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The log-linear return approximation, bubbles, and predictability

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Abstract

We study in detail the log-linear return approximation introduced by Campbell and Shiller (1988a). First, we derive an upper bound for the mean approximation error, given stationarity of the log dividend-price ratio. Next, we simulate various rational bubbles which have explosive conditional expectation, and we investigate the magnitude of the approximation error in those cases. We find that surprisingly the Campbell-Shiller approximation is very accurate even in the presence of large explosive bubbles. Only in very large samples do we find evidence that bubbles generate large approximation errors. Finally, we show that a bubble model in which expected returns are constant can explain the predictability of stock returns from the dividend-price ratio that many previous studies have documented.

Keywords: Stock return, Taylor expansion, bubble, simulation, predictability

JEL Codes: C32, C52, C65, G12.

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1 Introduction

Since the seminal paper by Campbell and Shiller (1988a), the log-linear return approximation, relating log stock returns linearly to log prices and log dividends, has become one of the central equations in empirical financial research on stock return predictability, tests of present value models, return variance decompositions, and discrete-time dynamic asset allocation, see e.g. the textbook treatments in Campbell et al. (1997), Campbell and Viceira (2002), and Cochrane (2005).

The log-linear approximation looks as follows

$$r_{t+1} \approx \rho p_{t+1} + (1 - \rho)d_{t+1} - p_t + k, \quad (1)$$

where $r_{t+1} \equiv \log((P_{t+1} + D_{t+1})/P_t)$, $p_t \equiv \log(P_t)$, and $d_t \equiv \log(D_t)$. ρ is a parameter slightly less than 1, and k is a constant. P_t is the stock price measured at the end of period t , while D_t is dividend paid during period t . The appealing feature of (1) is that even though return is stochastic and varies over time, it is related linearly to prices and dividends whereby standard econometric techniques for linear models can be applied directly to analyze (1) - and models building on (1) - empirically. Thus, financial models with time-varying expected returns are conveniently analyzed based on (1). Taking conditional expectations of (1), solving it recursively forward for p_t , and imposing a terminal condition, lead to a log-linear present value model with time-varying expected returns - the so-called *dynamic Gordon growth model*.

The log-linear return relation (1) is only an approximation. It is derived from a first-order Taylor expansion of the definition of the log gross stock return. The expansion is around the dividend-price ratio. The relation holds exactly if this ratio is constant. But if the dividend-price ratio varies over time, there is an approximation error which depends on the persistence and volatility of the ratio. An underlying assumption of the approximation is that $\delta_t \equiv d_t - p_t$ is mean stationary, and usually the expansion point is taken to be the unconditional mean of δ_t . Prices and dividends, p_t and d_t , are allowed to be non-stationary, but they need to cointegrate so that δ_t is stationary. In that case a convenient reparameterization of (1) is

$$r_{t+1} \approx \delta_t - \rho\delta_{t+1} + \Delta d_{t+1} + k, \quad (2)$$

which only contains stationary variables. Up to the 1980s the dividend-price

ratio - although highly persistent - appeared to be stationary, and the log-linear approximation was found to be very accurate, see Campbell and Shiller (1988a) and Cochrane (1992). However, since then the dramatic stock market boom of the 1990s decreased δ_t and increased its persistence, and standard unit root tests no longer reject the null hypothesis of non-stationarity at even high significance levels. This has led researchers to question the validity of the log-linear approximation when applied on recent data. Campbell (2008) argues that the log-linear approximation breaks down when δ_t has a unit root because in that case the unconditional mean does not exist, and he proposes an alternative model that is valid when δ_t is a random walk, dividend is known one period in advance, and log return and dividend growth are conditionally normally distributed. The model delivers return forecasts in the manner of the classic Gordon growth model.

A unit root in δ_t can only be rationalized if r_t and/or Δd_t have unit roots. Neither theory nor empirical evidence support such non-stationarity in returns and dividend growth. Hence, modeling δ_t as a unit root process should be regarded a finite-sample approximation in the case where the root is very close to - but strictly less than - unity. An interesting question is how accurate the log-linear approximation is in a finite sample when the autoregressive root in δ_t is very close to unity.

An even more intriguing case that in theory leads to a non-stationary dividend-price ratio is a rational speculative bubble. Such a bubble induces an *explosive* component in δ_t . Some researchers rule out such bubbles *a priori*, e.g. Lewellen (2004) and Cochrane (2008). But, in fact, explosive rational bubbles cannot be ruled out completely based on economic theory (see, e.g., Tirole, 1985, and Diba and Grossman, 1988a), and some recent studies do find evidence of explosiveness in stock prices. Engsted (2006), Engsted and Nielsen (2010), and Phillips et al. (2009), find that US stock prices contain an explosive component not found in dividends, and explosive roots are contained in the confidence intervals for the largest autoregressive root in the US log dividend-price ratio reported by Campbell and Yogo (2006).

The question is what happens to the log-linear approximation when δ_t is explosive? Does it break down completely? At first sight one would say yes because with an explosive dividend-price ratio its unconditional mean is not stable and it wanders arbitrarily away from any point very fast. However, Cochrane (2008, section 4.1) discusses bubbles in the context of the log-linear approximation and, although he chooses to rule out bubbles based on economic theory, common sense, and earlier empirical results, he does

not consider the log-linear approximation fundamentally inconsistent with the presence of bubbles. Recent empirical tests for rational bubbles (Wu, 1997; Balke and Wohar, 2009; Phillips et al., 2009) also use the log-linear approximation as the underlying theoretical framework.

In the present paper we investigate in detail the approximation error of the log-linear return approximation under both stationarity and explosiveness of the log dividend-price ratio, δ_t . We also investigate whether the presence of rational bubbles may explain the predictability of stock returns based on δ_t that several studies have found since the 1980s. First, we derive an upper bound for the mean approximation error, given stationarity of δ_t , and we show that the minimum upper bound is obtained by setting the linearization point equal to the unconditional mean of δ_t .

Next, we conduct a simulation study to investigate the finite-sample properties of the log-linear approximation in the presence of bubbles. In practice researchers may erroneously treat δ_t as a stationary process, either because of the downward finite-sample bias of autoregressive parameter estimates, or because of periodically collapsing rational bubbles. Evans (1991) showed that such bubbles, which have explosive conditional expectation, appear stationary in a finite sample. We simulate various bubbles of the Evans type and investigate the properties of the approximation error in each case. The underlying pricing model is one in which expected returns are constant and log dividends follow a random walk with drift. These assumptions imply that the bubble is the only cause for the approximation error. We find that surprisingly the approximation error is quite small, even for relatively large bubbles. For example, in a sample with 100 observations and a bubble that collapses with 15% probability every period when the bubble grows above some threshold, and where the bubble on average over the sample accounts for 50% of the stock price, actual and approximate log returns are extremely highly correlated, as are actual and approximate log dividend-price ratios (correlations above 0.999). Average actual and approximate log returns are very similar, as are average actual and approximate log dividend-price ratios, and the standard deviations of exact and approximate values are almost identical. As expected, the approximation error increases with the size of the bubble, but only slightly. Only with a very large sample (1,000 observations) do we find evidence of non-negligible approximation errors generated by the bubble. Overall, the simulation results indicate that the Campbell-Shiller log-linear return approximation is surprisingly accurate, even in the presence of large explosive bubbles.

Finally, we use the same simulated data to investigate whether stock returns are predictable from the dividend-price ratio. We conduct regressions similar to those in Cochrane (2008), i.e. r_{t+1} , Δd_{t+1} , and δ_{t+1} are regressed onto δ_t . In many cases we find predictive coefficients very similar to those reported by Cochrane on US data. In particular, for a number of plausible periodically collapsing bubbles we are able to match almost exactly the return predictability results reported by Cochrane. Cochrane interprets his results as evidence of time-varying expected returns. Since our simulated data are from a model with constant expected returns, the only cause for the found predictability is the presence of the bubble. Thus, our results point to an alternative interpretation of Cochrane's empirical findings: expected returns are constant, but bubbles make returns appear predictable from the dividend-price ratio in linear predictive regressions.

