Developing mathematical modelling competence: conceptual clarification and educational planning

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Abstract

In this paper we introduce the concept of mathematical modelling competence, by which we mean being able to carry through a whole mathematical modelling process in a certain context. Analysing the structure of this process, six sub-competences are identified. Mathematical modelling competence cannot be reduced to these six sub-competences, but they are necessary elements in the development of mathematical modelling competence. Experience from the development of a modelling course is used to illustrate how the different nature of the sub-competences can be used as a tool for finding the balance between different kinds of activities in a particular educational setting. Obstacles of social, cognitive and affective nature for the students' development of mathematical modelling competence are reported and discussed in relation to the sub-competences.

1. Introduction

Consider the formulation of the following two tasks in relation to a mathematical modelling course at university level:

1. How should anaesthetic be administered during surgery?
2. Anaesthetic is given during surgery to keep the patient unconscious and relaxed. Typically the drug is given by injection or drip (an intravenous line) into the bloodstream, or by inhalation. The drug spreads into the brain and muscle tissue where it functions, and with time is decomposed and eliminated. The graph in Fig. 1 shows the logarithm of the bloodstream concentration of pancuronium (an anaesthetic) as a function of time after injection of 4 mg into a person with an estimated blood volume of 5.2 l. During surgery the concentration of pancuronium in the muscle tissue must be between 250 and 300 μg/l. Build a model that describes the interchange of pancuronium between the blood and the tissue, and use the model to find a way to drug the patient so that the condition is fulfilled during a 2 h time slot.

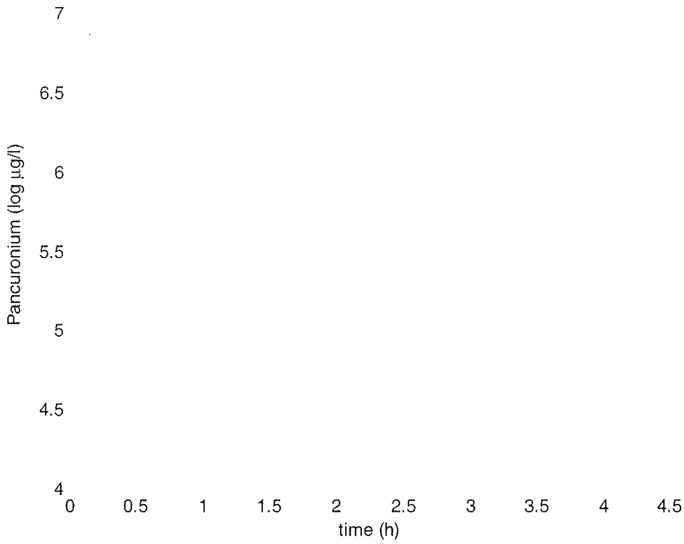

Fig 1. Concentration of pancuronium in the blood [log(µg/l)] as function of time since injection of 4 mg. The measured data are represented by the stars.

We use these two tasks when expanding on the following synopsis:

During the last few years we, together with two other colleagues (Johnny Ottesen and Tinne Hoff Kjeldsen, IMFUFA, Roskilde University), have developed and taught a course for freshmen at Roskilde University (see Blomhøj *et al.*, 2001 for a thorough characterisation and evaluation of the 1999/2000 course). The main objective of the course is to support the development of the students' mathematical modelling competence, which is a developmental parallel to one of our core research interests.

In this paper we concentrate on one aspect of the symbiosis; the interaction between conceptual clarification, dealt with in Section 2, and the characterization and weighing of different types of tasks as part of curriculum design, which we discuss in Section 3, first in general terms and then with explicit reference to our experiences from the developmental work. In Section 4 we briefly sum up our findings.

2. Conceptual clarification

In the two next sections we clarify our conceptual understanding of mathematical modelling and of mathematical modelling competence.

2.1. Mathematical modelling

In our work we refer to a description of the creation and use of a mathematical model consisting of the following six sub-processes (see Niss, 1989; Blomhøj, 1993; Gregersen and Jensen, 1998):

(a) Formulation of a task (more or less explicit) that guides you to identify the characteristics of the perceived reality that is to be modelled.

(b) Selection of the relevant objects, relations etc. from the resulting domain of inquiry, and idealisation of these in order to make possible a mathematical representation.

(c) Translation of these objects and relations from their initial mode of appearance to mathematics.

(d) Use of mathematical methods to achieve mathematical results and conclusions.

(e) Interpretation of these as results and conclusions regarding the initiating domain of inquiry.

(f) Evaluation of the validity of the model by comparison with observed or predicted data or with theoretically based knowledge.

