Economics Working Paper

2009-10

On the non-causal link between volatility and growth

Olaf Posch and Klaus Wälde

School of Economics and Management
Aarhus University
Bartholins Allé 10, Building 1322
DK-8000 Aarhus C - Denmark
Phone +45 8942 1610
Mail: oekonomi@econ.au.dk
Web: www.econ.au.dk

On the non-causal link between volatility and growth

Olaf Posch[(a)] and Klaus Wälde[(b)]
[(a)]Aarhus University and CREATES, CESifo
[(b)]University of Mainz, CESifo, and UCL Louvain la Neuve*

August 2009

Abstract

A model highlighting the endogeneity of both volatility and growth is presented. Volatility and growth are therefore correlated but there is no causal link from volatility to growth. This joint endogeneity is illustrated by working out the effects through which economies with different tax *levels* differ both in their volatility and growth. Using a continuous-time DSGE model with plausible parametric restrictions, we obtain closed-form measures of macro volatility based on cyclical components and output growth rates. Given our results, empirical volatility-growth analysis should include controls in the conditional variance equation. Otherwise an omitted variable bias is likely.

Keywords: Tax effects, Volatility measures, Poisson uncertainty, Endogenous cycles and growth, Continuous-time DSGE models
JEL classification numbers: E32, E62, H3, C65

*Corresponding author: Olaf Posch (oposch@econ.au.dk). The first author appreciates financial support from the Center for Research in Econometric Analysis of Time Series, CREATES, funded by The Danish National Research Foundation. We would like to thank Raouf Boucekkine, Peter Funk, Christian Kleiber, Ken Sennewald, Jonathan Schulz, Marcel Thum and Philippe Weil and seminar and conference participants in Bonn, Brussels, Cambridge, Florence, Geneva, Lausanne, Madrid, Munich and Rotterdam for discussions and comments on an earlier draft entitled 'Natural volatility, welfare and taxation'.

1 Introduction

Background. In a seminal paper, Ramey and Ramey (1995) find a strong negative correlation between the mean and volatility of output growth. Subsequent papers confirm this empirical link for many other datasets (Martin and Rogers, 1997; Acemoglu et al., 2003; Aghion et al., 2005; Imbs, 2007; Posch, 2008).[1] Complementary to Ramey and Ramey, these studies use additional controls such as exchange rate variability, financial development and various measures of openness, institutions or monetary and fiscal policy.

The open question. These studies are primarily of an empirical nature. Except for Aghion et al. (2005), these authors argue that the negative relation between volatility and growth observed in the cross-section of countries may reflect causality. These papers do not, however, inquire into the exact structural channels through which macro volatility and growth interact. It therefore remains an open question whether the link indeed reflects a causal relationship.

Our message. This paper plays the devil's advocate and argues that any measure of macro volatility used in the empirical literature measures an endogenous quantity. Endogeneity arises from propagation and the endogeneity of technological jumps. Volatility is endogenous both for measures that are based on cyclical components and for measures based on growth rates. According to our interpretation, there is never a causal link from volatility to growth. There is only a correlation.

Our framework. Our analysis builds on the DSGE models of endogenous cyclical growth (Bental and Peled, 1996; Matsuyama, 1999; Francois and Lloyd-Ellis, 2003, 2008; Wälde, 2002, 2005).[2] We use a version which has analytical solutions for plausible parametric restrictions and which comprises the continuous-time real business cycle (RBC) model as a special case. This allows us to work out the importance of the underlying theoretical background for empirical work. Under an RBC view with exogenous technology shocks, volatility can be 'more causal' for growth than under an cyclical growth view with endogenous jumps.

Using a plausible parameter restriction, we obtain two analytical volatility measures (borrowing heavily from García and Griego, 1994). The first measures we use is the standard deviation of output growth rates, the second one is based on stochastically detrended variables. While the first volatility measure is the common measure in the empirical literature, the latter is identical in spirit to the many empirical detrending methods where a time series is split into a growth trend and a stationary cyclical component.

Results. We illustrate how structural parameters affect our general-equilibrium volatility

[1] There is work suggesting that the link is not pronounced using time series evidence (Beaumont et al., 2008). At different levels of aggregation either no significant relationship is found using state data (Dawson and Stephenson, 1997), or a positive link is found at the sectoral level (Imbs, 2007).

[2] These papers in turn build on Aghion and Howitt (1992), Grossman and Helpman (1991) and Segerstrom et al. (1990). The present paper builds explicitly on the stochastic Aghion and Howitt model using the formulation for risk averse agents introduced by Wälde (1999).

and growth measures directly by changing the variance and intensity of the shocks, and indirectly by affecting the shock propagation.[3] In our theoretical model, we focus on tax rates as an example for economic policy parameters. A correlation between volatility and growth can be predicted if the growth and volatility measures in our model economy are considered for different (constant) tax rates. The volatility-growth link can even change sign if tax rates are altered. We identify three channels through which macro volatility can be affected by policy parameters, i.e., the speed of convergence (to the non-stochastic steady state), the jump size and the arrival rate.

We therefore argue that it is important to account for the joint endogeneity before one can address a potential causal relation between macro volatility and growth in any meaningful way. For empirical analysis of the Ramey and Ramey (1995) type we recommend the inclusion of control variables in the conditional variance equation to account for the endogeneity of volatility.

Table of contents. The paper proceeds as follows. Section 2 introduces the model of endogenous volatility and growth with taxes. Section 3 offers the solution for the equilibrium dynamics and illustrates the notion of cyclical growth. Section 4 contains our main theoretical contribution, the derivation of closed-form volatility measures. Section 5 comprises our main economic insights and points out potential pitfalls in applied research. Section 6 holds some concluding remarks.

2 The model

Production possibilities. Technological progress is labor augmenting and embodied in capital. All capital goods can be identified by a number denoting their date of manufacture and therefore their vintage. A capital good K_j of vintage j allows workers to produce with labor productivity A^j, where $A > 1$ is a constant parameter.[4] Hence, a more modern vintage $j+1$ implies a labor productivity that is A times higher than that of vintage j. The corresponding production function reads $Y_j = K_j^\alpha (A^j L_j)^{1-\alpha}$, where the amount of labor allocated to that vintage is L_j and $0 < \alpha < 1$ denotes the output elasticity of capital.

There is a very large number of research firms which operate under perfect competition.[5] Research costs are recovered by returns of a prototype which is the outcome of a successful project. This differs from standard modeling of R&D where successful research leads to a blueprint only. The prototype is a production unit - a machine - of size κ_t. This new prototype is owned by the individuals who financed the successful R&D project (as reflected

[3] An empirical investigation of the propagation link between taxes and volatility is in Posch (2008).

[4] As in Wälde (2005) and in contrast to Boucekkine et al. (2005) or Feichtinger et al. (2006), we have a discrete number of vintages. We share with the work of Boucekkine et al. the combined analysis of endogenous fluctuations and growth and with Feichtinger et al. the analysis of the effect of "breakthroughs" in technological progress.

[5] This is in the tradition of Hellwig and Irmen (2001); Boldrin and Levine (2001, 2002); Wälde (2002).

in the budget constraint below). The currently most advanced vintage is denoted by q and implies a labor productivity of A^q.[6] The new prototype yields a labor productivity of A^{q+1} for workers having access to this new technology.

Research is a risky activity. Uncertainty in research is captured by a Poisson process q_t where the arrival rate of success is denoted by λ_t. Resources employed for research are denoted by R_t. An exogenous function D_t captures the difficulty to make an invention (as in Segerstrom, 1998). This function captures the idea that an economy needs to put more effort into research for the next generation of capital goods, if new technologies are to appear at a constant rate. There are constant returns to scale at the firm level. On the sectoral level, however, an externality $h(\cdot)$ implies decreasing returns to scale,

$$\lambda_t = (R_t/D_t)h(R_t/D_t) \equiv (R_t/D_t)^{1-\gamma}, \quad 0 < \gamma < 1, \tag{1}$$

where the difficulty function D_t and the externality $h(\cdot)$ are taken as given by the firm.[7]

Given this research process, the capital stock of the next vintage follows

$$dK_{q+1} = \kappa_t dq_t, \tag{2}$$

which is a simple stochastic differential equation (SDE). The increment dq_t of the Poisson process q_t can either be 0 or 1. As successful research means $dq_t = 1$, this equation says that the capital stock increases from 0 to κ_t in the good outcome. When research is not successful, $dK_{q+1} = 0$ because $dq_t = 0$.

Capital accumulation of existing vintages 1 to q is riskless. When resources are used to accumulate existing capital, the capital stock of vintage j increases if investment in vintage j exceeds depreciation δ,

$$dK_j = (I_j - \delta K_j)\,dt, \quad j \leq q. \tag{3}$$

Given that value marginal productivity is highest for the most advanced vintage, investment takes place only in vintage q. As R&D takes place under perfect competition, there is no monopolist owning the new vintage and there is no patent protection. Thus, we observe $I_j = 0 \ \forall \ j < q$, and $I_q = I_t$ for the most advanced vintage. As soon as a new capital good is discovered through R&D, it is replicated by a large number of competing firms. In contrast to R&D, this is a deterministic process because capital accumulation simply means replicating existing machines. The process of capital accumulation is also - as in the standard Solow growth model - perfectly competitive.

Before we continue with the description of the model, we present a few equilibrium properties, some of them related to the vintage capital structure used here. They are useful

[6] More precisely, q_t denotes the Poisson process whereas q denotes the label of the most recent vintage (number of jumps up to time t). Though in principle interchangeable, after successful research, q_t increases by 1 while the label of older vintages remains like a stamp on the capital goods.

[7] Remember that arrival rates of Poisson processes can be added. Economically speaking, this means that there are many "small" arrival rates $\lambda_t^f = (R_t^f/D_t)h(R_t/D_t)$ where R_t^f stands for R&D investment in research firm f. Aggregating over all research firms leads to the economy wide arrival rate λ_t.

as they simplify the presentation of the government, preferences and the assumptions about the difficulty function as well as the size of the prototype. Each vintage of capital allows to produce a single output good, which is used for producing consumption goods, C_t, investment goods, I_t, as an input for research, R_t, and for government expenditures, G_t,

$$\sum_{j=0}^{q} Y_j \;=\; Y_t \;=\; C_t + I_t + R_t + G_t, \tag{4}$$

where the quantities denote *net* resources used for these activities, i.e., after taxation. All activities in the economy take place under perfect competition. Hence, the producer price of the production good, the consumption good, and both investment goods used for capital accumulation and research will therefore be identical,

$$p_t^Y = p_t^C = p_t^K = p_t^R. \tag{5}$$

Aggregate constant labor supply in this economy is L. Allowing labor to be mobile across all vintages such that wage rates equalize and assuming market clearing, $\sum_{j=0}^{q} L_j = L$, total output of the economy can be represented by a simple Cobb-Douglas production function,

$$Y_t = K_t^{\alpha} L^{1-\alpha}, \tag{6}$$

where vintage-specific capital has been aggregated to an aggregate capital index K_t,

$$K_t = K_0 + BK_1 + ... + B^q K_q = \sum_{j=0}^{q} B^j K_j, \quad B \equiv A^{\frac{1-\alpha}{\alpha}}. \tag{7}$$

This index can be thought of as counting the 'number of machines' of the first vintage, $j = 0$, that would be required to produce the same output Y_t as with the current mix of vintages.

Applying Itô's formula (or change of variable formula, cf. Sennewald (2007) for a rigorous background and Sennewald and Wälde (2006) for an introduction) to (7) using (2) and (3), the capital index K_t follows the SDE,

$$dK_t = (B^q I_t - \delta K_t)\, dt + B^{q+1} \kappa_t dq_t. \tag{8}$$

Because the capital index, K_t, is measured in units of the first vintage, it increases as a function of effective investment, $B^q I_t$, minus depreciation, δK_t. When an innovation occurs, the capital index increases by the effective size of the new prototype, $B^{q+1}\kappa_t$.

Government. The government levies taxes on income, τ_i, on wealth, τ_a, on consumption expenditures, τ_c, on investment expenditures, τ_k, and on research expenditures, τ_r. In our study, a positive tax either implies a real decrease in income or an increase in the effective price (consumer price), whereas a negative tax denotes a subsidy. The government uses all tax income (and does not save or run a debt) to provide basic government services G_t,

$$G_t = \tau_i(Y_t - \delta B^{-q} K_t) + \tau_k(I_t - \delta B^{-q} K_t) + \tau_r R_t + \tau_c C_t + \tau_a \left(1 + \tau_k\right) B^{-q} K_t \geq 0. \tag{9}$$

In order to focus on the effects of taxation from government expenditures, we assume that government expenditure does not affect household utility or the production possibilities of the economy. A myopic government simply provides basic government services without interest in neither stabilization policy nor optimal taxation. The tax structure thus is exogenously given to the model. Similarly, the absence of debt therefore is not relevant because we want to illustrate the incentive effects of distortionary taxation on output growth volatility in an endogenous cyclical growth model. Additional effects through the channel of fiscal debt might be interesting but beyond the scope of this paper.