The rest of the paper is organized as follows. In the next section we derive the upper bound for the mean approximation error in the log-linear return approximation given stationarity of the log dividend-price ratio. Next, in section 3, we investigate the properties of the approximation error in simulated data where stock prices are subject to various explosive bubbles. In section 4 we use the same simulated data to investigate whether such bubbles may generate predictability of stock returns from the dividend-price ratio. Finally, section 5 contains some concluding remarks.

2 The approximation error under stationarity

In this section we investigate in detail the approximation error in the log-linear return approximation. We derive (1) from a first-order Taylor expansion of the log gross stock return, and then we derive an upper bound for the mean approximation error, given stationarity of the log dividend-price ratio.

The one-period gross stock return from time t to $t + 1$ is defined as

$$1 + R_{t+1} \equiv \frac{P_{t+1} + D_{t+1}}{P_t}. \quad (3)$$

Taking logs to (3) gives

$$\begin{aligned}
r_{t+1} &\equiv \log(P_{t+1} + D_{t+1}) - \log(P_t) \\
&= \log\left[\left(1 + \frac{D_{t+1}}{P_{t+1}}\right) P_{t+1}\right] - \log(P_t) \\
&= \log[1 + e^{\delta_{t+1}}] + p_{t+1} - p_t.
\end{aligned}$$

The first term on the right-hand side of this expression is non-linear in the log dividend-price ratio. The first-order (linear) Taylor approximation of $f(\delta_{t+1}) \equiv \log[1+e^{\delta_{t+1}}]$ is

$$f(\delta_{t+1}) = f(\widehat{\delta}) + \left[\frac{1}{1 + e^{\widehat{\delta}}} e^{\widehat{\delta}} (\delta_{t+1} - \widehat{\delta}) \right], \quad (4)$$

where $\widehat{\delta}$ is the point around which the linearization is done. The approximation error in (4) is given as (see e.g. Sydsaeter and Hammond, 1995):

$$\text{Error}_{t+1} \equiv e_{t+1}^* = \frac{1}{2} \frac{e^{\widetilde{\delta}}}{[1 + e^{\widetilde{\delta}}]^2} (\delta_{t+1} - \widehat{\delta})^2, \quad (5)$$

where $\widetilde{\delta}$ is a number between $\widehat{\delta}$ and δ_{t+1} . Now (1) and (2) follow by setting $\rho = (1 + e^{\widehat{\delta}})^{-1}$ and collecting all constant terms in k ($k = \log\left(\frac{1}{\rho}\right) - (1 - \rho)\widehat{\delta} = -\log(\rho) - (1 - \rho)\log\left(\frac{1}{\rho} - 1\right)$):

$$r_{t+1} \equiv \rho p_{t+1} + (1 - \rho)d_{t+1} - p_t + k + e_{t+1}^* \quad (6)$$

$$\Rightarrow \delta_t \equiv \rho \delta_{t+1} + r_{t+1} - \Delta d_{t+1} - k - e_{t+1}^*. \quad (7)$$

How big is the approximation error e_{t+1}^* ? Notice first that since $e_{t+1}^* \geq 0$, the mean error, $E(e_{t+1}^*)$, is a valid measure of the magnitude of the approximation error. Notice also that since an upper limit for δ_t is 0, while there is no lower limit, it follows that $0 \leq \frac{e^{\widetilde{\delta}}}{[1 + e^{\widetilde{\delta}}]^2} \leq \frac{1}{4}$ for all $\widetilde{\delta}$. Thus, from (5) we can derive that $0 \leq e_{t+1}^* \leq \frac{1}{8}(\delta_{t+1} - \widehat{\delta})^2$. Assuming that δ_t is stationary, such that the first and second moments exist, we can set the point of linearization, $\widehat{\delta}$, equal to the unconditional mean of δ_t . Thereby we obtain the following upper bound for the mean approximation error in log returns:

$$E(e_{t+1}^*) \leq \frac{1}{8}V(\delta_{t+1}). \quad (8)$$

From (8) it is seen that the higher the variance of the dividend-price ratio, the higher the upper bound for the mean approximation error.

The mean approximation error upper bound is minimized by setting the point of linearization equal to the unconditional mean of the log dividend-price ratio, $E(\delta)$. This follows by writing $E(\delta_{t+1} - \widehat{\delta})^2$ as:

$$E(\delta_{t+1} - \widehat{\delta})^2 = E[\delta_{t+1} - E(\delta)]^2 + E(E(\delta) - \widehat{\delta})^2 + 2E[(\delta_{t+1} - E(\delta))(E(\delta) - \widehat{\delta})].$$

Since the final term in this expression equals 0, we have that $E(\delta_{t+1} - \widehat{\delta})^2 = V(\delta_{t+1}) + (E(\delta) - \widehat{\delta})^2$. Thus, from (5) it follows that $E(e_{t+1}^*) \leq \frac{1}{8}[V(\delta_{t+1}) + (E(\delta) - \widehat{\delta})^2]$ which is minimized for $\widehat{\delta} = E(\delta)$. This gives support to the standard practice in much of the literature of setting ρ equal to $(1 + e^{E(\delta)})^{-1}$. Many applications of the Campbell-Shiller log-linear model use the version of the model where the linearization point is $E(\delta)$. However, Campbell and Shiller (1988a) themselves derived the log-linear return relation by linearizing around the mean dividend growth and mean log return, i.e. $\rho = e^{E(\Delta d - r)}$. In the static Gordon growth model with constant dividend growth and constant returns, it holds that $e^{E(\Delta d - r)} = (1 + e^{E(\delta)})^{-1}$. However, if dividend growth and returns vary over time the two approaches do not necessarily lead to the same ρ value. Some studies just pick more or less arbitrarily a value of ρ close to one. Naturally, the Taylor expansion can be done around any value; our analysis above shows that by defining ρ in terms of $E(\delta)$, the approximation error is minimized in the sense that the upper bound for its mean value is minimized.

e_{t+1}^* is the approximation error in one-period approximative log return computed from the log-linear relation, $r_{t+1}^* = \rho p_{t+1} + (1 - \rho)d_{t+1} - p_t + k = \delta_t - \rho\delta_{t+1} + \Delta d_{t+1} + k$. Alternatively, we may consider the approximate log dividend-price ratio, δ_t^* , as a function of δ_{t+1}^* and actual log returns, r_{t+1} , and dividend growth, Δd_{t+1} : $\delta_t^* = \rho\delta_{t+1}^* + r_{t+1} - \Delta d_{t+1} - k$. Solving this equation recursively forward for δ_t^* , we obtain the approximative log dividend-price ratio:

$$\delta_t^* = -\frac{k(1 - \rho^{T-t})}{1 - \rho} + \sum_{j=0}^{T-t-1} \rho^j (r_{t+1+j} - \Delta d_{t+1+j}) + \rho^{T-t} \delta_T^*. \quad (9)$$

The actual log dividend-price ratio is then given as $\delta_t = \delta_t^* - \sum_{j=0}^{T-t-1} \rho^j e_{t+1+j}^*$. Since $e_{t+1+j}^* \geq 0, \forall j$, it follows that $\delta_t \leq \delta_t^*$, and the approximation error in the log dividend-price ratio is the summation of one-period log return approximation errors, discounted by ρ . Thus, by construction the log dividend-price ratio approximation error is larger in absolute value than the log return approximation error, and highly persistent since ρ is close to unity. However, since $\rho < 1$ and $E(e_{t+1}^*)$ has a finite upper bound, $E(\delta_t - \delta_t^*)$ also has a finite upper bound. The upper bound for $E(\delta_t - \delta_t^*)$ is directly related to the upper bound for $E(e_{t+1}^*)$.