Figure 2 is a depiction of this process. It contains a labelling of the sub-processes as well as an attempt to evaluate the six stages framing these. The sub-processes and stages can be illustrated with reference to the context of anaesthetic:

The 'perceived reality' could be a phenomenological understanding of the fact that drugging a patient during surgery is a dynamic process which must be balanced between what is needed in order to conduct the surgery without causing pain and in order not to overdose the patient. Formulation of a task, process (a), such as answering the question 'What dose gives an optimal drug concentration during surgery' narrows the focus on the perceived reality, thus guiding the mathematical modelling process. In this context the 'domain of inquiry' may be framed by the question: 'What may possibly influence the concentration of drug in the patient during surgery?'. The process of systematization, process (b), should identify what is really the essential mechanism of the diffusion and elimination of drug in the human

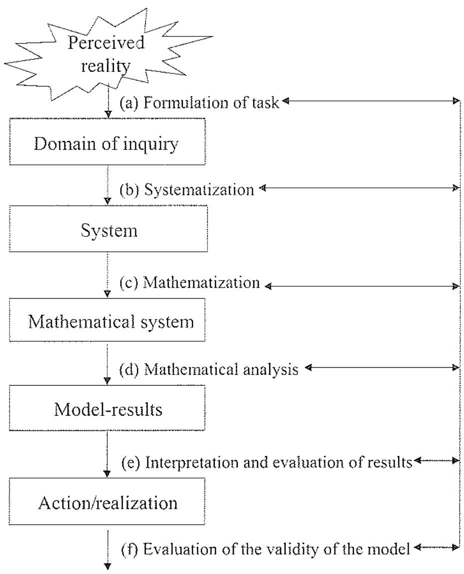

Fig 2. A graphic model of a mathematical modelling process.

body. In order to keep it simple the patient could then be described as a two (or more) compartment 'system', e.g. the blood volume and the volume of tissue, in which exchange of drugs can take place. A simple mathematisation, process (c), of this system would lead to a 'mathematical system' consisting of a pair of coupled linear differential equations. Analysing this 'mathematical system', process (d), could give rise to ideas about how to estimate model parameters and subsequently produce 'model results' in the form of numerical solutions showing the model responses—as graphs of drug concentration during surgery—to different anaesthetic strategies. The result must be interpreted and validated against empirical data, process (e), in order to support certain 'actions', e.g. a proposal for an anaesthetic strategy to, or the formulation of new insights into, the phenomena of anaesthetics. Last but not least, the validity of the entire mathematical modelling process should be evaluated, process (f). This includes questioning the extent and the validity of using the model (and the parameters) for other patients and/or with other drugs. New empirical data are needed for this process.

It must be emphasised that this is a model of an ideal mathematical modelling process focusing primarily on the structural aspects of the process. The model can be used both as a tool for analysing mathematical models and the mathematical modelling processes behind them, and, as shown in this paper, as a tool for normatively defining and analyzing 'mathematical modelling competence', see Fig. 2.

By 'mathematical modelling' we mean going through the entire process described above. Not necessarily as a 'one-way tour' from beginning to end; in fact it will often make more sense to go backwards and repeat some of the phases, or to go through all of them several times, as indicated by the arrows to the right hand side of Fig. 2. This is done not necessarily in a conscious and controlled manner; the better and more experienced you are at mathematical modelling, the more you do this automatically. But for us mathematical modelling means working with all aspects mentioned, one way or the other.

2.2. Mathematical modelling competence

By mathematical modelling competence we mean being able to autonomously and insightfully carry through all aspects of a mathematical modelling process in a certain context (cf. Fig. 2).

This phrasing coheres with our understanding of the concept competence, which we will define as someone's insightful readiness to act in a way that meets the challenges of a given situation (Jørgensen, 1999).

Several characteristics of this definition must be mentioned in order to prepare ourselves for the discussion to come. Firstly, competence is headed for action. We use 'action' in a broad sense, as the term 'readiness to act' in our definition of competence could imply a positive decision to refrain from performing a physical act, or indirectly being guided by one's awareness of certain features in a given situation. But no competence follows from being immensely insightful, if this insight cannot be activated in this broad interpretation of the word action.

Secondly, all competences have a sphere of exertion, i.e. a domain within which the competence can be brought to maturity. This does not mean that a competence is contextually tied to the use of a specific method for solving a given task. If this was the case, our attempt to define general competences would have no meaning. Competences are only contextual in the sense that they are framed by the historical, social, psychological etc. circumstances of the 'given situation' mentioned in our definition of competence (cf. Wedege, 1999).

