium characterization of the term structure', *Journal of Financial Economics* **5**, 177–188.

Zhou, H. (2001), 'Jump-diffusion term structure and ito conditional moment generator', *Finance and Economics Discussion Series, Board of Governors of the Federal Reserve System* **2001-28**.

A Appendix

A.1 The construction of the Schwartz space and its dual

We sketch the construction of the Schwartz space $\mathcal{S}(\mathbb{R})$ with the locally convex topology and its dual $\mathcal{S}'(\mathbb{R})$ with the weak topology. Let $\langle \omega, \eta \rangle$ denote the bilinear pairing with $\omega \in \mathcal{S}'(\mathbb{R})$ and $\eta \in \mathcal{S}(\mathbb{R})$. It follows that $\langle \cdot, \cdot \rangle$ is the inner product of $L^2(\mathbb{R})$ if $\omega, \eta \in L^2(\mathbb{R})$. The following construction of the Schwartz space and its dual is presented, for instance, in Kuo (1996, Chapter 3.2). Let $A := -\frac{d^2}{dx^2} + x^2 + 1$, so A is densely defined on $L^2(\mathbb{R})$. With the Hermite polynomial of degree n

$$H_n(x) := (-1)^n e^{x^2} \left(\frac{d}{dx} \right)^n e^{-x^2}$$

we define

$$e_n(x) := \frac{1}{\sqrt{\sqrt{\pi} 2^n n!}} H_n(x) e^{\frac{-x^2}{2}}.$$

The functions $e_n(x)$ are eigenfunctions of A and the corresponding eigenvalue is $2n+2$, $n \in \mathbb{N}_0$. The operator A^{-1} is bounded on $L^2(\mathbb{R})$, especially A^{-p} is a Hilbert-Schmidt operator for any $p > \frac{1}{2}$. Let for each $p \geq 0$, $|f|_p := |A^p f|_0$. The norm is given by the eigenvalues as

$$|f|_p = \left(\sum_{n=0}^{\infty} (2n+2)^{2p} \langle f, e_n \rangle^2 \right)^{1/2}.$$

We define

$$\mathcal{S}_p(\mathbb{R}) := \{ f ; f \in L^2(\mathbb{R}), |f|_p < \infty \}$$

and with these spaces the Schwartz space $\mathcal{S}(\mathbb{R})$ can be represented by $\mathcal{S}(\mathbb{R}) = \cap_{p \geq 0} \mathcal{S}_p(\mathbb{R})$. This construction leads to the Gelfand triple $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$. Furthermore, we get the following continuous inclusion maps

$$\mathcal{S}(\mathbb{R}) \subset \mathcal{S}_p(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}_p'(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}).$$

Let $\mathcal{B}$ denote the Borel σ-algebra on $\mathcal{S}'(\mathbb{R})$, i.e., the σ-algebra generated by the weak topology. One can show by the use of the Bochner-Minlos theorem that there is a unique Gaussian measure μ on $(\mathcal{S}'(\mathbb{R}), \mathcal{B})$. The space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$ is called *white noise*, and the space (L^2) denotes $L^2(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$. The bilinear form $\langle \omega, f \rangle$ with $f \in L^2(\mathbb{R})$ and $\omega \in \mathcal{S}'(\mathbb{R})$ is declared by

$$\lim_{k \to \infty} \langle \omega, \eta_k \rangle = \langle \omega, f \rangle$$

with $\eta_k \to f$ and $\{\eta_k\} \subset \mathcal{S}(\mathbb{R})$. It is possible to show that $\langle \cdot, f \rangle = \int_{\mathbb{R}} f(s)\, dB_s$ is a random variable in (L^2) for all $f \in L^2(\mathbb{R})$. The random variable $\langle \cdot, f \rangle$ has expectation value zero and variance $|f|_0^2$.

A.2 The construction of the Hida test and distribution space

In this section the construction of the Hida test and Hida distribution spaces $(\mathcal{S})$ and $(\mathcal{S})^*$ are outlined. These spaces are used to derive the Gaussian process B_t^v and to define a stochastic integral driven by these Gaussian processes. A more detailed description can be found e.g. in Kuo (1996, Chapter 3.3). Together with (L^2) they form also a Gelfand triple $(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$. Let $\langle\langle \Phi, \zeta \rangle\rangle$ denote the bilinear form of $\Phi \in (\mathcal{S})^*$ and $\zeta \in (\mathcal{S})$. For $f \in L^2(\mathbb{R}^n)$ the multiple Wiener integral with respect to the ordinary Brownian motion is defined by

$$I_n(f) := n! \int_{\mathbb{R}^n} f(t_1, t_2, t_3, \ldots, t_n) \, dB_{t_1} dB_{t_2} \ldots dB_{t_n}.$$

The following proposition is the chaos decomposition of (L^2), see Nualart (1995, Chapter 1).

Proposition A.1 *For all $F \in (L^2)$ there is a unique sequence $(f_n)_{n\in\mathbb{N}_0}$ such that $f_n \in L^2(\mathbb{R}^n)$ is symmetric and*

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

with convergence in (L^2).

For A as above the operator $\Gamma(A)$ on (L^2) is defined by

$$\Gamma(A)F = \sum_{n=0}^{\infty} I_n(A^{\otimes n} f_n)$$

with $F \in (L^2)$. Let $(\mathcal{S})_n = \Gamma(A)^n((L^2))$, we set $(\mathcal{S}) := \cap_{n\in\mathbb{N}}(\mathcal{S})_n$. With $(\mathcal{S})'_n := (\mathcal{S})_{-n}$, we get $(\mathcal{S})^* := \cup_{n\in\mathbb{N}}(\mathcal{S})_{-n}$. The topologies of $(\mathcal{S})$ and of $(\mathcal{S})^*$ are given by the projective limit topology and weak topology, respectively, see Kuo (1996, Chapter 2.2). As before there are continuous inclusion maps

$$(\mathcal{S}) \subset (\mathcal{S})_n \subset (L^2) \subset (\mathcal{S})_{-n} \subset (\mathcal{S})^*.$$

A.3 S-transform and Wick product

Now we introduce the S-transform from $(\mathcal{S})^*$ into the set of the functions from $\mathcal{S}(\mathbb{R})$ to $\mathbb{R}$.