Campbell and Shiller (1988a) find - on data samples that end in 1986, i.e. a period where δ_t appears to be stationary - that the approximation error in log returns is on average less than 10 percent of r_t and not highly variable (standard deviation less than 3 percent of the standard deviation of r_t). The approximation error in the log dividend-price ratio is on average less than 4 percent of δ_t , and its standard deviation is less than 10 percent of the standard deviation of δ_t . Actual and approximate log returns have a correlation higher than 0.999, and actual and approximate log dividend-price ratios have a correlation higher than 0.98. Campbell and Shiller conclude that the approximation error "appears to be small in practice" (Campbell and Shiller, 1988a, p.198).

It is noteworthy, however, that in the particular datasets used by Campbell and Shiller, the mean approximation errors are in fact very close to their upper bounds given by (8): Campbell and Shiller report, in their Tables A1 and A2 for Cowles/S&P data from 1871 to 1986, and NYSE data from 1926 to 1986, sample variances of δ_t equal to 0.059 and 0.069, respectively, for the two datasets. Thus, the upper bounds for the mean approximation errors are $\frac{1}{8} \cdot 0.059 = 0.007$ and $\frac{1}{8} \cdot 0.069 = 0.009$. The sample average of e_{t+1}^* is 0.005 and 0.008 in the two datasets, respectively, so the average approximation errors are very close to their upper bounds, given stationarity of δ_t .

After the 1980s, the dividend-price ratio fell dramatically and, as mentioned in the introduction, some recent research indicates that stock prices became explosive by including data from the 1990s, possibly as a result of a speculative bubble. We now turn to analyzing the approximation error in the presence of an explosive bubble.

3 The approximation error under an explosive bubble

Intuitively we expect that the approximation error increases when prices are explosive and subject to bubbles. This is because the error depends on the distance of δ_t from the point of linearization, and under bubbles this distance can be large. This is also seen in the formulas in section 2 where the expected error depends on $E(\delta_t)$ and $V(\delta_t)$. When δ_t is non-stationary these unconditional moments depend on t , and their sample estimates are inconsistent. Thus, $E(e_{t+1}^*)$ in (8) has no finite upper bound.

If δ_t follows an explosive linear autoregressive process, the largest root of the autoregressive polynomial will be larger than one. For example, for the AR(1) model in section 4 below, equation (17), $\phi > 1$. When estimating this AR(1) model in a finite sample one will not necessarily estimate ϕ to be larger than one, even if the true ϕ is larger than one. It is well-known that in a finite sample the least squares estimate of the autoregressive parameter of a persistent variable is downward biased. Hence, even with a bubble in prices, ϕ may well be estimated to be below one such that δ_t appears stationary. In addition, some rational bubbles that have explosive conditional expectation, do not follow linear autoregressive processes and such bubbles may make prices look stationary. This is, for example, the case with the periodically collapsing bubble process suggested by Evans (1991). Thus, the econometrician working with a finite data sample of bubble-inflated prices may not discover the bubble, and the approximation error in (2) may be small despite the bubble.

We would, however, expect the approximation error to be large when the sample size is large, when the bubble becomes large relative to fundamentals, and when the bubble does not often burst. In order to study this, we simulate various bubble processes and then compute and compare actual and approximate log returns and log dividend-price ratios in a way similar to what Campbell and Shiller (1988a) did in their appendix. Diba and Grossman (1988b) simulated non-bursting bubbles from a simple linear explosive AR(1) process. In our simulation study we use instead the Evans (1991) model with a partially bursting rational bubble. Using a model setup that allows the bubble to burst appears to be intuitively and empirically more reasonable. However, when the burst probability approaches 0 the Evans bubble will be very similar to the Diba-Grossman type of bubble.

3.1 The Evans bubble and the simulation setup

Rational bubbles were much analyzed in the 1980s based on a standard present value model with a constant discount factor, i.e. constant expected returns, e.g. West (1987), Diba and Grossman (1988a,b), and Evans (1991). According to this model, stock prices are determined as

$$\begin{aligned} P_t &= F_t + B_t, \quad \text{where} \\ F_t &= E_t \sum_{i=1}^{\infty} \left(\frac{1}{1+R} \right)^i D_{t+i}, \\ B_t &= \frac{1}{1+R} E_t B_{t+1}. \end{aligned} \tag{10}$$

R is the constant expected arithmetic return, and B_t is the rational bubble term which evolves explosively over time if $R > 0$: $E_t B_{t+1} = (1+R)B_t$. If $B_t = 0$, there is no bubble and prices are determined only by expected future discounted dividends. However, if $B_t > 0$ there is a bubble and this will induce explosiveness into P_t .

Using the reparameterization in Campbell and Shiller (1987), (10) can be rewritten as

$$P_t - \frac{1}{R} D_t = \frac{1+R}{R} E_t \sum_{i=1}^{\infty} \left(\frac{1}{1+R} \right)^i \Delta D_{t+i} + B_t. \tag{11}$$

This reparameterization is useful if dividends have a unit root, $D_t \sim I(1)$, and there is no bubble, i.e. $B_t = 0$. In this case (11) shows that P_t will also have a unit root and be cointegrated with D_t such that $P_t - \frac{1}{R} D_t$ is stationary (P_t and D_t have a common stochastic trend). What happens if $D_t \sim I(1)$ and there is a bubble, $B_t > 0$? Since the 1980s several empirical researchers have claimed that in this case P_t and D_t should not cointegrate because $P_t - \frac{1}{R} D_t$ will be explosive. However, this is only partially correct. According to (11), the linear combination $P_t - \frac{1}{R} D_t$ will contain the explosive root from B_t , but it will *not* contain a unit root. P_t still shares the unit root (i.e. stochastic trend) with D_t , so in *this* sense they are still cointegrated, even though the linear combination is not stationary.¹

¹Interestingly, standard single-equation residual based cointegration techniques cannot be used to capture the common stochastic trend in P_t and D_t when P_t contains a bubble...

We follow Evans (1991, appendix) and simulate periodically collapsing bubbles in a model where the expected return is constant, as in (10), and where log dividends, d_t , follow a random walk with drift.² An appealing feature of this setup in the present context is that with no bubble, because the dividend-price ratio is constant, there is no approximation error. The log-linear return relation holds exactly. Thus, by adding a bubble we know that the induced approximation error is due only to the bubble component.

Prices are generated according to (10) where R is set equal to 0.05 (i.e. a yearly expected return of 5%), and log dividends follow the random walk with drift:

$$d_t = \mu + d_{t-1} + \varepsilon_t \quad , \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2). \quad (12)$$

In accordance with Evans (1991, appendix), we set μ equal to 0.013 and σ_ε^2 equal to 0.016. The fundamentals component of prices (i.e. the present value of expected future dividends) is

$$F_t = \frac{1 + \mu + \frac{1}{2}\sigma_\varepsilon^2}{R - \mu + \frac{1}{2}\sigma_\varepsilon^2} D_t. \quad (13)$$

Thus, with no bubbles the price-dividend ratio is constant over time. The bubble component is modeled as follows:

$$B_{t+1} = \left\{ \begin{array}{ll} (1 + R)B_t u_{t+1} & \text{if } B_t \leq \alpha \\ \left[\omega + \frac{1}{\pi}(1 + R)\theta_{t+1}(B_t - \frac{1}{1+R}\omega) \right] u_{t+1} & \text{if } B_t > \alpha \end{array} \right\} \quad (14)$$

ω and α are positive parameters with $0 < \omega < (1 + R)\alpha$, and u_{t+1} is an exogenous *iid* positive random variable with $E_t u_{t+1} = 1$, and θ_{t+1} is an exogenous *iid* Bernoulli process which is independent of u_{t+1} ; it takes the values 1 and 0 with probabilities π and $1 - \pi$, respectively.