Thirdly, competence is an analytic concept with an inherent duality between a subjective and a social/cultural side (Wedege, 2000). Subjective because a competence always belongs to someone; competences do not exist by themselves—what exists are competent people. Social/cultural because the degree to which some actions 'meet the challenges' (cf. our definition of competence) is always relative to the surroundings, adding meaning and legitimacy to the actions (Jørgensen, 1999).

3. Educational planning and the development of mathematical modelling competence

3.1. A necessary balance

We now return to the tasks mentioned in the introduction in order to characterise these in the light of the given description of a mathematical modelling process.

The first task challenges the subject to work with all the mathematical modelling sub-processes, and we will therefore label it an invitation to mathematical modelling. As such the task is inherently 'underdetermined' and open-ended, which—we will argue—is the key characteristic of the challenge experienced by the subject when facing this kind of task. One is left with a feeling that can be characterized as 'perplexity due to too many roads to take and no compass given'.

Contrary to the above, we will argue that the second task only challenges the subject to work with sub-processes (c), (d) and (e). The delimitation of the context and task in sub-processes (a) and (b) is already dealt with in the formulation of the task, and the inclination to work with sub-process (f) comes from having worked with sub-processes (a) and (b). Tasks like this challenge the subject's ability to mathematise a more or less well-defined problem of a non-mathematical character, and therefore often give the subject a feeling of 'knowing what the goal is without knowing how to achieve it'. These characteristics make it relevant to turn to research done in the field of mathematical problem solving. The vast majority of this research approaches learning from the perspective of cognitive science (see e.g. Schoenfeld, 1987). Based on this one would expect mathematising and analysing mathematical models to be cognitively demanding activities for students—even in situations where the involved mathematical concepts are well known to them.

From a teaching point of view it is therefore reasonable to focus on sub-competences (c) and (d) of the mathematical modelling process. In doing so, the teacher will have to face the challenge of constructing teaching situations that enable the students to engage actively in mathematising and analysing models. Guided by the intention of working with problems that are realistic and representative in relation to mathematical modelling in general, the problems should therefore reflect the students' mathematical knowledge as well as their experience and knowledge about the contexts of the problems. With such problems at hand, some research demonstrates that it is difficult, but indeed possible, to carry out teaching that enables the students to improve significantly in their ability to perform mathematical problem solving (see Arcavi *et al.*, 1998).

However, full scale mathematical modelling also includes competences related to sub-processes (a), (b), (e) and (f). Teaching and learning difficulties connected to the development of these sub-competences cannot be fully analysed within the same theoretical framework as used in relation to processes (c) and (d). Formulation of problems, structuring complex situations and advancing critique of a mathematical modelling process and of the possible uses of a mathematical model bring forward another type of obstacle of a more social nature. This has to do with the students' difficulties in learning 'the game of mathematical modelling', so to speak. It can be illustrated by some of the students' questions: 'What is the meaning of this modelling business?', 'What kind of knowledge comes out of it?', 'What are we supposed to learn?', 'How do you know when you are on thin ice or on the right track?', 'How can one disregard important aspects just because one cannot describe them mathematically?', 'What makes one model better than the other?', and 'How can you put forward critique of your own model?'. This domain of obstacles could be analysed under the perspectives of the strong socialisation provided by the students' previous mathematics education. We find this very interesting and we want to investigate further how the problem of socialising the students into mathematical modelling as a special form of activity can be understood using the theory of situated learning (Lave and Wenger, 1991).

In this sense the two tasks mentioned in the introduction are basically different in nature. The two tasks may contribute in different but important ways to the development of mathematical modelling competence, and the connected learning problems may be analysed from different theoretical perspectives. In relation to teaching practice this leads to the question of how the two types of tasks are balanced against each other in a given educational setting and with certain educational goals in mind.

3.3. Two extreme positions

With the objective of developing the students' mathematical modelling competence through course teaching, two extreme positions seem possible. One argument—the holistic approach—could be that all available resources must be used in working with full-scale mathematical modelling processes. By definition mathematical modelling includes the processes (a)–(f) illustrated in Fig. 2, and therefore the students must have the opportunity to work with all these processes. If some processes are not apparent in the students' activities, one might expect that they will miss important sub-competences when it comes to mathematical modelling in new contexts. If the students always work with pre-structured problems, they cannot be expected to develop competences in structuring a complex domain of enquiry. Moreover, full-scale mathematical modelling may give authenticity to the students' work and this might be a motivating factor.

