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Modelling electricity forward markets by ambit fields

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Abstract

This paper proposes a new modelling framework for electricity forward markets, which is based on ambit fields. The new model can capture many of the stylised facts observed in energy markets. One of the main differences to the traditional models lies in the fact that we do not model the dynamics, but the forward price directly, where we focus on models which are *stationary* in time. We give a detailed account on the probabilistic properties of the new model and we discuss martingale conditions and change of measure within the new model class. Also, we derive a model for the spot price which is obtained from the forward model through a limiting argument.

Keywords: Electricity markets; forward prices; random fields; ambit fields; stochastic volatility.

JEL codes: C0, C1, C5, G1.

1 Introduction

This paper introduces a new model for electricity forward prices, which is based on *ambit fields* and *ambit processes*. Ambit fields and processes constitute a general probabilistic framework which is suitable for tempo–spatial modelling. Ambit processes are defined as stochastic integrals with respect to a multivariate random measure, where the integrand is given by a product of a deterministic kernel function and a stochastic volatility field and the integration is carried out over an *ambit set* describing the sphere of influence for the stochastic field.

Due to their very flexible structure, ambit processes have successfully been used for modelling turbulence in physics and cell growth in biology, see Barndorff-Nielsen & Schmiegel (2004, 2007, 2008*a*,*b*,*c*, 2009), Vedel Jensen et al. (2006). The aim of this paper is now to develop a new modelling framework for (electricity) forward markets based on the ambit concept.

Over the past two decades, the markets for power have been liberalised in many areas in the world. The typical electricity market, like for instance the Nordic Nord Pool market or the German EEX market, organises trade in spot, forward/futures contracts and European options on these. Although these assets are parallel to other markets, like traditional commodities or stock markets, electricity has its own distinctive features calling for new and more sophisticated stochastic models for risk management purposes, see Benth, Šaltytė Benth & Koekebakker (2008).

The electricity spot can not be stored directly except via reservoirs for hydro–generated power, or large and expensive batteries. This makes the supply of power very inelastic, and prices may rise by several magnitudes when demand increases, due to temperature drops, say. Since spot prices are determined by supply and demand, strong mean–reversion can be observed. The spot prices have clear deterministic patterns over the year, week and intra–day. The literature has focused on stochastic models for the spot price dynamics, which take some of the various stylised facts into account. Recently, a very general, yet analytically tractable class of models has been proposed in Barndorff-Nielsen et al. (2010*b*), based on Levy semistationary processes. This class nests most of the popular spot models for non–tradable commodities and is a special case of an ambit process.

One of the fundamental problems in power market modelling is to understand the formation of forward prices. Non–storability of the spot makes the usual buy–and–hold hedging arguments break down, and the notion of convenience yield is not relevant either. There is thus a highly complex relationship between spot and forwards.

A way around this would be to follow the so–called Heath–Jarrow–Morton approach, which has been introduced in the context of modelling interest rates, see Heath et al. (1992), and model the forward price dynamics directly (rather than modelling the spot price and deducing the forward price from the conditional expectation of the spot at delivery). There are many challenging problems connected to this way of modelling forward prices.

Firstly, standard models for the forward dynamics generally depend on the current time and the time to maturity. However, power market trades in contracts which deliver power over a *delivery period*, introducing a new dimension in the modelling. Hence comprehensive forward price models must be functions of both *time to* and *length of* delivery, which calls for random field models in time and space. Furthermore, since the market trades in contracts with overlapping delivery periods, specific no–arbitrage conditions must be satisfied which essentially puts restrictions on the space structure of the field. So far, the literature is not very rich on modelling power forward prices applying the Heath–Jarrow–Morton approach, presumably due to the lack of analytical tractability and empirical knowledge of the price evolution.

Empirical studies, see Frestad et al. (2010), have shown that the logarithmic returns of forward prices are non–normally distributed, with clear signs of heavy tails. Also, a principal component analysis by Koekebakker & Ollmar (2005) indicates a high degree of idiosyncratic risk in power forward markets. This strongly points towards random field models which, in addition, allow for stochastic volatility. Moreover, the structure determining the interdependencies between different contracts is by far not properly understood. Some empirical studies, see Andresen et al. (2010), suggest that the correlations between contracts are decreasing with time to maturity, whereas the exact form of this decay is not known. But how to take 'length of delivery' into account in modelling these interdependencies has been an open question. A first approach on how to tackle this problem will be presented later in this paper.

Ambit processes provide a flexible class of random field models, where one has a high degree of flexibility of modelling complex dependencies. These may be probabilistic coming from a driving Levy basis, or functional from a specification of an ambit set.

Our focus will be on ambit processes which are *stationary* in time. As such, our modelling framework differs from the traditional models, where stationary processes are (if at all) reached by limiting arguments. Modelling directly in stationarity seems in fact to be quite natural and is e.g. done in physics in the context of modelling turbulence, see e.g. Barndorff-Nielsen & Schmiegel (2007, 2009). Here we show that such an approach has strong potential in finance, too.

Due to their general structure, ambit processes easily incorporate the observed Samuelson effect in the volatility, leptokurtic behaviour in returns and possibly stochastic volatility and leverage effects. Note that the *Samuelson effect*, see Samuelson (1965), refers to the finding that, when the time to maturity approaches zero, the volatility of the forward increases and converges to the volatility of the underlying spot price.

Although many stylised facts of energy markets can easily be incorporated in an ambit framework, one may question whether ambit processes are not in fact *too* general to be a good building block for financial models. In particular, one property — the martingale property — is often violated by general ambit processes. However, we can and will formulate conditions which ensure that an ambit process is in fact a martingale. So, if we wish to stay within the martingale framework, we can do so by using a restricted subclass of ambit processes. On the other hand, in modelling terms, it is actually not so obvious whether we *should* stay within the martingale framework if our aim is to model electricity forward contracts. Given the illiquidity of electricity markets, it cannot be taken for granted that arbitrage opportunities arising from forward prices outside the martingale framework can be exercised. Also, we know from recent results in the mathematical finance literature, see e.g. Guasoni et al. (2008), Pakkanen (2010), that subclasses of non–(semi)–martingales can be used to model financial assets without necessarily giving rise to arbitrage opportunities in markets which exhibit market frictions, such as e.g. transaction costs.

Next, we will not work with the most general class of ambit processes since we are mainly interested in the time–stationary case.

Last but not least we will show that the ambit framework can shed some light on the connection between electricity spot and forward prices. Understanding the interdependencies between these two assets is crucial in many applications, e.g. in the hedging of exotic derivatives on the spot using forwards. A typical example in electricity markets is so–called user–time contracts, giving the holder the right to buy spot at a given price on a predefined number of hours in a year, say.

The outline for the remaining part of the paper is as follows. Section 2 gives an overview of the standard models used for forward markets. Section 3 reviews basic traits of the theory of ambit fields and processes. In Section 4, we describe the new model for (electricity) forward markets. In Section 5, we show how some of the traditional models for forward prices relate to ambit processes. Next we show what kind of spot model is implied by our new model for the forward price and we discuss that, under certain conditions, the implied spot price process equals in law a Lévy semistationary process, see Section 6. Section 7 presents the martingale conditions for our new model to ensure the absence of arbitrage. Also, since we do the modelling under the risk neutral measure, we discuss how a change of measure can be carried out to get back to the physical probability measure. Section 8 deals with extensions of our new modelling framework: While we mainly focus on arithmetic models for forward prices in this paper, we discuss briefly how geometric models can be constructed. Also, we give an outlook on how ambit field based models can be used to jointly model time and period of delivery. Finally, Section 9 concludes and Appendix A contains the proofs of our main results and some technical results on the correlation structure of the new class of models.

2 Overview on approaches to modelling forward prices

Before introducing ambit fields, let us review the exisiting literature on direct modelling of forward prices in commodity markets, i.e. the approach where one is not starting out with a specification of the underlying spot dynamics.

Although commodity markets have very distinct features, most models for energy forward contracts have been inspired by instantaneous forward rates models in the theory for the term structure of interest rates, see Koekebakker & Ollmar (2005) for an overview on the similarities between electricity forward markets and interest rates.

Hence, in order to get an overview on modelling concepts which have been developed in the context of the term structure of interest rates, but which can also be used in the context of electricity markets, we will now review these examples from the interest rate literature. However, later we will argue that, in order to account for the particular stylised facts of power markets, there is a case for leaving these models behind and focusing instead on ambit fields as a natural class for describing energy forward markets.

Throughout the paper, we denote by $t \in \mathbb{R}$ the current time, by $T \geq 0$ the time *of* maturity of a given forward contract, and by $x = T - t$ the corresponding time *to* maturity. We use $F_t(T)$ to denote the price of a forward contract at time t with time of maturity T. Likewise, we use f for the forward price at time t with time to maturity $x = T - t$, when we work with the Musiela parameterisation, i.e. we define f by

$$f_t(x) = f_t(T - t) = F_t(T).$$

2.1 Multi–factor models

Motivated by the classical Heath et al. (1992) framework, the dynamics of the forward rate under the risk neutral measure can be modelled by

$$df_t(x) = \sum_{i=1}^{n} \sigma_t^{(i)}(x) dW_t^{(i)}, \qquad \text{for } t \geq 0,$$

for $n \in I\!N$ and where $W^{(i)}$ are independent standard Brownian motions and $\sigma^{(i)}(x)$ are independent positive stochastic volatility process for $i = 1, \ldots, n$. The advantage of using these multi–factor models is that they are to a high degree analytically tractable. Extensions to allow for jumps in such models have also been studied in detail in the literature. However, a principal component analysis by Koekebakker & Ollmar (2005) has indicated that we need in fact many factors (large n) to model electricity forward prices. Hence it is natural to study extensions to infinite factor models which are also called random field models.

2.2 Random field models for the dynamics of forward rates

In order to overcome the shortcomings of the multifactor models, Kennedy (1994) has pioneered the approach of using random field models, in some cases called stochastic string models, for modelling the term structure of interest rates. Random field models have a continuum of state variables (in our case forward prices for all maturities) and, hence, are also called infinite factor models, but they are typically very parsimonious in the sense that they do not require many parameters. Note that finite–factor models can be accommodated by random field models as degenerate cases.

Kennedy (1994) proposed to model the forward rate by a centered, continuous Gaussian random field plus a continuous deterministic drift. Furthermore he specified a certain structure of the covariance function of the random field which ensured that it had independent increments in the time direction t (but not necessarily in the time to maturity direction x). Such models include as special cases the classical Heath et al. (1992) model when both the drift and the volatility functions are deterministic and also two–parameter models, such as models based on Brownian sheets. Kennedy (1994) derived suitable drift conditions which ensure the martingale properties of the corresponding discounted zero coupon bonds.

In a later article, Kennedy (1997) revisited the continuous Gaussian random field models and he showed that the structure of the covariance function of such models can be specified explicitly if one assumes a Markov property. Adding an additional stationarity condition, the correlation structure of such processes is already very limited and Kennedy (1997) proved that, in fact, under a strong Markov and stationarity assumption the Gaussian field is necessarily described by just three parameters.

The Gaussian assumption was relaxed later and Goldstein (2000) presented a term structure model based on non–Gaussian random fields. Such models incorporate in particular conditional volatility models, i.e. models which allow for more flexible (i.e. stochastic) behaviour of the (conditional) volatilities of the innovations to forward rates (in the traditional Kennedy approach such variances were just constant functions of maturity), and, hence, are particularly relevant for empirical applications. Also, Goldstein (2000) points out that one is interested in very *smooth* random field models in the context of modelling the term structure of interest rates. Such a smoothness (e.g. in the time to maturity direction) can be achieved by using *integrated random fields*, e.g. he proposes to integrate over an Ornstein–Uhlenbeck process. Goldstein (2000) derived drift conditions for the absence of arbitrage for such general non–Gaussian random field models.

While such models are quite general and, hence, appealing in practice, Kimmel (2004) points out that the models defined by Goldstein (2000) are generally specified as solutions to a set of stochastic differential equations, where it is difficult to prove the existence and uniqueness of solutions. The Goldstein (2000) models and many other conditional volatility random field models are in fact complex and often infinite dimensional processes, which lack the the key property of the Gaussian random field models introduced by Kennedy (1994): that the individual forward rates are low dimensional diffusion processes. The latter property is in fact important for model estimation and derivative pricing. Hence, Kimmel (2004) proposes a new approach to random field models which allows for conditional volatility and which preserves the key property of the Kennedy (1994) class of models: the class of *latent variable term structure models*. He proves that such models ensure that the forward rates and the latent variables (which are modelled as a joint diffusion) follow jointly a finite dimensional diffusion.

A different approach to generalising the Kennedy (1994) framework is proposed by Albeverio et al. (2004). They suggest to replace the Gaussian random field in the Kennedy (1994) model by a (pure jump) Lévy field. Special cases of such models are e.g. the Poisson and the Gamma sheet.

Finally, another approach for modelling forward rates has been proposed by Santa-Clara & Sornette (2001) who build their model on *stochastic string shocks*. We will review that class of models later in more detail since it is related (and under some assumptions even a special case) of the new modelling framework we present in this paper.