The bubble in (14) is rational since it satisfies the restriction $E_t B_{t+1} = (1 + R)B_t$, i.e. it has explosive conditional expectation. However, it periodically collapses in the sense that when $B_t > \alpha$, it bursts with probability

(*cont.*) By contrast, the VAR-based Johansen (1991) technique will capture both the explosive root and the common stochastic trend. See Engsted (2006) and Engsted and Nielsen (2010) for a detailed discussion of these things.

²Evans (1991) also simulates bubbles from a model where the level of dividends, D_t , instead of log dividends, d_t , follow a random walk with drift. However, that model has the counterintuitive implication that the growth rate of dividends decreases over time. Hence, we only consider the case with d_t being a random walk with drift.

$1 - \pi$ every period, but when it bursts it does so only partially: it falls to a mean value of $\omega > 0$ from which it starts growing again.

By varying the parameters ω , α , and π , one can alter the frequency with which the bubble erupts, the average length of time before collapse, and the scale of the bubble. We follow Evans and set $\alpha = 1$, $\omega = 0.5$, and initial $B_t = \omega$. Also in accordance with Evans, we model u_{t+1} as a log-normal *iid* variable, i.e. $u_t = \exp(y_t - \frac{1}{2}\tau^2)$, where $y_t \sim N(0, \tau^2)$ and where we set τ equal to 0.05. We examine first the case where $\pi = 0.85$ which means that the bubble collapses with 15% probability every period when it gets above α . In order to obtain a bubble with a reasonable magnitude relative to the total valuation, we initially multiply the bubble component by a factor $\lambda = 100$. With this value of λ , and a sample size of $T = 100$, the bubble on average over the T observations represents roughly 50% of the total valuation, i.e. $\lambda B/P \approx 50\%$. Subsequently we let λ vary in the range between 1 and 250, see below.³ We also vary π in the interval between 0.65 to 0.99.

For the simulation experiment we simulate from Evans' model 10,000 time series of length $T = 100$ or 1,000 for dividends and prices, from which we calculate exact and approximate log returns and dividend-price ratios. We then compute the approximation error as the difference between exact and approximate values. Approximate log returns are computed as $r_{t+1}^* = \rho p_{t+1} + (1 - \rho)d_{t+1} - p_t + k$, and approximate log dividend-price ratios are computed as δ_t^* in equation (9). We follow Campbell and Shiller (1988a) and set δ_T^* equal to the exact end-sample value, δ_T . In all cases we set ρ equal to $(1 + e^{\bar{\delta}})^{-1}$, where $\bar{\delta}$ is the sample mean in the particular simulated sample. In the tables below all numbers are averages over the 10,000 simulations.

3.2 Results from the simulation study

In Table 1 we report the simulated distributions of exact and approximate log returns and log dividend-price ratios, and the implied approximation errors, from a bubble model where $T = 100$, $\pi = 0.85$, and $\lambda = 100$. This bubble accounts for on average 51.7% of the stock price. We measure the magnitude of the approximation error in two different ways, either as the percentage average error (which we denote E1) or as the average percentage

³In his simulations Evans (1991, appendix) used a factor of $\lambda = 250$, which implies that $\text{Var}(\Delta B_t)$ is approximately three times the sample mean of $\text{Var}(\Delta F_t)$. We have experimented with this alternative measure of the magnitude of the bubble, but we found it to be quite erratic and therefore decided to use the measure $\lambda B/P$ instead.

error (denoted E2). The former is computed simply as the "Approx. error" in Table 1 divided by "Exact". It measures the average over the T observations of the approximation error relative to the exact value. E2 measures instead the percentage error at each observation t , $t = 1, \dots, T$, i.e. $\text{Error}_t/\text{Exact}_t$, averaged over the T observations. The two measures do not necessarily give the same results.

As seen from Panel A in Table 1, exact and approximate log returns have very similar means and medians and they are extremely highly correlated. The mean approximation error amounts to either 4.58% or 7.21% of the exact log return, depending on whether E1 or E2 is used. The average standard deviations of exact and approximate log returns are also almost identical ($\approx 25.9\%$), so that the percentage error is basically 0%.⁴ Panel B in Table 1 shows that exact and approximate log dividend-price ratios are also extremely highly correlated, and the average approximation error is around 1% in both the mean, median, and standard deviation, and as measured by both E1 and E2.

Compared to Campbell and Shiller's (1988a) Table A1 and A2, which report approximation errors in US data, the approximation errors in the bubble model do not appear large. Thus, a periodically collapsing bubble which has explosive conditional expectation, and which on average accounts for around 50% of the stock price, does not on average generate large approximation errors in the Campbell-Shiller log-linear approximation. Figure 1 shows the fundamentals price, F_t , and the bubble-inflated price, P_t , for one particular simulation out of the 10,000, and Figure 2 shows the associated exact and approximate log dividend-price ratios.

In Table 2 we let the average size of the bubble vary by varying λ in the interval from 0 (i.e no bubble) to 250. With $T = 100$, the size of the bubble then varies from 0% to 69%. As in Table 1, the no-burst probability is $\pi = 0.85$. We see that for all bubbles exact and approximate log returns are almost perfectly correlated, and the difference between their standard deviations is completely negligible. The percentage approximation error in average log returns increases with increasing λ , to around 6.0% if measured by E1 and to around 7.8% if measured by E2. It is noticeable that the mean approximation error for a bubble that accounts for 69% of the price

⁴In the simulated model the average arithmetic return is $\bar{R} = 0.05$. If returns are log-normally distributed it holds that $\bar{R} = \bar{r} + \frac{1}{2}\sigma_r^2$. From Table 1 we get $\bar{r} + \frac{1}{2}\sigma_r^2 = 0.0262 + \frac{1}{2}(0.2587)^2 = 0.06$. Thus, the simulated returns are far from being log-normally distributed.

($\lambda = 250$) is only marginally larger than the error for a bubble that accounts for 52% of the price ($\lambda = 100$). The same pattern is observed when looking at the approximation error in the log dividend-price ratio (Panel B in Table 2). The percentage error never gets much above 1%. Again, compared to the approximation errors on actual US data reported by Campbell and Shiller (1988a) in their Tables A1 and A2, the approximation errors in the simulated bubble data with $T = 100$ observations are by no means large, not even for relatively large bubbles.

In Table 3 we set the sample size equal to $T = 1,000$ in the simulations. This means that for a fixed π ($=0.85$), we will see more bubble bursts than when $T = 100$. Thus, over the 1,000 observations the average size of the bubble relative to the price will be lower than in Table 2. For log returns the results are basically the same as for $T = 100$. The approximation error in average log returns increases slightly with increasing λ above 50, but never gets above 9%. The percentage error in the standard deviation of log returns is always completely negligible ($\approx 0.2\%$). Looking instead at the log dividend-price ratio, we now begin to see larger approximation errors. The percentage error is around 3.5% for $\lambda = 250$, which is somewhat higher than the 1% with $T = 100$ in Table 2, although it is still not seriously large, and exact and approximate log dividend-price ratios remain extremely highly correlated also for very large λ values. However, when looking at the standard deviations of the ratios the approximation error is now around 12% which indicates some difference in the volatilities of exact and approximate log dividend yields.

In Tables 4 and 5 we vary the bubble burst probability, $1 - \pi$, while keeping λ fixed at 100. In the simulations with $T = 100$ (Table 4), the approximation error remains low in both log returns and log dividend-price ratios when we increase π from 0.65 to 0.99. Interestingly, the errors in the average log return and the average log dividend yield reach their maximum not for $\pi = 0.99$ but for $\pi = 0.95$. When the sample size is increased to $T = 1,000$ (Table 5), the approximation error now becomes relatively large for $\pi \geq 0.95$. For very high values of π the bubble rarely collapses and in a large sample this apparently generates larger errors. The approximation error is maximized for $\pi = 0.99$ where the percentage error is above 30% for both log returns and log dividend yields. As also seen, the correlation between exact and approximate log dividend yields drops to 0.95 for $\pi = 0.99$. With respect to the variability of the series, the standard deviations of exact and approximate log returns remain extremely similar, but for exact and approximate log dividend yields the standard deviations are noticeably different (by more than 13% in some

cases).