At the other extreme position—the atomistic approach—the argument could be that a course aimed at developing students' mathematical modelling competence must be concentrated on the processes of mathematising and analysing models mathematically. To support this position, one can point to the fact that activities connected to these elements of the mathematical modelling process can be seen as a way of learning mathematics. Through such activities the mathematical concepts in play gain new meaning for the students (Ottesen, 2000).

Moreover, working with full-scale mathematical modelling is a very time consuming way of learning. Due to both affective factors and lack of factual knowledge, insignificant experience with the real life phenomena often constitutes obstacles for the students' engagement in mathematical modelling activities. It takes time to develop the necessary common understanding of the phenomena among students. Full scale mathematical modelling therefore limits the time and effort spent on mathematisation and analysis of the mathematical system compared with the time spent on investigating the real-life problem at hand and on structuring the real life complexity into an object of mathematical modelling. Knowing that mathematisation is often experienced as cognitively demanding and frustrating, priority must be given to the students' work with this aspect of mathematical modelling.

3.3. Finding the balance in a particular educational setting

Coming back to the developmental project mentioned in the introduction, the course we are referring to, called BASE, is part of a 2 year introductory study programme in mathematics and natural science, which is the entrance to further studies in science and mathematics at Roskilde University, Denmark. As all other programmes at Roskilde University, this programme is project organised, meaning that half of the students' study activity is devoted to project work [see Legge (1997) for a visitor's reflections after 6 months of investigating this way of studying, and Niss (2000) for an introduction to the basic ideas of the introductory study programme in mathematics and natural science and the consecutive programme for those specialising in mathematics]. The other half is devoted to more or less traditional coursework.

Every semester each student participates in a project where a group, typically between four and six students, work together with a problem of their own choice under certain thematic limitations for about 3 months. Mathematical modelling plays a central role in the project work. For most students, building and/or analysing a mathematical model is a crucial part of two of their four projects.

The project work is conducted under the guidance of a supervisor, and it is documented with a written report, typically 50–100 pages. The report and the project are defended by the group at an oral exam.

The basis of the development of BASE in this context is the observation that many students experience severe difficulties when using mathematics as a tool for building and analysing mathematical models in their project work. This also counts for situations where the relevant mathematical concepts and methods are well known by the students. In particular, the competences related to sub-processes (c), (d) and (e) in the mathematical modelling process seem to require more learning effort and pedagogical support for most students than is possible to activate as an integrated part of the project work.

From a theoretical point of view this observation is not surprising. As pointed out, sub-competences connected to the 'inner parts' of mathematical modelling are cognitively demanding and therefore suitable for development through systematic course teaching rather than through project work. But from a theoretical point of view it is also clear that a course focusing on sub-processes (c), (d) and (e) would not be sufficient for the development of mathematical modelling competence. Therefore, the idea behind the developmental project was to design a course supplementing the students' project work regarding the development of mathematical modelling competence. The course was to give the students numerous experiences with mathematisation and with analysing mathematical models in meaningful contexts.

3.4. Course structure

The course is taught in classes of ~40 students and includes two sessions of 2.5 hours every week. A total of 50 sessions during the two semesters. The course is structured around six mini-projects, generally intended to make the students work on sub-process (c), (d) and (e) of the mathematical modelling process. The mini-project-groups (two or three students) work for 2 weeks (four course sessions), with building, analysing and discussing a mathematical model. The students produce written reports (~10 pages) for each mini-project. Two of these are evaluated at an individual oral exam.

Between the mini-projects, mathematical concepts and methods are treated more systematically through regular lectures and (group) work with applied mathematical problem solving—partly computer based.

The balance between these two types of activities is an important part of the design and further development of the course. Parallel to the development of the mathematical content, progression in relation to this balance is built into the course. At the beginning of the course, most time is spent on closed and well-arranged problems of mathematisation and analysis of simple, given models, while by the end of the course nearly all the students' work is concentrated on mini-projects with broader mathematical modelling perspectives.

Right from the start we emphasise that mathematical modelling is a complex process which draws on different kinds of competences. This is done by discussing the process of mathematical modelling in general, and in particular, by discussing how different versions of the same task challenge different elements of mathematical modelling competence—as illustrated in the Introduction. This, however, is by no means a simple task. Our experience from the course is that most students need extensive personal experience with the mathematical modelling process in order to develop a sound perception of the concept of mathematical modelling. However, discussing the nature of the different competences involved in mathematical modelling has a positive influence on the students' enthusiasm and on their ability to see and express the meaning of their study activities.