Producer prices from (5) are identical for all three production processes. When goods are sold, they are taxed differently such that consumer prices are $(1+\tau_c)\,p_t^C$, $(1+\tau_k)\,p_t^K$, $(1+\tau_r)\,p_t^R$, respectively. To rule out arbitrage between different types of goods, we assume that a unit of production is useless for other purposes once it is assigned for a special purpose: once a consumption good is acquired, it cannot be used for, e.g., capital accumulation.

Sales taxes have no theoretical upper bound. A 300% tax on the consumption good would imply that 3/4 of the price are taxes going to the government and 1/4 goes to the producer. Their lower bound is clearly -100%, when the good would be gratis. Similarly, the upper bound for taxes on income is 100% (instant confiscation of income), while there is no lower bound. Hence, we obtain $-1 < \tau_c, \tau_k, \tau_r$ and $\tau_i, \tau_a < 1$.

Preferences. The economy has a large number of representative households. Households maximize expected utility given by the integral over instantaneous utility, $u = u(c_t)$, resulting from consumption flows, c_t, and discounted at the subjective rate of time preference, ρ,

$$U_0 = E_0 \int_0^\infty e^{-\rho t} u(c_t) dt. \tag{10}$$

We assume that instantaneous utility is characterized by constant relative risk aversion,

$$u\,(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}, \quad \sigma > 0. \tag{11}$$

The budget constraint reflects investment possibilities in this economy, the impact of taxes and shows how real wealth, a_t, evolves over time. Households can invest in a risky asset by financing research, i_t, and in an (instantaneously) riskless asset by replicating capital. We measure wealth in units of the consumption good, priced at consumer prices. The household's budget constraint can best be illustrated from (A.12) in the appendix,

$$\begin{aligned} da_t \;\; = \;\; & \left(\frac{1-\tau_i}{1+\tau_c} \left(\sum_{j=0}^{q+1} w_j^K k_j + w_t \right) - c_t - \frac{1+\tau_r}{1+\tau_c} i_t \right) dt - \left(\frac{1-\tau_i}{1+\tau_k} \delta + \tau_a \right) a_t dt \\ & + \left(\frac{1+\tau_k}{1+\tau_c} \kappa_t \frac{i_t}{R_t} - \frac{B-1}{B} a_{t-} \right) dq_t, \end{aligned}$$

where $a_{t-} \equiv \lim_{s \to t} a_s$, $s < t$, denotes individual wealth an instant before a jump in t. Intuitively, capital rewards from all vintages j, $\sum_{j=0}^{q+1} w_j^K k_j$, taxed at the rate τ_i and divided by the consumer price gives after-tax capital income in units of the consumption good. The same reasoning applies to labor income w_t, consumption expenditures c_t, and investment into research i_t. Thus, the first bracket captures the increase in wealth a_t measured in units of the consumption good at consumer prices. The second term captures the deterministic wealth-reducing effect due to depreciation and the tax on wealth, where the tax rates in front of the depreciation rate ensure that only net capital rewards (after depreciation) are taxed. The third term is a stochastic component which increases the individual's wealth in case of successful research by the 'dividend payments' less 'economic depreciation'. Here, 'dividend payments' at the household level are given by the share i_t/R_t of a successful research project financed by the household times total payoffs determined by the size κ_t of the prototype times its value in units of the consumption good, i.e., $(1+\tau_k)/(1+\tau_c)$. The term $1+\tau_k$ implies that successful research yields an installed capital good. Moreover, 'economic depreciation' of $s \equiv (B-1)/B > 0$ percent emerges from the vintage capital structure as the most advanced vintage from (5) has a relative price of unity and all other vintages lose in value relative to the consumption good.

After some algebra, the budget constraint can be written as (cf. Appendix A.1)

$$da_t = \left(\left(\frac{1-\tau_i}{1+\tau_k}(r_t-\delta)-\tau_a\right)a_t + \frac{1-\tau_i}{1+\tau_c}w_t - c_t - \frac{1+\tau_r}{1+\tau_c}i_t\right)dt + \left(\frac{1+\tau_k}{1+\tau_c}\kappa_t\frac{i_t}{R_t} - sa_{t-}\right)dq_t, \tag{12}$$

where factor rewards

$$r_t = B^q\frac{\partial Y_t}{\partial K_t} \equiv B^q Y_K, \quad w_t = \frac{\partial Y_t}{\partial L} \equiv Y_L, \tag{13}$$

denote the rental rate of capital and the wage rate, respectively.

Assumptions. For the problem to be well defined, we need assumptions on the functional forms of the 'difficulty function' as well as on the 'size' of the new prototype. We capture the innovations in the past by the current (tax-independent) size of total wealth, $K_t^{obs} = La_t$,

$$D_t \ \equiv\ D\frac{1+\tau_c}{1+\tau_k}K_t^{obs} \ =\ DB^{-q}K_t, \quad D>0. \tag{14}$$

Measuring wealth in consumer prices, the price of the capital good increases by the tax τ_k and the price to be paid for one unit of the consumption good increases by τ_c. Through these channels taxes directly affect individual's real wealth, however, it seems plausible that taxes do *not* directly affect the difficulty level.

The size of the prototype is argued to increase in the amount of time and resources R_t spent on developing κ_t. Longer research could imply a larger prototype. We capture these aspects in a simple and tractable way by keeping κ_t proportional to the (tax-independent) size of total wealth an instant before a jump, $K_{t-}^{obs} = La_{t-}$,

$$\kappa_t \ \equiv\ \kappa\frac{1+\tau_c}{1+\tau_k}K_{t-}^{obs} \ =\ \kappa B^{-q}K_{t-}, \quad 0<\kappa\ll 1. \tag{15}$$

While it may be debatable whether or not the payoffs of the risky research project, as a kind of income, could be subject to taxation, it seems a plausible assumption that the payoff itself, that is the size of the prototype, does not directly depend on tax rates.

3 Equilibrium dynamics

Solving the model requires conditions for optimal consumption and research expenditure. These two conditions, together with the capital accumulation constraint (8), market clearing, and optimality conditions of competitive firms provide a system consisting of 6 equations that determines the time paths of variables of interest K_t, C_t, R_t, Y_t, w_t and r_t.

Such a system can best be understood by introducing auxiliary variables: In the classical Solow growth model, capital per effective worker (or efficiency unit) is shown to converge to a non-stochastic steady state and transitional dynamics can be separated from the analysis of long-run growth. In the present context, we define $\hat{K}_t$ and $\hat{C}_t$ as

$$\hat{K}_t \equiv K_t/A^{q/\alpha} = B^{-q_t}K_t/A_t^q, \quad \hat{C}_t \equiv C_t/A^q, \tag{16}$$

which is almost identical to capital and consumption per effective worker as labor supply is constant here. These variables allow us to separate the analysis of cyclical properties of the model from long-run growth. In what follows, we denote $\hat{K}_t$ and $\hat{C}_t$ as 'cyclical components' of K_t and C_t since $A^{q/\alpha}$ and A^q turn out to be the stochastic trends for the capital index in units of vintage 0 and in units of the most recent vintage q, respectively. All variables expressed in units of the consumption good (including the capital stock in units of the most recent vintage) share the same trend, A^q, as from (16). Thus dividing non-stationary variables such as Y_t, C_t, R_t, I_t , w_t and G_t by the common stochastic trend A^q, these 'cyclical variables' turn out to be stationary and within a bounded range (r_t is stationary by construction).

3.1 An explicit solution

It would be interesting to analyze such a system in all generality. One would run the risk, however, of losing the big picture and instead be overwhelmed by many small results. As the main objective of this paper is closed-form measures of volatility, we restrict ourselves to a particular parameter set of the model that allows very sharp analytical results.

Theorem 1 *If relative risk aversion equals the output elasticity of the capital stock, $\sigma = \alpha$, we obtain an equilibrium with optimal policy functions*

$$\hat{C}_t = \Psi\hat{K}_t, \quad \hat{R}_t = \Gamma\hat{K}_t, \tag{17}$$

where we define constants

$$\Psi \equiv \frac{1+\tau_k}{1+\tau_c}\left(\frac{1-\sigma}{\sigma}\left(\frac{1-\tau_i}{1+\tau_k}\delta+\tau_a\right)+\frac{\rho+\lambda-(1-s)\lambda\xi^{-\sigma}}{\sigma}-\frac{1+\tau_r}{1+\tau_k}\lambda^{1-\gamma}D\right), \quad (18)$$

$$\Gamma \equiv \lambda^{1-\gamma}D, \quad (19)$$

$$\xi \equiv 1+\kappa-s, \quad (20)$$

and the arrival rate becomes

$$\lambda = \left(\frac{1+\tau_k}{1+\tau_r}\frac{\kappa}{D}\xi^{-\sigma}\right)^{\frac{1-\gamma}{\gamma}}. \quad (21)$$

Proof. see Appendix B.3 ■

Suppose the technological improvement (or economic depreciation s) of an innovation is sufficiently large relative to the size of the new prototype $\kappa \ll 1$ such that $\xi \leq 1$, or $\kappa \leq s$. Intuitively this assumption ensures that cyclical variables are accumulated and not reduced over the cycle which seems the only empirically plausible assumption (cf. Wälde, 2005). It follows from (12) and (15) that wealth, a_t/a_{t-}, and thus consumption, C_t/C_{t-} or research R_t/R_{t-} jump by the factor ξ or equivalently by $\kappa - s$ percent, whereas output Y_t/Y_{t-} from (6), (8), and (15) increases by $(1+B\kappa)^\alpha$ immediately after successful research.

The parametric restriction $\sigma = \alpha$ implies a relatively high intertemporal elasticity of substitution above unity (or risk aversion below unity). While there is supporting empirical evidence (as in Vissing-Jørgensen, 2002; Gruber, 2006), the relevance, our fundamental insights about the presence of tax effects on volatility, as well as the channels through which taxes affect volatility will not depend on this assumption. This parametric restriction has proven useful in the macro literature to the study of equilibrium dynamics (e.g. Chang, 1988; Xie, 1991, 1994; Boucekkine and Tamarit, 2004; Smith, 2007; Posch, 2009).

3.2 Cyclical growth

Exploiting the implications of Theorem 1, we can obtain the general-equilibrium behavior of agents in a way as simple as in the deterministic Solow growth model with a constant saving rate, even though we have forward-looking agents and an uncertain environment.

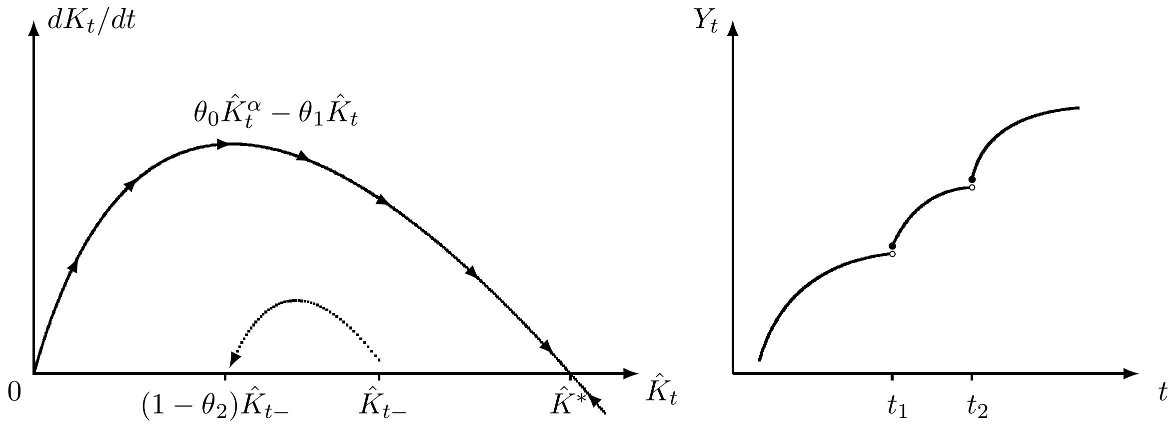
In terms of cyclical components, using Itô's formula (change of variables) together with capital accumulation in (8), the market clearing condition in (4) and the detrending rule (16), our capital index follows (cf. Appendix B.3)

$$d\hat{K}_t = \left(\hat{Y}_t - \hat{C}_t - \hat{R}_t - \hat{G}_t - \delta\hat{K}_t\right)dt + \left(A^{-1}\xi - 1\right)\hat{K}_{t-}dq_t.$$

Inserting optimal consumption and research expenditure from (17) of Theorem 1, as well as government revenues using $\hat{G}_t = A^{-q}G_t$ and government revenues from (A.16), yields

$$\begin{aligned} d\hat{K}_t &= \left(\frac{1-\tau_i}{1+\tau_k}\hat{Y}_t - \left(\frac{1+\tau_c}{1+\tau_k}\Psi + \frac{1+\tau_r}{1+\tau_k}\Gamma + \frac{1-\tau_i}{1+\tau_k}\delta + \tau_a\right)\hat{K}_t\right)dt + \left(A^{-1}\xi - 1\right)\hat{K}_{t-}dq_t \\ &\equiv \left(\theta_0\hat{K}_t^\alpha - \theta_1\hat{K}_t\right)dt - \theta_2\hat{K}_{t-}dq_t, \quad (22) \end{aligned}$$

Figure 1: Dynamics of cyclical capital and growth cycles



Note: This figure illustrates equilibrium dynamics of cyclical capital stock (intensive form) (left panel), and the resulting endogenous growth cycles for output (right panel), where jumps occur at t_1 and t_2, each starting a new growth cycle.

where we inserted $\hat{Y}_t = A^{-q}Y_t = \hat{K}_t^\alpha L^{1-\alpha}$ from (6) and defined parameters

$$\theta_0 \equiv \frac{1-\tau_i}{1+\tau_k}L^{1-\alpha}, \quad \theta_1 \equiv \frac{1}{\sigma}\left(\rho+\lambda-(1-s)\,\lambda\xi^{-\sigma}+\frac{1-\tau_i}{1+\tau_k}\delta+\tau_a\right), \quad \theta_2 \equiv 1-A^{-1}\xi.$$

As a result, similar to the Solow model our model implies a one-dimensional reducible SDE with non-linear drift in (22), but satisfying utility-maximizing behavior of agents for $\alpha=\sigma$. Note that θ_1 is obtained when inserting Ψ and Γ from (18) and (19), respectively.

