Chapter 3.4.3

CLASSROOM ACTIVITIES AND THE TEACHER

Soeren Antonius, Chris Haines, Tomas Højgaard Jensen and Mogens Niss
with Hugh Burkhardt

University of Southern Denmark, Denmark, Email: soeren.antonius@dig.sdu.dk /

City University, London, UK, Email: c.r.haines@city.ac.uk /

Danish University of Education, Denmark, Email: thje@dpu.dk /

Roskilde University, Denmark, Email: mn@mmf.ruc.dk

Abstract: In this section we discuss a broad range of classroom activities, and of teaching style, that are required to produce the benefits that modeling can provide to the student learning mathematics. We discuss support for teachers and the shortage of good modeling tasks that need to be developed into effective curriculum materials.

1. INTRODUCTION

The development of performance in modelling processes¹ requires a much richer range of learning activities than the *explanation-example-exercises* ‘ritual’ that dominates traditional imitative curricula, and classrooms. These activities are similar to those needed for non-routine problem solving in pure mathematics and so, through them, students (and teachers) develop many other mathematical competencies. In previous chapters we have demonstrated that modelling provides concrete embodiments of mathematical concepts, develops reliable computation and checking, develops multiple connections inside and outside mathematics, and so on. In this section we discuss the broader range of classroom activities, and of teaching style, that this requires. What about support for teachers? There is a shortage of good modelling tasks that have been developed into effective curriculum materials. These need to offer sufficient support for typical teachers, who are mostly inexperienced in teaching modelling, without undermining student autonomy. Since most teachers are used to working with very supportive

teaching materials, even in the familiar curriculum, few will do well in new and challenging teaching without similarly detailed support. Professional development activity will also be important, particularly in the early stages of teaching modelling.

2. THE PATTERN OF CLASSROOM ACTIVITIES

Among the characteristic differences from a traditional mathematics classroom, we have seen that students:

- spend much more time on each, more-substantial problem;
- discuss mathematics with each other;
- explore alternative solution pathways;
- choose appropriate mathematical ‘tools’ to employ;
- carry through extended chains of reasoning, reliably;
- use checking strategies to get their analyses technically correct;
- interpret and evaluate the reasonableness of arguments and solutions;
- explain both results and reasoning to others.

Overall, students take much more responsibility for their work and their solutions. (This is, of course, what they will need to do in life and work). This is a major shift in the beliefs of most students and teachers about the nature of “doing mathematics”, and thus of the implicit ‘classroom contract’ of mutual expectations.

The diversity of applications and modelling activities in schools, colleges and universities is considerable. Their variety and complexity can be considered as follows:

- **Tasks in mathematics and applications** are shorter activities, often within a single lesson. Many are familiar in standard curricula: in the primary school tasks related to lengths, areas, volumes or to data collection, representation and analysis; in the secondary school, dealing with word problems and other illustrative applications; in further and higher education, short modelling exercises. Many of these will rightly be *illustrative applications* but *active modelling* is vital. At every level they should include smaller tasks on mathematising, handling the mathematics, and the other modelling processes, as well as more substantial problems like the following:

Area of a Porsche asks the students to estimate the surface area of a car “for budgetting the paint shop”. It is useful in showing alternative approaches and the model improvement process;

Dead girl presents the stopping distance problem in a dramatic context –

at different speeds, how far will a car travel before being able to stop when someone steps out into the road;

Bus trip takes a different approach – it describes alternative answers under differing conditions for transporting a number of school children from one place to another. The social context also becomes important, introduced in contrast (Verschaffel et al., 2000) with the usual approach to word problems.

- **Investigations** are longer activities that may extend over periods from two or three classes to two or three weeks. The *Numeracy through Problem Solving* modules are well-supported examples of this kind of active modelling for secondary school.
- **Projects** are even more complex activities, extending over weeks or months, usually addressing a broader problem embedded in the real world. *Yatzy Oil Rig* (de Bock & Roelens, 1993) deals with the movement of an oil rig from its construction site down the River Scheldt to Rotterdam through hazards of overhead power cables, depth contours for the river bed, tidal variations and logistics.
- **Dissertations**, usually in the university sector, develop over periods up to one year. The subjects may be quite general, e.g. on cartography and conformal maps, with the direction of the work and the outcomes very much dependent on the student and on the supervisor.
- **Class/lecture demonstrations** are usually led by the teacher or lecturer, though the best examples involve the students in active roles. They are often physical demonstrations of mathematical models either in a laboratory situation or on a larger scale using everyday materials showing that models work. In a modelling cycle description they validate the model. In *Walking the plank* a scaffold plank is marked off in quarters along its length. It is then extended from the stage towards the audience. Participants from the audience come to the stage and counterbalance the demonstrator as he/she walks the plank. There has to be discussion and interaction during the activity on the number of children needed to balance the demonstrator. Strict safety measures need to be put in place. (Haines & Le Masurier, 1986)