3.5. Modelling the dynamics of an anaesthetic drug—an example

The second task from the introductory section is used to initiate the students' work with one of the last mini-projects in the course. The task is designed to challenge the students to build a model on the basis of given empirical (authentic) data, and to use the model to control the concentration of an anaesthetic drug during surgery and, finally, to make it possible to discuss the validity of the model.

3.5.1. *A first structuring of the work: working 'backwards' in a cyclic process*

By the formulation of the task, the domain of enquiry is already defined [process (a) in Fig. 2]. Moreover, an important help for the structuring of the system to be modelled [process (b) in Fig. 2] is given indirectly by supplying the graph illustrated in Fig. 1 (which could be an authentic point of departure for modelling the dynamics of anaesthetic drugs).

Working with the task as a mini-project, the students notice that after 1.5 hours the graph seems to be a straight line in the semi-logarithmic plot (cf. Fig. 1). They argue that from that moment the decrease in the blood concentration of pancuronium can be described by a simple exponential function. Building on previous experience the students estimate the two parameters, describing the exponential decrease by means of linear regression.

During the process of analysing the graph in Fig. 1 some groups develop the hypothesis that the entire graph can be modelled by the sum of two exponential functions. Therefore, they continue analysing the remaining parts of the data after having subtracted the contribution from the estimated exponential function. Applying linear regression on the remaining parts of the data supports the hypothesis and gives estimates of the parameters for the other exponential function.

Reaching this stage, the groups are usually able to reformulate their task and build a model of the interchange between blood and tissue, which has bi-exponential solutions. Drawing on their recent experience it is not too difficult for the students to think of a two-compartment model with linear rates, corresponding to two coupled linear differential equations, as a possible mathematical model.

In this first phase the students are working backwards in the mathematical modelling process from the supposedly bi-exponential function (considered as the mathematical system) to the sub-processes of mathematising and systematising [processes (c) and (b) in Fig. 2].

3.5.2. *Moving forwards*

After the first phase of analysing the given data the students are now ready to move forward in the mathematical modelling process by trying to describe the interchange of pancuronium between blood and tissue as a two-compartment system. In this process of describing the system that is to be mathematised the students may use their previous experience with compartment modelling. First of all it must be decided to use the amount or concentration of pancuronium in the blood and in the tissue as the two compartment variables. Furthermore, it must be considered how the decomposition of the drug is represented. Even though the students are aware of the fact that they are looking for a linear model, the representation of the rates of the interchange of pancuronium is not at all a trivial task.

Supported by dialogue with the teacher, a typical first outcome of the students' modelling is a compartment diagram (see Fig. 3) and a description stating that the interchange of drug between blood and tissue can be described as simple diffusion, meaning that the flow from a compartment is proportional with the concentration in the compartment (cf. Ficks law). In addition to this, the drug is supposedly eliminated at a rate proportional to the drug concentration in the bloodstream.

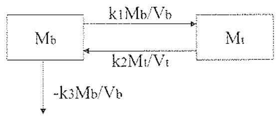

Fig 3. A model of the dynamic interchange of pancuronium between two compartments. M_b and M_t are the amount of drug in the bloodstream and in one aggregated pole of tissue respectively. V_b and V_t are the volume of blood and tissue respectively. k_1 and k_2 are constant rates of diffusion between blood and tissue, and k_3 is the constant rate of elimination from the bloodstream due to urination etc.

Determining the initial conditions of the system is another question that often provokes some discussion among the students, starting with questions such as 'When should the model be started—before or after the injection of the 4 mg of pancuronium?' and 'If we want to give some more during surgery, what then?'. $M_b(0) = 4$ mg and $M_t(0) = 0$, where $M_{(b)}$ and $M_{(t)}$ are the amounts of drug in the blood and tissue, respectively (the patient is assumed not to be drugged initially), is the obvious choice.

3.5.3. Mathematising the system

As can be seen from the description, the student activities in this phase are to do with both processes (b) and (c) of the modelling process as illustrated in Fig. 2. Setting up a compartment diagram typically involves both defining and mathematising the system. This could, on the one hand, give course to pedagogical difficulties when challenging the students to consider alternative mathematisations of the flows in a compartment diagram that is already mathematised. But on the other hand, the fact that compartment diagrams can be translated directly into differential equations using the principle of 'rate of change equals rate of inflow minus rate of outflow' is a great support for the students in the difficult process of writing up a set of differential equations as a mathematical representation of a system.