For $\eta \in \mathcal{S}(\mathbb{R})$ set $I(\eta) := I_1(\eta) = \int_{\mathbb{R}} \eta(s) \, dB_s$.

Proposition A.2 *For all $\eta \in \mathcal{S}(\mathbb{R})$ the random variable* $\exp(I(\eta) - 1/2|\eta|_0^2)$ *is a Hida test function and the set*

$$\left\{ e^{I(\eta)-1/2|\eta|_0^2} : \eta \in \mathcal{S}(\mathbb{R}) \right\}$$

is total in $(\mathcal{S})$.

Proof. See Kuo (1996, Proposition 5.10). ■

Definition A.3 *The S-transform of a Hida distribution $\Phi \in (\mathcal{S})^*$ is defined by*

$$(S\Phi)(\eta) := \left\langle\!\!\left\langle \Phi, e^{I(\eta)-1/2|\eta|_0^2} \right\rangle\!\!\right\rangle, \quad \eta \in \mathcal{S}(\mathbb{R}).$$

Proposition A.4 *The S-transform of $\langle \cdot, f \rangle$ with $f \in \mathcal{S}'(\mathbb{R})$ is given by*

$$S(\langle \cdot, f \rangle)(\eta) = \langle f, \eta \rangle$$

Proof. See Kuo (1996, Proposition 5.9). ■

This proposition will be used to define the stochastic integral driven by Gaussian processes with dependent increments. For further details of the S-transform, see Kuo (1996, Chapter 5), and for further explanation about the notation and the properties of the term $\langle \cdot, f \rangle$ with $f \in \mathcal{S}'(\mathbb{R})$, see Kuo (1996, Chapter 3.4).

Example A.5 *The S-transform of the ordinary Brownian motion B_t is*

$$\begin{aligned} S(B_t)(\eta) &= \left\langle\!\!\left\langle B_t, e^{I(\eta)-1/2|\eta|_0^2} \right\rangle\!\!\right\rangle \\ &= \int_{\mathcal{S}'(\mathbb{R})} B_t e^{I(\eta)-1/2|\eta|_0^2} \, d\mu \\ &= \int_{\mathbb{R}} 1_{[0,t]}(u)\eta(u) \, du = \int_0^t \eta(u) \, du, \end{aligned}$$

by the use of proposition A.4.

Example A.6 *We calculate the S-transform of the fractional Brownian motion B_t^H.*

$$\begin{aligned} S(B_t^H)(\eta) &= \left\langle\!\!\left\langle B_t^H, e^{I(\eta)-1/2|\eta|_0^2} \right\rangle\!\!\right\rangle \\ &= \int_{\mathcal{S}'(\mathbb{R})} B_t^H e^{I(\eta)-1/2|\eta|_0^2} \, d\mu \\ &= \int_{\mathcal{S}'(\mathbb{R})} \int_{\mathbb{R}} (M_-^H 1_{[0,t]})(s) \, dB_s \, e^{I(\eta)-1/2|\eta|_0^2} \, d\mu \\ &= \int_0^t (M_+^H \eta)(s) \, ds \end{aligned}$$

*where several steps like fractional integration by parts and the fact that the integrals are well-defined are used, see Bender (2003*b*).*

Example A.7 *For the short range Brownian motion B_t^s we get*

$$\begin{aligned} S(B_t^s)(\eta) &= \left\langle\!\left\langle B_t^s, e^{I(\eta)-1/2|\eta|_0^2} \right\rangle\!\right\rangle \\ &= \int_{\mathcal{S}'(\mathbb{R})} B_t^s e^{I(\eta)-1/2|\eta|_0^2}\, d\mu \\ &= \int_{\mathbb{R}} \eta(u) t^2 \exp(-(u-t)^2)\, du, \end{aligned}$$

by the use of proposition A.4.

Now one can define the Wick product as follows (see Kuo (1996, , Chapter 8.4, page 92))

Definition A.8 *The Wick product of two Hida distributions Φ and Ψ in $(\mathcal{S})^*$, denoted by $\Phi \diamond \Psi$, is the unique Hida distribution in $(\mathcal{S})^*$ such that $S(\Phi \diamond \Psi)(\eta) = S(\Phi)(\eta)S(\Psi)(\eta)$ for all $\eta \in \mathcal{S}(\mathbb{R})$.*

The stochastic integral with dependent increments will be defined in terms of the white noise integral by use of the Wick product.

A.4 The white noise $\mathbf{W_t^v}$

In this subsection we differentiate B_t^v.

Definition A.9 *Let I be an interval in $\mathbb{R}$. A mapping $X : I \to (\mathcal{S})^*$ is called a stochastic distribution process. A stochastic distribution process is called differentiable in the $(\mathcal{S})^*$-sense, if*

$$\lim_{h\to 0} \frac{X_{t+h} - X_t}{h}$$

exists in $(\mathcal{S})^*$.

Now we are prepared to compute dB_t^v/dt in the $(\mathcal{S})^*$-sense and to define the stochastic integral $\int_{\mathbb{R}} X_t\, dB_t^v$ as a white noise integral by use of the Wick product.

Theorem A.10 *The derivative of B_t^v is*

$$W_t^v := \frac{d}{dt} B_t^v = \left\langle \cdot, \frac{d}{dt} m(u,t) \right\rangle$$

, and the S-transform of W_t^v is

$$S(W_t^v)(\eta) = \left\langle \frac{d}{dt} m(u,t), \eta(u) \right\rangle.$$

Proof. See Dippon & Schiemert (2006*b*). ■

With δ_t denoting the Dirac distribution we get

Example A.11 (Ordinary Brownian motion) *For the derivative of the ordinary Brownian motion* B_t*, we use* $\frac{d}{dt}1_{[0,t]} = \delta_t$ *so* $dB_t/dt =: W_t = \langle \cdot, \delta_t \rangle$*, furthermore* $S(W_t)(\eta) = \eta(t)$*, see Kuo (1996, Chapter 3.1).*