Dimensional Analysis (for example, Giordano & Weir, 1991) is an extremely powerful, and neglected, technique in modelling and applications. It provides a way to introduce quite difficult physical models to those without a strong background in science and applied mathematics. For example, the motion of a simple pendulum can be discussed in terms of the physical quantities mass, length and time leading to the usual formula for its period. Similarly it is easy to establish that the wind force on a van travelling on a motorway or across a high bridge is proportional to the square of the speed at which the van is being driven. These two introductory models have been

used to enthuse and engage pupils at upper secondary and undergraduate levels with some success.

For investigations, projects and dissertations it is usual for there to be a formal reporting stage; sometimes this is also done for shorter tasks in schools but more informally. Reporting methods include written reports (Berry & Davies, 1996), oral presentations (Crouch & Le Masurier, 1996) and posters (Berry & Houston, 1995).

Group work contributes significantly to the engagement of students, increasing motivation, and leading to better understanding of both the real world context and the mathematical concepts and techniques required for success. Recent work (for example, Ikeda & Stephens, 2001), has shown significant improvements in performance where discussion between members of the group takes place at the outset.

Assessment that recognises student achievement in applications and modelling has led to the development of innovative practices including: teacher assessment; self- and peer assessment; written reports; posters and oral presentations (Haines & Dunthorne, 1996). In addition, written examinations on specific project areas can assess students' ability to *transfer* their understanding and skills to less- or more-remote situations. They can also meet concerns about plagiarism. Preparing for such examinations brings challenging issues of transfer into the classroom.

3. TASKS AND TEACHING MATERIALS

The choice of mathematical tasks for students to tackle, in the classroom and in assessment, epitomises any curriculum. In the previous sections we have summarised and exemplified the essential characteristics of good modelling tasks, and illustrative applications. There are further examples of good tasks throughout this book.

The emphasis in teaching differs for the different types of task. *Modelling* can *only* be taught through a teaching approach that is investigative and student-centred, with the teacher playing largely consultative rather than directive roles. *Illustrative applications*, including stylised *word problems*, can be taught in traditional teacher-centred ways (as is largely has been) – though they too benefit from the same broadening of teaching style, which develops students' ability to work independently on less-routine tasks. While there is plenty of teaching material for teacher-centred teaching, there is still surprisingly little that offers detailed support for the greater challenges of teaching modelling. As in the early days of teaching problem solving within mathematics, much of the available material provides good problems with only general guidance on teaching strategies. While this may be sufficient for in-

novative teachers of some experience, it is not enough support for typical teachers who are new to teaching modelling.

The first fully-engineered materials for teaching modelling in schools were provided in the 1960s by *Unified Sciences and Mathematics for Elementary Schools* (USMES), the pioneering US project from the Education Development Center; even so, these provided rather general guidance and proved too challenging for most teachers. In the 1980's, the Shell Centre developed the much more supportive *Numeracy through Problem Solving* (NTPS) materials for lower secondary students, with an associated public examination.

The materials that have most directly and effectively addressed the development of other mathematical competencies through modelling have been developed over many years by the team at the Freudenthal Institute (FI). Their *Realistic Mathematics Education* program (RME) is based on *emergent modelling*, where students develop mathematical concepts through the modelling process, much as discussed in this chapter – but with well-engineered teaching material.

In recent years in both the UK and the US, the curriculum has again narrowed its focus, so investigative activities of any kind are rare. However, the current interest in mathematical literacy offers some hope of further progress, and the development of more well-engineered teaching materials to support typical teachers at all levels. They are sorely needed.

4. MODES OF WORKING IN THE CLASSROOM

These can be analysed in various ways. Basically, students can work individually or in groups; however, this dichotomy covers a multitude of important variations. There is a substantial literature on group working; here we have space for only a few comments.