2.3 Intuitive description of an ambit field based model for forward prices

After we have reviewed the traditional models for the term structure of interest rates, which are (partially) also used for modelling forward prices of commodities, we wish to give an intuitive description of the new framework we propose in this paper before we present all the mathematical details.

As in the aforementioned models, we also propose to use a random field to account for the two temporal dimensions of current time and time to maturity. However, the main difference of our new modelling framework compared to the traditional ones is that we model the forward price *directly*.

This *direct* modelling approach is in fact twofold: First, we model the forward prices directly rather than the spot price, which is in line with the Heath et al. (1992) framework. Second, we do not specify the *dynamics* of the forward price, but we specify a random field, an ambit field, which explicitly describes the forward price. In particular, we propose to use random fields given by stochastic integrals of type

$$\int_{A_t(x)} h(\xi, s, x, t)\sigma_s(\xi)L(d\xi, ds), \tag{1}$$

as a building block for modelling $f_t(x)$. A natural choice for L — motivated by the use of Lévy processes in the one–dimensional framework — is the class of *Lévy bases*, which are infinitely divisible random measures as described in more detail below. Here the integrand is given by the product of a deterministic kernel function h and a random field σ describing the stochastic volatility.

We will describe in more detail below, how stochastic integrals of type (1) have to be understood. Note here that we integrate over a set $A_t(x)$, the *ambit set*, which can be chosen in many different ways. We will discuss the choice of such sets later in the paper.

An important motivation for the use of ambit processes is that we wish to work with processes which are *stationary* in time, i.e. in t, rather than formulating a model which *converges* to a stationary process. Hence, we work with stochastic integrals starting from $-\infty$ in the temporal dimension, more precisely, we choose ambit sets of the form $A_t(x) = \{(\xi, s) : -\infty < s \leq t, \xi \in I_t(s, x)\}$, where $I_t(s, x)$ is typically an interval including x, rather than integrating from 0, which is what the traditional models do which are constructed as solutions of stochastic partial differential equations (SPDEs). (In fact, many traditional models coming from SPDEs can be included in an ambit framework when choosing the ambit set $A_t(x) = [0, t] \times \{x\}$.)

In order to obtain models which are stationary in the time component t, but not necessarily in the time to maturity component x, we assume that the kernel function depends on t and s only through the difference $t - s$, so having that h is of the form $h(\xi, s, x, t) = k(\xi, t - s, x)$, that σ is stationary in time and that $A_t(x)$ has a certain structure, as described below. Then

$$\int_{A_t(x)} k(\xi, t - s, x)\sigma_s(\xi)L(d\xi, ds). \tag{2}$$

Note that Hikspoors & Jaimungal (2008) and Benth (2010) provide empirical evidence that both the spot and the forward price are influenced by a *stochastic volatility* field σ. Here we assume that σ describes the volatility of the forward market as a whole. More precisely, we will assume that the volatility of the forward depends on previous states of the volatility both in time and in space, where the spatial dimension reflects the time to maturity.

The general structure of ambit fields makes it possible to allow for general *dependencies between forward contracts*. In the electricity market, a forward contract has a close resemblance with its neighbouring contracts, meaning contracts which are close in maturity. Empirics (by principal component analysis) suggest that the electricity markets need many factors, see e.g. Koekebakker & Ollmar (2005), to explain all the noises, contrary to interest rate markets where one finds 3–4 sources of noise as relevant. Since electricity is a non–storable commodity, forward looking information plays a crucial role in settling forward prices. Different information at different maturities, such as plant maintenance, weather forecasts, political decisions etc., give rise to a high degree of idiosyncratic risk in the forward market, see Benth & Meyer-Brandis (2009). These empirical and theoretical findings justify a random field model in electricity and also indicate that there is a high degree of dependency around contracts which are close in maturity, but much weaker dependence when maturities are farther apart. The structure of the ambit field and the volatility field which we propose in this paper will allow us to “bundle” contracts together in a flexible fashion.

3 Ambit fields and processes

This section reviews the concept of ambit fields and ambit processes which form the building blocks of our new model for the electricity forward price. For a detailed account on this topic see Barndorff-Nielsen et al. (2010*a*) and Barndorff-Nielsen & Schmiegel (2007). Throughout the paper, we denote by $(\Omega, \mathcal{F}, P^*)$ our probability space. Note that we use the $*$ notation since we will later refer to this probability measure as a risk neutral probability measure.

3.1 Review of the theory of ambit fields and processes

The general framework for defining an ambit process is as follows. Let $Y = \{Y_t(x)\}$ with $Y_t(x) := Y(x,t)$ denote a stochastic field in space–time $\mathcal{X} \times \mathbb{R}$ and let $\tau(\theta) = (x(\theta), t(\theta))$ denote a curve in $\mathcal{X} \times \mathbb{R}$. The values of the field along the curve are then given by $X_\theta = Y_{t(\theta)}(x(\theta))$. Clearly, $X = \{X_\theta\}$ denotes a stochastic process. In most applications, the space $\mathcal{X}$ is chosen to be $\mathbb{R}^d$ for $d = 1, 2$ or 3. Further, the stochastic field is assumed to be generated by innovations in space–time with values $Y_t(x)$ which are supposed to depend only on innovations that occur prior to or at time t and in general only on a restricted set of the corresponding part of space–time. I.e., at each point (x,t), the value of $Y_t(x)$ is only determined by innovations in some subset $A_t(x)$ of $\mathcal{X} \times \mathbb{R}_t$ (where $\mathbb{R}_t = (-\infty, t]$), which we call the *ambit set* associated to (x,t). Furthermore, we refer to Y and X as an *ambit field* and an *ambit process*, respectively.

In order to use such general ambit processes in applications, we have to impose some structural assumptions. More precisely, we will define $Y_t(x)$ as a stochastic integral plus a smooth term, where the integrand in the stochastic integral will consist of a deterministic kernel times a positive random variate which is taken to embody the *volatility* of the field Y. More precisely, we think of ambit fields as being of the form

$$Y_t(x) = \mu + \int_{A_t(x)} h(\xi, s, x, t)\, \sigma_s(\xi)\, L(\mathrm{d}\xi, \mathrm{d}s) + \int_{D_t(x)} q(\xi, s, x, t)\, a_s(\xi)\, \mathrm{d}\xi \mathrm{d}s, \tag{3}$$

where $A_t(x)$, and $D_t(x)$ are ambit sets, h and q are deterministic function, $\sigma \geq 0$ is a stochastic field referred to as *volatility*, a is also a stochastic field, and L is a *Lévy basis*.

Note that the corresponding ambit process X along the curve τ is then given by

$$X_\theta = \mu + \int_{A(\theta)} h(\xi, s, \tau(\theta))\sigma_s(\xi)L(d\xi, ds) + \int_{D(\theta)} q(\xi, s, \tau(\theta))a_s(\xi)d\xi ds, \tag{4}$$

where $A(\theta) = A_{t(\theta)}(x(\theta))$ and $D(\theta) = D_{t(\theta)}(x(\theta))$.

Of particular interest in many applications are ambit processes that are stationary in time and nonanticipative. More specifically, they may be derived from ambit fields Y of the form

$$Y_t(x) = \mu + \int_{A_t(x)} h(\xi, t-s, x)\, \sigma_s(\xi)\, L(\mathrm{d}\xi, \mathrm{d}s) + \int_{D_t(x)} q(\xi, t-s, x)\, a_s(\xi)\, \mathrm{d}\xi \mathrm{d}s. \tag{5}$$

Here the ambit sets $A_t(x)$ and $D_t(x)$ are taken to be *homogeneous* and *nonanticipative* i.e. $A_t(x)$ is of the form $A_t(x) = A + (x,t)$ where A only involves negative time coordinates, and similarly for $D_t(x)$. We assume further that $h(\xi, u, x) = q(\xi, u, x) = 0$ for $u \leq 0$.

3.2 Background on Lévy bases

Let $\mathcal{S}$ denote the δ–ring of subsets of an arbitrary non–empty set S, such that there exists an increasing sequence $\{S_n\}$ of sets in $\mathcal{S}$ with $\cup_n S_n = S$, see Rajput & Rosinski (1989). Recall from e.g. Barndorff-Nielsen & Shephard (2010), Pedersen (2003), Rajput & Rosinski (1989) that a Lévy basis

$L = \{L(B), B \in \mathcal{S}\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is an independently scattered random measure with Lévy–Khinchin representation

$$\psi_{L(B)}(v) = \log\left(\mathbb{E}(\exp(ivL(B)))\right),$$

given by

$$\psi_{L(B)}(v) = iva(B) - \frac{1}{2}v^2 b(B) + \int_{\mathbb{R}} \left(e^{ivr} - 1 - ivr\mathbb{I}_{[-1,1]}(r)\right) l(dr, B), \tag{6}$$

where a is a signed measure on $\mathcal{S}$, b is a measure on $\mathcal{S}$, $l(\cdot,\cdot)$ is the generalised Lévy measure such that $l(dr, B)$ is a Lévy measure on $\mathbb{R}$ for fixed $B \in \mathcal{S}$ and a measure on $\mathcal{S}$ for fixed dr. Without loss of generality we can assume that the generalised Lévy measure factorises as $l(dr, d\eta) = U(dr, \eta)\mu(d\eta)$, where μ is a measure on $\mathcal{S}$. Concretely, we take μ to be the *control measure*, see Rajput & Rosinski (1989), defined by

$$\mu(B) = |a|(B) + b(B) + \int_{\mathbb{R}} \min(1, r^2) l(dr, B), \tag{7}$$

where $|\cdot|$ denotes the total variation. Further, $U(dr, \eta)$ is a Lévy measure for fixed η. If $U(dr, \eta)$ does not depend on η, we call l and L *factorisable*. Note that a and b are absolutely continuous with respect to μ: $a(d\eta) = \widetilde{a}(\eta)\mu(d\eta)$, and $b(d\eta) = \widetilde{b}(\eta)\mu(d\eta)$.

For $\eta \in \mathcal{S}$, let $L'(\eta)$ be an infinitely divisible random variable such that

$$\psi_{L'(\eta)}(v) = \log\left(\mathbb{E}(\exp(ivL'(\eta)))\right),$$

with

$$\psi_{L'(\eta)}(v) = iv\widetilde{a}(\eta) - \frac{1}{2}v^2\widetilde{b}(\eta) + \int_{\mathbb{R}} \left(e^{ivr} - 1 - ivr\mathbb{I}_{[-1,1]}(r)\right) U(dr, \eta), \tag{8}$$

then we have

$$\psi_{L(d\eta)}(v) = \psi_{L'(\eta)}(v)\mu(d\eta). \tag{9}$$

In the following, we will refer to $L'(\eta)$ as the *Levy seed* of L at η. If L is factorisable, with $\mathcal{S} \subset \mathbb{R}^n$ and if μ is proportional to the Lebesgue measure, then L is called *homogeneous*.

Furthermore, we get from Rajput & Rosinski (1989, Proposition 2.6) that

$$\begin{aligned}\psi_{\int \overline{f} dL}(v) = \log\left(\mathbb{E}\left(\exp\left(iv\int \overline{f} dL\right)\right)\right) &= \int \log\left(\mathbb{E}(\exp(iv\overline{f}(\eta)L'(\eta)))\right)\mu(d\eta)\\ &= \int \psi_{L'(\eta)}(v\overline{f}(\eta))\mu(d\eta), \end{aligned}\tag{10}$$

for a deterministic function $\overline{f}$ which is integrable with respect to the Lévy basis.

In order to be able to compute moments of integrals with respect to a Lévy basis, we invoke a generalised Lévy–Itô decomposition, see Pedersen (2003). Corresponding to the generalised Lévy–Khintchine formula, (6), the Lévy basis can be written as

$$\begin{aligned} L(B) &= a(B) + \sqrt{b(B)}W(B) + \int_{\{|y|<1\}} y(N(dy, B) - \nu(dy, B)) + \int_{\{|y|\geq 1\}} yN(dy, B)\\ &= a(B) + \sqrt{b(B)}W(B) + \int_{\{|y|<1\}} y(N - \nu)(dy, B) + \int_{\{|y|\geq 1\}} yN(dy, B), \end{aligned}$$

for a Gaussian basis W and a Poisson basis N with intensity ν.

Now we have all the tools at hand which we need to compute the conditional characteristic function of ambit fields defined in (3) where σ and L are assumed independent and where we condition on the path of σ. The conditional cumulant function is denoted by ψ^σ and has the form

$$\psi^\sigma_{\int_{A_t(x)} h(\xi,s,x,t)\sigma_s(\xi)L(d\xi,ds)}(v) = \log\left(\mathbb{E}\left(\exp\left(iv\int_{A_t(x)} h(\xi,s,x,t)\sigma_s(\xi)L(d\xi,ds)\right)\middle|\,\sigma\right)\right)$$
$$= \int_{A_t(x)} \psi^\sigma_{L'(\xi,s)}\left(vh(\xi,s,x,t)\sigma_s(\xi)\right)\mu(d\xi ds), \quad (11)$$

where $\psi^\sigma_{L'}$ denotes, the characteristic exponent of the Lévy seed of L conditionally on σ and μ is the control measure associated with the Lévy basis L, cf. (8) and (7).