Example A.12 (Fractional Brownian motion) *For* $H \in (0,1)$ *the fractional Brownian motion* $B^H_\cdot : \mathbb{R} \to (\mathcal{S})^*$ *is differentiable in the* $(\mathcal{S})^*$*-sense and*

$$W^H_t := \frac{d}{dt} B^H_t = I(\delta_t \circ M^H_+)$$

and for all $\eta \in \mathcal{S}(\mathbb{R})$

$$\langle \delta_t \circ M^H_+, \eta \rangle = (M^H_+ \eta)(t).$$

The S*-transform of* W^H_t *is given by*

$$S(W^H_t)(\eta) = \left\langle\!\!\left\langle \frac{d}{dt} B^H_t, e^{I(\eta) - 1/2|\eta|_0^2} \right\rangle\!\!\right\rangle = \frac{d}{dt}(SB^H_t)(\eta) = (M^H_+ \eta)(t).$$

Example A.13 (Short range Brownian motion) *The derivative of* B^s_t *is given by*

$$\left\langle \cdot, \frac{d}{dt} t^2 \exp(-(t-u)^2) \right\rangle = \left\langle \cdot, 2t \exp(-(t-u)^2) + t^2(-2(u-t)) \exp(-(u-t)^2) \right\rangle.$$

The stochastic distribution process W^v_t is called white noise of B^v_t. With the white noise W^v_t we can define $\int_{\mathbb{R}} X_t \, dB^v_t$.

A.5 White noise integral and stochastic differential equations driven by $\mathbf{B^v_t}$

We start with the definition of the white noise integral, see Kuo (1996, Chapter 13).

Definition A.14 *The stochastic distribution process* $X_\cdot : \mathbb{R} \to (\mathcal{S})^*$ *is white noise integrable, if there is a* $\Psi \in (\mathcal{S})^*$ *such that, for all* $\eta \in \mathcal{S}(\mathbb{R})$*,* $(SX_\cdot)(\eta) \in L^1(\mathbb{R})$ *and*

$$S(\Psi)(\eta) = \int_{\mathbb{R}} (SX_t)(\eta) \, dt.$$

This definition makes sense as the S-transform is injective.

Definition A.15 *If* $X_t \diamond W^v_t$ *is white noise integrable, the stochastic integral of the stochastic distribution process* X_t *is given by*

$$\int_{\mathbb{R}} X_t \, dB^v_t := \int_{\mathbb{R}} X_t \diamond W^v_t \, dt.$$

This definition coincides in the case of the fractional Brownian motion with the definition of the fractional Itô integral, see Bender (2003*b*) and Hu & Øksendal (2003). Furthermore consider a continuously differentiable stochastic distribution process Z_t.

Definition A.16 *If $X_t \diamond (dZ_t)/(dt)$ is white noise integrable, then the stochastic distribution process X_t has a stochastic integral $\int_{\mathbb{R}} X_t\, dZ_t := \int_{\mathbb{R}} X_t \diamond (dZ_t)/(dt)dt$.*

Theorem A.17 *Let $a, b \in \mathbb{R}$, $X_\cdot : [a, b] \to (\mathcal{S})_{-p}$ be continuous for some $p \in \mathbb{N}$, and $W_\cdot^v : \mathbb{R} \to (\mathcal{S})_{-q}$ be continuous for some $q \in \mathbb{N}$. Then $\int_a^b X_t\, dB_t^v$ exists. Further for any sequence of tagged partitions $\tau_n = (\pi_k^{(n)}, t_k^{(n)})$ of $[a, b]$ with $\lim_{n\to\infty} \max\{|\pi_k - \pi_{k-1}|; k = 1, .., n\} = 0$, we have*

$$\lim_{n\to\infty} \sum_{k=0}^{n} X_{t_{k-1}^{(n)}} \diamond \left(B^v_{\pi_k^{(n)}} - B^v_{\pi_{k-1}^{(n)}} \right) = \int_a^b X_t\, dB_t^v$$

with limit in $(\mathcal{S})^*$. Let $\int_0^t X_s\, dB_s^v$ exist and $\int_0^t X_s\, dB_s^v \in (L^2)$. Then $E(\int_0^t X_s\, dB_s^v) = 0$.

Proof. See Dippon & Schiemert (2006*b*) for proof. ■

The S-transform of $\int_{\mathbb{R}} X_t\, dB_t^v$ is therefore given by

$$S\left(\int_{\mathbb{R}} X_t\, dB_t^v\right)(\eta) = \int_{\mathbb{R}} S(X_t \diamond W_t^v)(\eta)\, dt = \int_{\mathbb{R}} S(X_t)(\eta) \left\langle \frac{d}{dt} m(u, t), \eta(u) \right\rangle dt.$$

Example A.18 *We calculate*

$$\int_0^t B_s^v\, dB_s^v = \int_0^t B_s^v \diamond W_s^v\, ds = \frac{1}{2} (B_t^v)^{\diamond 2},$$

where $(\cdot)^{\diamond 2}$ is the Wick square. By the Wick calculus presented in Kuo (1996, Chapter 8), we get

$$\frac{1}{2} (B_t^v)^{\diamond 2} = \frac{1}{2} (B_t^v)^2 - \frac{1}{2} |m(u, t)|_0^2.$$

Note that this example coincides with the case of ordinary Brownian motion with the Itô-integral and with the case of fractional Brownian motion with the fractional Itô integral.

Example A.19 *Consider the stochastic differential equation*

$$dX_t = \mu X_t\, dt + \sigma X_t\, dB_t^v,$$

with constants $\mu \in \mathbb{R}$ and $\sigma > 0$, which is declared by the integral equation

$$\begin{aligned} X_t &= \int_0^t \mu X_s\, ds + \int_0^t \sigma X_s\, dB_s^v \\ &= \int_0^t (\mu X_s + \sigma X_s \diamond W_s^v)\, ds \\ &= \int_0^t X_s \diamond (\mu + \sigma W_s^v)\, ds, \end{aligned}$$

Its solution is given by

$$X_t = \exp^{\diamond}(\mu t + \sigma B_t^v) = \exp\left(\mu t + \sigma B_t^v - \frac{1}{2}\sigma^2 |m(u,t)|_0^2\right),$$

where $\exp^{\diamond}(\langle\cdot, f\rangle) = \sum_{k=0}^{\infty} (\langle\cdot, f\rangle)^{\diamond k}/k!$, *and Wick calculus is used in the last equation, see e.g. Kuo (1996, Chapter 8).*

Definition A.20 *Let D be a subset of $(\mathcal{S})^*$. A function $f : \mathbb{R}\times D \to (\mathcal{S})^*$ admits a Wick representation in D, if there exists a sequence of Wick polynomials $\{a_k(t)X^{\diamond k}\}_{k=0}^{\infty}$ with continuously differentiable $a_k(t)$, such that for all $X \in D$*

$$f(t,X) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k(t) X^{\diamond k}$$

with convergence in $(\mathcal{S})^*$.