The terms in (22) containing parameters θ_0 through θ_2 have an economic interpretation: $\theta_0\hat{K}_t^\alpha$ represents cyclical output of this economy reduced by taxation, $\theta_1\hat{K}_t$ denotes effective resource allocation to research, private and government consumption, as well as physical depreciation. From (22), $\theta_1-\alpha\theta_0\hat{K}_t^{\alpha-1}$ is the speed of convergence towards $\hat{K}^*$. The parameter θ_2 denotes the proportional size of the jump in the cyclical capital index.

For illustration, Figure 1 plots $\hat{K}_t$ against the deterministic part of the stochastic growth $d\hat{K}_t/dt$ (left panel). Similar to the steady-state in the Solow model, the non-stochastic steady state, $\hat{K}^*$, is

$$\hat{K}^* = \left(\frac{\theta_0}{\theta_1}\right)^{\frac{1}{1-\alpha}} = \left(\frac{\frac{1-\tau_i}{1+\tau_k}\sigma}{\rho+\lambda-(1-s)\,\lambda\xi^{-\sigma}+\frac{1-\tau_i}{1+\tau_k}\delta+\tau_a}\right)^{\frac{1}{1-\alpha}} L, \tag{23}$$

where we used the definitions from (22). Note that the non-linear deterministic part in equation (22) implies that the speed of convergence (the slope in Figure 1) depends on the level of $\hat{K}_t$, thus changes as $\hat{K}_t$ moves towards $(1-\alpha)\theta_1$.

We can now start our analysis as in deterministic models. Suppose $\hat{K}_0$ is the initial capital stock, $0<\hat{K}_0<\hat{K}^*$.[8] Households optimally allocate parts of their savings between research

[8]Without loss of generality, we abstract from the case where $\hat{K}_0 > \hat{K}^*$. Given that $\theta_2 < 1$, at some point in time cyclical capital stock will be below its non-stochastic steady state with probability one.

and capital accumulation. Assuming a certain length of time without jumps, i.e., without successful innovation, the economy grows due to capital accumulation and converges to the non-stochastic steady state, $\hat{K}^*$. As in the Solow model, growth rates are initially high and approach zero. Once a jump occurs, $q_t = q_{t-} + 1$, the capital stock of the new vintage $q+1$ increases discretely by κ_t from (2). This leads to a discrete increase of the capital index by the effective size, $B^{q+1}\kappa_t$. Although capital increases by the size of the new prototype, our assumption about κ being sufficiently small ensures $\xi \leq 1$ in (20), and the cyclical capital $\hat{K}_t$ unambiguously decreases because the frontier technology shifts outwards (cf. Figure 1). Because of higher marginal products, capital accumulation becomes more profitable, growth rates jump to a higher level approaching zero again until the next innovation occurs.

The discrete increases of labor productivity by A imply a step function in *vintage-specific* total factor productivity (TFP), in contrast to the smooth evolution in traditional balanced growth models à la Romer (1990). As a result, output in this economy is growing through cycles as illustrated in Figure 1 and fluctuations are a natural phenomenon in a growing economy. However, this step function of vintage-specific TFP does *not* imply that there are discrete jumps in aggregate TFP. As we show in (6), vintages of capital goods can easily be aggregated to an index (7) which weights them such that prices fully reflect differences in productivity and the *aggregate* TFP is constant and equal to unity.

4 Volatility measures and endogeneity

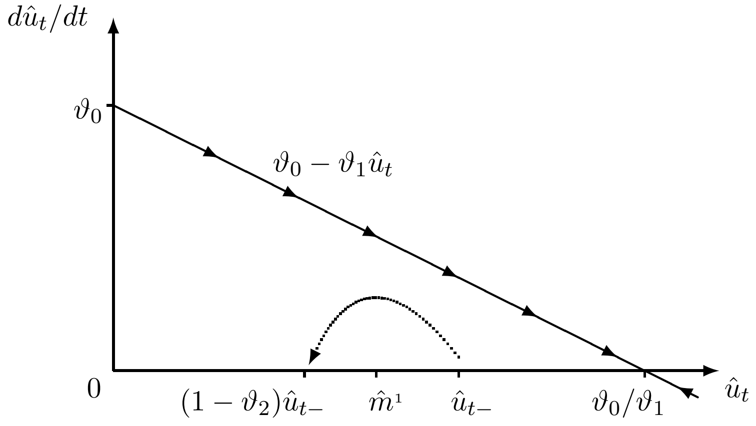
Volatility can be measured in many ways. The empirical literature focuses on either the standard deviation of output growth rates or the variance of cyclical variables. In this study, we obtain closed-form measures based on both cyclical components and growth rates. We show that both are closely related. Our limiting properties are based on cyclical components, so a measure based directly on cyclical components is appealing. Measures based on growth rates are more complicated, but straightforward to apply in empirical research.

4.1 Cyclical components

The empirical literature offers a large number of techniques to obtain stationary variables. Given their complexity, virtually none of these filters allows us to derive cyclical components which imply closed-form measures of volatility. Moreover, deterministic filters, e.g. removing a deterministic trend, would give no meaningful cyclical variables as the second moment is not bounded. We therefore use a very simple stochastic filter, the Solow-type detrending rule used in (16), to compute our cyclical components. It captures the trend by a step function A^q, caused by the discrete increases of q_t. In fact, we decompose the series into a stochastic trend and a stationary cyclical component.

Cyclical utility. For analytical tractability we work with cyclical utility. In most empir-

Figure 2: Detrended instantaneous utility



Note: This figure illustrates the dynamics of cyclical utility with constant speed of convergence (the slope of $d\hat{u}/dt$). Otherwise the dynamics are similar to those of cyclical capital stock (compare with Figure 1).

ical studies, the measures of volatility are based on aggregates such as output. While the dynamics of cyclical output are very similar to that of cyclical utility, its non-linear drift results in moments that would not allow us to derive analytic expression. Based on cyclical utility, we are able to compute higher moments explicitly as they denote the solution to a reducible ordinary differential equation (ODE). Moreover, we show that it is reasonable to assume that the qualitative effects on volatility are equivalent because the channels are the same.[9] For $\alpha = \sigma = .5$, even the quantitative effects for cyclical output and cyclical utility are the same (cf. also Appendix 7.1).

We define individual cyclical utility, in analogy to (11), as the component of utility that stems from the cyclical component of consumption in (16),

$$\hat{u}_t = \frac{(\hat{C}_t/L)^{1-\sigma}}{1-\sigma}. \tag{24}$$

Using Itô's formula, cyclical utility follows

$$\begin{aligned} d\hat{u}_t &= \left(\theta_0(\Psi/L)^{1-\sigma} - \theta_1(1-\sigma)\hat{u}_t\right) dt + \left((1-\theta_2)^{1-\sigma} - 1\right) \hat{u}_{t-} dq_t \\ &\equiv (\vartheta_0 - \vartheta_1 \hat{u}_t)\, dt - \vartheta_2 \hat{u}_{t-} dq_t, \end{aligned} \tag{25}$$

where we defined

$$\vartheta_0 \equiv (\Psi/L)^{1-\sigma}\theta_0, \quad \vartheta_1 \equiv (1-\sigma)\theta_1, \quad \vartheta_2 \equiv 1 - (1-\theta_2)^{1-\sigma}.$$

Most notably, the structure is similar to the evolution of the cyclical capital (22), only the speed of convergence, ϑ_1, is constant (the slope in Figure 2). Now the SDE in (25)

[9] In fact, there are approximation rules which allow to compute e.g. the coefficient of variation (cv) of consumption once the cv of utility (a monotone transformation of consumption) is known.

is reducible with a linear drift. Again, we can gain insights from plotting $\vartheta_0 - \vartheta_1 \hat{u}_t$ on the vertical axis, while $\hat{u}_t$ is depicted on the horizontal axis (Figure 2). Obviously, cyclical utility has support between 0 and its non-stochastic steady state, $0 < \hat{u}_0 < \hat{u}^*$, which from (25) is given by ϑ_0/ϑ_1. Starting from $\hat{u}_0$, as long as no innovation takes place, the cyclical component approaches its upper bound. Each successful research project reduces cyclical utility by $\vartheta_2 \hat{u}_{t-}$, i.e., ϑ_2 percent of its level an instant before the innovation, which ensures that cyclical utility always remains positive.

Computing moments. Exploiting the methods in García and Griego (1994), we can compute moments of the cyclical component as follows. Using the integral version of (25), $\hat{u}_t = \hat{u}_0 + \int_0^t (\vartheta_0 - \vartheta_1 \hat{u}_s)\, ds - \int_0^{t-} \vartheta_2 \hat{u}_s dq_s$, and the martingale property (cf. Appendix 7.2), we obtain

$$E_0(\hat{u}_t) = \hat{u}_0 + \int_0^t (\vartheta_0 - \vartheta_1 E_0(\hat{u}_s))\, ds - \lambda \int_0^{t-} \vartheta_2 E_0(\hat{u}_s) ds, \tag{26}$$

which gives the evolution of the first moment of $\hat{u}_t$ as a linear ODE which can be solved and is shown to converge to a constant. Using a similar approach, higher order moments can be computed easily.[10] In fact, denoting the nth moment by

$$\hat{m}_t^n \equiv E_0(\hat{u}_t^n), \tag{27}$$

the first and second moment of the stationary distribution are given by

$$\hat{m}^1 \equiv \lim_{t\to\infty} \hat{m}_t^1 = \frac{\vartheta_0}{\vartheta_1 + \lambda\vartheta_2}, \tag{28}$$

$$\hat{m}^2 \equiv \lim_{t\to\infty} \hat{m}_t^2 = \frac{2\vartheta_0}{2\vartheta_1 + \lambda(1 - (1-\vartheta_2)^2)} \hat{m}^1. \tag{29}$$

To understand the moments, we go back to Figure 2. Observe that the first moment $\hat{m}^1$ lies between 0 and the non-stochastic steady state ϑ_0/ϑ_1. As the process $\hat{u}_t$ is described completely by (25), given an arrival rate λ, only the parameters of this process, ϑ_0, ϑ_1, ϑ_2 and λ, can show up in its moments. A larger ϑ_0 and a smaller ϑ_1 shifts the mean $\hat{m}^1$ to the right as it moves the $d\hat{u}_t/dt$ line to the right (cf. Figure 2). When ϑ_2 or λ increases, the mean shifts to the left as either jumps are larger or more frequent. From (29), the second moment has properties similar to $\hat{m}^1$ with respect to ϑ_0, ϑ_1, ϑ_2 and λ. Thus, a larger range and more frequent jumps increase the dispersion and thus the second moment.