In fact, the students easily translate the compartment diagram in Fig. 3 into two coupled linear homogenous differential equations. However, this model does not directly give a solution for the concentration in blood and tissue. One way to overcome this problem is to divide the two equations with V_b and V_t (volumes of blood and tissue), respectively, which again provokes discussions among the students of how to estimate the parameters. V_b can be estimated directly from the data, since the concentration in the blood immediately after injection of 4 mg of pancuronium is known from the data. V_t cannot be estimated directly and some simplifications have to be made in order to reduce the number of parameters. One possibility is to assume that V_t and V_b are of the same magnitude. This yields the following system of differential equations:

$$c_{b'}(t) = -(a_1 + a_3)c_b(t) + a_2 c_t(t)$$

$$c_{t'}(t) = a_1 c_b(t) - a_2 c_t(t)$$

where t denotes the time, c_b and c_t the concentration of pancuronium in blood and tissue respectively (given by M_b/V_b and M_t/V_b). The a's correspond to the k's in Fig. 3: a_1

describes the rate of flow from the bloodstream into the muscle tissue, a_2 the reverse rate and a_3 the elimination of the drug from the bloodstream.

Together with the initial conditions [$c_b(0) = 4\,mg/V_b$ and $c_t(0) = 0$] these differential equations form a mathematical system that becomes the object of the students' mathematical analysis [process (d) in Fig. 2].

From here it is natural for the students to ask if and how the parameters can be estimated from the empirical data and to what extent the model can reproduce these data. Building on their knowledge about the form of the general solution to a system of two coupled first order linear differential equations and their previous analysis of the empirical data, the students are well prepared to look for the eigenvalues of the system. It turns out that the system has two negative eigenvalues of a different magnitude (say $\beta < \alpha$), and that they relate simply to the parameters of the model through the following two equations:

$$\alpha\beta = a_2 a_3$$

$$\alpha + \beta = -(a_1 + a_2 + a_3)$$

Since α and β are already estimated from analysing the data by means of linear regression, this leaves the students with only one free parameter. Using MatLab to solve the system numerically for different values of a_3 the students are given the possibility of trying to fit solutions of c_b to the empirical graph in Fig. 1. Of course the students also have access to the data in numerical form and it is quite easy for them to obtain a very nice match between the model result and the empirical graph. In fact, the curve shown in Fig. 1 is obtained in this way.

In general the students are quite excited about the fact that this is possible, but at the same time they are also quite uncertain about the legitimacy of this process. This experience, therefore, gives a basis for discussing the validity of the model with the students.

During the process of estimating the parameters the students' work can be described as a cyclic process between (d) and (e) in the mathematical modelling process. These activities build on and further develop the students' conceptual understanding of the relation between 'the system of differential equations', 'the eigenvalues of the system', 'the parameters of the model', 'the form of the (numerical) solutions' and 'the empirical data'. These relations are depicted in Fig. 5.

3.5.4. Using the model

Having reached this far in the working process, it is now meaningful for the students to ask how the patient is drugged in order to fulfill the condition given the pancuronium concentration in the tissue. First of all they draw the solution curve for c_t that corresponds to the blood concentration, shown as a dashed curve in Fig. 4.

As can be seen, the concentration in the tissue reaches a level of $250\,\mu g/l$ after $\sim$25 min. Maximum is reached $\sim$30 min after the injection and the concentration already drops below $250\,\mu g/l$ only 1 hour after surgery. Observing this, the students think of ways to maintain the level of c_t between 250 and $300\,\mu g/l$ as prescribed in the formulation of the modelling task. Different suggestions are put forward: a larger dose in the initial injection, a second injection with a smaller dose after 30 min or giving the patient a drip with a continuous supply of drug.

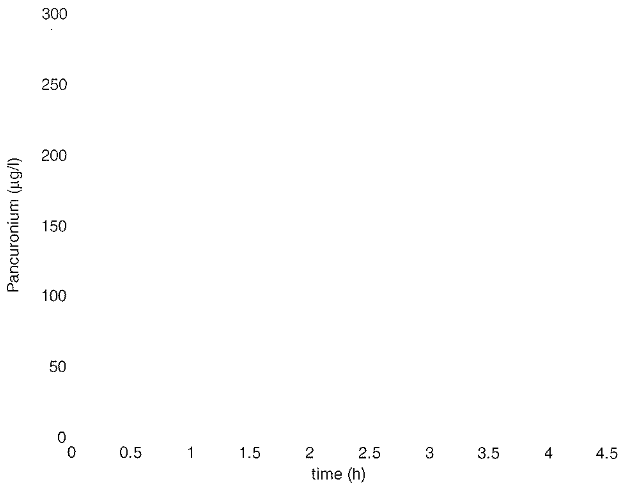

Fig 4. Shows two different model results for the tissue concentration, c_t. The dashed curve shows the correct match of c_b to the empirical data, i.e. 4 mg of pancuronium given by injection at $t = 0$. The same goes for the solid curve, except for the fact that the patient is given an additional drip, which gives close to 262 µg/l/min during the time slot from 30 min to 2 hours.