The main conclusion from this analysis is that only in very large samples do we find evidence that rational bubbles, that have explosive conditional expectation, generate large approximation errors in the Campbell-Shiller log-linear return approximation.

4 Return predictability under bubbles

An interesting question is whether bubbles may explain the predictability of stock returns by the dividend-price ratio that several studies have reported over the years, e.g. Fama and French (1988), Campbell and Shiller (1988a,b), Lewellen (2004), Campbell and Yogo (2006), Cochrane (2008), Chen (2009), and Engsted and Pedersen (2010). Shiller (2000) argues that bubbles are the main reason for predictability of stock returns based on valuation ratios. However, most studies in this area consider return predictability a result of time-varying expected returns due to e.g. time-varying risk-premia, and rule out bubbles from the outset, see e.g. Lewellen (2004) and Cochrane (2008).

To investigate this issue, we will use the simulated bubble data from the previous section to analyze whether returns in these data are predictable from the dividend-price ratio. The predictability regressions are identical to those used by Cochrane (2008), which form a restricted first-order VAR system for log returns, log dividend growth, and the log dividend-price ratio. We now briefly describe Cochrane's setup before reporting our results using the simulated bubble data.

4.1 Cochrane's VAR setup with a bubble

Cochrane (2008) derives a number of implications that the Campbell-Shiller log-linear return approximation implies for a VAR model that looks as follows:

$$r_{t+1} = a_r + b_r \delta_t + \varepsilon_{t+1}^r, \quad (15)$$

$$\Delta d_{t+1} = a_d + b_d \delta_t + \varepsilon_{t+1}^d, \quad (16)$$

$$\delta_{t+1} = a_\delta + \phi \delta_t + \varepsilon_{t+1}^\delta. \quad (17)$$

Projecting on δ_t , the approximate identity (2) implies that the coefficients in (15) - (17) obey the linear relationship (ignoring the constant term, k)

$$b_r \approx 1 - \phi\rho + b_d. \quad (18)$$

On annual US data from 1926 to 2004, Cochrane finds the following parameter estimates, with standard errors in parentheses: $\widehat{b}_r = 0.097$ (0.050), $\widehat{b}_d = 0.008$ (0.044), and $\widehat{\phi} = 0.941$ (0.047). The estimated value of ρ is 0.9638. Based on these estimates it is seen that (18) holds almost exactly, indicating that the approximation error in (2) is negligible. The estimates also imply that the log dividend-price ratio is stationary (implying no bubbles), and that dividend growth is completely unpredictable from the dividend-price ratio while returns are predictable, which means that all variation in dividend yields comes from time-varying expected returns; nothing comes from time-varying expected dividend growth or bubbles. This is the interpretation offered by Cochrane.⁵

In section 4.1 in his paper, Cochrane discusses what a bubble would imply for the VAR system (15) - (17). From the ex ante version of the approximate identity (2), we can solve recursively forward for δ_t to get (again abstracting from the constant term k , and the fact that (2) only holds approximately)

$$\delta_t = E_t \sum_{i=0}^{\infty} \rho^i (r_{t+1+i} - \Delta d_{t+1+i}) + \lim_{h \rightarrow \infty} \rho^h E_t \delta_{t+h}. \quad (19)$$

If the last term in (19) goes to 0 when h goes to ∞ , there is no bubble. However, if the last term does not go to 0, then there is a bubble in prices in which case the log dividend-price ratio explodes to $-\infty$. δ_t will evolve as an explosive process with autoregressive parameter $1/\rho > 1$. Cochrane argues that this is so because under an explosive bubble $b_r = b_d = 0$, but being an identity (18) still has to hold whereby $\phi = 1/\rho$. With these parameter restrictions, (19) becomes

$$\begin{aligned} \delta_t &= \lim_{h \rightarrow \infty} \rho^h E_t \delta_{t+h} \\ &= \lim_{h \rightarrow \infty} \rho^h \phi^h \delta_t = \lim_{h \rightarrow \infty} \rho^h \left(\frac{1}{\rho} \right)^h \delta_t = \delta_t. \end{aligned}$$

⁵The b_r estimate of 0.097 is only marginally significant (and not significant based on a simulated p -value) if equation (15) is treated in isolation. However, Cochrane (2008) shows that given $b_d = 0$ and $\phi < 1$, the b_r estimate is highly statistically significant.

Thus, according to Cochrane, in principle a rational bubble could be consistent with the approximate identity (2), namely for $\phi = 1/\rho > 1$. However, Cochrane chooses to rule out bubbles based on both theoretical, empirical, and 'common sense' arguments: "This view [bubbles] is hard to hold as a matter of economic theory, so I rule it out on that basis" (p.1554). And: "do we really believe that dividend yields will wander arbitrarily far in either the positive or negative direction? Are we likely to see a market price-dividend ratio of one, or one thousand, in the next century or two?" (p.1555). Cochrane says "no" and, therefore, either expected dividends or expected returns (or both) must be the cause of price movements. In his empirical analysis Cochrane finds that dividend yields predict returns but do not predict dividend growth, and based on these findings he concludes that all price variation is due to changing expected returns; nothing comes from dividends or bubbles. In the next subsection we challenge this conclusion.

4.2 Return predictability in the simulated bubble data

We interpret the theoretical bubble literature differently than Cochrane, and we offer an alternative interpretation of the empirical results reported by Cochrane. First, rational bubbles are not completely ruled out in theoretical models. The restrictions that economic theory put on such bubbles are very tight (see e.g. Diba and Grossmann, 1988a), but in dynamically inefficient overlapping generations economies bubbles are not inconsistent with optimizing behavior, see e.g. Tirole (1985) and Abel et al. (1989). See also Santos and Woodford (1997) and Abreu and Brunnermeier (2003) for models in which bubbles may arise as an equilibrium phenomenon. Leroy (2004) provides a rational bubble interpretation of the behavior of US stock prices in the 1990s. Brunnermeier (2008) gives a brief survey of the literature on bubbles.

Second, bubbles do not necessarily imply price-dividend ratios as extreme as conjectured by Cochrane. For example, the Evans' (1991) type periodically collapsing stochastic rational bubble that we investigated in section 3, does not - despite having explosive conditional expectation - imply highly implausible price-dividend ratios. As Evans points out, in order to be empirically plausible a bubble needs to have a high chance of collapsing when reaching high levels. The characterizing feature of Evans' bubble is that it successively grows and bursts, but when it bursts it does so only partially so that it can continue to grow. Some of the rational bubbles that Evans and

we simulate give price-dividend ratios that stay within 'reasonable' limits.

In addition, recent empirical research does find explosiveness in US stock prices (Engsted, 2006; Engsted and Nielsen, 2010; Phillips et al., 2009), and explosive roots are contained in the confidence intervals for the largest autoregressive root in the US log price-dividend ratio reported by Campbell and Yogo (2006).

In order to examine whether Cochrane's finding of return (but no dividend) predictability by a stationary dividend-price ratio may in fact be the result of a rational bubble in a model with a constant expected return, we will estimate the three equations (15) - (17) on the bubble data that we simulated in section 3. Since the underlying bubble model implies unpredictable dividend growth, c.f. equation (12), and a constant expected return, R , any time-variation in δ_t and predictability of r_{t+1} from δ_t must come from the bubble. In Tables 6 and 7 we report the results of this exercise. Table 6 gives results for a sample size of $T = 100$, two different values of the bubble multiplication factor ($\lambda = 100$ or 250), and three different values of the no-burst probability ($\pi = 0.85, 0.65, \text{ or } 0.99$). Table 7 reports results for the same values of λ and π , but for a sample size of $T = 1,000$. The tables also report $\lambda B/P$ as a measure of the average size of the bubble relative to the total valuation of the stock. The "implied" column computes each of the regression coefficients b_r , b_d , and ϕ from the other two and the identity (18). By comparing the estimated and implied values we get a sense of the importance of the approximation error for the predictability results. The tables are directly comparable to Cochrane's Table 2 which reports estimates of the system (15) - (17) on annual US data from 1926 to 2004.