Now the question arises which functions f have this property. A sufficient condition for this is given in the following corollary.

Corollary A.21 *Suppose that $g : \mathbb{R} \to \mathbb{R}$ has a power series representation with $g(x) = \sum_{k=0}^{\infty} g_k x^k$ for all $x \in \mathbb{R}$, and that there exist positive constants C, p and a such that for $X \in (\mathcal{S})^*$ and for all $\eta \in \mathcal{S}(\mathbb{R})$*

$$|g(S(X)(\eta))| \leq C \exp(a|\eta|_p^2). \tag{22}$$

Then

$$g^{\diamond}(X) := \left(\sum_{k=0}^{\infty} g_k X^{\diamond k}\right) \in (\mathcal{S})^*.$$

Proof. See Dippon & Schiemert (2006*b*). ■

In order to show where condition (22) may fail, we give the following example.

Example A.22 *Let $h \in L^2(\mathbb{R})$. Then $\exp^{\diamond}(\langle\cdot,h\rangle^{\diamond k}) \notin (\mathcal{S})^*$ for $k > 2$. We calculate the S-transform and get*

$$S\left(\sum_{n=0}^{\infty} \frac{(\langle\cdot,h\rangle)^{\diamond(kn)}}{n!}\right)(\eta) = \sum_{n=0}^{\infty} \frac{(\langle h,\eta\rangle^k)^n}{n!} = \exp(\langle h,\eta\rangle^k).$$

Now choose $h = \eta$, so condition (22) fails.

Theorem A.23 *Let $D := \{\langle\cdot,h\rangle,\ h \in \mathcal{S}'(\mathbb{R})\}$. Suppose that $f : \mathbb{R}\times D \to (\mathcal{S})^*$ admits a Wick representation in D, then the chain rule for f holds with $b > a$*

$$f(b, B_b^v) - f(a, B_a^v) = \int_a^b \left(\frac{\partial f(t,B_t^v)}{\partial t} + \frac{\partial f(t,B_t^v)}{\partial x} \diamond W_t^v\right) dt.$$

Proof. See Dippon & Schiemert (2006*b*). ■

Theorem A.24 (Chain rule) *Suppose $f : \mathbb{R} \times D \to (\mathcal{S})^*$ admits a Wick representation and let $X_\cdot : \mathbb{R} \to D$ be a continuously differentiable stochastic distribution process. Then for $b > a$ it holds*

$$f(b, X_b) - f(a, X_a) = \int_a^b \frac{\partial f(u, X_u)}{\partial t} + \frac{\partial f(u, X_u)}{\partial x} \diamond \frac{dX_u}{du}\, du.$$

Proof. See Dippon & Schiemert (2006*b*). ■

Proposition A.25 *With $b > a$ let $f(\cdot) : [a, b] \to \mathbb{C}$ be a continuously differentiable function. Then it holds that*

$$\left\| \int_a^b f(s) dB_s^v \right\|_0^2 = (f(b))^2 v(b, b) - 2f(b)f(a)v(b, a) + (f(a))^2 v(a, a) \tag{23}$$

$$- 2\int_a^b f(b)\frac{df(s)}{ds} v(b, s)ds + 2\int_a^b f(a)\frac{df(s)}{ds} v(a, s)ds + \int_a^b \int_a^b \frac{df(s)}{ds}\frac{df(t)}{dt} v(s, t)dsdt. \tag{24}$$

Proof. Use the formula for partial integration from Dippon & Schiemert (2006*b*) and derive

$$\int_a^b f(s)dB_s^v = f(b)B_b^v - f(a)B_a^v - \int_a^b \frac{df(s)}{ds} B_s^v ds.$$

Now by regarding the bilinear mapping and using Fubinis Theorem

$$\left\langle\!\left\langle f(b)B_b^v - f(a)B_a^v - \int_a^b \frac{df(s)}{ds} B_s^v ds, f(b)B_b^v - f(a)B_a^v - \int_a^b \frac{df(s)}{ds} B_s^v ds \right\rangle\!\right\rangle$$
$$= (f(b))^2 v(b, b) - 2f(b)f(a)v(b, a) + (f(a))^2 v(a, a) - 2\int_a^b f(b)\frac{df(s)}{ds} v(b, s)ds$$
$$+ 2\int_a^b f(a)\frac{df(s)}{ds} v(a, s)ds + \int_a^b \int_a^b \frac{df(s)}{ds}\frac{df(t)}{dt} v(s, t)dsdt.$$

■

□

B Appendix

Proof of Theorem 4.2

As the set $\{\exp^\diamond(\langle \cdot, f(u)\rangle, f \in L^2(\mathbb{R})\}$ is total in (L^2) it follows that

$$\varphi_A^o(t) = \sum_{k=0}^{\infty} a_k(t) \exp^\diamond(\langle \cdot, f_k(u, t)\rangle)$$

with $a_k(\cdot) : [0,T] \to \mathbb{C}$ and $f_k(\cdot,t) \in L^2(\mathbb{R})$ for all $k \geq 0$ and for all $t \in [0,T]$. Similarly for $\varphi_T^o(t)$ it holds that

$$\varphi_T^o(t) = \sum_{k=0}^{\infty} b_k(t) \exp^\diamond(\langle \cdot, g_k(u,t)\rangle)$$

with $b_k(\cdot) : [0,T] \to \mathbb{C}$ and $g_k(\cdot,t) \in L^2(\mathbb{R})$ for all $k \geq 0$ and for all $t \in [0,T]$. Regarding the wealth process $V^o(t)$ it follows