Berry and Houston (2004, p. 36) use the concept *mathematical working styles* to capture how students work in modelling courses for undergraduates. It is a three-dimensional concept with the following components: the role of IT-tools; working with representations; and a sequence of actions that forms the approach to solving the problem. Ferri (2004, p. 47) uses the concept *mathematical thinking styles* and defines it as the way in which an individual prefers to present, to understand and to think through mathematical facts and connections by certain internal imaginations and/or external representations². This has several elements organised in five stimuli strands: *environmental* (sound, light, temperature, school- and room-design), *emotional* (motivation, persistence, responsibility, need for structure), *sociological* (self-work, pair-work, team-work, adult support, varied), *physiological* (perceptual, intake,

time, mobility), and *psychological* (global/analytical, hemisphericity, impulsive/reflective).

We shall focus on the sociological stimuli specified in the following question: What kind of interpersonal working modes can be observed in the classroom when students are working with modelling?

Individual work is (even when students sit in groups) the most common working style for traditional teacher-guided instruction. Students listen to the teacher, take notes, ask questions which the teacher answers, and do exercises. They also solve routine word problems at school and/or at home, which are later assessed by the teacher.

Group work seems to be an appropriate mode of working for modelling problems. Groups can be formed by the students themselves or by the teacher. They can be randomly or deliberately formed and may be homogeneous (group members have similar capabilities) or heterogeneous in composition. Groups may function in a variety of ways.

- Groups are formed and dissolved constantly. If a student has a problem, he/she tries to form a group (with another student) or tries to be admitted to an existing group in order to solve this specific problem. When the job is done, the group is dissolved.
- The group functions as a pleasant sociological environment. Students small-talk – not about mathematical problems, but in parallel to their individual work. The group seems not to be a learning facilitator, but one should not neglect the fact that these environmental factors can play a significant role for some students' learning process (cf. the environmental stimuli).
- The group members work parallel to each other, on the same problems, but approaches, methods and results are constantly discussed, negotiated and checked in order to reach an agreement. Sometimes an agreement is reached, sometimes not (Matsuzaki, 2004) which could mean that different answers to the same problem are produced. This way of functioning involves the risk that a student, who is less able, un-motivated, frivolous, or lazy, can become a 'passenger' and not take any responsibility for the group work.
- The problem is split into sub-problems and distributed to (or by) the group members in such a way that each group member is responsible for a specific part, and the complete answer to the problem is composed from the individual contributions. This is often the case with major problems and project work. It forces students to collaborate. The outcome is highly dependent on each student's work. If just one student fails, the work will not be completed in time.

Note that in all these cases individual work is also involved; as in everyday life, the balance between different modes of working is important.

Most students seem to prefer working collaboratively when modelling. As in everyday life, discussions are often helpful when tackling more complex problems. When one student is stuck, another may be able to supply a suggestion, which a third may be able to elaborate. In this, the teacher acts as a guide, asking questions to promote thinking and learning³. The group can do more than the sum of group members' contributions.

Group work can also be seen as a way of entering a *landscape of investigation* (Alrø & Skovsmose, 2003). This landscape is explored by a particular form of student-student and student-teacher relationship, and the theoretical background for this way of working and communicating is specified in the so-called *Inquiry-Co-operation Model*.

Group work is not unproblematic, however. If the group work relies on each group member, the work may fall apart if just one member is absent one day, or if he/she is not able to do his/her job. This may cause frustration and irritation, and these are just some of the problems that the group must be able to deal with. Group-work also raises the question of fairness; the outcome is highly dependent on the group members' willingness and ability to co-operate. The ability to co-operate in groups is not inherent in students; it has to be learned (Laborde, 1994).

5. THE ROLES OF THE TEACHER

When learning mathematical modelling, the traditional teacher's role as the prime source of explanation, demonstration and correct answers is no longer appropriate. The teacher can no longer micro-manage the students' thinking or they will not develop the strategic capabilities that are at the centre of modelling; guidance remains essential but it, too, must focus on strategic questions (Shell Centre, 1984), with:

- *More metacognitive prompts*: "What have you tried?", "What did you find?", "What are you going to try next?", "What will that tell you?".
- *Some prompts focused on specific strategies*: "Have you looked at some specific cases?", "Did you see any patterns that you recognize?", "It may help if you represent this in a different way", "Have you tried to check that using another method?"
- *Little detailed guidance*: "Isn't that the difference of two squares?", "Why don't you try a linear fit?", "That's wrong".

Developing the mathematics

While students work on a modelling problem, their main objective is to produce an interesting and useful analysis and report, not to develop particular mathematical techniques. The mathematics is used as a tool to facilitate

this process and is not seen as an end in itself. These are appropriate priorities.