Our measure. Using both moments, computing the variance would be straightforward. As a measure of volatility, however, it seems less suitable because of scale-dependence. A scale-independent measure is the variance of the percentage deviations from some non-stochastic steady state or from expected value. Such a relative measure of dispersion is coefficient of variation (cv). Given that the variance of a random variable is the difference between its

[10] The structure of the moments is remarkable as it shows that the distribution of $\hat{u}_t$ exists, is unique and represents a generalization of the β-distribution (thanks to Christian Kleiber for pointing this out). As shown in Appendix 7.3, fairly complex expressions appear for state dependent moments.

second moment and the square of its mean, it is defined by

$$cv^2 \equiv \lim_{t\to\infty} \frac{Var_0(\hat{u}_t)}{(E_0(\hat{u}_t))^2} = \frac{\vartheta_2^2}{2\vartheta_1/\lambda + 2\vartheta_2 - \vartheta_2^2}, \tag{30}$$

where for the second equality we inserted the moments from (28) and (29), respectively.

Computing the cv shows that it is independent of ϑ_0. This is not surprising as ϑ_0 is a scaling parameter and the cv is scale-independent. This can intuitively be understood from Figure 2 where the effect of ϑ_0 on the cyclical component could be removed by scaling both axes with $1/\vartheta_0$. A lower speed of convergence, ϑ_1, implies a higher measure of relative dispersion, cv. Clearly, the slower the economy approaches its non-stochastic steady state, the higher is the overall variability of cyclical components. The jump term ϑ_2 and the arrival rate, λ (note that ϑ_1/λ decreases in λ), have a positive effect on cv, meaning that larger and more frequent jumps imply a higher measure of relative dispersion.

Our channels in growth rates. In order to relate our measures in (30) to empirical measures, it is very useful to study the growth rates of cyclical variables. To obtain moments of growth rates, we use integral equations for the log-variables and exploit the martingale property. For cyclical capital, the growth rates are obtained from

$$\begin{aligned} d\ln\hat{K}_t &= \left(\theta_0\hat{K}_t^{\alpha-1} - \theta_1\right)dt + \left(\ln\hat{K}_t - \ln\hat{K}_{t-}\right)dq_t \\ &= \left(\theta_0\hat{K}_t^{\alpha-1} - \theta_1\right)dt + \ln(1-\theta_2)dq_t. \end{aligned} \tag{31}$$

Integrating gives the growth rate of cyclical capital per unit of time Δ as

$$\ln\hat{K}_t - \ln\hat{K}_{t-\Delta} = \int_{t-\Delta}^{t} \frac{1-\tau_i}{1+\tau_k} r_s/\alpha ds - \theta_1\Delta + \ln(1-\theta_2)(q_t - q_{t-\Delta}), \tag{32}$$

where we relate growth rates to the integrated process of capital rewards, $r_t = \alpha\hat{K}_t^{\alpha-1}L^{1-\alpha}$. Similarly, the growth rate of cyclical output is $\Delta\hat{y}_t \equiv \ln\hat{Y}_t - \ln\hat{Y}_{t-\Delta} = \alpha(\ln\hat{K}_t - \ln\hat{K}_{t-\Delta})$. By inspection of (32), the expected growth rate of cyclical variables per unit of time is zero. This result is intuitive because $\hat{K}_t$ is bounded between 0 and $\hat{K}$, which implies a stationary distribution (as illustrated in Figure 1). In order to calculate the variance of growth rates the following lemma is very useful.

Lemma 1 *Suppose that* $\ln\hat{K}_t$ *follows (31), then*

$$\lim_{t\to\infty} Cov_0\left(\ln\hat{K}_t - \ln\hat{K}_{t-\Delta}, q_t - q_{t-\Delta}\right) = \ln(1-\theta_2)\lambda\Delta.$$

Proof. Appendix C.2 ■

After some algebra, we obtain the asymptotic variance as (cf. Appendix 7.6)

$$\lim_{t\to\infty} Var_0(\Delta\hat{y}_t) = \lim_{t\to\infty} Var_0\left(\frac{1-\tau_i}{1+\tau_k}\int_{t-\Delta}^{t} r_s ds\right) + (\alpha\ln(1-\theta_2))^2\lambda\Delta. \tag{33}$$

This result is remarkable because it shows that the variance of growth rates depends on the variance of the (integrated) process of capital rewards, which in turn follow

$$dr_t = c_1 r_t (c_2 - r_t)dt + c_3 r_{t-} dq_t, \tag{34}$$

where $c_1 \equiv \frac{1-\alpha}{\alpha}\frac{1-\tau_i}{1+\tau_k}$, $c_2 \equiv \vartheta_1/c_1$, and $c_3 \equiv \vartheta_2/(1-\vartheta_2)$. In fact, this SDE describes the (transitional) equilibrium dynamics of capital rewards, often referred to as the stochastic Verhulst equation. It is shown that r has a unique limiting distribution, and the moments of the limiting distribution are available in closed-form (cf. Appendix 7.5)

$$E(r) = \frac{c_1 c_2 + \ln(1+c_3)\lambda}{c_1}, \quad Var(r) = \frac{c_3\lambda - \ln(1+c_3)\lambda}{c_1} E(r).$$

Unfortunately, the variance of the integrated process in (33) is complicated because of the non-linear dynamics in (34).[11] In order to obtain a closed-form expression, we propose to approximate the asymptotic variance of the integrated process by

$$\lim_{t\to\infty} Var_0\left(\int_{t-\Delta}^{t} r_s ds\right) \approx \lim_{t\to\infty} Var_0\left(r_t\Delta\right) = Var(r)\Delta^2. \tag{35}$$

Two observations give support to the usefulness of this approximation. First, in simulations for reasonable calibrations we find only negligible differences. Second, we are not interested in the variance of the growth rate *per se*, but in the asymptotic effects of taxes. A precise measure would take into account the auto-covariance function based on asymptotic moments $\lim_{t\to\infty} E_0(r_s r_u)$. Because of the non-linear structure of capital reward dynamics in (34), joint moments depend on higher-order moments, and thus are difficult to compute analytically. Hence, our approximate measure focuses on the tax effects on the variance neglecting the auto-correlation structure of capital rewards in our comparative static analysis below.

To summarize, we can define a measure of volatility based on growth rates as

$$\begin{aligned} Var(\Delta\hat{y}_t) &\equiv \lim_{t\to\infty} Var_0\left(\frac{1-\tau_i}{1+\tau_k} r_t\right)\Delta^2 + (\alpha\ln(1-\theta_2))^2\lambda\Delta \\ &= \left(\tfrac{\alpha}{1-\alpha}\right)^2 \left(\tfrac{\vartheta_2}{1-\vartheta_2} + \ln(1-\vartheta_2)\right)\left(\vartheta_1 - \ln(1-\vartheta_2)\lambda\right)\lambda\Delta^2 + \left(\tfrac{\alpha}{1-\alpha}\ln(1-\vartheta_2)\right)^2\lambda\Delta, \end{aligned} \tag{36}$$

where we inserted the asymptotic moments for r and collected terms. Obviously, this measure shares the property of scale independence with the *cv* because we consider growth rates, which by construction are scale independent.

[11] For a similar mean-reverting model of the spot rate dynamics, namely the Ornstein-Uhlenbeck process, $dr_t = c_1(c_2 - r_t)dt + c_3 dq_t$, the auto-covariance function and the measure are available in closed form. For this linear model, $Var(\int_{t-\Delta}^{t} r_s ds)$ is proportional to $Var(r) \equiv \lim_{t\to\infty} Var_0(r_t)$, and $Var(\int_{t-\Delta}^{t} r_s ds) \approx Var(r)\Delta^2$ coincides with its second-order Taylor approximation about $\Delta = 0$.

4.2 Output growth rates

An empirically more obvious measure is based on observed output growth rates. According to the detrending rule (16), we may write logarithmic output as

$$\begin{aligned} \ln Y_t &= \alpha \ln K_t + (1-\alpha)\ln L \\ &= \alpha \ln \hat{K}_t + (1-\alpha)\ln L + q_t \ln A, \end{aligned} \tag{37}$$

i.e., we split our time series $\ln Y_t$ into a trend component, $q_t \ln A$, and a stationary component, $\alpha \ln \hat{K}_t + (1-\alpha)L,$.[12] Both the trend component and the stationary component are stochastic. Even though our model is formulated in continuous time, we can relate our trend component to a discrete-time random walk as $q_t \equiv q_{t-\Delta} + \Delta q_t$, where $\Delta q_t \sim (\lambda\Delta, \lambda\Delta)$, describes a pure random walk with drift. Hence, the trend component $q_t \ln A$ has a unit root and the cyclical component $\hat{K}_t$ is stationary by construction.

Let the growth rates per unit of time be $\Delta y_t \equiv \ln Y_t - \ln Y_{t-\Delta}$, from (37) we obtain

$$\Delta y_t = \alpha\big(\ln \hat{K}_t - \ln \hat{K}_{t-\Delta}\big) + (q_t - q_{t-\Delta})\ln A \;=\; \Delta \hat{y}_t + \Delta q_t \ln A. \tag{38}$$

Using Lemma 1 and (36), we define our second measure based on output growth rates as

$$\begin{aligned} Var(\Delta y_t) &\equiv Var(\Delta \hat{y}_t) + \lim_{t\to\infty} Cov_0(\ln \hat{K}_t - \ln \hat{K}_{t-\Delta}, q_t - q_{t-\Delta}) + Var(\Delta q_t \ln A) \\ &= Var(\Delta \hat{y}_t) + 2\tfrac{\alpha}{1-\alpha}\ln A \ln(1-\vartheta_2)\lambda\Delta + (\ln A)^2\lambda\Delta \\ &= \left(\tfrac{\alpha}{1-\alpha}\right)^2 \left(\tfrac{\vartheta_2}{1-\vartheta_2} + \ln(1-\vartheta_2)\right)(\vartheta_1 - \ln(1-\vartheta_2)\lambda)\,\lambda\Delta^2 \\ &\quad + \left(\tfrac{\alpha}{1-\alpha}\ln(1-\vartheta_2) + \ln A\right)^2 \lambda\Delta. \end{aligned} \tag{39}$$

Similarly, using (38) and the expectation operator,

$$E(\Delta y_t) \;\equiv\; \lim_{t\to\infty} E_0(\Delta y_t) \;=\; E(\Delta \hat{y}_t) + E(\Delta q_t)\ln A \;=\; \lambda \ln A \Delta. \tag{40}$$

Hence, the long-run expected growth rate of the common stochastic trend is determined by the arrival rate of new technologies. From (21), λ increases in the investment tax, τ_k, and decreases in the tax on research, τ_r. Below we study the effects of taxes on volatility.

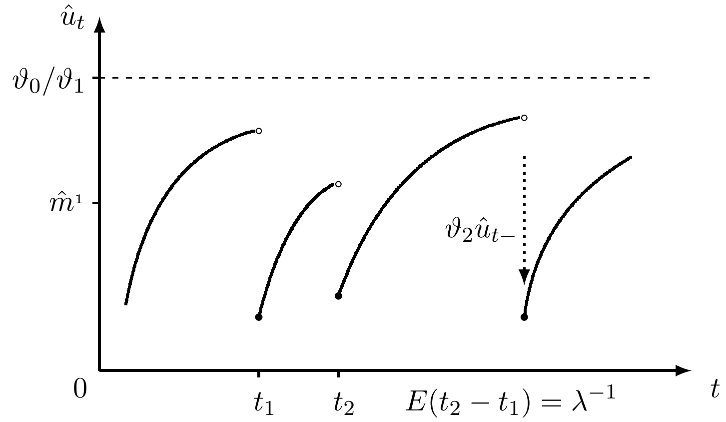
5 Volatility and taxation

5.1 Theoretical findings

Our measure of volatility in (30) is affected through three channels, the speed of convergence ϑ_1, the jump size ϑ_2, and the arrival rate λ. As shown, these determinants appear in the

[12] Other models of endogenous fluctuations and growth are of a deterministic nature. An exception is Bental and Peled (1996) who first studied endogenous fluctuations and growth. Unfortunately, their model is fairly complex making an explicit analysis of stochastic properties of trends and cycles a difficult task.