$$\begin{aligned}
&V^o(t)\\
&= \varphi_A^o(t) \cdot A(t) + \varphi_T^o(t) \cdot P^T(t,r(t))\\
&= \sum_{k=0}^{\infty} a_k(t) \exp^\diamond\left(\langle \cdot, f_k(u,t)\rangle\right) \cdot \exp^\diamond\left(\int_0^t r(s)ds + \gamma(t)\right)\\
&\quad + \sum_{k=0}^{\infty} b_k(t) \exp^\diamond(\langle \cdot, g_k(u,t)\rangle) \cdot \exp^\diamond(a(t) + b(t)r(t))\\
&= \sum_{k=0}^{\infty} \exp\left(\frac{1}{2}\left(-|f_k(u,t)|_0^2 - 2\gamma(t) + \left| f_k(u,t) + \int_0^t \int_0^s \sigma \exp(-\kappa(s-w))\frac{dm(u,w)}{dw} dwds\right|_0^2\right)\right)\\
&\quad \cdot a_k(t) \exp^\diamond\left(\langle \cdot, f_k(u,t)\rangle\right) \diamond \exp^\diamond\left(\int_0^t r(s)ds + \gamma(t)\right)\\
&\quad + \sum_{k=0}^{\infty} \exp\left(\frac{1}{2}\left(-|g_k(u,t)|_0^2\right.\right.\\
&- \left| b(t) \int_0^t \sigma \exp(-\kappa(t-s)) \frac{dm(u,s)}{ds} ds\right|_0^2 + \left.\left.\left| f_k(u,t) + \int_0^t b(t)\sigma \exp(-\kappa(t-s))\frac{dm(u,s)}{ds} ds\right|_0^2\right)\right)\\
&\quad \cdot b_k(t) \exp^\diamond(\langle \cdot, g_k(u,t)\rangle) \diamond \exp^\diamond(a(t) + b(t)r(t))
\end{aligned}$$

So set

$$\begin{aligned}
\varphi_A^w(t) := \sum_{k=0}^{\infty} &\exp\left(\frac{1}{2}\left(-|f_k(u,t)|_0^2 - 2\gamma(t) + \left| f_k(u,t) + \int_0^t \int_0^s \sigma \exp(-\kappa(s-w))\frac{dm(u,w)}{dw} dwds\right|_0^2\right)\right)\\
&\cdot a_k(t) \exp^\diamond(\langle \cdot, f_k(u,t)\rangle)
\end{aligned}$$

and

$$\begin{aligned}
\varphi_T^w(t) := \sum_{k=0}^{\infty} \exp&\left(\frac{1}{2}\left(-|g_k(u,t)|_0^2 - \left| b(t) \int_0^t \sigma \exp(-\kappa(t-s))\frac{dm(u,s)}{ds} ds\right|_0^2\right.\right.\\
&\left.\left. + \left| f_k(u,t) + b(t)\int_0^t \sigma \exp(-\kappa(t-s))\frac{dm(u,s)}{ds} ds\right|_0^2\right)\right) b_k(t) \exp^\diamond(\langle \cdot, g_k(u,t)\rangle)
\end{aligned}$$

and the Wick portfolio is derived. The ordinary portfolio is obtained from a Wick portfolio in the following way. Let

$$\varphi_A^w(t) := \sum_{k=0}^{\infty} a_k(t) \exp^{\diamond}(\langle \cdot, f_k(u,t) \rangle)$$

and

$$\varphi_T^o(t) := \sum_{k=0}^{\infty} b_k(t) \exp^{\diamond}(\langle \cdot, g_k(u,t) \rangle).$$

Set

$$\varphi_A^o(t) := \sum_{k=0}^{\infty} \exp\left(\frac{1}{2}\left(|f_k(u,t)|_0^2 + 2\gamma(t) - \left|f_k(u,t) + \int_0^t \int_0^s \sigma \exp(-\kappa(s-w)) \frac{dm(u,w)}{dw} dw ds\right|_0^2\right)\right)$$
$$\cdot\, a_k(t) \exp^{\diamond}(\langle \cdot, f_k(u,t) \rangle)$$

and

$$\varphi_T^o(t) := \sum_{k=0}^{\infty} \exp\left(\frac{1}{2}\left(|g_k(u,t)|_0^2 + \left|b(t) \int_0^t \sigma \exp(-\kappa(t-s)) \frac{dm(u,s)}{ds} ds\right|_0^2\right.\right.$$
$$\left.\left. - \left|f_k(u,t) + \int_0^t b(t)\sigma \exp(-\kappa(t-s)) \frac{dm(u,s)}{ds} ds\right|_0^2\right)\right) b_k(t) \exp^{\diamond}(\langle \cdot, g_k(u,t) \rangle),$$

in the calculus before one derives the ordinary portfolio.

□

Proof of Theorem 4.3

Apply

$$V(t) = \varphi^A(t) \diamond A(t) + \varphi^T(t) \diamond P^T(t, r(t))$$

in equation (4) and the chain rule for P^T and for $A(t)$, it is obtained that

$$\int_0^T r(t) \diamond V(t, r(t)) + V(t, r(t))\psi(t) - r(t) \diamond \varphi^T \diamond P^T - (\varphi^T \diamond P^T)\psi(t)$$
$$+ \varphi^T \diamond (P_t^T + P_r^T \diamond (\kappa(\theta - r(t)))dt + \int_0^T \varphi_T \diamond P_r^T dB_t^v$$
$$= \int_0^T r(t) \diamond V(t, r(t)) + V(t, r(t))\psi(t)dt$$
$$+ \int_0^T \varphi^T \diamond (P_r^T \diamond (\kappa(\theta - r(t))) - r(t) \diamond P^T - \psi(t)P^T + P_t^T + \sigma P_r^T \diamond W_t^v)dt.$$

If there exists an $\eta \in \mathcal{S}(\mathbb{R})$ such that

$$S\left(P_r^T \diamond (\kappa(\theta - r(t))) - r(t) \diamond P^T - \psi(t)P^T + P_t^T + \sigma P_r^T \diamond W_t^v\right)(\eta) = 0 \qquad (25)$$

then it follows by applying the change of S-transform and integration that

$$S(V(T))(\eta) - S(V(0))(\eta) = \int_0^T S(V(t, r(t))(\eta) S(r(t) + \psi(t))(\eta) dt.$$

This is an integral equation for $S(V(t, r(t)))(\eta)$ and has the solution

$$S(V(T, r(T))(\eta) = S(V(0, r(0)))(\eta) \exp\left(S\left(\int_0^T r(t)dt + \int_0^T \psi(t)dt\right)(\eta)\right).$$