The teacher can, however, use the many opportunities provided by the work to motivate the learning of mathematical techniques in a more explicit way. This section offers suggestions, taken from the *Numeracy through Problem Solving* (NTPS) modules, as to how this can be achieved without destroying the essential flow of the modelling activity.

How and when may mathematical topics be introduced?

Mathematical activity may be initiated by either the student or by the teacher. For example:

A student may become aware of the need to acquire a particular skill in order to complete her consumer report. “What is the best way to present this data?” “Should I draw a pie chart, bar chart or what?” This kind of situation can lead to an invaluable learning experience because the student wants to know something. Such opportunities occur rather unpredictably. Also, if you have a large class, it is unwise to spend a great deal of time helping one person. One approach is to ask the student to describe the problem to the whole class and invite help and advice from other students.

The teacher may wish to use some ideas from the problem to support a more intensive piece of work on a particular topic. ‘Today, we are going to look at the topic of ratio, using the shopping surveys you carried out. Which type of chocolate gives you the most for your money?’ Do not expect students to use, autonomously, mathematics that they have only recently been taught. There is a gap, typically of several years, between first ‘learning’ a skill and being able to use it with flexibility and fluency. Students will tend only to use skills that they have mastered. (There is evidence that this “few year gap” can be reduced – it requires a more ‘rounded’ approach to learning, with a variety of applications and non-routine problem solving to supplement and give meaning to technical exercises.)

Before, during or after – a timing dilemma

Teacher-initiated work on mathematical techniques related to the theme may occur before, during, or after the modelling itself. Each has advantages and drawbacks.

- *Before:* “I’ll give them some practice at drawing pie charts now, so that they will be more inclined to use them later on, when they begin work on the module.” This timing has the advantage that the student will, if all goes well, have the technique polished and ready to be used, but it seems artificial to learn a new technique just before you need for it. Students soon assume that the ‘problem’ is an illustrative application – merely a vehicle for practising the new technique, rather than developing their modelling competency.

- *During*: “They seem to be having difficulty in organising their data. We’ll take a break from this module for a few lessons and do some work on this, together using data that I’ll provide”. This timing allows you to respond to needs as they arise. But, if students always expect you to produce the method or solution when the going gets difficult, you undermine autonomy. If such teaching is done too often then the work on the module will drag on too long and become boring.
- *After*: “When we have finished the problem, we will look at the techniques we have used in greater depth.” This does not threaten student autonomy, and working on the module may motivate and enable them to perceive the value of techniques when they are taught. However, students may still not be able to use techniques autonomously unless they are given further opportunities to select and apply them in other modelling contexts.

Whatever you decide at each stage, it is important to be vigilant about sustaining the students’ strategic control of their work; it is too easy to allow them to revert to the imitative role that the traditional curriculum encourages. If the balance of support is inappropriate for modelling, the students soon lose the feeling of ‘ownership’ of the problem to the teacher and revert to the traditional passive, imitative role that inhibits learning of all kinds.

Steen and Forman’s “Principles of Best Practice” (2001) summarise all this, and is adapted here.

Effective teachers of modelling employ pedagogy that is:

Active:

- Encourage students to explore a variety of strategies.
- Stimulate discussion of available data in relation to what is being asked.
- Require students to seek out missing information needed to solve problems.
- Make hands-on use of concrete materials.

Student-centered:

- Focus on problems that students see as relevant and interesting.
- Help students learn to work with others.
- Developed strong technical communication skills among students.
- Provide opportunities for students to use their own knowledge and experience.

Contextual:

- Ask students to engage problems first in context, then with mathematical formalities.
- Suggest resources that might provide additional information.
- Require that students verify the reasonableness of solutions in the context of the original problem.
- Encourage students to see connections of mathematics to work and life.

This book illustrates these principles from work in various countries on the teaching of modelling. The published proceedings of the series of biennial ICTMA conferences (ICTMA) are also an important source of examples and analysis of the development of this field.

6. HOW MUCH GUIDANCE?

This question is at the heart of the teaching of modelling – indeed of all non-routine problem solving. However, as noted above, there are specific issues when other mathematical competencies are a learning goal. If the teacher allows students to select their own skills to deploy, then they are likely to choose only those with which they are most familiar and secure. They will tend to avoid the more challenging and difficult ideas, even when imitatively fluent⁴. On the other hand, if the teacher tells students which mathematical techniques to use, then the teacher removes the strategic demand and the problems become exercises in using the given techniques.