Panel (a) in Table 6 shows that in a sample with 100 observations and a bubble that collapses with probability 15% every period when it gets above some threshold value, and that on average accounts for 52% of the stock price, regressing log return, r_{t+1} , onto the log-dividend-price ratio, δ_t , gives an average coefficient of $b_r = 0.117$ with standard error $\sigma = 0.065$ over the 10,000 simulation runs. These estimates match very closely Cochrane's estimates on US data ($b_r = 0.097$, $\sigma = 0.050$). The estimates of the dividend growth coefficients in the simulated data are $b_d = -0.026$ and $\sigma = 0.039$, to be compared to Cochrane's, $b_d = 0.008$ and $\sigma = 0.044$. In both cases the b_d estimates are close to 0 and statistically insignificant. Our estimate of δ_t 's autoregressive coefficient ($\phi = 0.866$ with $\sigma = 0.062$) is, however, lower than Cochrane's estimate ($\phi = 0.941$ with $\sigma = 0.047$), although a value of 0.866 still implies a high degree of persistence. The estimate of ϕ below 1

illustrates Evans' (1991) point that in a finite sample it will be difficult to detect a rational periodically collapsing bubble that has explosive conditional expectation.

Panels (b) and (c) in Table 6, and Panels (a) to (c) in Table 7, show that the above results also hold for larger bubbles, for bubbles that collapse more often, and when the sample size increases. In all cases the b_r estimates are positive and statistically significant at the 5% level (or marginally significant), the b_d estimates are close to 0 and insignificant, and the ϕ estimates are slightly below 1. Only if the bubble burst probability $1 - \pi$ approaches 0 does return predictability disappear, as seen in Panels (d) in the tables where $\pi = 0.99$. These results imply that plausible periodically collapsing rational bubbles in a world where expected returns are constant, may generate return predictability in linear regressions of log returns onto log dividend-price ratios, thus offering an alternative interpretation of the findings of Cochrane and many others in which the predictability is interpreted as reflecting time-varying expected returns.

By comparing the estimated coefficients with the "implied" coefficients in Tables 6 and 7, we see that in all cases the differences are very small. The implied coefficients are computed using the relation $b_r = 1 - \phi\rho + b_d$ which only holds approximately, c.f. equation (18). In the results with $T = 100$, the implied coefficients deviate from the estimated coefficients by only 0.001. In the results with $T = 1,000$, implied and estimated coefficients deviate by at most 0.008. Thus, it seems that with respect to predictability regressions, the approximation error in the log-linear return approximation is completely negligible, even in cases with substantial volatility in the log dividend-price ratio.

5 Concluding remarks

In this paper we have investigated in detail the log-linear return approximation of Campbell and Shiller (1988a) which has become one of the cornerstones of empirical finance. First, we have derived an upper bound for the mean approximation error, given stationarity of the log dividend-price ratio, δ_t , and we have shown that the minimum upper bound is obtained by setting the point of linearization equal to the unconditional mean of δ_t . This gives support to the usual practice in the empirical literature using the Campbell-Shiller approximation of defining the 'discount factor', ρ , in the

approximation in term of the sample mean of δ_t .

Second, we have studied the properties of the Campbell-Shiller approximation in the presence of rational explosive bubbles. We have done that by simulating various periodically collapsing explosive bubbles of the Evans (1991) type, and then investigated the approximation error in each case. We find - perhaps surprisingly - that unless the sample size is very large, such bubbles do not induce large approximation errors in the log-linear relation. Thus, in practice in a given finite sample, the presence of a speculative bubble which induces explosive components into prices does not necessarily destroy the Campbell-Shiller approximative relation as a useful framework for empirical analysis.

Third, within the log-linear approximative framework we have investigated whether periodically collapsing explosive bubbles may explain the common finding of predictability of log stock returns from the log dividend-price ratio. Using the simulated bubble data from a model where expected returns are constant, we find that indeed log returns, r_{t+1} , appear significantly predictable from δ_t , and that δ_t appears stationary. In fact, for a number of empirically plausible bubbles, we are able to match very closely the predictability results reported recently by Cochrane (2008), thus offering an alternative interpretation of Cochrane's results. Cochrane interprets his results as implying that variation in dividend yields reflects only time-varying expected returns, with nothing coming from dividends or bubbles. Our simulations show that Cochrane's results may equally well be explained as a result of bubbles.

Naturally, in reality Cochrane may well be correct in arguing that expected returns are time-varying and that bubbles play no role. Our results just show that in principle the presence of a bubble in a model with constant expected returns may also lead to the dividend yield appearing stationary and having predictive ability for future stock returns. Thus, Cochrane's findings cannot be used to rule out bubbles.

Finally, our predictability results on the simulated data show that with regard to common return predictability regressions, the approximation error in the Campbell-Shiller log-linear approximation is completely negligible.

The main conclusion from our analyses is that the Campbell-Shiller approximation appears highly accurate and robust, even when the log dividend-price ratio is highly volatile and contains non-stationary components.

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7 Tables

	Exact	Approx.	Approx. error	Percent error, E1	Percent error, E2
A: Log return					
Mean	0.0262	0.0250	0.0012	4.58%	7.21%
Median	0.0252	0.0239	0.0013	5.16%	4.06%
Std.dev.	0.2587	0.2586	0.0001	0.04%	0.07%
Corr(r, r^*) = 1.0000					
B: Log dividend-price					
Mean	-4.4810	-4.4306	-0.0504	1.12%	1.09%
Median	-4.3925	-4.3458	-0.0467	1.06%	0.77%
Std.dev.	0.5342	0.5284	0.0058	1.09%	1.20%
Corr(δ, δ^*) = 0.9992					

Notes: The table reports the mean, median, standard deviation, and correlation of exact and approximate log returns (r_t and r_t^*) and exact and approximate log dividend-price ratios (δ_t and δ_t^*), using the simulated data from the bubble model (10), (12), (13), and (14). Approximate log returns are computed as $r_{t+1}^* = \rho p_{t+1} + (1 - \rho)d_{t+1} - p_t + k$, and approximate log dividend-price ratios are computed as δ_t^* in equation (9). ρ is calculated as $\rho = (1 + \exp(\bar{\delta}))^{-1}$, where $\bar{\delta}$ is the average log dividend-price ratio in the particular simulation run. "Approx. error" is obtained as "Exact" minus "Approx.". "Percent error, E1" gives the percentage average error, computed as "Approx. error" divided by "Exact". "Percent error, E2" gives the average percentage error, computed as the percentage error at each observation averaged over the $T = 100$ observations. The numbers in the table are averages over 10,000 simulations.

Table 1: Simulated distribution of the approximation error. No-burst probability $\pi = 0.85$; bubble factor $\lambda = 100$; sample size $T = 100$.

	Bubble factor, λ						
	0	1	50	100	150	200	250
$\lambda B/P$	0.00%	2.59%	38.2%	51.7%	59.7%	65.1%	69.0%
A: Log return:							
Corr(r, r^*)	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
E1 (avg r)	0.00%	0.25%	3.92%	4.58%	5.53%	6.02%	5.97%
E2 (avg r)	0.00%	0.61%	6.49%	7.21%	6.45%	7.75%	7.79%
E1 (sd r)	0.00%	0.00%	0.09%	0.04%	0.04%	0.03%	0.00%
E2 (sd r)	0.00%	0.02%	0.09%	0.07%	0.05%	0.04%	0.03%
B: Log div/price:							
Corr(δ, δ^*)	-	0.9998	0.9991	0.9992	0.9993	0.9993	0.9994
E1 (avg δ)	0.00%	0.11%	0.99%	1.12%	1.15%	1.13%	1.10%
E2 (avg δ)	0.00%	0.11%	0.93%	1.09%	1.11%	1.12%	1.11%
E1 (sd δ)	0.00%	0.78%	0.99%	1.09%	1.16%	1.17%	1.20%
E2 (sd δ)	0.00%	0.27%	1.10%	1.20%	1.24%	1.25%	1.25%

Notes: The table reports percentage approximation errors (E1 and E2) for the mean and standard deviation of log returns ("avg r " and "sd r "), and for the mean and standard deviation of log dividend-price ratios ("avg δ " and "sd δ "). The table also reports the correlation between exact and approximate values. $\lambda B/P$ is the average size of the bubble relative to the total valuation of the stock. Otherwise see the notes to Table 1.