3.3 Walsh–type integration with respect to a Lévy basis

Since ambit processes are defined as stochastic integrals with respect to a Lévy basis, we briefly review in this section in which sense this stochastic integration should be understood. Throughout the rest of the paper, we work with stochastic integration with respect to martingale measures as defined by Walsh (1986). We will review this theory here briefly and refer to Barndorff-Nielsen et al. (2010*a*) for a detailed overview on integration concepts with respect to Lévy bases.

In the following we will present the integration theory on a bounded domain and comment later on how one can extend the theory to the case of an unbounded domain.

Let S denote a bounded set in $\mathcal{X} = \mathbb{R}^d$ for a $d \in \mathbb{N}$ and let $(S, \mathcal{S}, leb)$ denote a measurable space, where $\mathcal{S}$ denotes the Borel σ–algebra on S and leb is the Lebesgue measure.

Let L denote a Lévy basis on $S \times [0,T] \in \mathcal{B}(\mathbb{R}^{d+1})$ for some $T > 0$. For any $A \in \mathcal{B}_b(S)$ and $0 \le t \le T$, we define

$$L_t(A) = L(A,t) = L(A \times (0,t]),$$

which is a measure–valued process. Note that for a fixed set $A \in \mathcal{B}_b(S)$, $L_t(A)$ is an additive process in law.

In the following, we want to use the $L_t(A)$ as integrators as in Walsh (1986). In order to do that, we work under the square–integrability assumption, i.e.:

Assumption (A1): For each $A \in \mathcal{B}_b(S)$, we have that $L_t(A) \in L^2(\Omega, \mathcal{F}, P^*)$.

Note that, in particular, assumption (A1) excludes α–stable Lévy bases for $\alpha < 2$.

Next, we define the filtration $\mathcal{F}_t$ by

$$\mathcal{F}_t = \cap_{n=1}^{\infty}\mathcal{F}^0_{t+1/n}, \qquad \text{where} \qquad \mathcal{F}^0_t = \sigma\{L_s(A) : A \in \mathcal{B}_b(S), 0 < s \le t\} \vee \mathcal{N}, \quad (12)$$

and where $\mathcal{N}$ denotes the P–null sets of $\mathcal{F}$. Note that $\mathcal{F}_t$ is right–continuous by construction.

In the following, we will work without loss of generality under the zero–mean assumption on L, i.e.

Assumption (A2): For each $A \in \mathcal{B}_b(S)$, we have that $\mathbb{E}(L_t(A)) = 0$.

One can show that under the assumptions (A1) and (A2), $L_t(A)$ is a (square–integrable) *martingale* with respect to the filtration $(\mathcal{F}_t)_{0 \le t \le T}$. Note that these two properties together with the fact that $L_0(A) = 0$ a.s. ensure that $(L_t(A))_{t \ge 0, A \in \mathcal{B}(R^d)}$ is a *martingale measure* with respect to $(\mathcal{F}_t)_{0 \le t \le T}$ in the sense of Walsh (1986). Furthermore, we have the following orthogonality property: If $A, B \in \mathcal{B}_b(S)$ with $A \cap B = \emptyset$, then $L_t(A)$ and $L_t(B)$ are independent. Martingale measures which satisfy such an orthogonality property are referred to as *orthogonal martingale measures* by Walsh (1986), see also Barndorff-Nielsen et al. (2010*a*) for more details.

For such measures, Walsh (1986) introduces a *covariance measure* Q by

$$Q(A \times [0,t]) = < L(A) >_t, \tag{13}$$

for $A \in \mathcal{B}(R^d)$. Note that Q is a positive measure and is used by Walsh (1986) when defining stochastic integration with respect to L.

Walsh (1986) defines stochastic integration in the following way. Let $\zeta(\xi, s)$ be an *elementary* random field $\zeta(\xi, s)$, i.e. it has the form

$$\zeta(\xi, s, \omega) = X(\omega)\mathbb{I}_{(a,b]}(s)\mathbb{I}_A(x)\,, \tag{14}$$

where $0 \leq a < t$, $a \leq b$, X is bounded and $\mathcal{F}_a$–measurable, and $A \in \mathcal{S}$. For such elementary functions, the stochastic integral with respect to L can be defined as

$$\int_0^t \int_B \zeta(\xi, s)\, L(dx, ds) := X\left(L_{t\wedge a}(A \cap B) - L_{t\wedge b}(A \cap B)\right)\,, \tag{15}$$

for every $B \in \mathcal{S}$. It turns out that the stochastic integral becomes a martingale measure itself. Clearly, the above integral can easily be generalised to allow for integrands given by *simple* random fields, i.e. finite linear combinations of elementary random fields. Let $\mathcal{T}$ denote the set of simple random fields and let the *predictable* σ–algebra $\mathcal{P}$ be the σ–algebra generated by $\mathcal{T}$. Then we call a random field *predictable* provided it is $\mathcal{P}$–measurable. The aim is now to define stochastic integrals with respect to L where the integrand is given by a predictable random field.

In order to do that Walsh (1986) defines a norm $\|\cdot\|_L$ on the predictable random fields ζ by

$$\|\zeta\|_L^2 := \mathbb{E}\left[\int_{[0,T]\times S} \zeta^2(\xi, s)\, Q(d\xi, ds)\right]\,, \tag{16}$$

which determines the Hilbert space $\mathcal{P}_L := L^2(\Omega \times [0,T] \times S, \mathcal{P}, Q)$, and he shows that $\mathcal{T}$ is dense in $\mathcal{P}_L$. Hence, in order to define the stochastic integral of $\zeta \in \mathcal{P}_L$, one can choose an approximating sequence $\{\zeta_n\}_n \subset \mathcal{T}$ such that $\|\zeta - \zeta_n\|_L \to 0$ as $n \to \infty$. Clearly, for each $A \in \mathcal{S}$, $\int_{[0,t]\times A} \zeta_n(\xi, s)\, L(d\xi, ds)$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, P)$, and thus there exists a limit which is defined as the stochastic integral of ζ.

Then, this stochastic integral is again a martingale measure and satisfies the following *Itô–type isometry*:

$$\mathbb{E}\left[\left(\int_{[0,t]\times A} \zeta(\xi, s) L(d\xi, ds)\right)^2\right] = \|\zeta\|_L^2\,, \tag{17}$$

see (Walsh 1986, Theorem 2.5) for more details.

Remark 1. In order to use Walsh–type integration in the context of ambit fields, we note the following:

- General ambit sets $A_t(x)$ are not necessarily bounded. However, the stochastic integration concept reviewed above can be extended to unbounded ambit sets using standard arguments, cf. Walsh (1986, p. 289).

- For ambit fields with ambit sets $A_t(x) \subset \mathcal{X} \times (-\infty, t]$, we define Walsh–type integrals for integrands of the form

$$\zeta(\xi, s) = \zeta(\xi, s, x, t) = \mathbb{I}_{A_t(x)}(\xi, s) h(\xi, s, x, t)\sigma_s(\xi). \tag{18}$$

- The original Walsh's integration theory covers integrands which do not depend on the time index t. Clearly, the integrand given in (18) generally exhibits t–dependence due to the choice of the ambit set $A_t(x)$ and due to the deterministic kernel function h. In order to allow for time dependence in the integrand, we can define the integrals in the Walsh sense for any *fixed* t.

In order to ensure that the ambit fields (as defined in (3)) are well–defined (in the Walsh–sense), throughout the rest of the paper, we will work under the following assumption:

Assumption (A3): Let L denote a Lévy basis on $S \times [-\infty, T]$, where S denotes a not necessarily bounded set S in $\mathcal{X} = \mathbb{R}^d$ for some $d \in I\!N$. We define a covariance measure Q for an unbounded domain (extending definition (13)) and, next, we define a Hilbert space $\mathcal{P}_L$ with norm $|| \cdot ||_L$ as in (16) (extended to an unbounded domain) and, hence, we have an Itô isometry of type (17) extended to an unbounded domain. We assume that, for fixed x and t,

$$\zeta(\xi, s) = \mathbb{I}_{A_t(x)}(\xi, s) h(\xi, s, x, t) \sigma_s(\xi)$$

satisfies

1. $\zeta \in \mathcal{P}_L$,
2. $||\zeta||_L^2 = \mathbb{E}\left[\int_{\mathbb{R}\times\mathcal{X}} \zeta^2(\xi, s) Q(d\xi, ds)\right] < \infty$.

Note that in our forward price model we will discard the drift term from the general ambit field defined in (3) and hence we do not add an integrability condition for the drift.

3.4 Lévy Semistationary Processes ($\mathcal{LSS}$)

After having reviewed the basic traits of ambit fields, we briefly mention the null–spatial case of semi–stationary ambit fields, i.e. the case when we only have a temporal component and when the kernel function depends on t and s only through the difference $t - s$. This determines the class of Lévy semistationary processes ($\mathcal{LSS}$), see Barndorff-Nielsen et al. (2010*b*). Specifically, let $Z = (Z_t)_{t\in\mathbb{R}}$ denote a general Lévy process on $\mathbb{R}$. Then, we write $Y = \{Y_t\}_{t\in\mathbb{R}}$, where

$$Y_t = \mu + \int_{-\infty}^{t} g(t-s)\omega_s \mathrm{d}Z_s + \int_{-\infty}^{t} p(t-s) a_s \mathrm{d}s, \tag{19}$$

where μ is a constant, g and p are nonnegative deterministic functions on $\mathbb{R}$, with $g(t) = p(t) = 0$ for $t \leq 0$, and ω and a are càdlàg, stationary processes. The reason for here denoting the volatility by ω rather than σ will become apparent later. In abbreviation the above formula is written as

$$Y = \mu + g * \omega \bullet Z + p * a \bullet leb, \tag{20}$$

where leb denotes Lebesgue measure. In the case that Z is a Brownian motion, we call Y a *Brownian semistationary* ($\mathcal{BSS}$) *process*, see Barndorff-Nielsen & Schmiegel (2009). The stochastic integration is here assumed to be in the sense of Basse-O'Connor et al. (2010) since we start the integration at $-\infty$ rather than computing the integral over a compact interval. Note that this general integration concept specialises to Itô integration in the semimartingale framework.

In the following, we will often, for simplicity, work within the set–up that both $\mu = 0$ and $q \equiv 0$, hence

$$Y_t = \int_{-\infty}^{t} g(t-s)\omega_s dZ_s. \tag{21}$$

For integrability conditions on ω and g, we refer to Barndorff-Nielsen et al. (2010*b*). Note that the stationary dynamics of Y defined in (21) is a special case of a volatility modulated Lévy–driven Volterra process, which has the form

$$Y_t = \int_{-\infty}^{t} g(t,s)\omega_s \, dZ_s \,, \tag{22}$$

where Z is a Lévy process and g is a real–valued measurable function on $\mathbb{R}^2$, such that the integral with respect to Z exists.

4 Modelling the forward market by ambit fields

After having reviewed the basic definitions of ambit fields and the stochastic integration concept due to Walsh (1986), we proceed now by introducing a general model for (deseasonalised) electricity forward prices based on ambit fields.

We consider a probability space $(\Omega, \mathcal{F}, P^*)$. We set $\mathbb{R}_+ = [0,\infty)$ and define a Lévy basis $L = (L(s,A))_{s\in\mathbb{R}, A\in\mathcal{B}(\mathbb{R}_+)}$ and a stochastic volatility field $\sigma = (\sigma_s(A))_{s\in\mathbb{R}, A\in\mathcal{B}(\mathbb{R}_+)}$. Throughout the remaining part of the paper, we define the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ by

$$\mathcal{F}_t = \cap_{n=1}^{\infty} \mathcal{F}_{t+1/n}^0, \qquad \text{where} \qquad \mathcal{F}_t^0 = \sigma\{L(s,A) : A \in \mathcal{B}(\mathbb{R}_+), s \le t\} \vee \mathcal{N}, \tag{23}$$

and where $\mathcal{N}$ denotes the P–null sets of $\mathcal{F}$. Note that $\mathcal{F}_t$ is right–continuous by construction. Also, we define the enlarged filtration $\{\overline{\mathcal{F}}_t\}_{t\in\mathbb{R}}$ by

$$\overline{\mathcal{F}}_t = \cap_{n=1}^{\infty} \overline{\mathcal{F}}_{t+1/n}^0, \qquad \text{where} \qquad \overline{\mathcal{F}}_t^0 = \sigma\{(L(s,A), \sigma_s(A)) : A \in \mathcal{B}(\mathbb{R}_+), s \le t\} \vee \mathcal{N}. \tag{24}$$

4.1 The model

The new model for the forward price $f_t(x)$ is defined for fixed $t \in \mathbb{R}$ and for $x > 0$ by

$$f_t(x) = \int_{A_t(x)} k(\xi, t-s, x)\sigma_s(\xi) L(d\xi, ds), \tag{25}$$

where

(i) the Lévy basis L is square integrable and has zero mean (this is an extension of assumptions (A1) and (A2) to an unbounded domain);

(ii) the stochastic volatility field σ is assumed to be adapted to $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ and independent of the Lévy basis L and in order to ensure stationarity in time, we assume that $\sigma_s(\xi)$ is stationary in s;

(iii) the kernel function k is assumed to be non–negative and chosen such that $k(\xi,u,x) = 0$ for $u < 0$;

(iv) the convolution $k \star \sigma$ is integrable w.r.t. L, i.e. it satisfies (A3);

(v) the ambit set is chosen to be

$$A_t(x) = A_t = \{(\xi, s) : \xi > 0, \ s \le t\}, \tag{26}$$

for $t \in \mathbb{R}, \ x \ge 0$, see Figure 1. Note that the ambit set is of the type $A_t(x) = A_0(x) + (0,t)$ for $A_0(x) = \{(\xi,s) : \xi \ge 0, \ s \le 0\}$. In the following, we will drop the (x) in the notation of the ambit set, i.e. $A_t(x) = A_t$, since the particular choice of the ambit set defined in (26) does not depend on the spatial component x.