We illustrate this issue with two complementary case studies of classroom ‘trajectories’ starting from the same very broad task statement:

Which means of transport is the best?

They show that, while teachers face important strategic decisions, the situation does not need to be so polarised. The teacher doesn’t have to choose between allowing complete strategic freedom and none. It is possible to manage the process so that students begin by tackling problems unaided, then compare advantages and disadvantages of the approaches and strategies used, and then refine these into more powerful methods or introduce new methods in a tentative ‘maybe this will help’ manner. In this way the teacher acts as a co-constructor of the mathematics. Some of the examples above show how this can be done – and the kind of support that helps teachers.

However, the constraints that the teachers applies through the initial problem is important. Here the first teacher takes a more open approach than the second. In the analysis we focus on two issues:

- How is the teacher dealing with the balance between mathematical modelling as a goal in itself, and as a means to develop some of the other mathematical competencies?
- What opportunities for the teacher does the approach offer, and what obstacles and difficulties does it carry with them?

First classroom trajectory

This comes from an experimental two-year teaching programme in a Danish upper secondary school (Jensen, to appear). The mathematics teaching was guided by the KOM set of mathematical competencies. The class-

room activity structure enabled students to get experience with all aspects of the mathematical modelling process.

In the first phase of the work the teacher

- made the mathematical competencies explicit, but emphasised modelling;
- gave a list of interesting openly-formulated problem areas to choose from.

Within the task statement above, the students chose to focus on the following question:

What is the average use of energy in moving x people from one floor to another using respectively a staircase, an escalator and a lift? Which option uses the most?

This focus was much narrower than what the teacher had in mind – comparing cars, trains, planes etc from various perspectives: time; price; pollution.

The students continued the detailed specification of the problem by focusing on the relation between the height of the stairs of a staircase and the effectiveness of the use of energy in the leg muscles. However, after some discussion with the teacher, the physiology involved was deemed too demanding, so they simplified the problem to calculating the difference in potential energy between being on the two floors. The lift was dealt with satisfactorily, but the escalator turned out to be a real mathematisation challenge. How can one make a mathematical representation of the way a group of people enter and leave an escalator, the number of people on the escalator at a certain time and the use of energy this adds up to?

Modelling vs. other mathematics: In this case the teacher had a clearly specified focus: to develop the students modelling competency. It turned out that several other mathematical competencies were also used during the project, not least among these *communication* (through the negotiations in the problem specification and mathematisation processes) and *problem tackling* (through struggling with both formulating and solving the model problem). However, because of the openness of the task, one could not ensure the use of specific mathematical topics (Blomhøj & Jensen, 2003). For that, the teacher must take more control, reducing student autonomy.

Opportunities and obstacles: This teaching style helped the teacher to create an open, trustful and mutually supportive atmosphere in the classroom. This came from being honest about his role as mediator between the goals of the curriculum and the autonomous interests of the students. The second opportunity follows from this. The teacher helped develop modelling competency directly: *Here is a goal – go for it, and I will help you!* There seemed to be three difficult issues for the teacher to deal with:

- The demands, both personal and mathematical, of this style of teaching;

- the general difficulty of competency-guided teaching, not least due to the students' problems in grasping the essence of the different competencies.
- The distribution of time among the different projects and the various more teacher-controlled modes of teaching taking place.

Second classroom trajectory

In this example, the initial task was more closely specified by the teacher:

“Which is the best means of transport in the metropolitan area of Copenhagen amongst bicycle, bus, car, metro, or train for an individual living and working in this area?”

It is up to you to specify the conditions for answering this question, including what “best” means, and what assumptions are being made. You have all four lesson-hours per week (plus home-work time) for three weeks at your disposal.”

The students tackled this task in five groups of 3-5, each of which made its own specification. Then each group consulted the teacher, who had to approve the specification and the approach chosen, bearing in mind the mathematical tractability at their level, the time frame at their disposal, and information and data to be identified and collected. As the teacher wanted to give the students as much independence as possible, she did not share her deliberations on these matters with the students; she made her own assessment of the task, the capabilities of the students in each group, and her opportunities to provide guidance at crucial points.

One group of students decided to keep a dual view of the term “best”, because they found that both “time consumption” and “cost” were objective functions of relevance. They wanted to be able to look at the trade-off between the two, should they find (as they assumed) that different means of transport would be optimal.