Figure 3: The cyclical components and their determinants ϑ_1, ϑ_2 and λ



Note: This figure illustrates the determinants of cyclical components and the coefficient of variation (*cv*) of cyclical utility, using an arbitrary realization of the SDE in (25). Our measure in (30) is determined by the speed of convergence, ϑ_1 (or the scale independent range), the arrival rate, λ, and the jump size, ϑ_2.

measures based on growth rates in (36) and (39). For illustration, the interpretation of these channels is based on (25). Consider an arbitrary realization of the cyclical component in Figure 3. In line with our previous results, the speed of convergence, ϑ_1, determines the range of cyclical utility $(0, \vartheta_0/\vartheta_1)$. The upper limit corresponds to the non-stochastic steady state for cyclical capital, $\hat{K}^* = (\theta_0/\theta_1)^{\frac{1}{1-\alpha}}$. However, the only parameter which matters for the relative dispersion of cyclical utility is ϑ_1 (recalling that ϑ_0 is a scaling parameter only). From its definition in (25) and the discussion of (22), it is clear that the parameter contains the effective resource allocation to both research expenditures and total consumption. Moreover, the arrival rate λ measures the frequency of jumps (the inverse measures the expected length). Finally, the size of the jump is measured by ϑ_2.[13] Hence we find that macro volatility depends on the *level* of taxes if at least one of the three channels, (i) the speed of convergence, (ii) the jump size or (iii) the arrival rate, depend on taxes.[14]

To understand the effects of taxation on macro volatility, we may restrict attention to the speed of convergence and the arrival rate (or jump probability), because the jump size does not depend on taxes. The independence of ϑ_2 follows from the fact that the jump in consumption, $\xi = 1 - s + \kappa$, from (25) is not affected by taxes. Economically, this result is obtained because payoffs κ are not taxed and economic depreciation, s, does not imply tax-exemption as does physical depreciation, δ. The tax effects on the arrival rate λ are

[13] A similar decomposition of the channels through which measures based on cyclical output, $\hat{Y}_t = \hat{K}_t^{\alpha} L^{1-\alpha}$, or cyclical consumption, $\hat{C}_t = \Psi \hat{K}_t$, are affected by taxes is provided in Appendix 7.1. As the speed of convergence is not constant for either variable, these effects are analytically intractable. However, close to the non-stochastic steady state, the speed of convergence is the same for all three variables, $\vartheta_1 = (1-\alpha)\theta_1$.

[14] If growth and cycles are exogenous, i.e., if there is an exogenous arrival rate λ without research, the model describes a continuous-time RBC model with vintage-specific capital. In this case, macro volatility is partly endogenous and affected by taxation through the speed of convergence, ϑ_1.

Table 1: Qualitative tax effects on composite parameters, macro volatility and growth

		Taxes				
		τ_i (income)	τ_c (consumption)	τ_r (research)	τ_k (investment)	τ_a (wealth)
ϑ_1	(speed of convergence)	$-$	0	$-$	$+^\dagger$	$+$
ϑ_2	(jump size)	0	0	0	0	0
λ	(arrival rate)	0	0	$-$	$+$	0
$E(\Delta y_t)$	(mean growth rate)	0	0	$-$	$+$	0
$sd(\Delta y_t)$	(s.d. of growth rates)	$-$	0	$-$	$+^\dagger$	$+$
$cv(\hat{u})$	(coefficient of variation)	$+$	0	$-$	$+$	$-$

† for δ sufficiently small

Note: This table shows the qualitative tax effects of time-invariant tax rates on macro volatility and growth and their components. The measures include the speed of convergence of cyclical utility, ϑ_1, the jump size, ϑ_2, and the arrival rate, λ, which determine the long-run expected growth rate, Δy_t, and the coefficient of variation of cyclical utility, $cv(\hat{u})$.

obtained from (21). The parameter ϑ_1 in (25) depends on taxes both directly and indirectly through the arrival rate. The direct effect reflects the effective rate of physical depreciation, $\frac{1-\tau_i}{1+\tau_k}\delta+\tau_a$, whereas the indirect effect reflects tax effects on the arrival rate, λ, which in turn are due to changes in private consumption, $\hat{C}_t$, research expenditures, $\hat{R}_t$, and government consumption, $\hat{G}_t$. Inserting λ into ϑ_1 gives unambiguous results (cf. Appendix C.3). For reading convenience, the qualitative results are summarized in Table 1.

5.2 Comparative statics

Let us now combine the effects of our three channels on volatility in a comparative static analysis. As we have only two tax-dependent channels, the speed of convergence, ϑ_1, and the arrival rate, λ, taxes affect the variance of the limiting distribution of stationary macro variables by either changing the speed of convergence (without affecting λ in ϑ_1), the arrival rate, or both. Clearly, a tax which has no effect on ϑ_1 and λ, does not affect our measures either. The tax on consumption expenditures, τ_c, is such a tax because government consumption offsets changes in private consumption.

When taxing wealth, τ_a, the arrival rate λ is not affected. The speed of convergence, ϑ_1, increases which causes cv in (30) to decline. Economically, τ_a decreases the households' return on savings, or equivalently, increases the effective rate of depreciation. This in turn implies a lower non-stochastic steady-state, $\hat{K}^*$, and more resources are used for consumption and research. Holding constant the length of a cycle but 'squeezing' the cyclical components in Figure 3, the relative dispersion of the components must be lower.

An increase in the income tax, τ_i, reduces the speed of convergence ϑ_1 but does not affect the jump probability, λ. As a consequence, volatility unambiguously increases in this tax.

How can this result be understood? The parameter ϑ_1 in (25) decreases for the following reason: Only *net* investment is taxed (as discussed above), this means that a higher tax on income increases the positive effect of the refunding policy and reduces the impact of the depreciation rate, δ. A lower effective depreciation rate increases incentives for capital accumulation, and the non-stochastic steady-state capital stock, $\hat{K}^*$, increases.

For the taxes on research, τ_r, and investment, τ_k, the results are less clear-cut. With these taxes the arrival rate λ, is affected, which in turn changes cv directly and indirectly through ϑ_1. The direct effect of λ on cv is unambiguously positive. Computing the derivatives, however, we obtain the results for our measures of volatility as in Table 1. A higher tax on research depresses the arrival rate and the ratio ϑ_1/λ increases, which in turn decreases cv in (30). Intuitively, higher rates τ_r make investment in research less profitable and the arrival rate falls. Less frequent jumps imply a lower relative dispersion of cyclical variables. A lower λ also decreases ϑ_1, thus less resources are used for consumption and research. This implies a larger range $1/\vartheta_1$ in Figure 2 and higher volatility. The indirect effect through the lower speed of convergence does not compensate the direct effect through the lower arrival rate. Hence, the ratio ϑ_1/λ increases and cv in (30) decreases (as shown in Appendix C.3).

Similarly, the results for τ_k are as follows: A higher tax on the accumulation of physical capital shifts resources towards consumption and research. It increases the arrival rate which in turn increases volatility and ϑ_1. An additional effect comes about through a negative effect on the effective rate of depreciation, which makes the effect on ϑ_1 ambiguous. Nonetheless, the ratio ϑ_1/λ unambiguously decreases, thus cv in (30) increases.

Given the discussion above, we can now understand why empirical measures such as the *sd* of output growth rates may also depend on taxes. Consider the speed of convergence ϑ_1. As shown above, an increase in ϑ_1 decreases the range of the cyclical component. Intuitively, this decreases the cv and variables in efficiency units, but increases the variance of capital rewards. This in turn implies a higher variance of output growth rates (using Lemma 1). Hence, the tax effects implied through the propagation of shocks is reversed for measures based on growth rates. The qualitative effects on the arrival rate, however, are identical to measures of relative dispersion of cyclical components.

5.3 The non-causality between volatility and growth

We are now prepared to make our main point. For a given tax policy, our economy follows a certain cyclical growth path. Now imagine a second economy with a different tax policy and a third one with yet another tax policy and so on. Given our comparative static results, it is straightforward to understand why growth and volatility are correlated and that this correlation can take any sign - depending on cross-country differences in tax systems and which measure one uses for volatility.

Suppose differences across countries exist only in the investment tax (value added tax on physical investment goods). Table 1 shows that both growth and volatility increase in the investment tax (independently of the measure of volatility), as resources are shifted to R&D. In a cross-section of countries, one would observe a positive correlation between volatility and growth. The same positive correlation would exist if countries differed only in their research tax. A negative correlation between volatility and growth is predicted by our model for various combinations of tax rates. One example is when countries with a high tax on investment also have a high tax on income. The investment tax increases volatility and growth, the income tax decreases volatility (focusing on the empirical measure, i.e., the *sd* of growth rates in Table 1). If the negative effect is stronger than the volatility-increasing effect of the investment tax, there is a negative correlation between investment and growth.

The reader may want to discuss our examples on tax policy structures as well as the generality of our parametric restriction $\alpha = \sigma$. Yet, our general point remains: Differences in economic policies across countries imply differences in growth rates and volatility. Depending on cross-country differences, correlations of any magnitude and sign can occur and in no case is there a causal relationship between the two.

5.4 Implications for empirical research

Given our theoretical findings, there are at least two messages for empirical work: First, a volatility-growth analysis in the spirit of Ramey and Ramey (1995) needs to add - relative to the original Ramey and Ramey setup - additional controls to the conditional variance equation. Second, once these additional variables are added, the existing omitted variable bias will be reduced. Even then, however, the estimated coefficient for the volatility-growth link remains a measure of correlation and not a measure of causality.

To elaborate on the first point, consider the following extension of Ramey and Ramey,

$$\Delta y_{it} = \nu\sigma_{it} + \theta X_{it} + \varepsilon_{it}, \quad \text{where} \quad \varepsilon_{it} \sim N(0, \sigma_{it}^2) \tag{41a}$$

$$\sigma_{it}^2 = \alpha_i + \beta Z_{it} \tag{41b}$$

where Δy_{it} is the growth rate of output per capita for country i in year t, σ_{it} is the standard deviation of the residuals; X_{it} is a vector of control variables (the Levine-Renelt variables); Z_{it} (a subset of X_{it}) is a vector of control variables which affect both growth and volatility; α_i are country fixed effects (i.e., country dummy variables); whereas θ and β are vectors of coefficients. The key parameter of interest is ν, which links growth to volatility.

For $\beta = 0$ this specification is equivalent to the original Ramey and Ramey formulation. Given our theoretical arguments, the conditional variance equation also needs to include additional controls, i.e., the variables included in Z_{it}. As we have seen that the level of taxes can have a strong effect on volatility, we would expect β to be significant. According to our model, Z_{it} measures the level of taxes. If taxes were constant over time, $Z_{it} = Z_i$, our proposed extension would be equivalent to country-specific fixed effects - as already included in Ramey and Ramey (1995, p.1141). A successful empirical implementation of our argument therefore requires sufficient variation in taxes over time. As is well-known from Mendoza-Razin-Tesar tax rates, such variation is present in the data indeed.

On the second point, once additional controls Z_{it} are added, the estimator for ν - the volatility-growth link - will become a function of Z_{it}. To the extend that our theoretical model captures some aspect of the true data-generating process, adding Z_{it} removes an omitted variable bias. Proceeding in this direction (see Posch, 2008) shows that the estimate for ν becomes more precise, and ν becomes more negative than in Ramey and Ramey. Yet, given our theoretical point of view, the relation between volatility and growth does still *not* reflect a causality, it is only a correlation.

6 Conclusion

There is a growing literature which analyzes the joint endogeneity of volatility and growth. This paper emphasizes the implication of this approach for the volatility-growth link: This link is a pure correlation, there is no causality running from volatility to growth.

We illustrate our point by identifying tax rates as the truly fundamental parameters which determine the growth rate and the degree of volatility of a country. One of our main theoretical contribution lies in the fact that all measures of volatility are obtained analytically. This allows us to follow an analytical approach in understanding the channels through which the level of tax rates affects volatility and growth of a country.

We find that tax rates determine the sign of the correlation between volatility and growth. For example, if taxes on wealth are used to facilitate R&D investment, growth and volatility are positively correlated. In contrast, if taxes on wealth are used to promote physical capital investment, a negative link can occur. For empirical work building on Ramey and Ramey (1995), we conclude that additional control variables are needed in the conditional variance equation. Neglecting to do so results in biased estimates of the growth-volatility link.

As always, there are limitations in this paper opening up interesting future research avenues. Given that we want to push our argument to the extreme, we do not allow for any exogenous source of volatility. In an extended calibrated framework with both endogenous and exogenous shocks, one can identify which share of volatility is caused endogenously. This would allow to take a more balanced point of view and would lead to a conclusion on the degree (i.e., a certain percentage) to which volatility is causal for growth. In our framework, there is only correlation. In this extended framework (and probably in the real world), volatility could be partly causal for growth.

More broadly speaking, the concept of causality and the extent to which causality can be identified by a framework in the tradition of Ramey and Ramey (1995) can be investigated. One can start with a theoretical model with both endogenous and exogenous sources of volatility. Simulating this model would allow regressions of the Ramey and Ramey type with this artificial data. As the theoretical world would tell us 'how causal' volatility is for growth, one would understand which specification we need for a regression in order to be able to identify causality. We leave all of this for future work.