Note that $f_t(x)$ is a stochastic process in time for each fixed x. Also, it is important to note that for *fixed* x, $f_t(x)$ is stationary in t, more precisely $f_t(\cdot)$ is a stationary field in time. However, as soon as we replace x by a function of t, $x(t)$ say, in our case by $x(t) = T - t$, $f_t(x(t))$ is generally not stationary any more. The intuition behind this approach is that we do not believe that the price of an individual forward contract is necessarily stationary in time, but different forward contracts which have the same time to maturity left (e.g. a certain choice of a yearly and a quarterly forward contract) are modelled in stationarity.

In order to construct a specific model for the forward price, we need to specify the kernel function k, the stochastic volatility field $\sigma_s(\xi)$ and L.

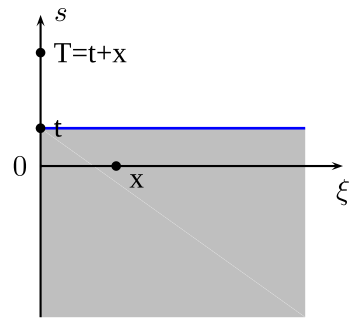

Figure 1: The ambit set $A_t(x) = A_t$.

It is important to note that, when working with general ambit processes as defined in (25), in modelling terms we can play around with both the ambit set, the weight function k, the volatility field σ and the Lévy basis in order to achieve a dependence structure we want to have. As such there is generally not a unique choice of the ambit set or the weight function or the volatility field to achieve a particular type of dependence structure and the choice will be based on stylised features, market intuition and considerations of mathematical/statistical tractability.

In order to make the model specification easier in practice, we have decided to work with the encompassing ambit set defined in (26). The kernel function k should be chosen in such a way that the Samuelson effect can be captured well. We will come back to this issue later again.

Also the choice of the volatility field is very important in our modelling framework. We propose to model the volatility field itself by an ambit field of the form

$$\sigma_t^2(x) = \int_{A_t(x)} j(\xi, t-s, x) L^*(d\xi, ds), \tag{27}$$

for a nonnegative Lévy basis L^*, a non–negative deterministic kernel function j satisfying $j(\xi, u, x) = 0$ for $u < 0$ and an ambit set $A_t(x)$ defined as in (26) (but it would be possible to choose an ambit set which is different from $A_t(x)$ for the model specification of σ). Clearly, the tempo–spatial volatility field σ defined by (27) is stationary with respect to the temporal dimension. In order to ensure that forward contracts close in maturity dates are strongly correlated with each other (as indicated by empirical studies), we can choose the kernel j such that

$$Cor(\sigma_t^2(x), \sigma_t^2(\bar{x}))$$

is high for values of x and $\bar{x}$ which are close to 0 (i.e. closeness to delivery). Also, the specification of the volatility field σ can be used to incorporate the Samuelson effect in our model.

Remark 2. Note that the forward price at time 0 implied by the model is given as

$$f_0(T) = \int_{A_0} k(\xi, -s, x)\sigma_s(\xi) L(d\xi, ds). \tag{28}$$

Hence, we view the *observed* forward price as a realisation of the random variable $f_0(T)$ given in (28), contrary to most other models where $f_0(T)$ is considered as deterministic and *put equal to* the observed price.

Remark 3. We have chosen to model the forward price in (25) as an arithmetic model. One could of course interpret $f_t(x)$ in (25) as the *logarithmic* forward price, and from time to time in the discussion below this is the natural context. However, in the theoretical considerations, we stick to the arithmetic model, and leave the analysis of the geometric case to Section 8.1. We note that Bernhardt et al. (2008) proposed and argued statistically for an arithmetic spot price model for Singapore electricity data. An arithmetic spot model will naturally lead to an arithmetic dynamics for the forward price. Benth et al. (2007) proposed an arithmetic model for spot electricity, and derived an arithmetic forward price dynamics. In Benth, Cartea & Kiesel (2008) arithmetic spot and forward price models are used to investigate the risk premium theoretically and empirically for the German EEX market.

We conclude this section by studying a simple example which is included in our new modelling framework. Since Gaussian models are straightforward to construct, we focus on a non–Gaussian case here.

Example 1. *Let L be a homogeneous symmetric normal inverse Gaussian (NIG) Lévy basis, more specifically having Lévy seed with density*

$$\pi^{-1}\delta|y|^{-1}\gamma K_1(\gamma|y|),$$

where K denotes the modified Bessel function of the second kind and where $\delta, \gamma > 0$, see Barndorff-Nielsen (1998). Then

$$\psi_{L'}(\theta) = \delta\gamma - \delta\left(\gamma^2 + \theta^2\right)^{1/2}.$$

Further, for simplicity, we set $\sigma_s(\xi) \equiv 1$ (since the NIG distribution already incorporates some sort of stochastic volatility compared to Brownian white noise) and choose an exponential kernel function

$$k(\xi, t-s, x) = \exp(-\alpha(\xi + (t-s) + x)) = \exp(-\alpha(\xi - s))\exp(-\alpha T),$$

for $\alpha > 0$. Then

$$\begin{aligned}\log(\mathbb{E}(iv f_t(x))) &= \int_{A_t} \psi_{L'}(vk(\xi, t-s, x))d\xi ds \\ &= \delta\gamma \int_{-\infty}^{t}\int_0^{\infty} \left(1 - \sqrt{1 + c^2\exp(-2\alpha(\xi - s))}\right) d\xi ds,\end{aligned}$$

for $c = v\exp(-2\alpha T)/\gamma$. This integral can be expressed in terms of standard functions, see Section A.1 in the appendix.

5 Relating traditional model classes to the ambit framework

As already mentioned, the use of ambit fields for constructing models for forward prices is entirely new to the literature and extends the use of correlated random fields to allow for both functional and statistical dependence, as described in more detail below.

In the following, we will describe how some of the traditional models can be related to the ambit framework.

5.1 Heath–Jarrow–Morton model

In the *geometric* Heath et al. (1992) framework, the dynamics of the logarithmic forward price under the risk neutral measure are modelled by

$$d\log(f_t(x)) = \sigma_t(x)dW_t, \qquad \text{for } t \geq 0,$$

where W is a standard Brownian motion and σ is a positive stochastic volatility process. Note that we start at time 0 here. Hence, the explicit formula for the forward price is given by

$$f_t(x) = f_0(x) \exp\left(\int_0^t \sigma_s(x) dW_s - \frac{1}{2} \int_0^t \sigma_s^2(x) ds \right).$$

Clearly, such a model is a special case of an ambit field defined in (3), where $A_t(x) = [0,t] \times \{x\}$, L is a Gaussian Lévy basis and the kernel function h satisfies $h \equiv 1$.

5.2 Random field models

Ambit processes embed the Gaussian and Lévy field models proposed in Albeverio et al. (2004), Kennedy (1994, 1997). To see that note that we can set $\sigma \equiv 1$ and we can choose $A_t(x)$ to be an interval.

If we allow for a non–trivial kernel function h or stochastic volatility field σ we can obtain some of the conditional volatility models proposed in Goldstein (2000), Kimmel (2004).

5.3 Stochastic string shock model

Also, the *stochastic string shock* model by Santa-Clara & Sornette (2001), which was designed to model the term structure of interest rates, is related to the ambit framework. Their modelling framework is given as follows. The dynamics of the forward rate are given by

$$d_t f_t(x) = \alpha_t(x) dt + \sigma_t(x) d_t Z(t,x),$$

for adapted processes a and σ and a stochastic string shock Z. Note here that the notation d_t is taken from Santa-Clara & Sornette (2001) and refers to the fact that we look at the differential operator w.r.t. t. A string shock is defined as random field $(Z(t,x))_{t,x\geq 0}$ which is continuous in both t and x and is a martingale in t. Furthermore the variance of the t–increments has to equal the time change, i.e. $Var(d_t Z(t,x)) = dt$ for all $x \geq 0$, and the correlation of the t–increments, i.e. $Cor(d_t Z(t,x), d_t Z(t,y))$, does not depend on t. Santa-Clara & Sornette (2001) show that such stochastic strings can be obtained as solutions to second order linear stochastic partial differential equations (SPDEs). It is well–know that such SPDEs have a unique solution (under some boundary conditions), see Morse & Feshbach (1953) and the references in Barndorff-Nielsen et al. (2010*a*), and the solution is representable in terms of an integral, often of convolution type, of a Green function with respect to the random noise. The class of stochastic strings given by solutions to SPDEs is large and includes in particular (rescaled) Brownian sheets and Ornstein–Uhlenbeck sheets. Similarly to the procedure presented in Goldstein (2000), Santa-Clara & Sornette (2001) argue that it might also be useful to smoothen the string shocks further, so that they are particularly smooth in direction of time to delivery x. Again, this can be achieved by integrating a stochastic string shock with respect to its second component. Stochastic string shock models are true generalisations of the Heath et al. (1992) framework which do not lose the parsimonious structure of the original HJM model. Also, due to their general structure, string models can give rise to a variety of different correlation functions and, hence, are very flexible tools for modelling various stylised facts without needing many parameters.

The main element in the stochastic string model, see Santa-Clara & Sornette (2001, p. 159), is the term

$$\int_0^t \sigma_v(T-v) d_v Z(v, T-v) = \int_0^t \int_0^\infty \sigma_s(T-s) G(T-s,z) \eta(s,z) dz ds, \tag{29}$$

where Z is a stochastic string shock, η is white noise, σ is an adapted process and G is the corresponding Green function. The derivation by Santa-Clara & Sornette (2001) is partly heuristic. However, rigorous mathematical meaning can be given to the integral in (29) by the Walsh (1986) concept of martingale measures, see Section 3.3.

This may be compared to a special case of our ambit process where the integration is carried out with respect to a Gaussian Lévy basis, i.e. by choosing

$$\int_0^t \sigma_v(T-v)d_vZ(v,T-v) = \int_0^t \int_0^\infty \sigma_s(T-s)G(T-s,z)d_sW_s(dz).$$

So, for a deterministic function σ the product of σ and G is what we can model by the function h in the ambit framework, i.e.

$$h(\xi,s,T) = \sigma_s(T-s)G(T-s,\xi).$$

The main difference between the approach advocated in the present paper and the stochastic string shock approach lies in the fact that the ambit fields focused on here are constructed as stationary processes in time where the integration of the temporal component starts at $-\infty$ and not at 0 and, also, we consider general Lévy bases with a wide range of infinitely divisible distributions and do not restrict ourselves to the continuous Gaussian case.

5.4 Audet et al. (2004) model

Consider the model by Audet et al. (2004) written in the Musiela parameterisation. They study the electricity market on a finite time horizon $[0,T^*]$ and model the dynamics of the forward price $f_t(x)$ by

$$df_t(x) = f_t(x)e^{-\alpha x}\sigma_{x+t}dB_{x+t}(t),$$

for a deterministic, bounded volatility curve $\sigma : [0,T^*] \to \mathbb{R}_+$, a constant $\alpha > 0$ and where B_{x+t} denotes a Brownian motion for the forward price with time of maturity $x+t$. Further, the correlation structure between the Brownian motions is given by

$$\mathrm{corr}(dB_{x'}(t), dB_x(t)) = \exp(-\rho(x-x'))\,dt = \exp(-\rho|T-T'|)\,dt, \quad \text{for all } 0 \leq x, x' \leq T^*-t, \tag{30}$$

where $x' = T'-t, x = T-t$. Such a model implies that the volatility of the forward price is lower than the volatility of the spot price, an effect which is described by the parameter α. Also, forward contracts which are close in maturity can be modelled to be strongly correlated, an effect which is reflected by the choice of the parameter ρ.

We observe that the above model for the logarithmic forward price is in fact another special case of an ambit process, with deterministic volatility and an ambit set $\mathcal{A}_t(x) = [0,t]\times\{x\}$, and the Lévy basis being a Gaussian random field which is Brownian in time and has a spatial correlation structure in space as specified in (30).

5.5 Forward model implied by the spot model described in Barndorff-Nielsen et al. (2010*b*)

If the deseasonalised log–spot price is given by (22), then we know from (Barndorff-Nielsen et al. 2010*b*, Proposition 4) that in the case when $L = W$ is a standard Brownian motion and the stochastic volatility process is given by

$$\omega_t = \int_{-\infty}^t G(t,s)\widetilde{Z}_s,$$

for a Lévy process $\widetilde{Z}$, then the corresponding forward price $f_t(x)$ is given by

$$\ln f_t(x) = \Xi_t(x) + \int_{-\infty}^{t} g(t+x,s)\omega_s \, dW_s + \frac{1}{2}\int_{-\infty}^{t}\int_0^x g^2(t+x,t+u)G(t+u,s)\, du \, d\widetilde{Z}_s,$$

for some deterministic function $\Xi_t(x)$. We observe that, apart from the term $\Xi_t(x)$ this is the superposition of two ambit processes.