So

$$V(T, r(T)) = V(0, r(0)) \exp^{\diamond}\left(\int_0^T r(t)dt + \int_0^T \psi(t)dt\right) = V(0, r(0)) \exp\left(\int_0^T r(t)dt\right).$$

Define the measure $\mu_\eta(\mathcal{B}) = S(1_{\mathcal{B}})(\eta)$. As the measure μ of the probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$ and the measure μ_η, which is induced by the S-transform are equivalent, it follows that if $V(0) = 0$ in the measure μ, so $V(0) = 0$ in the measure μ_η, then $V(T) = 0$ in the measure the μ_η and so $V(0) = 0$ in the measure μ. Regarding the partial derivatives of $P^T(t, r(t))$

$$P_t^T(t, r(t)) = P^T(t, r(t)) \diamond \left(\frac{da(t)}{dt} + \frac{db(t)}{dt} r(t)\right)$$
$$P_r^T(t, r(t)) = P^T(r, (t)) b(t).$$

Plugging these partial derivatives into the equation (25) and the differential equations for a and b

$$S(P_r^T)(\eta) S(\kappa(\theta - r(t)))(\eta) - S(r(t))(\eta) S(P^T)(\eta) - S(\psi(t) P^T)(\eta) + S(P_t^T)(\eta)$$
$$= -\sigma S(P_r^T)(\eta) \left\langle \frac{dm(u, t)}{dt}, \eta(u) \right\rangle,$$
$$S(P^T)(\eta)(b(t)\kappa(\theta - S(r(t))(\eta)) - S(r(t))(\eta) - \psi(t) + \frac{da(t)}{dt} + \frac{b(t)}{dt} S(r(t))(\eta))$$
$$= -\sigma b(t) S(P^T)(\eta) \left\langle \frac{dm(u, t)}{dt}, \eta(u) \right\rangle.$$

Thus

$$
\begin{aligned}
S(r(t))(\eta)(\frac{db(t)}{dt} - \kappa b(t) - 1) + b(t)\kappa\theta - \psi(t) + \frac{da(t)}{dt} &= -b(t)\sigma \left\langle \frac{dm(u,t)}{dt}, \eta(u) \right\rangle, \\
b(t)\kappa\theta - \psi(t) + (\zeta - \kappa\theta)b(t) + \psi(t) &= -b(t) \left\langle \frac{dm(u,t)}{dt}, \eta(u) \right\rangle, \\
\zeta b(t) &= -b(t)\sigma \left\langle \frac{dm(u,t)}{dt}, \eta(u) \right\rangle, \\
-\zeta &= \left\langle \frac{dm(u,t)}{dt}, \sigma\eta(u) \right\rangle.
\end{aligned}
$$

Due to equation (2) there exists an η such that the last equation is satisfied.

□

Proof that the market price of risk is independent of the maturity

Let the bonds P^T and P^S have two different λ^T and λ^S with $\lambda^T = \lambda^S + p$, where $p > 0$. Regard the self-financing portfolio

$$
V(t) = \varphi^T \diamond P^T + \varphi^S \diamond P^S
$$

applying this equation and the property that it is a self-financing strategy one obtains

$$
dV = \left(\varphi^T \diamond P^T \diamond (\alpha^T + \sigma^T \diamond W_t^v) + \varphi^S \diamond P^S \diamond (\alpha^S + \sigma^S \diamond W_t^v)\right) dt.
$$

Inserting the terms for λ^T, λ^S and the difference of them gives

$$
\begin{aligned}
dV &= (\varphi^T \diamond P^T \diamond (\lambda^T \diamond \sigma^T + r(t) + \psi(t) + \sigma^T \diamond W_t^v) \\
&\quad + \varphi^S \diamond P^S \diamond (\lambda^S \diamond \sigma^S + r(t) + \psi(t) + \sigma^S \diamond W_t^v))dt \\
&= ((r(t) + \psi(t)) \diamond (\varphi^T \diamond P^T + \varphi^S \diamond P^S) + \\
&\quad ((\lambda^S + W_t^v) \diamond (\varphi^T \diamond P^T \diamond \sigma^T + \varphi^S \diamond P^S \diamond \sigma^S)) + p\,\sigma b(t)\varphi^T \diamond P^T)dt \\
&= (r(t) + \psi(t)) \diamond V dt + \left((\lambda^S + W_t^v) \diamond (\varphi^T \diamond P^T \diamond \sigma^T + \varphi^S \diamond P^S \diamond \sigma^S)\right) dt + p\,\sigma b(t)\varphi^T \diamond P^T dt.
\end{aligned}
$$

For the term $\lambda^S + W_t^v$ it holds that

$$
S(\lambda^S + W_t^v)(\eta) = 0
$$

with the η of equation (25). Thus it holds with the same η in the integral representation of the differential forms that

$$
S(V(T) - V(0))(\eta) = \int_0^T S((r(t) + \psi(t)))(\eta) S(V(t))(\eta) dt + \int_0^T \sigma b(t) p\, S(\varphi^T)(\eta) S(P^T)(\eta) dt.
$$

Take a trading strategy such that $(\sigma b(t)\varphi^T \diamond P^T) > 0$, then there is an arbitrage. Thus $p = 0$.

$\square$

PROOF OF THEOREM 4.4

Regard

$$dP^T(t, r(t)) = P_t^T + P_r^T \diamond (\kappa(\theta - r(t))) + \sigma P_r^T \diamond W_t^v dt$$

and

$$dP^t(t, r(t)) = \alpha^T \diamond P^T + \sigma^T \diamond P^T \diamond W_t^v dt.$$

Subtracting the equations and using the market price of risk λ, one gets

$$(\lambda(t) \diamond \sigma^T + r(t) + \psi(t)) \diamond P^T = P_t^T + P_r^T \diamond (\kappa(\theta - r(t))).$$