7 Appendix

7.1 Cyclical components

As from (24), we have $d\hat{u}_t = (1/L)^{1-\sigma}/(1-\sigma)d\hat{C}_t^{1-\sigma}$. With $\hat{C}_t = \Psi\hat{K}_t$ from (17) we obtain $d\hat{C}_t^{1-\sigma} = \Psi^{1-\sigma}d\hat{K}_t^{1-\sigma}$. Using (22) and Itô's formula,

$$
\begin{aligned}
d\hat{K}_t^{1-\sigma} &= (1-\sigma)\big(\theta_0\hat{K}_t^{\alpha} - \theta_1\hat{K}_t\big)\hat{K}_t^{-\sigma}dt + \Big(\big(\hat{K}_{t-} - \theta_2\hat{K}_{t-}\big)^{1-\sigma} - \hat{K}_{t-}^{1-\sigma}\Big)dq_t \\
&= (1-\sigma)\big(\theta_0 - \theta_1\hat{K}_t^{1-\sigma}\big)dt - \big(1-(1-\theta_2)^{1-\sigma}\big)\hat{K}_{t-}^{1-\sigma}dq_t. \qquad (42)
\end{aligned}
$$

The non-stochastic steady state (range of the cyclical component) is $\hat{K}^{1-\sigma*} = \theta_0/\theta_1$, the speed of convergence is $(1-\sigma)\theta_1$, and jump term is $1-(1-\theta_2)^{1-\sigma}$. Note that the speed of convergence is constant as long as the drift component is linear.

Similarly, for obtaining cyclical output, $d\hat{Y}_t = L^{1-\alpha}d\hat{K}_t^{\alpha}$, we compute

$$
d\hat{K}_t^{\alpha} = \alpha\big(\theta_0\hat{K}_t^{\alpha-1} - \theta_1\big)\hat{K}_t^{\alpha}dt - \left(1-(1-\theta_2)^{\alpha}\right)\hat{K}_{t-}^{\alpha}dq_t, \qquad (43)
$$

which denotes a reducible SDE in $\hat{K}_t^{\alpha}$ with non-linear drift. The non-stochastic steady state is $\hat{K}^{\alpha*} = (\theta_0/\theta_1)^{\frac{\alpha}{1-\alpha}}$, the speed of convergence is $\alpha\theta_1 - (2\alpha-1)\theta_0\hat{K}_t^{\alpha-1}$ and *not* constant unless $\alpha = .5$, and the jump term $1-(1-\theta_2)^{\alpha}$, increases (decreases) relative to $\hat{K}_t^{1-\sigma}$ for $\alpha > .5$ ($\alpha < .5$) and is the same for $\alpha = \sigma = .5$.

For utility, we use the scaled version of (42), $\hat{u}_t = \hat{K}_t^{1-\sigma}(\Psi/L)^{1-\sigma}/(1-\sigma)$, to obtain

$$
\begin{aligned}
d\hat{u}_t &= \big(\theta_0(\Psi/L)^{1-\sigma} - \theta_1(1-\sigma)\hat{u}_t\big)dt + \Big((1-\theta_2)^{1-\sigma} - 1\Big)\hat{u}_{t-}dq_t \\
&\equiv (\vartheta_0 - \vartheta_1\hat{u}_t)\,dt - \vartheta_2\hat{u}_{t-}dq_t,
\end{aligned}
$$

where ϑ_0 through ϑ_2 are defined as in (25).

7.2 Properties of the Poisson process

We use the martingale property of various expressions. These expressions are special cases of $\int_0^t f(q_s,s)\,dq_s - \lambda\int_0^t f(q_s,s)\,ds$, which is a martingale (cf. García and Griego, 1994), i.e.,

$$
E_0\left(\int_0^t f(q_s,s)\,dq_s - \lambda\int_0^t f(q_s,s)\,ds\right) = 0, \qquad (44)
$$

where λ is the (constant) arrival rate of q_t.

7.3 Computing moments

Expressing the integral version in (26) as a differential equation and using the definition in (27), we obtain $d\,\hat{m}_t^1 = (\vartheta_0 - (\vartheta_1 + \lambda\vartheta_2)\hat{m}_t^1)\,dt$, which is a linear ODE with solution

$$\hat{m}_t^1 \quad = \quad e^{-(\vartheta_1+\lambda\vartheta_2)t}\left(\hat{m}_0^1 + \int_0^t e^{(\vartheta_1+\lambda\vartheta_2)s}\vartheta_0 ds\right) = e^{-(\vartheta_1+\lambda\vartheta_2)t}\left(\hat{m}_0^1 + \vartheta_0\frac{e^{(\vartheta_1+\lambda\vartheta_2)t}-1}{\vartheta_1+\lambda\vartheta_2}\right).$$

It can be simplified to

$$\hat{m}_t^1 = e^{-(\vartheta_1+\lambda\vartheta_2)t}\left(\hat{m}_0^1 - \frac{\vartheta_0}{\vartheta_1+\lambda\vartheta_2}\right) + \frac{\vartheta_0}{\vartheta_1+\lambda\vartheta_2}. \tag{45}$$

As $\vartheta_1 + \lambda\vartheta_2 > 0$, the first moment of $\hat{u}_t$ is in the long run given by

$$\hat{m}^1 \equiv \lim_{t\to\infty} \hat{m}_t^1 = \frac{\vartheta_0}{\vartheta_1+\lambda\vartheta_2}.$$

For higher moments, the basic ODE determining the evolution of $\hat{u}_t^n$ is from (25)

$$\begin{aligned} d\hat{u}_t^n &= n\hat{u}_t^{n-1}(\vartheta_0 - \vartheta_1\hat{u}_t)\,dt - (1-(1-\vartheta_2)^n)\,\hat{u}_{t-}^n dq_t \\ &= n\left(\vartheta_0\hat{u}_t^{n-1} - \vartheta_1\hat{u}_t^n\right)dt - (1-(1-\vartheta_2)^n)\,\hat{u}_{t-}^n dq_t. \end{aligned} \tag{46}$$

Using the integral version, applying expectations and the martingale result (44), we obtain $dE_0\hat{u}_t^n = \left(n\vartheta_0 E_0\hat{u}_t^{n-1} - (n\vartheta_1 + \lambda(1-(1-\vartheta_2)^n))\,E_0\hat{u}_t^n\right)dt$. Using the definition in (27),

$$d\hat{m}_t^n = (n\vartheta_0\hat{m}_t^{n-1} - (n\vartheta_1 + \lambda(1-(1-\vartheta_2)^n))\,\hat{m}_t^n)\,dt. \tag{47}$$

It shows that all moments converge to finite limits for $t\to\infty$. For the first moment, this follows from (45) (see Appendix C.1 for the second moment). The proofs for higher moments follow an identical approach. In short, for asymptotic moments where $d\hat{m}_t^n/dt = 0$, we obtain from (47)

$$\hat{m}^n = \frac{n\vartheta_0}{n\vartheta_1 + \lambda(1-(1-\vartheta_2)^n)}\hat{m}^{n-1}. \tag{48}$$

By inserting $n = 2$, it implies (29), with $n = 1$, it becomes (28), and by definition $\hat{m}^0 = 1$.

7.4 Limiting distribution

If the nth moment $\hat{m}_t^n \equiv E_0(\hat{u}^n)$ has bounded support, then $\hat{m}^n \equiv \lim_{t\to\infty} E_0(\hat{u}_t^j)$ is the jth moment of the limiting distribution for any $j < n$, and the moments in (28) and (29) converge to the moments of the limiting distribution. Moreover, $\hat{u}_t$ has a unique limiting distribution (Rao, 1973, p.121, Casella and Berger, 2001, Theorem 2.3.11.). Hence, the sequence $\{\hat{u}_t\}_{t=t_0}^{\infty}$ converges in distribution to a random variable $\hat{u}$,

$$\hat{u}_t \xrightarrow{\mathcal{D}} u \quad \text{where} \quad 0 < \hat{u}_t < \hat{u}^*. \tag{49}$$

In fact, the limiting density of any smooth transformation of $\hat{u}_t$ is determined by the change of variable formula for densities (cf. Merton, 1975).

By inspection of moments in (48), $\hat{u}$ has a generalized β-distribution. For $\vartheta_2 = 1$, the moments in (48) are $\hat{m}^n = \frac{n\vartheta_0}{n\vartheta_1+\lambda}\hat{m}^{n-1}$. Starting from $\hat{m}^0 = 1$, repeated inserting yields

$$\hat{m}^n = \frac{\vartheta_0^n n!}{\prod_{i=1}^n (i\vartheta_1 + \lambda)} = \left(\frac{\vartheta_0}{\vartheta_1}\right)^n \frac{\Gamma(n+1)}{\prod_{i=1}^n (i + \lambda/\vartheta_1)} = \left(\frac{\vartheta_0}{\vartheta_1}\right)^n \frac{\Gamma(n+1)\Gamma(1+\lambda/\vartheta_1)}{\Gamma(n+1+\lambda/\vartheta_1)},$$

where $\Gamma(\cdot)$ is the gamma function. Apart from the scaling factor $(\vartheta_0/\vartheta_1)^n$, the last expression denotes the nth moment of a β-distribution with parameters 1 and λ/ϑ_1. Hence, $\hat{u}$ has the asymptotic representation $\hat{u} = (\vartheta_0/\vartheta_1)^n X$, where $X \sim \text{Beta}(1, \lambda/\vartheta_1)$. For $\vartheta_2 \neq 1$, we obtain a generalized β-distribution which, to the best of our knowledge, has not been encountered before. Analyzing its properties in detail should be done in future research.

7.5 Moments of the rental rate of capital

In a seminal paper Merton (1975) shows that the output-to-capital ratio in the Solow model under Normal uncertainty has a Gamma distribution. We obtain the asymptotic moments of the rental rate of capital (output-to-capital ratio times output elasticity of capital) in a model of endogenous growth through cycles under Poisson uncertainty.

Because $r_t = \frac{\alpha}{1-\alpha}\Phi^{1-\sigma}\hat{u}_t^{-1}$ is a smooth transformation of $\hat{u}_t$, from (49) the sequence $\{r_t\}_{t=t_0}^{\infty}$ converges in distribution to a random variable r (cf. Posch, 2009),

$$r_t \overset{\mathcal{D}}{\to} r \quad \text{where} \quad r^* < r_t < \infty. \tag{50}$$

Using (22) and $r_t = B^q \alpha K_t^{\alpha-1} L^{1-\alpha} = \alpha \hat{K}_t^{\alpha-1} L^{1-\alpha}$ which gives

$$\begin{aligned} d\hat{K}_t^{\alpha-1} &= (\alpha-1)K_t^{\alpha-2}\big(\theta_0 \hat{K}_t^{\alpha} - \theta_1 \hat{K}_t\big)dt + \big(\hat{K}_t^{\alpha-1} - \hat{K}_{t-}^{\alpha-1}\big)dq_t \\ &= (\alpha-1)\big(\theta_0 \hat{K}_t^{2\alpha-2} - \theta_1 \hat{K}_t^{\alpha-1}\big)dt - \big(1-(1-\theta_2)^{\alpha-1}\big)\hat{K}_{t-}^{\alpha-1}dq_t, \end{aligned}$$

which implies defining c_1 to c_3 as in (34),

$$\begin{aligned} dr_t &= (\alpha-1)r\left(\frac{1-\tau_i}{1+\tau_k}r_t/\alpha - \theta_1\right)dt + \big((1-\theta_2)^{\alpha-1} - 1\big)r_{t-}dq_t \\ &= c_1 r_t (c_2 - r_t)\,dt + c_3 r_{t-}dq_t, \end{aligned}$$

We use the smooth transformation $\ln r_t$,

$$\ln r_t \overset{\mathcal{D}}{\to} \ln r \quad \text{where} \quad \ln r^* < \ln r_t < \infty, \tag{51}$$

to obtain $d\ln r_t = c_1(c_2 - r_t)dt + \ln(1+c_3)dq_t$, which has the solution

$$\ln r_t - \ln r_{t_0} = \int_{t_0}^{t} c_1(c_2 - r_s)ds + \ln(1+c_3)(q_t - q_{t_0}).$$

Employing the property that $\ln r_t$ and $\ln r_{t-\Delta}$ share the same asymptotic mean as from (51),

$$\begin{aligned} \lim_{t\to\infty} E_0(\ln r_t) - \lim_{t\to\infty} E_0(\ln r_{t_0}) &= c_1 c_2 \Delta - c_1 \lim_{t\to\infty} \int_{t-\Delta}^{t} E_0(r_s)ds + \ln(1+c_3)\lim_{t\to\infty} E_0(q_\Delta) \\ \Rightarrow \quad E(r) &\equiv \lim_{t\to\infty} E_0(r_t) = \frac{c_1 c_2 + \ln(1+c_3)\lambda}{c_1}. \end{aligned} \tag{52}$$

For the second moment, we use the integral equation applying the expectation operator,

$$dE_0(r_t) = c_1\left(c_2E_0(r_t) - E_0(r_t^2)\right)dt + c_3E_0(r_{t-})\lambda dt.$$

Letting $t \to \infty$ we obtain for the integral equation

$$\begin{aligned}
\lim_{t\to\infty} E_0(r_t) - \lim_{t\to\infty} E_0(r_{t_0}) &= \lim_{t\to\infty}\int_{t-\Delta}^{t}(c_1c_2 + c_3\lambda)E_0(r_s)ds - \lim_{t\to\infty}\int_{t-\Delta}^{t} c_1E_0(r_s^2)ds \\
\Leftrightarrow\ 0 &= \lim_{t\to\infty} E_0(r_t)(c_1c_2 + c_3\lambda)\Delta - \lim_{t\to\infty} E_0(r_t^2)c_1\Delta \\
\Rightarrow\quad E(r^2) &\equiv \lim_{t\to\infty} E_0(r_t^2) = E(r)\frac{c_1c_2 + c_3\lambda}{c_1}. \qquad (53)
\end{aligned}$$

Hence the asymptotic variance, i.e., the variance of the limiting distribution for r_t is

$$Var(r) \equiv \lim_{t\to\infty} Var_0(r_t) = E(r^2) - (E(r))^2 = \frac{c_3\lambda - \ln(1 + c_3)\lambda}{c_1}E(r).$$

Note that the variance is proportional to the mean, which seems plausible given the geometric structure of the stochastic differential in (34).