6 Constructing a spot model from the forward model

After we have presented our new model for the forward price, which is based on time–stationary ambit fields, we turn our attention now to the question of which model for the electricity spot price is implied by our new modelling framework.

In order to answer this question, we are interested in the limiting behaviour of the forward price $f_t(T-t)$ when t tends to time of delivery T.

Let us assume the existence of function κ such that the kernel function k satisfies (pointwise)

$$\lim_{t\to T} k(\xi, t-s, T-t) =: \kappa(\xi, T-s).$$

A natural candidate for the spot price is then

$$S_T = \int_{A_T} \kappa(\xi, T-s)\sigma_s(\xi)L(d\xi, ds) = \int_{\mathbb{R}_+\times\mathbb{R}} \mathbb{I}_{A_T}(\xi,s)\kappa(\xi,T-s)\sigma_s(\xi)L(d\xi,ds), \tag{31}$$

provided the integral exists. In fact, under a mild further condition we will have

$$f_t(T-t) \xrightarrow{\mathbb{P}} S_T, \qquad \text{as } t \to T.$$

Equivalently, the question is whether

$$f_t(T-t) - S_T = \int_{\mathbb{R}_+\times\mathbb{R}} \varkappa(\xi,s,t,T)\sigma_s(\xi)L(d\xi,ds) \xrightarrow{\mathbb{P}} 0, \tag{32}$$

as $t \to T$, where

$$\varkappa(\xi,s,t,T) = \mathbb{I}_{A_t}(\xi,s)k(\xi,t-s,T-t) - \mathbb{I}_{A_T}(\xi,s)\kappa(\xi,T-s).$$

From Section A.3 in the appendix, we know that

$$Var(f_t(T-t) - S_T) = C\int_{\mathbb{R}_+\times\mathbb{R}} \varkappa^2(\xi,s,t,T)\mathbb{E}(\sigma_s^2(\xi))d\xi ds,$$

for a constant $C > 0$ (specified by the variance of L).

Hence, provided the integral defined in (31) exists, a sufficient condition for (32) is given by

$$\varkappa(\xi,s,t,T) \xrightarrow{L^2} 0, \qquad \text{as } t \to T. \tag{33}$$

So, if the limit above exists, then it is natural to interpret the limiting process S defined in (31) as a *spot price process*.

Note that the spot price process implied by our ambit field–based forward price model is driven by a tempo–spatial Lévy base, more precisely by a two–parameter random field and not just by a Brownian motion or a Lévy process.

6.1 The Gaussian case

A case of some special interest is the situation where the driving Lévy basis L of the ambit field is Gaussian.

Under the assumption that σ is independent of L, we obtain that $f_t(x)$ is mixed normal, in particular

$$f_t\left(x\right)|\,\sigma \sim N\left(0, \int_{A_t} k(\xi, t-s, x)^2 \sigma_s^2\left(\xi\right) d\xi ds\right).$$

Let W be a homogeneous Gaussian Lévy basis and let k be of the form

$$k(\xi, t-s, x) = k_0^*(t-s, x) k_1(\xi) \tag{34}$$

with $k_0^*(T-s, 0) = k_0(T-s)$, for some function k_0 with $k_0^*(t-s, T-t) \to k_0(T-s)$ for $t \to T$. Hence

$$f_t(x) = \int_{A_t} k_1(\xi) k_0^*(x, t-s) \sigma_s(\xi) W(d\xi, ds).$$

Suppose that

$$\mathbb{I}_{A_t}(\xi, s) k_1(\xi) k_0^*(x, t-s) - \mathbb{I}_{A_T}(\xi, s) k_1(\xi) k_0(T-s) \xrightarrow{L^2} 0,$$

as $t \to T$. Then we obtain for $x \to 0$ that

$$f_t\left(x\right) \xrightarrow{\mathbb{P}} S_T = \int_{A_T} k_1(\xi) k_0(T-s) \sigma_s(\xi) W(d\xi, ds).$$

Clearly,

$$S_T|\sigma \sim N\left(0, \int_{-\infty}^{T} k_0^2(T-s) \left(\int_0^{\infty} k_1^2(\xi)\sigma_s^2(\xi) d\xi\right) ds\right).$$

Remark 4. Note that if we define a positive kernel function g and a positive stochastic process ω by

$$g^2(T-s) = k_0^2(T-s), \qquad \omega_s^2 = \int_0^{\infty} k_1^2(\xi)\sigma_s^2(\xi) d\xi, \quad \text{for all } s \leq T,$$

then

$$S_T \stackrel{law}{=} \int_{-\infty}^{T} g(T-s)\omega_s dB_s, \tag{35}$$

where B is a standard Brownian motion and ω^2 is a stationary process. Hence the spot price equals in law a $\mathcal{BSS}$ process. The latter has been studied in Barndorff-Nielsen et al. (2010*b*) as a model for electricity spot prices.

Let us consider an example:

Example 2. *Motivated by the standard OU models, we choose*

$$k_0^*(t-s, x) = \sigma \exp(-\alpha(t-s+x)),$$

for some $\alpha > 0$. Thus, the spot price becomes

$$Y_t = \int_{-\infty}^{t} \sigma \exp(-\alpha(t-s))\, dB_s\,,$$

which we recognise as the stationary solution of the OU process

$$dY_t = -\alpha Y_t \, dt + \sigma \, dB_t \, .$$

*The choice of k_0^* can also be motivated from continuous time ARMA processes, see Brockwell (2001*a,b*). For $\alpha_i > 0, i = 1, \dots, p, p \geq 1$, introduce the matrix*

$$A = \begin{bmatrix} 0 & I_{p-1} \\ -\alpha_p & -\alpha_{p-1} \cdots - \alpha_1 \end{bmatrix} , \tag{36}$$

where I_n denotes the $n \times n$ identity matrix. Define the p–dimensional vector $\mathbf{b}' = (b_0, b_1, \dots, b_{p-1})$, where $b_q = 1$ and $b_j = 0$ for $q < j < p$, and introduce

$$k_0^*(t-s, x) = \mathbf{b}' \exp(A(t-s+x)\mathbf{e}_p \, ,$$

with $\mathbf{e}_k$ being the kth canonical unit vector in $\mathbb{R}^p$. Then, as long as the eigenvalues of A all have negative real part, the stationary spot price is given as

$$Y_t = \int_{-\infty}^{t} \mathbf{b}' \exp(A(t-s))\mathbf{e}_p \, dB_s \, .$$

This is a continuous–time ARMA process. Such spot models for electricity have been suggested by Bernhardt et al. (2008), with the driving noise being an α–stable Lévy process.

Remark 5. Recall that the Samuelson effect describes the empirical fact that the volatility of the forward price converges to the volatility of the spot price when the time to maturity approaches zero.

This finding is in fact naturally included in our modelling framework when we model the (logarithmic) forward price by an ambit field and the (logarithmic) spot price by an $\mathcal{LSS}$ process as long as the assumptions above are satisfied. Recall that the conditional variance of the forward contract at time t is given by

$$\int_0^\infty k_0^*(v,x)^2 \int_0^\infty k_1(\xi)^2 \sigma_{t-v}^2(\xi) d\xi dv,$$

which converges by construction and as $x \to 0$, and hence $t \to T$, to

$$\int_0^\infty k_0^2(v) \int_0^\infty k_1(\xi)^2 \sigma_{T-v}^2(\xi) d\xi dv,$$

which is the conditional variance of the spot at maturity.

Example 3. *Bjerksund et al. (2000) propose a geometric Brownian motion model for the electricity forward price with volatility given by $\eta(t,T) = a/(T-t+b)$ for a, b two positive constants. They argue that the Samuelson effect in electricity markets are much steeper than in other commodity markets, defending the choice of a hyperbolic function rather than exponential. The volatility $\eta(t,T)$ motivates the choice*

$$k^*(t-s,x) = \frac{a}{t-s+x+b} \, ,$$

which yields

$$g(t-s) = \frac{a}{t-s+b} \, .$$

Hence, we have from above results that the forward dynamics converges to the spot under the additional hypothesis on the link between k_1, σ_t, and ω. Furthermore, the Samuelson effect holds for this. (Note that $\int_{-\infty}^t g^2(t-s)ds < \infty$ for $a, b > 0$, hence the integrability condition (A3) is satisfied.)

7 Martingale conditions and change of measure

7.1 Martingale conditions

According to the traditional modelling framework, the forward price is modelled such that it is a semimartingale and so that there exists a P^*–measure under which the price dynamics becomes a (local) martingale. In the standard HJM framework in interest rate theory this is stated as a drift condition on the dynamics. However, here we have an explicit dynamics, and the semimartingale property is connected to the regularity of the input in the stochastic integral.

First, we will formulate the martingale conditions for more general ambit fields as defined in (3), where the ambit set $A_t(x) = A_t$ is chosen as in (26). Next, we show how such conditions simplify in the new modelling framework described in (25).

Note that all proofs will be given in the appendix.

Theorem 1. *Let $x = T - t$ for some $T > 0$ and for a fixed $t \in \mathbb{R}$ write*

$$Y_t(x) = Y_t(T-t) = \int_{A_t} h(\xi, s, T-t, t)\sigma_s(\xi)L(d\xi, ds), \quad \textit{where } A_t = \{(\xi, s) : \xi > 0, s \leq t\},$$

for a deterministic kernel function h, an adapted, non–negative random field σ and a Lévy basis L satisfying both (A1) and (A2) on an unbounded domain and (A3).

Then $(Y_t(T-t))_{t\in\mathbb{R}}$ is a martingale w.r.t. $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ if and only if for all $\xi > 0$, $s \leq t \leq T$ we have

$$h(\xi, s, T-t, t) = \tilde{h}(\xi, s, T), \tag{37}$$

for some deterministic kernel function $\tilde{h}$.

Remark 6. If we would like to work with Lévy bases L which do not have zero mean, then the martingale conditions have to be extended by an additional drift condition.

Corollary 1. *In the special case of the new model defined in (25), we get that $(f_t(T-t))_{t\in\mathbb{R}}$ is a martingale w.r.t. $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ if and only if for all $\xi > 0$, $s \leq t \leq T$ we have*

$$k(\xi, t-s, T-t) = \tilde{k}(\xi, s, T)\,, \tag{38}$$

for a deterministic kernel function $\tilde{k}$.

Remark 7. Note that we have stated the martingale property for all t on the real line (which does not include $-\infty$). We refer to Basse-O'Connor et al. (2009) for a study on martingale properties at $-\infty$.

However, in practical terms, we are mainly interested in the martingale property for $t \geq 0$ since this is when the market is active. Negative times are *only* a modelling device in order to have stationary models.

Remark 8. Note that when condition (38) is satisfied, then $f_t(x) = \int_{A_t} \tilde{k}(\xi, s, T)\sigma_s(\xi)L(ds, d\xi)$. If we construct a spot model from such a forward model as in Section 6 (under suitable conditions), then we obtain a spot of the form $S_T = \int_{A_T} \tilde{k}(\xi, s, T)\sigma_s(\xi)L(ds, d\xi)$, which includes the specification in (31) as a special case. Further, similar to Remark 5, we can show that also within the martingale framework, we can account for the Samuelson effect.

Clearly, the martingale condition is rather strong and it is hence necessary to check whether there are actually any relevant cases left, which are not excluded by condition (38). Hence, let us study some examples.

First we show that the condition (38) covers the standard Heath et al. (1992) models, that come from stochastic partial differential equations.

Example 4. *The traditional way to model the forward dynamics using the Musiela parameterisation with $x = T - t$, is given by*

$$df_t(x) = \frac{\partial f_t}{\partial x}(x)\,dt + h(x,t)\,dW_t\,,$$

where, for simplicity, we disregard any spatial dependency in the Gaussian field W so that it is indeed a Brownian motion. Under appropriate (weak) conditions, the mild solution of this stochastic partial differential equation (SPDE) is given by

$$f_t(x) = S_t f_0(x) + \int_0^t S_{t-s} h(x,s)\,dW_s\,,$$

where S_t is the right–shift operator, $S_t g(x) = g(x+t)$, see Carmona & Tehranchi (2006), Da Prato & Zabczyk (1992) for more details. Hence,

$$f_t(x) = f_0(x+t) + \int_0^t h(s,(t+x)-s)\,dW_s = f_t(T-t) = f_0(T) + \int_0^t h(s, T-s)\,dW_s.$$

Thus, we see that the martingale condition (38) is satisfied.

Another important example is motivated by the Audet et al. (2004) model.

Example 5. *In our modelling framework defined in (25), we choose k to be of the form*

$$k(\xi, t-s, x) = k(\xi, t-s, T-t) = \exp(-\alpha((\xi + x) + (t-s))) = \exp(-\alpha((\xi + T - s))),$$

for some $\alpha > 0$. Then the martingale condition is clearly satisfied.