Insert that $P^T(t, r(t)) = \exp^\diamond(a(t) + b(t)r(t))$ and get the partial derivatives of P^T

$$P_t^T = P^T \diamond (a'(t) + b'(t)r(t)), \tag{26}$$

$$P_r^T = P^T b(t). \tag{27}$$

These results and the definition of σ^T leads to

$$\begin{aligned}
&(\lambda(t) \diamond \sigma^T + r(t) + \psi(t)) \diamond P^T = P^T \diamond (a'(t) + b'(t)r(t)) + P^T \diamond (\kappa(\theta - r(t)))b(t) \\
&P^T \diamond \big(\lambda(t) \diamond \sigma^T + r(t) + \psi(t) - a'(t) - b'(t)r(t) - \kappa(\theta - r(t))b(t)\big) = 0 \\
&P^T \diamond (\lambda(t)\sigma b(t) + r(t) + \psi(t) - a'(t) - b'(t)r(t) - \kappa(\theta - r(t))b(t)) = 0.
\end{aligned}$$

Now set $\lambda(t) =: \lambda$ constant and knowing that P^T is positive a.s. it follows that

$$\begin{aligned}
(\lambda\sigma b(t) + r(t) + \psi(t) - a'(t) - b'(t)r(t) - \kappa(\theta - r(t))b(t)) = 0 \\
\lambda\sigma b(t) + \psi(t) - a'(t) - \kappa\theta b(t) + r(t)(1 - b'(t) + \kappa b(t)) = 0.
\end{aligned}$$

From this equation the ODEs with the conditions $b(T) = 0$ and $a(T) = 0$ are

$$b'(t) = 1 + \kappa b(t)$$

and

$$a'(t) = (\lambda\sigma - \kappa\theta)b(t) + \psi(t).$$

The solutions are

$$b(t) = \frac{\exp(-k(T - t)) - 1}{k}$$

and

$$a(t) = (\lambda\sigma - \kappa\theta)\frac{1/k(\exp(-k(T-t)) - 1) + T - t}{k}$$
$$+ \frac{\left\| \int_0^t \int_0^s \sigma \exp(-\kappa(s-u)) dB_u^v ds \right\|_0^2}{2} - \frac{\left\| \int_0^T \int_0^s \sigma \exp(-\kappa(s-u)) dB_u^v ds \right\|_0^2}{2}$$
$$= (\frac{\lambda\sigma}{\kappa} - \theta)(b(t) + T - t) + \frac{\left\| \int_0^t \int_0^s \sigma \exp(-\kappa(s-u)) dB_u^v ds \right\|_0^2}{2}$$
$$- \frac{\left\| \int_0^T \int_0^s \sigma \exp(-\kappa(s-u)) dB_u^v ds \right\|_0^2}{2},$$

where

$$\frac{\left\| \int_0^t \int_0^s \sigma \exp(-\kappa(s-u)) dB_u^v ds \right\|_0^2}{2} = \frac{\left\| \int_0^t \int_u^t \sigma \exp(-\kappa(s-u)) ds dB_u^v \right\|_0^2}{2}$$
$$= \frac{\left\| \int_0^t \sigma \frac{-1}{k} (\exp(-\kappa(t-u)) - 1) dB_u^v \right\|_0^2}{2}$$
$$= \frac{1}{2} \int_0^t \int_0^t \sigma^2 \exp(-2\kappa t + \kappa(u+s)) v(u,s)\, du\, ds,$$

by the use of equation (23).

Setting $\lambda\sigma = \zeta$ one has proven the equations (5) and (6).

Furthermore, we get

$$P^T = \exp^{\diamond}(a(t) + b(t)r(t)) = \exp\left(a(t) + b(t)r(t) - \frac{1}{2} b^2(t) \left\| \int_0^t \sigma \exp(-\kappa(t-s)) dB_s^v \right\|_0^2 \right).$$

□

The sketch of the proof of theorem 4.2 with $r_n(t)$ instead of $r(t)$.

Set

$$\gamma_n(t) := \frac{1}{2} \left\| \int_0^t \int_0^s \sum_{l=1}^n \alpha_l \sigma_l \exp(-k_l(s-u))\, dB_u^{v_l}\, ds \right\|_0^2 \tag{28}$$

and

$$E := \int_0^t \int_0^s \sum_{l=1}^n \alpha_l \sigma_l \exp(-k_l(s-w)) \frac{dm_l(u,w)}{dw} dw ds$$
$$P := \int_0^t \sum_{l=1}^n b_l(t) \sigma_l \exp(-k_l(t-s)) \frac{dm_l(u,s)}{ds} ds.$$

Regarding the proof of theorem 4.2 one has to replace $\gamma(t)$ by $\gamma_n(t)$,

$$\int_0^t \int_0^s \sigma \exp(-\kappa(s-w)) \frac{dm(u,w)}{dw} dwds$$

by E and

$$\int_0^t b(t)\sigma \exp(-\kappa(t-s)) \frac{dm(u,s)}{ds}\, ds$$

by P and the proof is done.

□

Proof of absence of arbitrage in the multi-factor Ornstein-Uhlenbeck model

Consider the self-financing portfolio $V_n(t) = \varphi^A \diamond A^n(t) + \varphi^T \diamond P^{T,n}$, where $A^n(t) := \exp(\int_0^t r_n(s)ds)$. As before one gets $A^n(t) = \exp^{\diamond}(\int_0^t r_n(s)ds + \gamma_n(t))$ with $\gamma_n(t)$ as in equation (28). We skip the index n in the following. So by the self-financing property it follows similarly as before that

$$\begin{aligned} V(T) - V(0) &= \int_0^T V(t) \diamond (r(t) + \psi(t))dt \\ &\quad + \int_0^T \varphi^T \diamond \left(P_t^T - P^T \diamond (r(t) + \psi(t)) + \sum_{l=1}^n P_{p_l}^T \diamond \frac{dp_l(t)}{dt} \right) dt \\ &= \int_0^T V(t) \diamond (r(t) + \psi(t))dt \\ &\quad + \int_0^T \varphi^T \diamond P^T \diamond (\sum_{l=1}^n (b'(t)p_l(t)) + a'(t) - (\alpha_0 + \sum_{l=1}^n \alpha_l p_l(t) \\ &\quad + \psi(t)) + \sum_{l=1}^n b_l(t)(k_l(\theta_l - p_l(t)) + \sigma_l W_t^{v_l}))dt. \end{aligned}$$

The last integral is

$$\int_0^T \varphi^T \diamond P^T \diamond (\sum_{l=1}^n (p_l(t)(b'(t) - \alpha_l - k_l b_l(t))) + a'(t) - \alpha_0 - \psi(t)) + \sum_{l=1}^n (b_l(t)k_l\theta_l + b_l(t)\sigma_l W_t^{v_l})dt.$$

Now insert equations (13) and (12), and it follows that

$$\int_0^T \varphi^T \diamond P^T \diamond (\sum_{l=1}^n (\zeta_l b_l(t) - b_l(t)\sigma_l W_t^{v_l}))dt. \tag{29}$$

By equation (9) there is an $\eta \in \mathcal{S}(\mathbb{R})$ such that the integral (29) is 0. By the same argument as in the proof to theorem 4.3 it follows that there is no arbitrage in the class of self-financing portfolios with the short rate $r_n(t)$ instead of the short rate $r(t)$.