Table 2: Approximation error in the simulated data with varying bubble size (λ). No-burst probability $\pi = 0.85$; sample size $T = 100$.

	Bubble factor, λ						
	0	1	50	100	150	200	250
$\lambda B/P$	0.00%	0.70%	9.62%	13.5%	16.0%	17.9%	19.4%
A: Log return:							
Corr(r, r^*)	1.0000	1.0000	0.9998	0.9997	0.9997	0.9996	0.9996
E1 (avg r)	0.00%	0.25%	2.67%	4.16%	5.68%	6.96%	7.96%
E2 (avg r)	0.00%	0.14%	2.99%	4.92%	6.45%	7.73%	8.84%
E1 (sd r)	0.00%	0.00%	0.18%	0.22%	0.26%	0.20%	0.19%
E2 (sd r)	0.00%	0.02%	0.17%	0.20%	0.21%	0.22%	0.22%
B: Log div/price:							
Corr(δ, δ^*)	-	0.9981	0.9890	0.9873	0.9864	0.9858	0.9854
E1 (avg δ)	0.00%	0.05%	1.13%	1.90%	2.53%	3.08%	3.57%
E2 (avg δ)	0.00%	0.05%	1.08%	1.79%	2.38%	2.89%	3.34%
E1 (sd δ)	0.00%	5.30%	10.2%	11.3%	11.7%	12.0%	12.0%
E2 (sd δ)	0.00%	1.59%	9.17%	11.1%	12.0%	12.5%	12.8%

Notes: The table reports percentage approximation errors (E1 and E2) for the mean and standard deviation of log returns ("avg r " and "sd r "), and for the mean and standard deviation of log dividend-price ratios ("avg δ " and "sd δ "). The table also reports the correlation between exact and approximate values. $\lambda B/P$ is the average size of the bubble relative to the total valuation of the stock. Otherwise see the notes to Table 1.

Table 3. Approximation error in the simulated data with varying bubble size (λ). No-burst probability $\pi = 0.85$; sample size $T = 1,000$.

	No-burst probability, π				
	0.65	0.75	0.85	0.95	0.99
$\lambda B/P$	48.2%	49.3%	51.7%	60.0%	72.3%
A: Log return:					
Corr(r, r^*)	1.0000	1.0000	1.0000	0.9999	0.9997
E1 (avg r)	3.49%	3.86%	4.58%	5.80%	4.26%
E2 (avg r)	4.59%	5.54%	7.21%	8.74%	6.15%
E1 (sd r)	0.12%	0.12%	0.04%	0.04%	1.41%
E2 (sd r)	0.11%	0.10%	0.07%	0.06%	0.15%
B: Log div/price:					
Corr(δ, δ^*)	0.9994	0.9993	0.9992	0.9991	0.9995
E1 (avg δ)	0.70%	0.84%	1.12%	1.55%	1.21%
E2 (avg δ)	0.68%	0.82%	1.09%	1.49%	1.22%
E1 (sd δ)	1.02%	1.07%	1.09%	0.51%	1.22%
E2 (sd δ)	1.06%	1.13%	1.20%	0.93%	0.72%

Notes: The table reports percentage approximation errors (E1 and E2) for the mean and standard deviation of log returns ("avg r " and "sd r "), and for the mean and standard deviation of log dividend-price ratios ("avg δ " and "sd δ "). The table also reports the correlation between exact and approximate values. $\lambda B/P$ is the average size of the bubble relative to the total valuation of the stock. Otherwise see the notes to Table 1.

Table 4: Approximation error in the simulated data with varying no-burst probability (π). Bubble factor $\lambda = 100$; sample size $T = 100$.

	No-burst probability, π				
	0.65	0.75	0.85	0.95	0.99
$\lambda B/P$	11.8%	12.3%	13.5%	18.8%	41.5%
A: Log return:					
Corr(r, r^*)	0.9998	0.9998	0.9997	0.9994	0.9996
E1 (avg r)	3.01%	3.30%	4.16%	9.51%	29.6%
E2 (avg r)	3.37%	3.82%	4.92%	10.8%	32.1%
E1 (sd r)	0.23%	0.23%	0.22%	0.14%	0.05%
E2 (sd r)	0.21%	0.21%	0.20%	0.14%	0.03%
B: Log div/price:					
Corr(δ, δ^*)	0.9924	0.9908	0.9873	0.9759	0.9529
E1 (avg δ)	1.22%	1.41%	1.90%	5.35%	37.1%
E2 (avg δ)	1.15%	1.33%	1.79%	4.98%	32.4%
E1 (sd δ)	10.4%	10.6%	11.3%	13.6%	9.50%
E2 (sd δ)	10.1%	10.4%	11.1%	13.5%	11.6%

Notes: The table reports percentage approximation errors (E1 and E2) for the mean and standard deviation of log returns ("avg r " and "sd r "), and for the mean and standard deviation of log dividend-price ratios ("avg δ " and "sd δ "). The table also reports the correlation between exact and approximate values. $\lambda B/P$ is the average size of the bubble relative to the total valuation of the stock. Otherwise see the notes to Table 1.

Table 5: Approximation error in the simulated data with varying no-burst probability (π). Bubble factor $\lambda = 100$; sample size $T = 1,000$.

	(a)			(b)		
	$\lambda = 100$	$\pi = 0.85$	$\lambda B/P = 52\%$	$\lambda = 250$	$\pi = 0.85$	$\lambda B/P = 69\%$
	$\widehat{b}, \widehat{\phi}$	$\widehat{\sigma}$	implied	$\widehat{b}, \widehat{\phi}$	$\widehat{\sigma}$	implied
r	0.117	0.065	0.118	0.113	0.057	0.113
Δd	-0.026	0.039	-0.027	-0.019	0.026	-0.020
δ	0.866	0.062	0.867	0.873	0.060	0.874

	(c)			(d)		
	$\lambda = 100$	$\pi = 0.65$	$\lambda B/P = 48\%$	$\lambda = 100$	$\pi = 0.99$	$\lambda B/P = 72\%$
	$\widehat{b}, \widehat{\phi}$	$\widehat{\sigma}$	implied	$\widehat{b}, \widehat{\phi}$	$\widehat{\sigma}$	implied
r	0.167	0.090	0.169	0.023	0.041	0.022
Δd	-0.037	0.045	-0.038	-0.014	0.027	-0.013
δ	0.805	0.087	0.806	0.968	0.041	0.967

Note: The table reports estimates of b_r , b_d , and ϕ (and associated standard errors, σ) in the system (15)-(17), using the simulated data from the bubble model (10), (12), (13), and (14). The numbers are averages of regressions over 10,000 simulated series with $T = 100$ observations in each. "implied" denotes the calculated coefficient based on the other two coefficients and the identity (18), using $\rho = (1 + \exp(\bar{\delta}))^{-1}$. The values of ρ in parts (a), (b), (c), and (d) are: 0.9888, 0.9935, 0.9872, and 0.9954. λ is the bubble multiplication factor. $1 - \pi$ is the probability that the bubble will burst every period. $\lambda B/P$ is the average size of the bubble relative to the total valuation of the stock.

Table 6: Predictability regressions on the simulated bubble data. Sample size $T = 100$.