Further important examples of kernel functions which satisfy the martingale condition can be constructed as follows.

Example 6. *We can focus on kernel functions k which factorise like*

$$k(\xi, t-s, x) = k_0^*(t-s, x) k_1(\xi),$$

as described in (34). Clearly, the choice of the function k_1 does not have any impact on the question whether the ambit field is a martingale. This is determined by the choice of the function k_0^*.

Motivated from the Bjerksund et al. (2000) model, we could choose

$$k_0^*(t-s, x) = \frac{a}{t-s+x+b} = \frac{a}{T-s+b},$$

for $a, b > 0$.

Also, motivated by the CARMA models discussed in Example 2, the following choice of k_0^* is also interesting:

$$k_0^*(t-s, x) = \mathbf{b}' \exp(A(t-s+x)\mathbf{e}_p\,,$$

for the p–dimensional vector $\mathbf{b}' = (b_0, b_1, \ldots, b_{p-1})$, where $b_q = 1$ and $b_j = 0$ for $q < j < p$, with $\mathbf{e}_k$ being the kth canonical unit vector in $\mathbb{R}^p$ and where the matrix A is defined as in (36).

So we have seen that it is possible to formulate martingale conditions for ambit fields and we have studied some relevant examples of forward price models which satisfy the martingale condition. However, the martingale condition (37) implies that we cannot have t–dependence in the kernel function. This unfortunately rules out many interesting more general ambit fields.

In the energy context, however, it might not be as crucial that $f_t(T-t)$ is a martingale as it is in the context of modelling interest rates. In fact, as already indicated in the Introduction, one can argue that from a liquidity point of view, it would be possible to use non–martingales for modelling forward prices since in many emerging electricity markets, one may not be able to find any buyer to get rid of a forward, nor a seller when one wants to enter into one. Hence, the illiquidity prevents possible arbitrage opportunities from being exercised.

7.2 Change of measure

If our forward price model is formulated under a risk–neutral pricing measure, it is of interest to understand how to get back to the physical measure in order to have a model for the observed prices. We introduce an Esscher transform to accommodate this.

Throughout this section we will assume that the Lévy basis is *homogeneous* to simplify the notation.

Remark 9. Note that in order to define the change of measure we work on a market with finite time horizon $T^* > 0$, hence we define our model on $\mathbb{R}_{T^*} = (-\infty, T^*]$ rather than on $\mathbb{R}$.

Define the process

$$M_t^\theta = \exp\left(\int_{A_t} \theta(s,\xi)\, L(d\xi, ds) - \int_{A_t} \psi_{L'}(-i\theta(s,\xi))\, d\xi\, ds\right)\,, \tag{39}$$

where $\psi_{L'}$ is the characteristic exponent of the seed of L (and related to ψ_L through equation (9)). The deterministic function $\theta : A_t \mapsto \mathbb{R}_{T^*}$ is supposed to be integrable with respect to the Lévy basis L in the sense of Walsh. Assume that

$$\mathbb{E}\left(\exp\left(\int_{A_t} \psi_{L'}(-i\theta(s,\xi)\, d\xi\, ds\right)\right) < \infty, \text{ for all } t \in \mathbb{R}_{T^*}. \tag{40}$$

Then we see that M_t^θ is a martingale with respect to $\mathcal{F}_t$ with $M_0^\theta = 1$. We use this in order to define an equivalent probability P by

$$\frac{dP}{dP^*}\Big|_{\mathcal{F}_t} = M_t^\theta\,, \tag{41}$$

for $t \geq 0$. Hence, we have a change of measure from the risk neutral probability P^* under which the forward price is defined to a real world probability P. In effect, the function θ is an additional parameter to be modelled and estimated, and it will play the role as the *market price of risk*, as it models the difference between the risk–neutral and objective price dynamics.

We compute the characteristic exponent of an integral of L under P: For any $v \in \mathbb{R}$, and Walsh–integrable function f with respect to L, it holds that

$$\begin{aligned}\log \mathbb{E}_P\left[\exp(iv\int_{A_t} f(s,\xi)L(d\xi,ds)\right] &= \log \mathbb{E}\left[\exp\left(\int_{A_t} ivf(s,\xi) + \theta(s,\xi)\, L(d\xi,ds)\right)\right] \\ &\quad \times \exp\left(-\int_{A_t} \psi_{L'}(-i\theta(s,\xi))\, d\xi\, ds\right) \\ &= \int_{A_t} \psi_{L'}(vf(s,\xi) - i\theta(s,\xi)) - \psi_{L'}(-i\theta(s,\xi))\, d\xi\, ds\,.\end{aligned}$$

Note that the transform above is a simple generalization of the Esscher transform of Lévy processes, see Shiryaev (1999) and Benth, Šaltytė Benth & Koekebakker (2008) for more details on this aspect.

8 Extensions

We consider various extensions of our model, in particular, a geometric forward model and the question of how to model forwards with delivery period.

8.1 Geometric modelling framework

So far, we have worked with an arithmetic model for the forward price since this is a very natural model choice and is in line with the traditional random field based models where the forward rate is directly modelled by e.g. a Gaussian random field. However, standard critical arguments include that such models can in principal produce negative prices and hence might not be realistic in practice. One way to overcome that problem would be to work with positive Lévy bases (recall that the kernel function and the stochastic volatility component in the ambit field are by definition positive). Clearly, in such a set–up we would have to relax the zero–mean assumption. But this is straightforward to do. An alternative and more traditional approach would be to work with geometric models, i.e. we model the forward price as the exponential of an ambit processes. Most of the results we derived before can be directly carried over to the geometric set–up. E.g. when we study the link between the forward price and the spot price, this has to be interpreted as the link between the logarithmic forward price and the logarithmic spot price. Likewise, when looking at probabilistic properties such as the moments and cumulants of the processes, they can be regarded as the moments/cumulants of the logarithmic forward price.

The only result, which indeed needs some adjustment, is in fact the martingale property. The condition on the kernel function h stays the same as in Theorem 1 when we go to the geometric model framework, but on top of that there will be an additional drift condition. In order to keep the exposition as simple as possible, we will focus on *homogeneous* Lévy bases, see Section 3.2, in this section.

Before we formulate the martingale condition, we specify an additional integrability assumption.

Assumption (A4) Let Y be defined as in (3), where we assume that L is a *homogeneous* Lévy basis with conditional characteristic exponent $\psi_{L'}^{\sigma}$ and h satisfies the condition of Theorem 1. We assume that

$$\mathbb{E}\left(\exp\left(\int_{A_t}\psi_{L'}^{\sigma}(-i\widetilde{h}(\xi,s,T)\sigma_s(\xi))d\xi ds\right)\right)<\infty,\ \text{for all } t\in\mathbb{R}.$$

Now we can formulate the martingale conditions for the geometric forward price model.

Theorem 2. *Let $A_t=\{(\xi,s): s\leq t; x\geq 0\}$ and let $\psi_{L'}$ denote the characteristic exponent of the homogeneous Lévy basis L and $\psi_{L'}^{\sigma}$ the corresponding conditional characteristic exponent. Further, we assume that the integrability condition (A4) is satisfied. Then, the forward price at time t with delivery at time $t\leq T$, $f_t(T)=(f_t(T))_{t\leq T}$ with*

$$f_t(T)=\exp\left(\int_{A_t}\widetilde{h}(\xi,s,T)\sigma_s(\xi)L(d\xi,ds)-\int_{A_t}\psi_{L'}^{\sigma}(-i\widetilde{h}(\xi,s,T)\sigma_s(\xi))d\xi ds\right),$$

is a martingale with respect to $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$.

Consider the example of a Gaussian Lévy basis:

Example 7. *In the special case that $L=W$ is a standardised, homogeneous Gaussian Lévy basis and that (A4) is satisfied, we have that*

$$f_t(T)=\exp\left(\int_{A_t}\widetilde{h}(\xi,s,T)\sigma_s(\xi)W(d\xi,ds)-\frac{1}{2}\int_{A_t}\widetilde{h}^2(\xi,s,T)\sigma_s^2(\xi)d\xi ds\right),$$

is a martingale with respect to $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$.

8.2 Outlook on how to include period of delivery into the modelling framework

So far, we have focused on forward prices with fixed delivery time, i.e. on $f_t(x) = f_t(T-t)$. However, in energy markets, there is not just a time of delivery T, but typically a *delivery period*, i.e. at time of delivery $T = T_1$ a certain amount of electricity, say, gets delivered until time T_2 for some $T_2 \geq T_1$, see e.g. Benth, Šaltytė Benth & Koekebakker (2008, Chapter 6) and Barth & Benth (2010). The forward price $F_t(T_1, T_2)$ at time t with delivery period $[T_1, T_2]$ is defined by (see e.g. Benth, Šaltytė Benth & Koekebakker (2008))

$$F_t(T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f_t(u - t)du.$$

Hence, given an ambit model of $f_t(x)$, we simply average it over the delivery period in order to have the forward price for a contract with delivery period.

Alternatively, we could think of modelling $F_t(T_1, T_2)$ directly – by an ambit field. The main idea here is to include the length of the delivery period $\tau := T_2 - T_1$ as an additional spatial component. E.g. we could think of using

$$\int_{A_t(x,\tau)} k(\xi, \chi, t - s, \tau, x)\sigma_s(\xi, \chi)L(d\xi, d\chi, ds),$$

as a building block for $F_t(T_1, T_2)$. The main obstacle in building such models is the no–arbitrage condition between contracts with overlapping delivery periods. In fact, any model for $F_t(T_1, T_2)$ must satisfy (see Benth, Šaltytė Benth & Koekebakker (2008))

$$F_t(T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} F_t(\tau, \tau)\, d\tau\,,$$

which puts serious restrictions on the degrees of freedom in modelling.

It will be interesting to study the analytical properties of such models in more detail in future research.

8.3 A short note on the relationship between spot, forward and delivery period

We have previously discussed how a spot model can be constructed from our general forward model. However, it is well–known that there is no convergence of electricity forward prices to the spot as time to *start of delivery* approaches. That is, if the delivery period is $[T_1, T_2], T_1 < T_2$, then the forward price $F_t(T_1, T_2)$ at time t does not converge to the spot price as $t \to T_1$. One could mimic such a behaviour with the model class we study here, by choosing the 'delivery time' T as the mid–point, say, in the delivery interval $[T_1, T_2]$, $T = (T_1 + T_2)/2$. Then we can still associate a spot price to the forward dynamics $f_t(x)$, but we will never actually observe the convergence in the market since at start of delivery we have $x = (T_2 - T_1)/2$. On the other hand, we will get a model where there is an explicit connection between the forward at "maturity" $t = T_1$ and the spot Y_{T_1}. This opens for modelling spot and forward jointly, taking into account their dependency structure.

9 Conclusion

This paper presents a new modelling framework for electricity forward prices. We propose to use ambit fields which are special types of tempo–spatial random fields as the building block for the new modelling class. Ambit fields are constructed by stochastic integration with respect to Lévy bases and we have argued in favour of the integration concept of Walsh (1986) in the context of financial applications since it enables us to derive martingale conditions for the forward prices. Furthermore, we have shown that forward and spot prices can be linked to each other within the ambit field framework. Also, we have discussed relevant examples of model specifications within the new modelling framework and have related them to the traditional modelling concepts. In addition, we have discussed how a change of measure between the risk–neutral and physical probability measure can be carried out, so that our model can be used both for option pricing purposes and for statistical studies under the physical measure.

A natural next step to take is to test our new model empirically and to study statistical aspects related to our ambit–field models, such as model estimation and model specification tests etc.. We plan to address these issues in detail in future research.

Another interesting aspect, which we leave for future research, is to adapt our modelling framework for applications to the term structure of interest rates.

A Proofs and some further results

A.1 Explicit results for Example 1

Note that

$$\int_0^\infty \left(1-\sqrt{1+c^2\exp(-2\alpha(\xi-s))}\right)d\xi = -\frac{1}{2\alpha}\left[\left(2\sqrt{1+c^2\exp(2\alpha s)}-2\right)\right.$$
$$\left.+\left(2\log(2)-\log\left(\frac{\left(\sqrt{1+c^2\exp(2\alpha s)}+1\right)c^2\exp(2\alpha s)}{\sqrt{1+c^2\exp(2\alpha s)}-1}\right)\right)\right].$$

Hence, we get

$$\begin{aligned}
&\frac{-8\alpha^2}{\delta\gamma}\log(\mathbb{E}(iv f_t(x))) = -8+8\ln(2)-4\ln\left(c^2\right)-2\left(\ln(2)\right)^2+4\ln(2)\ln\left(c^2\right)\\
&+8\sqrt{1+c^2\mathrm{e}^{2t\alpha}}+4\ln\left(-1+\sqrt{1+c^2\mathrm{e}^{2t\alpha}}\right)-4\ln\left(1+\sqrt{1+c^2\mathrm{e}^{2t\alpha}}\right)\\
&+\left(\ln\left(-1+\sqrt{1+c^2\mathrm{e}^{2t\alpha}}\right)\right)^2+4\,\mathit{dilog}\left(1/2+1/2\sqrt{1+c^2\mathrm{e}^{2t\alpha}}\right)\\
&-4\ln\left(-1+\sqrt{1+c^2\mathrm{e}^{2t\alpha}}\right)\ln(2)+2\ln\left(-1+\sqrt{1+c^2\mathrm{e}^{2t\alpha}}\right)\ln\left(1+\sqrt{1+c^2\mathrm{e}^{2t\alpha}}\right)\\
&-\left(\ln\left(1+\sqrt{1+c^2\mathrm{e}^{2t\alpha}}\right)\right)^2-8t\alpha+8\ln(2)t\alpha-\left(\ln\left(c^2\mathrm{e}^{2t\alpha}\right)\right)^2,
\end{aligned}$$

where the dilogarithm function is defined by $\mathrm{dilog}(t)=\int_1^t\frac{\log(x)}{1-x}dx$ for $t>1$.