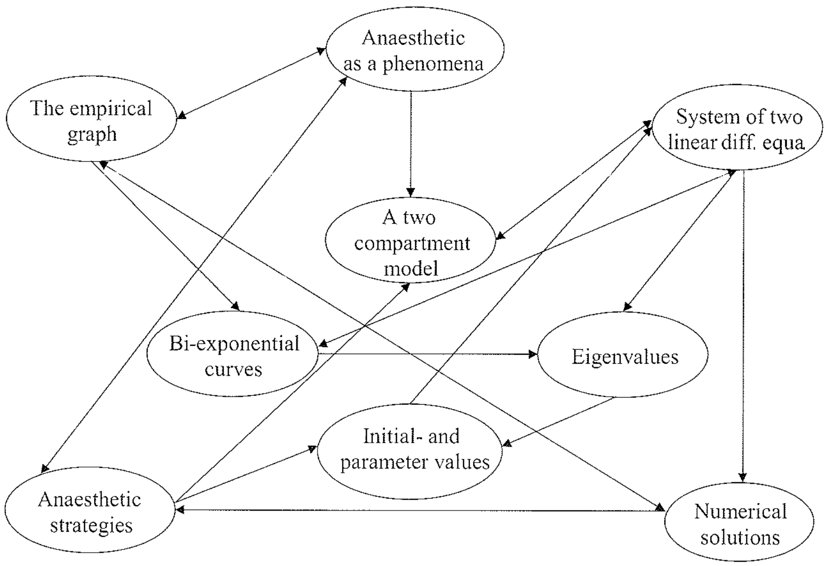

Fig 5. The diagram shows the conceptual relations of the students' modelling work.

The first idea is very easy for the students to investigate and reject because trying to keep the concentration above 250 µg/l during a 2 hour timeslot results in considerable excess.

Technically it is harder for the students to investigate the second option, as this requires further programming in MatLab in order to include a break before the second injection in the mathematical representation of the system. This gives rise to discussions of how the mathematical model is actually constituted. It becomes clear to the students that it is not merely a system of differential equations with parameters and initial values; a description of how to use the model and for what purpose is also an important part of the model. Following the strategy of two injections it is possible to stay within the given boundaries for 2 hours, even though c_t oscillates somewhat.

The third option forces the students to reconsider part of the mathematical modelling by changing the system to include an inflow to the compartment of the amount of pancuronium in the bloodstream. Eventually this leads to the addition of an inhomogeneous term, I(t), in the differential equation for c_b. Assuming I(t) is constant it is in fact possible for the students to control the equilibrium of the model by choosing the right value; $I(t) = I_0$. Solving the equations $c_b' = 0$ and $c_t' = 0$ in the extended model yields the equation

$$I_0 = (a_a 2/a_3)c_{t0}$$

where c_{t0} is the decidable level of concentration in the tissue. Referring to the formulation of the task, c_{t0} could be the mean of the boundaries; 275 µg/l/min.

Extending the model to include a drip (an intravenous line), it is quite natural for the students to analyse the equilibrium of the model. They often find it difficult to decide on the level of c_{t0}, and therefore need to be challenged by the teacher in order to use the calculated value of I(t) to design and test different drug programmes. One idea is of course only to use the drip with constant inflow of I_0 during the surgery. This is easily tested by a numerical analysis of the mathematical system. However, using this programme, c_t only reaches the minimum level after 6 hours which obviously is not optimal either for the patient or for the hospital system.

Working with the model and discussing the results, the students gradually take control of the process of designing and testing different drug programmes. Some groups develop quite sophisticated programmes. Figure 4 is the result of one of them, where the students argued that the drip is only to be used when c_b starts decreasing after the initial injection, and that it should be turned off just before surgery is over.

In general the students become quite enthusiastic when attempting to alternate between using the model and subsequently making alterations, i.e. working with sub-processes (d) and (e) in the mathematical modelling process which takes them back to (b) again (cf. Fig. 2). They find ways to implement their drug programmes in their MatLab programme, and they become quite thoughtful and precise in their argumentation pro and contra the different solutions suggested.