At first this group wanted to tackle a well-defined simple problem: to select two points, A and B, in the Copenhagen area and then look at all realistic routes between the two for cars and bicycles, and at all the public transport options connecting small neighbourhoods of A and B, respectively. However, the teacher thought that this would not be challenging enough in mathematical terms – just comparison of a few costs and riding times. The work would be nearly all data collection – reading of time-tables, perhaps taking measurements etc. This educational challenge was not sufficient, so the teacher urged the group to specify its task in a different way.

Thus prompted, the group decided to *narrow* its task down to comparing transport by car with transport by public buses, *but* without specifying the points of departure and arrival. They did so by modelling the average cost and duration of car rides per km during rush hours and “quiet hours”, respectively, taking into account the average number of stoplights per km and the

expected waiting time at stoplights at different hours. Similarly, they used the official timetables of buses to compute the average fare and duration of a bus ride per km. In order to do all this they collected data samples of distance data and the places of stoplights, and they collected information from the traffic offices of the city of Copenhagen concerning the waiting times at different categories of stoplights in the city. They also attempted to model the time costs entailed by deceleration and acceleration at stoplights.

At the end of the second week, their model was almost complete, and conclusions were about to be drawn with regard to the main question. The teacher then asked students to *improve* the model by analysing the sensitivity of the model and its conclusions to inaccuracies and uncertainties in the data and assumptions made. This led students to pursue the effects of errors by means of interval arithmetic, a technique to which their teacher gave the group a brief introduction. Students finished their work by producing a poster, and defended this at an oral “poster session”.

Modelling vs. other mathematics: In this trajectory, the teacher had a clear vision of the goals of the modelling activities:

- fostering students’ competence in mathematical modelling in highly messy and blurred real world contexts;
- the making of decisions, idealisations, simplifications, and improvements;
- the collection of appropriate data (a primary goal); but also
- to activate, in the modelling process, mathematical knowledge and skills that they could deal with but at the edge of these students’ ability.

Thus the project stimulated students’ activation and consolidation, to different degrees, of the entire spectrum of mathematical competencies, as well as specific subject matter knowledge and skills. The emphasis was on the competencies of problem tackling, representation, communication and above all, of course, modelling. This teacher achieved this balance by guidance at a few crucial points so as to ensure that their work would be mathematically relevant and challenging, and relevant to the initial questions.

Opportunities and obstacles: As usual, the most significant challenge to the teacher is to strike a proper balance between student autonomy and teacher intervention. Too little feedback from the teacher can cause frustration and insecurity with the students – the teacher had to provide inputs to students that allowed them to take responsibility for their own work but also helped them over hurdles and decisions that were likely to carry them astray.

Achieving this balance presents every teacher of modelling with formidable demands. The Steen-Forman *Principles of Best Practice* (2001) has a useful list of pitfalls to avoid:

- Selecting tasks in order to cover mathematics rather than to explore and solve interesting problems.

- Overlooking interesting or challenging mathematics that lies embedded beneath the surface of many “real-world” examples.
- Imposing unwarranted structure on a contextually rich problem in the interest of ensuring appropriate mathematical coverage.
- Believing that complex problems require sophisticated mathematics and that there is something wrong with solutions that use elementary techniques.
- Choosing tasks that fail to help students prepare for higher achievement in mathematics.
- Presenting tasks in the form of mathematics worksheets, thereby sterilizing the context of everything that makes it problematic.
- Lacking conceptual continuity and intellectual growth in the sequencing of tasks in which mathematical activities are embedded.
- Failing to bring mathematical closure (including concepts, vocabulary, methods, generalizations) at the conclusion of an open-ended project.
- Not allowing sufficient reflection on the process of mathematical modeling.

In addition to reflective experience of supervising and guiding the processes involved, teachers also need a deep understanding of many different kinds of subject matter that allows them to predict the possible obstacles and outcomes of different paths students may follow. This requires new kinds of teacher decisions and interventions. The teacher has to be able to live with uncertainty, continually gathering and processing information about the state and development of each student, and making appropriate decisions on intervention. This is a *mathematics teacher competence* that cannot be acquired by means of pre-service preparation only. It has to be developed in service, but the seeds should be planted in pre-service education.

¹ *comprehension, formulation, transformation, interpretation, evaluation and communication*, through cycles of *improvement* from an initial *simple* model.

² c.f. learning styles of Dunn & Dunn (1998).

³ In this way, group work is one way of giving substance to Vygotsky's *zone of proximal development*.

⁴ Treilibs (1979) for example, invited 120 very able 17-18 year old students to solve a variety of mathematical problems set in realistic contexts. None used algebra.