A.2 Proof of the martingale condition

Proof of Theorem 1. From the definition of a martingale, we must show that

$$\mathbb{E}[Y_t(T-t)\,|\,\mathcal{F}_{\widetilde{t}}]=Y_{\widetilde{t}}(T-\widetilde{t}), \qquad \text{for all } \widetilde{t}\le t.$$

Note that for $\widetilde{t}\le t$, we have that $A_{\widetilde{t}}\subseteq A_t$. Using the independence property of σ and L and the fact that L is a zero mean process, we find

$$\begin{aligned}
&\mathbb{E}[Y_t(T-t)\,|\,\mathcal{F}_{\widetilde{t}}]\\
&=\mathbb{E}\left[\int_{A_{\widetilde{t}}}h(\xi,s,T-t,t)\sigma_s(\xi)L(d\xi,ds)+\int_{A_t\setminus A_{\widetilde{t}}}h(\xi,s,T-t,t)\sigma_s(\xi)L(d\xi,ds)\,\middle|\,\mathcal{F}_{\widetilde{t}}\right]\\
&=\int_{A_{\widetilde{t}}}h(\xi,s,t,T-t)\sigma_s(\xi)L(d\xi,ds)=Y_{\widetilde{t}}(T-\widetilde{t})+I_{\widetilde{t}}(T-\widetilde{t}),
\end{aligned}$$

where

$$I_{\widetilde{t}}(T-\widetilde{t}) = \int_{A_{\widetilde{t}}} \left\{ h(\xi,s,T-t,t) - h(\xi,s,T-\widetilde{t},\widetilde{t}) \right\} \sigma_s(\xi) L(d\xi,ds) \, .$$

Without loss of generality we assume that $Var(L)=1$. Since L is a Lévy basis with zero mean, we know that $\mathbb{E}(I_{\widetilde{t}}(T-\widetilde{t}))=0$ and from the Itô isometry we therefore get that

$$Var(I_{\widetilde{t}}(T-\widetilde{t})) = \int_{A_{\widetilde{t}}} \left\{ h(\xi,s,T-t,t) - h(\xi,s,T-\widetilde{t},\widetilde{t}) \right\}^2 \mathbb{E}(\sigma_s^2(\xi)) d\xi ds.$$

Thus, in order to obtain $I_{\widetilde{t}}(T-\widetilde{t})=0$, we need that for all $0 \leq \xi$, $s \leq \widetilde{t} \leq t \leq T$

$$h(\xi,s,t,T-t) = h(\xi,s,\widetilde{t},T-\widetilde{t}) \, . \tag{42}$$

When we look at condition (42) more closely, then we observe that there is in fact only one class of functions, which satisfy such a condition, i.e. functions of the form

$$h(\xi,s,T-t,t) = \tilde{h}(\xi,s,T),$$

for all $\xi \geq 0$, $s \leq \widetilde{t} \leq t \leq T$ for some deterministic kernel function $\tilde{h}$.

$\square$

Proof of Theorem 2. We show that $M=(M_t)_{t\in\mathbb{R}}$ with $M_t = \exp(Y_t(T-t)-d_t)$ is a martingale with respect to $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ where

$$Y_t(T-t) = \int_{A_t} \widetilde{h}(\xi,s,T)\sigma_s(\xi)L(ds,d\xi), \qquad d_t = \int_{A_t} \psi_{L'}^{\sigma}(-i\widetilde{h}(\xi,s,T)\sigma_s(\xi))d\xi ds,$$

where $\psi_{L'}$ is the characteristic exponent of L conditional on σ and $A_t = \{(\xi,s) : s \leq t; x \geq 0\}$. Clearly, M is measurable with respect to $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ and also integrable due to the integrability assumption (A4). Further, for all $\widetilde{t} \leq t$, we have that

$$\begin{aligned}
&\mathbb{E}(M_t|\mathcal{F}_{\widetilde{t}}) = \mathbb{E}\left(\exp(Y_t(T-t)-d_t)\middle|\,\mathcal{F}_{\widetilde{t}}\right) \\
&= \mathbb{E}\left(\exp\left(\int_{A_{\widetilde{t}}} \widetilde{h}(\xi,s,T)\sigma_s(\xi)L(ds,d\xi) + \int_{A_t\setminus A_{\widetilde{t}}} \widetilde{h}(\xi,s,T)\sigma_s(\xi)L(ds,d\xi) - d_{\widetilde{t}} + d_{\widetilde{t}} - d_t\right)\middle|\,\mathcal{F}_{\widetilde{t}}\right) \\
&= M_{\widetilde{t}}\,\mathbb{E}\left(\exp\left(\int_{A_t\setminus A_{\widetilde{t}}} \widetilde{h}(\xi,s,T)\sigma_s(\xi)L(ds,d\xi) - (d_t - d_{\widetilde{t}})\right)\middle|\,\mathcal{F}_{\widetilde{t}}\right).
\end{aligned}$$

Using the formula for the characteristic functions of integrals with respect to Lévy bases, see Rajput & Rosinski (1989) and Section 3.2, we get

$$\begin{aligned}
&\mathbb{E}\left(\exp\left(\int_{A_t\setminus A_{\widetilde{t}}} \widetilde{h}(\xi,s,T)\sigma_s(\xi)L(ds,d\xi)\right)\middle|\,\mathcal{F}_{\widetilde{t}}\right) \\
&= \mathbb{E}\left(\mathbb{E}\left(\exp\left(\int_{A_t\setminus A_{\widetilde{t}}} \widetilde{h}(\xi,s,T)\sigma_s(\xi)L(d\xi,ds)\right)\middle|\,\overline{\mathcal{F}}_{\widetilde{t}}\right)\middle|\,\mathcal{F}_{\widetilde{t}}\right) \\
&= \mathbb{E}\left(\exp\left(\int_{A_t\setminus A_{\widetilde{t}}} \psi_{L'}^{\sigma}(\widetilde{h}(\xi,s,T)\sigma_s(\xi))d\xi ds\right)\middle|\,\mathcal{F}_{\widetilde{t}}\right) = \mathbb{E}\left(\exp\left(d_t - d_{\widetilde{t}}\right)\middle|\,\mathcal{F}_{\widetilde{t}}\right).
\end{aligned}$$

Hence the result follows.

In the special case that L is a standardised, homogeneous Gaussian Lévy basis, the drift is given by

$$d_t = \frac{1}{2}\int_{A_t} \widetilde{h}^2(\xi,s,T)\sigma_s^2(\xi)d\xi ds.$$

$\square$

A.3 Second order structure of ambit fields and cross correlations

We provide some results on the probabilistic properties of the ambit fields which are useful in modelling.

A.3.1 Second order structure of ambit fields

Now we study the second order properties of a general ambit field given by

$$Y_t(x) = \int_{A_t(x)} h(s,\xi,t,x)\sigma_s(\xi)L(ds,d\xi), \tag{43}$$

for a Lévy basis L (not necessarily with zero mean), a homogeneous ambit set $A_t(x)$ (as defined above) and a process σ which is independent of L and where h denotes a damping function (ensuring that the integral exists). In order to compute various moments of the ambit field, we work with the Lévy–Itô decomposition:

$$\begin{aligned} Y_t(x) = &\int_{A_t(x)} h(s,\xi,t,x)\sigma_s(\xi)\sqrt{b}W(d\xi,ds) + \int_{A_t(x)}\int_{\{|y|\leq 1\}} yh(s,\xi,t,x)\sigma_s(\xi)(N-\nu)(dy,ds,d\xi) \\ &+ \int_{A_t(x)}\int_{\{|y|\geq 1\}} yh(s,\xi,t,x)\sigma_s(\xi)N(dy,ds,d\xi), \end{aligned}$$

where $b > 0$ (w.l.o.g. we choose b to be a constant and not depending on (ξ, s)) and N is a Poisson random measure with compensator ν. Hence, $N(A) \sim Poisson(\nu(A))$ and, in particular,

$$\mathbb{E}(N(A)) = \nu(A) = Var(N(A)), \qquad \mathbb{E}((N(A)^2) = \nu(A) + \nu(A)^2.$$

Furthermore, we know that

$$\mathbb{E}\left(N(A) - \nu(A)\right) = 0, \qquad Var(N(A) - \nu(A)) = \mathbb{E}(N(A) - \nu(A))^2 = \nu(A).$$

Assumption (H) In the following, we work under the assumptions that

- The generalised Lévy measure ν is factorisable, i.e. $\nu(dy, d\eta) = U(dy)\mu(d\eta)$, for $\eta = (\xi, s)$,
- the measure μ is homogeneous, i.e. $\mu(d\eta) = cd\eta$, for a constant $c \in \mathbb{R}$. For ease of exposition, we choose $c = 1$. Hence, we have $\nu(dy, ds, d\xi) = U(dy)dsd\xi$.

Furthermore, we use the following notation. Let $\kappa_1 = \int_{\{|y|\geq 1\}} yU(dy)$ and $\kappa_2 = \int_{\mathbb{R}} y^2U(dy)$ and define a function $\rho : \mathbb{R}^4 \to \mathbb{R}$ by

$$\rho(s,\tilde{s},\xi,\tilde{\xi}) = \mathbb{E}\left(\sigma_s(\xi)\sigma_{\tilde{s}}(\tilde{\xi})\right) - \mathbb{E}\left(\sigma_s(\xi)\right)\mathbb{E}\left(\sigma_{\tilde{s}}(\tilde{\xi})\right), \tag{44}$$

for $s, \tilde{s}, \xi, \tilde{\xi} \geq 0$.

Theorem 3. *Let $t, \tilde{t}, x, \tilde{x} \geq 0$ and let $Y_t(x)$ be an ambit field as defined in (43) and assume that assumption (H) holds. The second order structure is then given by*

$$\begin{aligned} \mathbb{E}\left(Y_t(x)|\,\sigma\right) &= \kappa_1 \int_{A_t(x)} h(s,\xi,t,x)\sigma_s(\xi)dsd\xi, \\ \mathbb{E}\left(Y_t(x)\right) &= \kappa_1 \int_{A_t(x)} h(s,\xi,t,x)\mathbb{E}\left(\sigma_s(\xi)\right)dsd\xi. \end{aligned}$$

The variance is given by

$$Var\;(Y_t(x)|\,\sigma) = (b+\kappa_2)\int_{A_t(x)} h^2(s,\xi,t,x)\sigma_s^2(\xi)dsd\xi,$$

$$Var\,(Y_t(x)) = (b+\kappa_2)\int_{A_t(x)} h^2(s,\xi,t,x)\mathbb{E}\left(\sigma_s^2(\xi)\right)dsd\xi + \kappa_1^2\int_{A_t(x)}\int_{A_t(x)} h(s,\xi,t,x)h(\tilde{s},\tilde{\xi},t,x)\rho(s,\tilde{s},\xi,\tilde{\xi})dsd\xi d\tilde{s}d\tilde{\xi}.$$

The covariance is given by

$$Cov\;(Y_t(x),Y_{\tilde{t}}(\tilde{x})|\,\sigma) = (b+\kappa_2)\int_{A_t(x)\cap A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(s,\xi,\tilde{t},\tilde{x})\sigma_s^2(\xi)d\xi ds,$$

$$Cov\,(Y_t(x),Y_{\tilde{t}}(\tilde{x})) = (b+\kappa_2)\int_{A_t(x)\cap A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(s,\xi,\tilde{t},\tilde{x})\mathbb{E}\left(\sigma_s^2(\xi)\right)d\xi ds + \kappa_1^2\int_{A_t(x)}\int_{A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(\tilde{s},\tilde{\xi},\tilde{t},\tilde{x})\rho(s,\tilde{s},\xi,\tilde{\xi})d\tilde{s}d\tilde{\xi}dsd\xi.$$

Corollary 2. *The conditional correlation is given by*

$$Cor\;(Y_t(x),Y_{\tilde{t}}(\tilde{x})|\,\sigma) = \frac{\int_{A_t(x)\cap A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(s,\xi,\tilde{t},\tilde{x})\sigma_s^2(\xi)d\xi ds}{\sqrt{\int_{A_t(x)} h^2(s,\xi,t,x)\sigma_s^2(\xi)d\xi ds\int_{A_{\tilde{t}}(\tilde{x})} h^2(\tilde{s},\tilde{\xi},\tilde{t},\tilde{x})\sigma_{\tilde{s}}^2(\tilde{\xi})d\tilde{\xi}d\tilde{s}}}.$$