□

Proof of theorem that market price of risk is independent of T with $r_n(t)$

We construct the same contradiction as in the corresponding proof before. Let

$$\alpha_l^T := (a'(t)/n + b_l'(t)p_l(t) + b_l(t)k_l(\theta_l - p_l(t)))$$

and

$$\sigma_l^T := b_l(t)\sigma_l,$$

so it holds that

$$dP^T = P^T \diamond \left(\sum_{l=1}^{n} \alpha_l^T + \sigma_l^T W_t^{v_l}\right) dt.$$

The market price of risk λ_l, which is assumed to be constant, is therefore defined as

$$\lambda_l \sigma_l^T = \alpha_l^T - \frac{r_n(t) + \psi_n(t)}{n}.$$

Now let $\lambda_m^T - \lambda_m^S = p$ with $p > 0$ for some m, then it is possible to create arbitrage with $r_n(t) = p_m(t)$ due to the Ornstein-Uhlenbeck model with one factor, so $\lambda_m^T = \lambda_m^S$.

□

Proof of theorem 5.1

Using the notation from before one gets

$$dP^T = P^T \diamond \left(\sum_{l=1}^{n} \alpha_l^T + \sigma_l^T W_t^{v_l}\right) dt$$

and

$$dP^T = (P_t^T + \sum_{l=1}^{n} P_{p_l}^T \diamond (k_l(\theta_l - p_l(t)) + \sigma_l W_t^{v_l}))dt.$$

Using the assumption to P^T in equation (11) it follows that

$$P^T \diamond \left(\sum_{l=1}^{n} \alpha_l^T\right) = P^T \diamond \left(a'(t) + \sum_{l=1}^{n} (b_l'(t)p_l(t) + b_l(t)k_l(\theta_l - p_l(t)))\right)$$

$$r(t) + \psi(t) + \sum_{l=1}^{n} \lambda_l^T \sigma_l b_l(t) = \left(a'(t) + \sum_{l=1}^{n} (b_l'(t)p_l(t) + b_l(t)k_l(\theta_l - p_l(t)))\right).$$

The ordinary differential equations (13) and (12) follow from this equation.

□

Working Papers from Finance Research Group

F-2008-07 Esben Høg, Per Frederiksen & Daniel Schiemert: On the Generalized Brownian Motion and its Applications in Finance.

F-2008-06 Esben Høg: Volatility and realized quadratic variation of differenced returns. A wavelet method approach.

F-2008-05 Peter Løchte Jørgensen & Domenico De Giovanni: Time Charters with Purchase Options in Shipping: Valuation and Risk Management.

F-2008-04 Stig V. Møller: Habit persistence: Explaining cross-sectional variation in returns and time-varying expected returns.

F-2008-03 Thomas Poulsen: Private benefits in corporate control transactions.

F-2008-02 Thomas Poulsen: Investment decisions with benefits of control.

F-2008-01 Thomas Kokholm: Pricing of Traffic Light Options and other Correlation Derivatives.

F-2007-03 Domenico De Giovanni: Lapse Rate Modeling: A Rational Expectation Approach.

F-2007-02 Andrea Consiglio & Domenico De Giovanni: Pricing the Option to Surrender in Incomplete Markets.

F-2006-09 Peter Løchte Jørgensen: Lognormal Approximation of Complex Path-dependent Pension Scheme Payoffs.

F-2006-08 Peter Løchte Jørgensen: Traffic Light Options.

F-2006-07 David C. Porter, Carsten Tanggaard, Daniel G. Weaver & Wei Yu: Dispersed Trading and the Prevention of Market Failure: The Case of the Copenhagen Stock Exhange.

F-2006-06 Amber Anand, Carsten Tanggaard & Daniel G. Weaver: Paying for Market Quality.

F-2006-05 Anne-Sofie Reng Rasmussen: How well do financial and macroeconomic variables predict stock returns: Time-series and cross-sectional evidence.

F-2006-04 Anne-Sofie Reng Rasmussen: Improving the asset pricing ability of the Consumption-Capital Asset Pricing Model.

F-2006-03 Jan Bartholdy, Dennis Olson & Paula Peare: Conducting event studies on a small stock exchange.

F-2006-02 Jan Bartholdy & Cesário Mateus: Debt and Taxes: Evidence from bank-financed unlisted firms.

F-2006-01 Esben P. Høg & Per H. Frederiksen: The Fractional Ornstein-Uhlenbeck Process: Term Structure Theory and Application.

F-2005-05 Charlotte Christiansen & Angelo Ranaldo: Realized bond-stock correlation: macroeconomic announcement effects.

F-2005-04 Søren Willemann: GSE funding advantages and mortgagor benefits: Answers from asset pricing.

F-2005-03 Charlotte Christiansen: Level-ARCH short rate models with regime switching: Bivariate modeling of US and European short rates.

F-2005-02 Charlotte Christiansen, Juanna Schröter Joensen and Jesper Rangvid: Do more economists hold stocks?

F-2005-01 Michael Christensen: Danish mutual fund performance - selectivity, market timing and persistence.

F-2004-01 Charlotte Christiansen: Decomposing European bond and equity volatility.

Handelshøjskolen i Århus

Aarhus School of Business

ISBN 9788778823649

Department of Business Studies

Aarhus School of Business
University of Aarhus
Fuglesangs Allé 4
DK-8210 Aarhus V - Denmark

Tel. +45 89 48 66 88
Fax +45 86 15 01 88

www.asb.dk