	(a)			(b)		
	$\lambda = 100$	$\pi = 0.85$	$\lambda B/P = 13\%$	$\lambda = 250$	$\pi = 0.85$	$\lambda B/P = 19\%$
	$\widehat{b}, \widehat{\phi}$	$\widehat{\sigma}$	implied	$\widehat{b}, \widehat{\phi}$	$\widehat{\sigma}$	implied
r	0.059	0.021	0.067	0.048	0.014	0.055
Δd	-0.009	0.017	-0.017	-0.005	0.010	-0.012
δ	0.945	0.017	0.953	0.958	0.014	0.965

	(c)			(d)		
	$\lambda = 100$	$\pi = 0.65$	$\lambda B/P = 12\%$	$\lambda = 100$	$\pi = 0.99$	$\lambda B/P = 86\%$
	$\widehat{b}, \widehat{\phi}$	$\widehat{\sigma}$	implied	$\widehat{b}, \widehat{\phi}$	$\widehat{\sigma}$	implied
r	0.077	0.034	0.083	0.017	0.008	0.015
Δd	-0.012	0.020	-0.018	-0.0015	0.004	-0.0002
δ	0.926	0.032	0.932	0.987	0.006	0.986

Note: The table reports estimates of \widehat{b}_r , \widehat{b}_d , and $\widehat{\phi}$ (and associated standard errors, σ) in the system (15)-(17), using the simulated data from the bubble model (10), (12), (13), and (14). The numbers are averages of regressions over 10,000 simulated series with $T = 1,000$ observations in each. "implied" denotes the calculated coefficient based on the other two coefficients and the identity (18), using $\rho = (1 + \exp(\overline{\delta}))^{-1}$. The values of ρ in parts (a), (b), (c), and (d) are: 0.9781, 0.9811, 0.9771, and 0.9960. λ is the bubble multiplication factor. $1 - \pi$ is the probability that the bubble will burst every period. $\lambda B/P$ is the average size of the bubble relative to the total valuation of the stock.

Table 7: Predictability regressions on the simulated bubble data. Sample size $T = 1,000$.

Fundamentals and Price

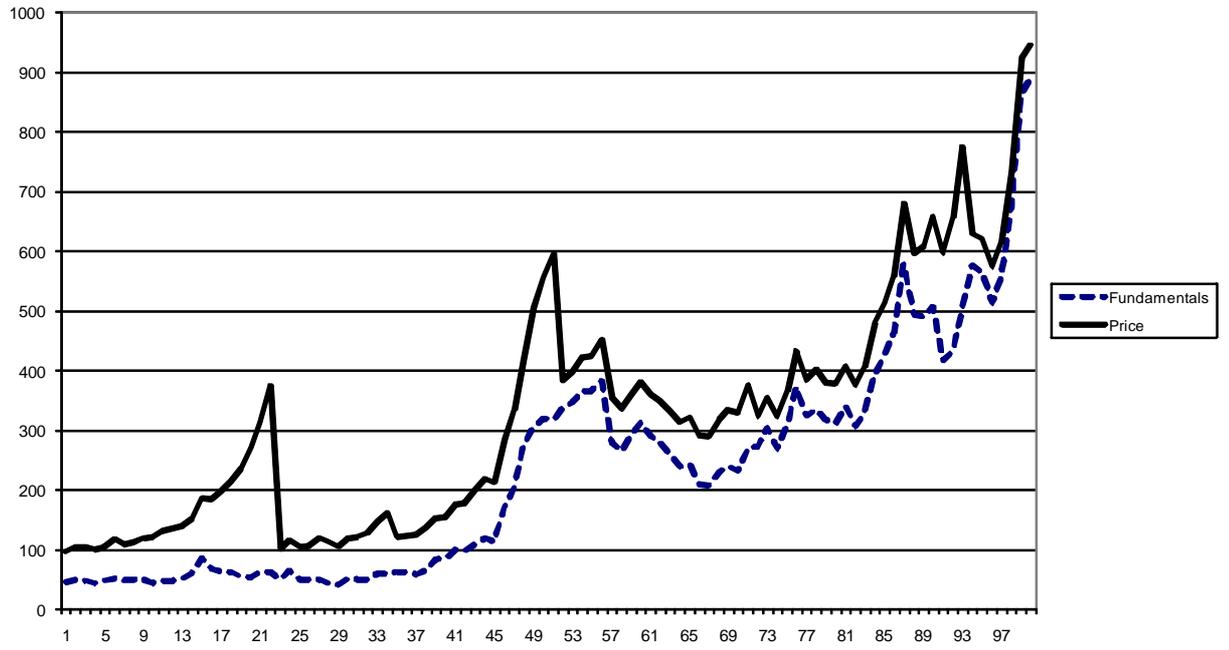


Figure 1: Simulated periodically collapsing bubble

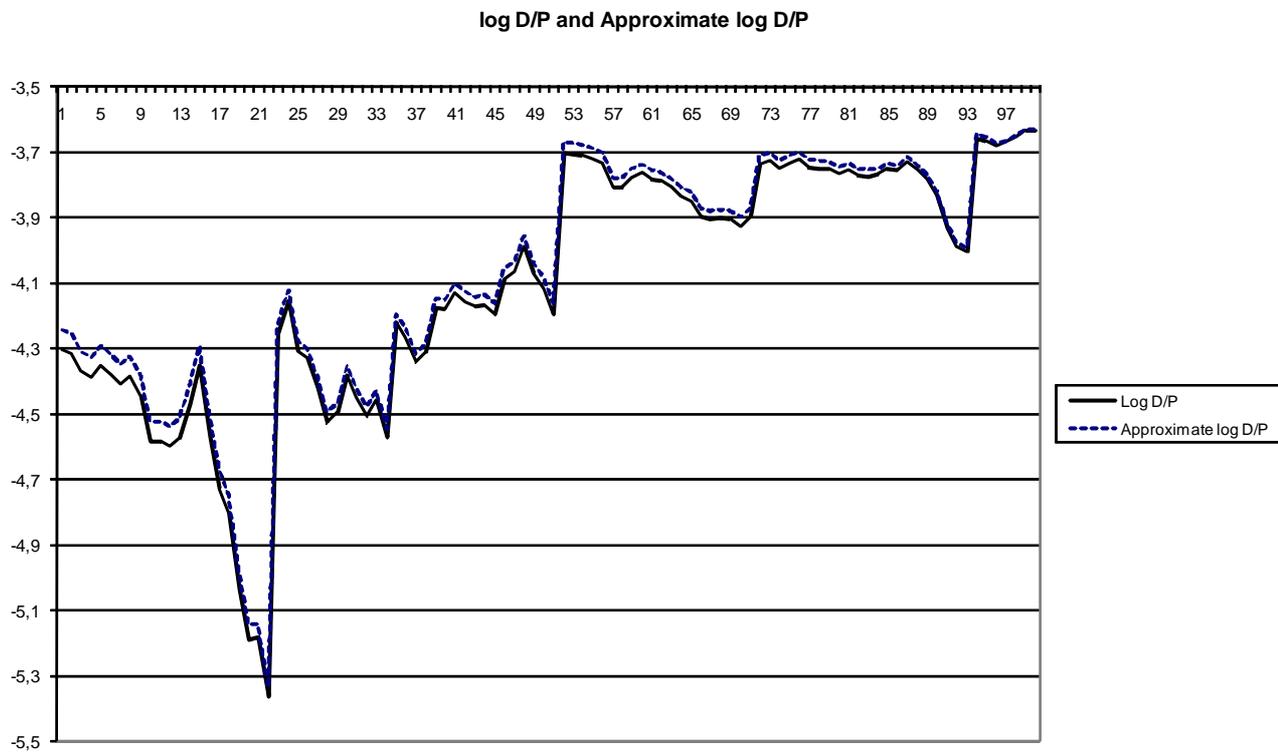


Figure 2: Exact and approximate log dividend-price ratio, computed from the simulated periodically collapsing bubble

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