For $\kappa_1 = 0$, the unconditional correlation is given by

$$Cor\,(Y_t(x),Y_{\tilde{t}}(\tilde{x})) = \frac{\int_{A_t(x)\cap A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(s,\xi,\tilde{t},\tilde{x})\mathbb{E}\left(\sigma_s^2(\xi)\right)d\xi ds}{\sqrt{\int_{A_t(x)} h^2(s,\xi,t,x)\mathbb{E}\left(\sigma_s^2(\xi)\right)d\xi ds\int_{A_{\tilde{t}}(\tilde{x})} h^2(\tilde{s},\tilde{\xi},\tilde{t},\tilde{x})\mathbb{E}\left(\sigma_{\tilde{s}}^2(\tilde{\xi})\right)d\tilde{\xi}d\tilde{s}}}.$$

A.3.2 Cross correlation

Next, we study the cross correlation, when we have a pair of ambit fields, i.e.

$$Y_t^{(i)}(x) = \int_{A_t^{(i)}(x)} h^{(i)}(s,\xi,t,x)\sigma_s^{(i)}(\xi)L^{(i)}(ds,d\xi),$$

for $i = 1,2$, where $h^{(i)}$, $\sigma^{(i)}$ and $L^{(i)}$ are defined as above. The corresponding Lévy–Itô decomposition is then given by

$$Y_t^{(i)}(x) = \int_{A_t^{(i)}(x)} h^{(i)}(s,\xi,t,x)\sigma_s^{(i)}(\xi)\sqrt{b^{(i)}}W^{(i)}(d\xi,ds) + \int_{A_t^{(i)}(x)}\int_{\{|y|\leq 1\}} yh^{(i)}(s,\xi,t,x)\sigma_s^{(i)}(\xi)(N^{(i)}-\nu^{(i)})(dy,ds,d\xi) + \int_{A_t(x)^{(i)}}\int_{\{|y|\geq 1\}} yh^{(i)}(s,\xi,t,x)\sigma_s^{(i)}(\xi)N^{(i)}(dy,ds,d\xi),$$

where $b^{I)} > 0$ and $N^{(i)}$ is a Poisson random measure with compensator $\nu^{(i)}$. We assume that $(L^{(1)},L^{(2)})$ is a homogeneous Lévy basis with generalised Lévy measure

$$\nu(y_1,y_2,s_1,s_2,\xi_1,\xi_2) = U(y_1,y_2)\mu(s_1,s_2,\xi_1,\xi_2).$$

Since we consider only homogeneous Lévy bases, we get

$$\nu(dy_1,dy_2,ds_1,ds_2,d\xi_1,d\xi_2) = U(dy_1,dy_2)ds_1ds_2d\xi_1d\xi_2,$$

where we set the proportionality constant to 1.

Correlation between $Y^{(1)}$ and $Y^{(2)}$

We distinguish two cases:

- Y is a driven by a Gaussian Lévy basis.
- Y is a driven by a pure jumps Lévy basis.

Theorem 4. *Under the assumptions above, we get the following covariation functions.*

Gaussian Lévy base *Let $L^{(i)}$ be a Gaussian Lévy base for $i = 1, 2$ and let ρ denote the corresponding correlation coefficient, i.e. $\rho d\xi ds = Cov\left(W^{(1)}(d\xi, ds), W^{(2)}(d\xi, ds)\right)$. Then*

$$\begin{aligned} &Cov\left(Y_t^{(1)}(x), Y_{\tilde{t}}^{(2)}(\tilde{x}) \middle| \sigma^{(1)}, \sigma^{(2)}\right) \\ &\qquad = \rho\sqrt{b^{(1)}b^{(2)}} \int_{A_t^{(1)}(x)\cap A_{\tilde{t}}^{(2)}(\tilde{x})} h^{(1)}(s,\xi,t,x)h^{(2)}(s,\xi,\tilde{t},\tilde{x})\sigma_s^{(1)}(\xi)\sigma_s^{(2)}(\xi)dsd\xi. \end{aligned}$$

The unconditional covariation is given by

$$Cov\left(Y_t^{(1)}(x), Y_{\tilde{t}}^{(2)}(\tilde{x})\right) = \rho\sqrt{b^{(1)}b^{(2)}} \int_{A_t^{(1)}(x)\cap A_{\tilde{t}}^{(2)}(\tilde{x})} h^{(1)}(s,\xi,t,x)h^{(2)}(s,\xi,\tilde{t},\tilde{x})\Upsilon(s,\xi)dsd\xi,$$

where

$$\Upsilon(s,\xi) = \mathbb{E}\left(\sigma_s^{(1)}(\xi)\sigma_s^{(2)}(\xi)\right) - \mathbb{E}\left(\sigma_s^{(1)}(\xi)\right)\mathbb{E}\left(\sigma_s^{(2)}(\xi)\right).$$

The pure jump case *Let $L^{(i)}$ be a pure jump Lévy base for $i = 1, 2$ and let $\kappa_{1,1} = \int_{|y|\geq 1}\int_{|y'|\geq 1} yy'U(dy, dy')$. Then*

$$\begin{aligned} &Cov\left(Y_t^{(1)}(x), Y_{\tilde{t}}^{(2)}(\tilde{x}) \middle| \sigma^{(1)}, \sigma^{(2)}\right) \\ &\qquad = \left(\kappa_{1,1} - \kappa_1^{(1)}\kappa_1^{(2)}\right) \int_{A_t^{(1)}(x)\cap A_{\tilde{t}}^{(2)}(\tilde{x})} h^{(1)}(s,\xi,t,x)h^{(2)}(s,\xi,\tilde{t},\tilde{x})\sigma_s^{(1)}(\xi)\sigma_s^{(2)}(\xi)dsd\xi. \end{aligned}$$

The unconditional variance is then given by

$$\begin{aligned} Cov\left(Y_t^{(1)}(x), Y_{\tilde{t}}^{(2)}(\tilde{x})\right) &= \int_{A_t^{(1)}(x)\cap A_{\tilde{t}}^{(2)}(\tilde{x})} h^{(1)}(s,\xi,t,x)h^{(2)}(s,\xi,\tilde{t},\tilde{x})\widetilde{\Upsilon}(s,s,\xi,\xi)dsd\xi \\ &\quad + \int_{A_t^{(1)}(x)}\int_{A_t^{(2)}(x)} h^{(1)}(s,\xi,t,x)h^{(2)}(\tilde{s},\tilde{\xi},\tilde{t},\tilde{x})\widetilde{\Upsilon}(s,\tilde{s},\xi,\tilde{\xi})d\tilde{s}d\tilde{\xi}dsd\xi, \end{aligned}$$

where

$$\widetilde{\Upsilon}(s,\tilde{s},\xi,\tilde{\xi}) = \kappa_{1,1}\mathbb{E}\left(\sigma_s^{(1)}(\xi)\sigma_{\tilde{s}}^{(2)}(\tilde{\xi})\right) - \kappa_1^{(1)}\kappa_1^{(2)}\mathbb{E}\left(\sigma_s^{(1)}(\xi)\right)\mathbb{E}\left(\sigma_{\tilde{s}}^{(2)}(\tilde{\xi})\right).$$

A.3.3 Proofs of the second order properties

Proof of Theorem 3. Recall that $\kappa_1 = \int_{|y|\geq 1} yU(dy)$ and $\kappa_2 = \int_{\mathbb{R}} y^2 U(dy)$. Then

$$\begin{aligned} \mathbb{E}\left(Y_t(x)\middle|\, \sigma\right) &= \int_{A_t(x)}\int_{\{|y|\geq 1\}} yh(s,\xi,t,x)\sigma_s(\xi)U(dy)dsd\xi = \kappa_1\int_{A_t(x)} h(s,\xi,t,x)\sigma_s(\xi)dsd\xi, \\ \mathbb{E}\left(Y_t(x)\right) &= \kappa_1\int_{A_t(x)} h(s,\xi,t,x)\mathbb{E}\left(\sigma_s(\xi)\right)dsd\xi. \end{aligned}$$

For the second moment, we get

$$\begin{aligned}
\mathbb{E}\left(Y_t(x)^2\middle|\,\sigma\right) &= (b+\kappa_2)\int_{A_t(x)} h^2(s,\xi,t,x)\sigma_s^2(\xi)dsd\xi + \kappa_1^2\left(\int_{A_t(x)} h(s,\xi,t,x)\sigma_s(\xi)dsd\xi\right)^2, \\
\mathbb{E}\left(Y_t(x)^2\right) &= (b+\kappa_2)\int_{A_t(x)} h^2(s,\xi,t,x)\mathbb{E}\left(\sigma_s^2(\xi)\right)dsd\xi \\
&\quad + \kappa_1^2\int_{A_t(x)}\int_{A_t(x)} h(s,\xi,t,x)h(\tilde{s},\tilde{\xi},t,x)\mathbb{E}\left(\sigma_s(\xi)\sigma_{\tilde{s}}(\tilde{\xi})\right)dsd\xi d\tilde{s}d\tilde{\xi}.
\end{aligned}$$

The conditional and unconditional variance is then given by

$$\begin{aligned}
Var\left(Y_t(x)|\,\sigma\right) &= (b+\kappa_2)\int_{A_t(x)} h^2(s,\xi,t,x)\sigma_s^2(\xi)dsd\xi, \\
Var\left(Y_t(x)\right) &= \mathbb{E}\left(Var\left(Y_t(x)|\,\sigma\right)\right) + Var\left(\mathbb{E}\left(Y_t(x)|\,\sigma\right)\right) \\
&= (b+\kappa_2)\int_{A_t(x)} h^2(s,\xi,t,x)\mathbb{E}\left(\sigma_s^2(\xi)\right)dsd\xi \\
&\quad + \kappa_1^2\int_{A_t(x)}\int_{A_t(x)} h(s,\xi,t,x)h(\tilde{s},\tilde{\xi},t,x)\rho(s,\tilde{s},\xi,\tilde{\xi})dsd\xi d\tilde{s}d\tilde{\xi}.
\end{aligned}$$

Next, we compute the covariance. In order to do that, we use throughout that for $y,\tilde{y}\in\mathbb{R}$ and $(s,\xi),(\tilde{s},\tilde{\xi})\in A_t(x)\cap A_{\tilde{t}}(\tilde{x})$:

$$\mathbb{E}\left(N(dy,ds,d\xi)N(d\tilde{y},d\tilde{s},d\tilde{\xi})\right) = \nu(dy,ds,d\xi)\nu(d\tilde{y},d\tilde{s},d\tilde{\xi}) + \nu(d\min(y,\tilde{y}),d\min(s,\tilde{s}),d\min(\xi,\tilde{\xi})),$$

and

$$\mathbb{E}\left((N-\nu)(dy,ds,d\xi)(N-\nu)(d\tilde{y},d\tilde{s},d\tilde{\xi})\right) = \nu(d\min(y,\tilde{y}),d\min(s,\tilde{s}),d\min(\xi,\tilde{\xi})).$$

For the product, we get

$$\begin{aligned}
\mathbb{E}\left(Y_t(x)Y_{\tilde{t}}(\tilde{x})|\,\sigma\right) &= (b+\kappa_2)\int_{A_t(x)\cap A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(s,\xi,\tilde{t},\tilde{x})\sigma_s^2(\xi)d\xi ds \\
&\quad + \kappa_1^2\int_{A_t(x)}\int_{A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(\tilde{s},\tilde{\xi},\tilde{t},\tilde{x})\sigma_s(\xi)\sigma_{\tilde{s}}(\tilde{\xi})d\tilde{s}d\tilde{\xi}dsd\xi, \\
\mathbb{E}\left(Y_t(x)Y_{\tilde{t}}(\tilde{x})\right) &= (b+\kappa_2)\int_{A_t(x)\cap A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(s,\xi,\tilde{t},\tilde{x})\mathbb{E}\left(\sigma_s^2(\xi)\right)d\xi ds \\
&\quad + \kappa_1^2\int_{A_t(x)}\int_{A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(\tilde{s},\tilde{\xi},\tilde{t},\tilde{x})\mathbb{E}\left(\sigma_s(\xi)\sigma_{\tilde{s}}(\tilde{\xi})\right)d\tilde{s}d\tilde{\xi}dsd\xi.
\end{aligned}$$

Therefore, the covariance is given by

$$\begin{aligned}
Cov\left(Y_t(x),Y_{\tilde{t}}(\tilde{x})|\,\sigma\right) &= (b+\kappa_2)\int_{A_t(x)\cap A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(s,\xi,\tilde{t},\tilde{x})\sigma_s^2(\xi)d\xi ds, \\
Cov\left(Y_t(x),Y_{\tilde{t}}(\tilde{x})\right) &= (b+\kappa_2)\int_{A_t(x)\cap A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(s,\xi,\tilde{t},\tilde{x})\mathbb{E}\left(\sigma_s^2(\xi)\right)d\xi ds \\
&\quad + \kappa_1^2\int_{A_t(x)}\int_{A_{\tilde{t}}(\tilde{x})} h(s,\xi,t,x)h(\tilde{s},\tilde{\xi},\tilde{t},\tilde{x})\rho(s,\tilde{s},\xi,\tilde{\xi})d\tilde{s}d\tilde{\xi}dsd\xi.
\end{aligned}$$

□

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