7.6 Moments of growth rates

Because ($\ln \hat{K}_t = \frac{1}{1-\sigma}\ln \hat{u}_t + \text{constant}$) is a smooth transformation of $\hat{u}_t$ in (49) the sequence $\{\ln \hat{K}_t\}_{t=t_0}^{\infty}$ converges in distribution to a random variable $\ln \hat{K}$,

$$\ln \hat{K}_t \xrightarrow{\mathcal{D}} \ln \hat{K} \quad \text{where} \quad -\infty < \ln \hat{K}_t < \ln \hat{K}^*. \qquad (54)$$

Intuitively, cyclical variables $\ln \hat{K}_t$ and $\ln \hat{K}_{t-\Delta}$ share the same asymptotic mean, which is the mean of the limiting distribution, $E(\ln \hat{K})$. Therefore, defining

$$\begin{aligned}
E(\Delta y_t) \equiv \lim_{t\to\infty} E_0(\Delta y_t) &= \lim_{t\to\infty} E_0\left(\ln \hat{K}_t - \ln \hat{K}_{t-\Delta}\right)\alpha + \lim_{t\to\infty} E_0(q_t - q_{t-\Delta})\ln A \\
&= E_0(q_\Delta)\ln A \;=\; \lambda\Delta \ln A
\end{aligned}$$

for any $t_0 > 0$, is the asymptotic mean of output growth rates. Economically, it employs an large sequence of growth rates of length Δ.

Lemma 2 *Given capital rewards as in (34), then*

$$\lim_{t\to\infty} Cov\left(\int_{t-\Delta}^{t}\left(\frac{1-\tau_i}{1+\tau_k}r_s/\alpha\right)ds, q_t - q_{t-\Delta}\right) = 0$$

is asymptotically uncorrelated.

Proof. Observe that from $Cov(aX + bY, Z) = aCov(X, Z) + bCov(Y, Z)$ we have

$$\begin{aligned}
Cov\left(\ln \hat{K}_t - \ln \hat{K}_{t-\Delta}, q_t - q_{t-\Delta}\right) &= Cov\left(\int_{t-\Delta}^{t}\left(\frac{1-\tau_i}{1+\tau_k}r_s/\alpha\right)ds, q_t - q_{t-\Delta}\right) \\
&\quad + \ln(1-\theta_2)\, Cov\left(q_t - q_{t-\Delta}, q_t - q_{t-\Delta}\right)
\end{aligned}$$

Employing Lemma 1 and the property $Var(q_\Delta) = \lambda\Delta$ gives the asymptotic result. ■

We now compute the second-order moments of growth rates (38). Observe that using growth rates of cyclical capital stock in (32) and Lemma 2,

$$\begin{aligned} Var(\hat{y}_t) &= \lim_{t\to\infty} Var_0\left(\frac{1-\tau_i}{1+\tau_k}\int_{t-\Delta}^{t} r_s ds\right) + \alpha^2(\ln(1-\theta_2))^2 \lim_{t\to\infty} Var_0(q_\Delta) \\ &\quad +\alpha^2 \ln(1-\theta_2) \lim_{t\to\infty} Cov_0\left(\frac{1-\tau_i}{1+\tau_k}\int_{t-\Delta}^{t} r_s/\alpha ds, q_t - q_{t-\Delta}\right). \end{aligned}$$

Using Lemma 2 we obtain the measure in (33). Similarly, using output growth rates in (38) together with Lemma 1, we obtain our measure in (39).

References

Acemoglu, D., S. Johnson, J. Robinson, and Y. Thaicharoen, "Institutional causes, macroeconomic symptoms: Volatility, crises and growth," *J. Monet. Econ.*, 2003, *50*, 49–123.

Aghion, P. and P. Howitt, "A model of growth through creative destruction," *Econometrica*, 1992, *60* (2), 323–351.

Aghion, Philippe, George-Marios Angeletos, Abhijit Banerjee, and Kalina Manova, "Volatility and Growth: Credit Constraints and Productivity-Enhancing Investment," *NBER*, 2005, *11349.*

Beaumont, Paul M., Stefan C. Norrbin, and F. Pinar Yigit, "Time series evidence on the linkage between volatility and growth of output," *Appl. Econ. Letters*, 2008, *15*, 45–48.

Bental, B. and D. Peled, "The accumulation of wealth and the cyclical generation of new technologies: A search theoretic approach," *Int. Econ. Rev.*, 1996, *37* (3), 687–718.

Boldrin, Michele and David K. Levine, "Factor Saving Innovation," *J. Econ. Theory*, 2002, *105*, 18–41.

— and David Levine, "Growth Cycles and Market Crashes," *J. Econ. Theory*, 2001, *96*, 13–39.

Boucekkine, Raouf and Ramon Ruiz Tamarit, "Imbalance effects in the Lucas model: an analytical exploration," *Topics in Macroecon.*, 2004, *4* (1), 1–17.

— , Omar Licandro, Luis A. Puch, and Fernando del Rio, "Vintage capital and the dynamics of the AK model," *J. Econ. Theory*, 2005, *120*, 39–72.

Casella, George and Roger L. Berger, *Statistical Inference*, Duxbury, 2001.

Chang, Fwu-Ranq, "The Inverse Optimal Problem: A Dynamic Programming Approach," *Econometrica*, 1988, *56* (1), 147–172.

Dawson, John W. and E. Frank Stephenson, "The link between volatility and growth: Evidence from the States," *Econ. Letters*, 1997, *55*, 365–369.

Feichtinger, Gustav, Richard F. Hartl, Peter M. Kort, and Vladimir M. Veliov, "Anticipation effects of technological progress on capital accumulation: a vintage capital approach," *J. Econ. Theory*, 2006, *126*, 143–164.

Francois, Patrick and Huw Lloyd-Ellis, "Animal spirits trough creative destruction," *Amer. Econ. Rev.*, 2003, *93* (3), 530–550.

_ and _ , "Implementation Cycles, Investment, and Growth," *Int. Econ. Rev.*, 2008, *49* (3), 901–942.

García, M. A. and R. J. Griego, "An elementary theory of stochastic differential equations driven by a poisson process," *Commun. Statist.-Stochastic Models*, 1994, *10*, 335–363.

Grossman, Gene M. and Elhanan Helpman, *Innovation and Growth in the Global Economy*, Cambridge, Massachusetts: The MIT Press, 1991.

Gruber, Jonathan, "A tax-based estimate of the elasticity of intertemporal substitution," *NBER*, 2006, *w11945.*

Hellwig, Martin and Andreas Irmen, "Endogenous Technical Change in a Competitive Economy," *J. Econ. Theory*, 2001, *101*, 1–39.

Imbs, Jean, "Growth and volatility," *J. Monet. Econ.*, 2007, *54*, 1848–1862.

Martin, Philippe and Carol Ann Rogers, "Stabilization Policy, Learning-By-Doing, and Economic Growth," *Oxford Econ. Pap.*, 1997, *49*, 152–166.

Matsuyama, Kiminori, "Growing through cycles," *Econometrica*, 1999, *67* (2), 335–347.

Merton, Robert C., "An asymptotic theory of growth under uncertainty," *Rev. Econ. Stud.*, 1975, *42* (3), 375–393.

Posch, Olaf, "Explaining volatility: The case of taxation," *Aarhus University, CREATES Research Paper*, 2008, *2008-4, http://www.poschs.com/data/Posch2009Jul.pdf.*

_ , "Structural estimation of jump-diffusion processes in macroeconomics," *J. Econometrics*, 2009, *forthcoming.*

Ramey, Gary and Valerie A. Ramey, "Cross-country evidence on the link between volatility and growth," *Amer. Econ. Rev.*, 1995, *85* (5), 1138–1151.

Rao, C. Radhakrishna, *Linear Statistical Inference and its Applications*, Wiley, 1973.

Romer, Paul M., "Endogenous technological change," *J. Polit. Economy*, 1990, *98*, 71–102.

Segerstrom, Paul S., "Endogenous Growth without Scale Effects," *Amer. Econ. Rev.*, 1998, *88* (5), 1290–1310.

_ , T.C.A. Anant, and Elias Dinopoulos, "A Schumpeterian Model of the Product Life Cycle," *Amer. Econ. Rev.*, 1990, *80* (5), 1077–1091.

Sennewald, Ken, "Controlled stochastic differential equations under Poisson uncertainty and with unbounded utility," *J. Econ. Dynam. Control*, 2007, *31* (4), 1106–1131.

_ and Klaus Wälde, "Itô's Lemma and the Bellman equation for Poisson processes: An applied view," *Journal of Economics*, 2006, *89* (1), 1–36.

Smith, William T., "Inspecting the Mechanism Exactly: A Closed-form solution to a stochastic growth model," *B.E. Journal of Macroeconomics*, 2007, *7* (30), 1–31.

Vissing-Jørgensen, Annette, "Limited asset market participation and the elasticity of intertemporal substitution," *J. Polit. Economy*, 2002, *110* (4), 825–853.

Wälde, Klaus, "A model of creative destruction with undiversifiable risk and optimising households," *Econ. J.*, 1999, *109*, 156–171.

_ , "The economic determinants of technology shocks in a real business cycle model," *J. Econ. Dynam. Control*, 2002, *27*, 1–28.

_ , "Endogenous growth cycles," *Int. Econ. Rev.*, 2005, *46* (3), 867–894.

Xie, Danyang, "Increasing returns and increasing rates of growth," *J. Polit. Economy*, 1991, *99* (2), 429–435.

_ , "Divergence in Economic Performance: Transitional Dynamics with Multiple Equilibria," *J. Econ. Theory*, 1994, *63*, 97–112.

Referees' appendix to

On the non-causal link between volatility and growth

The appendix follows the same structure as the paper (not for publication).

A The model

A.1 The household's budget constraint

Wealth is measured in units of the consumption good, priced at consumer prices. Nominal wealth is given by $(1+\tau_c)p_t^C a_t = \sum_{j=0}^{q+1} k_j v_j$, where k_j is the individual's physical capital stock and v_t is the value of one unit of the capital stock both of vintage j, respectively, and p_t^C is the producer price of the consumption good. Then, real wealth, a_t, is

$$a_t \equiv \frac{1}{1+\tau_c} \sum_{j=0}^{q+1} \frac{k_j v_j}{p_t^C}, \tag{A.1}$$

where q denotes the most advanced vintage. Despite the fact that households cannot own any capital of vintage $q+1$, $k_{q+1}=0$, the sum applies also to the next vintage to discover. The reason will become clear in a moment.

Households receive net capital payments $(1-\tau_i)\sum_{j=0}^{q+1} p_t^Y w_j^K k_j$, i.e., net dividends per unit of vintage capital j (marginal products) times the amount k_j of all vintages net labor income $(1-\tau_i)p_t^Y w_t$ used for saving and consumption purposes. Nominal savings are

$$s_t = (1-\tau_i)\left(\sum_{j=0}^{q+1} p_t^Y w_j^K k_j + p_t^Y w_t\right) - (1+\tau_c)\, p_t^C c_t.$$

Saving is used for financing research, $(1+\tau_r)p_t^R i_t$, and for accumulating existing capital goods. Households trade capital goods of the most recent vintage q only, the allocation of older capital goods is fixed (households are indifferent about trading in equilibrium). Beside the tax on wealth, τ_a, a fraction $\frac{1-\tau_i}{1+\tau_k}\delta$ of the capital stock depreciates, which implies that only net (and not gross) capital rewards are taxed. Capital held by households follows for *older vintages*

$$dk_j = -\left\{\frac{1-\tau_i}{1+\tau_k}\delta + \tau_a\right\} k_j dt, \quad j<q. \tag{A.2}$$

The relationship in (A.2) shows that a positive tax on wealth, $\tau_a > 0$, simply increases the rate of effective depreciation. We show below that this tax really applies to wealth a_t and not the number of machines or stocks k_j. A no-arbitrage condition demands that, as long as investment is positive, the value of an installed good equals the price of a new investment good, so that $v_q = (1+\tau_k)p_t^K$. Thus, for the *most recent* vintage, $j=q$, we obtain

$$dk_q = \left\{\frac{s_t - (1+\tau_r)p_t^R i_t}{(1+\tau_k)p_t^K} - \left(\frac{1-\tau_i}{1+\tau_k}\delta + \tau_a\right) k_q\right\} dt. \tag{A.3}$$

The household's capital stock k_q of the most advanced vintage q in (A.3) increases when the difference between net income and spending for consumption plus risky investment divided by the effective price v_q of an installed unit of capital, exceeds effective depreciation.