3.5.5. Validating the model

Using the students' experience with this work, it is now possible to challenge the students on the question of validity of the model they used for designing the drug programme, i.e. to work with process (f) in Fig. 2. One way of initiating such reflections is to ask the students if they recommend their drug programme for general use, or if it is only applicable for the patient from whom the empirical data is measured. This question often starts a discussion about the generality of the model parameters. After some considerations the students are able to defend the argument that parameters must be based on personal characteristics such as body volume and metabolic rate. This implies that a model applicable for practical work with anaesthetics must estimate different sets of parameters for different types of persons, and this can only be done on the basis of a very large number of empirical time series.

3.6. Reflections on task guidance versus dialogue-based guidance

The learning potential of working with sub-processes (c), (d) and (e) in relation to the mini-project on anaesthetics is illustrated by the diagram in Fig. 5. After working with the mini-project the students can ideally formulate in words the relations represented by the arrows in the diagram.

Only a few groups of students on our course will carry through the ways of reasoning described above. Most of the groups need substantial guidance. We want to emphasise that in the example, and during the course, the guidance given to the students in order to make them work with all the relations in Fig. 5 is not communicated directly by the formulation of the task itself. Whether or not the students meet these challenges depends very much on the dialogue between the teacher and the students during their work, and the ongoing discussion between the students in a group.

It is our experience that dialogue is much better than the use of more elaborate formulations of the mathematical modelling task with a lot of guiding questions. The students must accept the challenge to model the phenomena presented to them in order to develop their mathematical modelling competence (cf. our definition of this term). It is, therefore, important that the formulation of the task invites the students to accept this challenge and that the necessary indirect support is given through the teacher's dialogue with the students during their work. Having paid so much attention to this aspect is probably the main reason for the successful outcome of our course: at the end of the course it is possible to create situations where the students actually work with and investigate all the relations depicted in Fig. 5.

Recapping this experience we talk about the fruitfulness—possibly even the necessity—of replacing task guidance with dialogue-based guidance when attempting to assist the development of mathematical modelling competence.

3.7. Findings concerning the students' learning

On the basis of our experiences from teaching the course, the student evaluations of the course, the assessments of the students' learning—particularly the oral examinations—and from the close observations of, and interviews with, eight students (of the 74 students finishing the course in 2000), we emphasise the following aspects of the students' development of mathematical modelling competence in relation to conceptual clarification and course planning:

- In accordance with our conceptual analysis of the mathematical modelling process the challenge that the students met did in fact change as we varied the degree of pre-structuring the tasks. It is necessary to spend some time on developing the students competences related to the inner parts of the mathematical modelling process [i.e. sub-processes (c), (d) and (e)]. Pre-structured tasks that give the students the feeling of 'knowing what the goal is without knowing how to achieve it' are appropriate for this purpose. In order to develop mathematical modelling competence it is also necessary to challenge the students on the outer sub-processes, and for this purpose one needs open-ended tasks placing the students in a situation where they feel 'perplexity due to too many roads to take and no compass given'. The balancing of tasks within this framework was, and still is, a major theme of discussion among the group of teachers involved in our mathematical modelling course.

- During the course the students needed help to see their activities as parts of the mathematical modelling processes. Consequently, specific modelling-related activities had to be reflected in a comprehensive mathematical modelling process.
- The students' knowledge of the different competences of mathematical modelling seemed to be important for the structuring of their experiences and for their development of mathematical modelling competence. Consequently, we attempted to pay much attention to the development of metacognitive awareness as part of the entire learning experience that we aimed at providing for the students. An important part hereof was the strong emphasis placed on dialogue-based guidance instead of task guidance.

4. Perspectives

Seen from a research perspective we believe our work has two indications. The first is that in relation to analysing mathematical modelling competence, it is beneficial to make a conceptual clarification. In general, acknowledgement of the complexity and ambition of the concept of competence in relation to education may inspire a discussion of the potentials and limitations of different ways of organising teaching. More specifically, the progress in and obstacles for developing of mathematical modelling competence may be understood with reference to a description of the sub-processes in mathematical modelling.

The second indication is that more research is needed on the integration of different theoretical approaches to learning. In the specific analysis of the development of mathematical modelling competence we have experienced a need for integrating approaches based on findings from cognitive psychology and sociology. However, we also believe that there is a more general need for research looking at mathematics education with a holistic approach to learning.

Seen from a developmental perspective, we believe that one aspect in particular should be lifted out of the present context: a balance between the holistic approach and the atomistic approach is necessary when considering the design of an entire educational programme aiming at (among other things) developing the students' mathematical modelling competence. Neither of the two approaches alone is adequate. Special attention must be paid to the inadequacy of the atomistic approach since this is tempting to adopt due to its conformity with traditional teaching strategies in mathematics education.

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