In case of successful research, i.e., $dq_t = 1$, the household obtains the share i_t/R_t of total payoffs κ_t. Therefore, for the *next vintage*, $j = q + 1$, we obtain

$$dk_{q+1} = i_t/R_t \kappa_t dq_t. \tag{A.4}$$

In words, the individual payoff depends on individual investment i_t relative to total investment R_t into the successful project. After that, (A.3) applies to vintage $q + 1$.

As different vintages are perfect substitutes in production, from (7) prices are linked by

$$(1 + \tau_k)p_t^K = v_q = B^{q-j} v_j, \quad j \leq q. \tag{A.5}$$

Immediately after the innovation, the price of older vintages relative to the producer price of the consumption good using $p_t^K = p_t^C$ from (5) falls by $v_j/p_t^C = (1 + \tau_k)B^{-(q-j)}$, $j \leq q$.

Assuming that vintage prices generally evolve as

$$d(v_j/p_t^C) = a_j v_j/p_t^C dt + c_j v_j/p_{t-}^C dq_t, \tag{A.6}$$

the deterministic part must be zero, i.e., $a_j = 0$ for all vintages $j \leq q$. When research is successful, the price of a unit of given vintage j in terms of the consumption good drops by $v_j/p_t^C = (1 + \tau_k)B^{-(q+1-j)}$ from (A.5). Thus, using (A.6) and $s \equiv (B - 1)/B$ we obtain

$$c_j = \frac{d(v_j/p_t^C)}{v_j/p_{t-}^C} = \frac{v_j/p_t^C - v_j/p_{t-}^C}{v_j/p_{t-}^C} = \frac{(1 + \tau_k)B^{-(q+1-j)} - (1 + \tau_k)B^{-(q-j)}}{(1 + \tau_k)B^{-(q-j)}} = -s < 0,$$

which is identical for all vintages $j \leq q$, and prices therefore evolve according to

$$d(v_j/p_t^C) = -s v_j/p_{t-}^C dq_t, \quad j \leq q. \tag{A.7}$$

This equation reflects the devaluation (henceforth economic depreciation) of older vintages relative to the consumption good when a new vintage has been developed.

The budget constraint can now be derived by computing the differential

$$da = \frac{1}{1 + \tau_c} \sum_{j=0}^{q+1} d(k_j v_j/p_t^C). \tag{A.8}$$

Note that is implicitly assumed that tax rates are considered to be constant. For *older* vintages $0 < j < q$, we obtain using (A.2), (A.7) and applying Itô's formula

$$\begin{aligned} d(k_j v_j/p_t^C) &= -\frac{v_j}{p_t^C}\left\{\frac{1-\tau_i}{1+\tau_k}\delta + \tau_a\right\} k_j dt + \left\{\left(\frac{v_j}{p_{t-}^C} - s\frac{v_j}{p_{t-}^C}\right) k_j - \frac{v_j}{p_{t-}^C} k_j\right\} dq_t \\ &= -\left\{\frac{1-\tau_i}{1+\tau_k}\delta + \tau_a\right\} k_j v_j/p_t^C dt - s k_j v_j/p_{t-}^C dq_t. \end{aligned} \tag{A.9}$$

Similarly, for the *most advanced* vintage, q, we use (A.3), (A.7) and apply Itô's formula,

$$d(k_q v_q/p_t^C) = \frac{v_q}{p_t^C}\left\{\frac{s_t-(1+\tau_r)p_t^R i_t}{(1+\tau_k)p_t^K} - \left(\frac{1-\tau_i}{1+\tau_k}\delta+\tau_a\right)k_q\right\}dt - s\frac{v_q}{p_{t-}^C}k_q dq_t$$
$$= \left\{\frac{s_t}{p_t^C} - (1+\tau_r)i_t - \left(\frac{1-\tau_i}{1+\tau_k}\delta+\tau_a\right)k_q v_q/p_t^C\right\}dt - sk_q v_q/p_{t-}^C dq_t, \tag{A.10}$$

where we used (5) in the last step. Finally for the *next* vintage $q+1$ to come, using equation (A.4) with $v_{q+1}/p_t^C = (1+\tau_k)$ for the prototype after successful research, yields

$$d(k_{q+1}v_{q+1}/p_t^C) = v_{q+1}/p_t^C i_t/R_t\kappa_t dq_t = (1+\tau_k)i_t/R_t\kappa_t dq_t. \tag{A.11}$$

This is in accordance to the definition of *real* wealth in (A.1) since it denotes the consumption goods that can be exchanged for the prototype. Hence, as we want to include the evolution of κ_t, assets a_t need to equal the sum over all vintages including the not yet existing one.

Summarizing (A.9) to (A.11), the budget constraint (A.8) becomes

$$da_t = \frac{1}{1+\tau_c}\sum_{j=0}^{q-1}\left(-\left\{\frac{1-\tau_i}{1+\tau_k}\delta+\tau_a\right\}k_j v_j/p_t^C dt - sk_j v_j/p_{t-}^C dq_t\right) + \frac{1+\tau_k}{1+\tau_c}i_t/R_t\kappa_t dq_t$$
$$+\frac{1}{1+\tau_c}\left\{s_t/p_t^C-(1+\tau_r)i_t-\left(\frac{1-\tau_i}{1+\tau_k}\delta+\tau_a\right)k_q v_q/p_t^C\right\}dt - \frac{1}{1+\tau_c}sk_q v_q/p_{t-}^C dq_t.$$

Using the definition of *real* wealth (A.1) and inserting s_t yields

$$da_t = \left\{\frac{1-\tau_i}{1+\tau_c}\left(\sum_{j=0}^{q+1}w_j^K k_j + w_t\right) - c_t - \frac{1+\tau_r}{1+\tau_c}i_t\right\}dt - a_t\left\{\frac{1-\tau_i}{1+\tau_k}\delta+\tau_a\right\}dt$$
$$+\left\{\frac{1+\tau_k}{1+\tau_c}i_t/R_t\kappa_t - sa_{t-}\right\}dq_t. \tag{A.12}$$

With value marginal product, $w_j^K = Y_K B^j$ from equation (7) and (6), as well as inserting $B^j = B^q v_j/v_q$ from (A.5) with $v_q = (1+\tau_k)p_K$ we obtain

$$da_t = \left\{\frac{1-\tau_i}{1+\tau_c}\left(\sum_{j=0}^{q+1}B^q Y_K\frac{v_j k_j}{(1+\tau_k)p_t^K} + w_t\right) - c_t - \frac{1+\tau_r}{1+\tau_c}i_t\right\}dt - a_t\left\{\frac{1-\tau_i}{1+\tau_k}\delta+\tau_a\right\}dt$$
$$+\left\{\frac{1+\tau_k}{1+\tau_c}i_t/R_t\kappa_t - sa_{t-}\right\}dq_t.$$

Finally, using the definition of *real* wealth (A.1) and the definition of r_t in (13) gives

$$da_t = \left\{\left(\frac{1-\tau_i}{1+\tau_k}(r_t-\delta)-\tau_a\right)a_t + \frac{1-\tau_i}{1+\tau_c}w_t - c_t - \frac{1+\tau_r}{1+\tau_c}i_t\right\}dt + \left\{\frac{1+\tau_k}{1+\tau_c}i_t/R_t\kappa_t - sa_{t-}\right\}dq_t.$$

A.2 Total wealth and government expenditure

We show that aggregating the budget constraints of households yields the aggregate resource constraint. It also gives us the government's budget constraint.

Starting by summing up the budget constraint (12) of the representative consumer using $\sum_{i=1}^{L} a_t = La_t$, we obtain

$$\begin{aligned} dLa_t &= \left\{ \left(\frac{1-\tau_i}{1+\tau_k}(r_t-\delta) - \tau_a \right) La_t + \frac{1-\tau_i}{1+\tau_c} w_t L - \frac{1+\tau_r}{1+\tau_c} R_t - C_t \right\} dt \\ &\quad + \left\{ \frac{1+\tau_k}{1+\tau_c} \kappa_t - sLa_{t-} \right\} dq_t, \end{aligned}$$

where C_t and R_t denote $C_t = Lc_t$ and $R_t = Li_t$, respectively. Using the definition of wealth in (A.1) together with (A.5), we obtain total wealth as

$$La_t = \frac{1}{1+\tau_c} \sum_{j=0}^{q+1} \frac{k_j v_j}{p_t^C} L = \frac{1+\tau_k}{1+\tau_c} B^{-q} \sum_{j=0}^{q+1} K_j B^j \;=\; \frac{1+\tau_k}{1+\tau_c} B^{-q} K_t \equiv K_t^{obs}, \tag{A.13}$$

where we used the definition of the capital index from (7) in the last step. Inserting yields

$$\begin{aligned} d\left(\frac{1+\tau_k}{1+\tau_c} B^{-q} K_t \right) &= \left\{ \left(\frac{1-\tau_i}{1+\tau_k}(r_t-\delta) - \tau_a \right) \frac{1+\tau_k}{1+\tau_c} B^{-q} K_t + \frac{1-\tau_i}{1+\tau_c} w_t L - \frac{1+\tau_r}{1+\tau_c} R_t - C_t \right\} dt \\ &\quad + \left\{ \frac{1+\tau_k}{1+\tau_c} \kappa_t - s \frac{1+\tau_k}{1+\tau_c} B^{-q} K_{t-} \right\} dq_t \\ \Leftrightarrow \; d\left(B^{-q} K_t \right) &= \left\{ \left(\frac{1-\tau_i}{1+\tau_k}(r_t-\delta) - \tau_a \right) B^{-q} K_t + \frac{1-\tau_i}{1+\tau_k} w_t L - \frac{1+\tau_r}{1+\tau_k} R_t - \frac{1+\tau_c}{1+\tau_k} C_t \right\} dt \\ &\quad + \left\{ \kappa_t - sB^{-q} K_{t-} \right\} dq_t. \end{aligned}$$

Using Itô's formula we obtain

$$\begin{aligned} dK_t &= \left\{ \left(\frac{1-\tau_i}{1+\tau_k}(r_t-\delta) - \tau_a \right) B^{-q} K_t + \frac{1-\tau_i}{1+\tau_k} w_t L - \frac{1+\tau_r}{1+\tau_k} R_t - \frac{1+\tau_c}{1+\tau_k} C_t \right\} B^q dt \\ &\quad + \left\{ \left(B^{-q} K_{t-} + \kappa_t - sB^{-q} K_{t-} \right) B^{q+1} - \left(B^{-q} K_{t-} \right) B^q \right\} dq_t \\ &= \left\{ \left(\frac{1-\tau_i}{1+\tau_k}(r_t-\delta) - \tau_a \right) B^{-q} K_t + \frac{1-\tau_i}{1+\tau_k} w_t L - \frac{1+\tau_r}{1+\tau_k} R_t - \frac{1+\tau_c}{1+\tau_k} C_t \right\} B^q dt \\ &\quad + \left\{ (1-s) BK_{t-} + B^{q+1} \kappa_t - K_{t-} \right\} dq_t. \end{aligned}$$

Inserting factor rewards from (13) and $s = (B-1)/B$ yields

$$\begin{aligned} dK_t &= B^q \left\{ \left(\frac{1-\tau_i}{1+\tau_k} \left(B^q Y_K - \delta \right) - \tau_a \right) B^{-q} K_t + \frac{1-\tau_i}{1+\tau_k} Y_L L \right. \\ &\quad \left. - \frac{1+\tau_r}{1+\tau_k} R_t - \frac{1+\tau_c}{1+\tau_k} C_t \right\} dt + B^{q+1} \kappa_t dq_t. \end{aligned} \tag{A.14}$$

Now observe that

$$\frac{1-\tau_i}{1+\tau_k} Y_K K_t + \frac{1-\tau_i}{1+\tau_k} Y_L L = Y_K K_t + Y_L L - \frac{\tau_k+\tau_i}{1+\tau_k} \left(Y_K K_t + Y_L L \right) = Y_t - \frac{\tau_k+\tau_i}{1+\tau_k} Y_t,$$

where we used Euler's theorem, $Y_t = Y_K K_t + Y_L L$. Moreover, observe that

$$\begin{aligned} & -\frac{1-\tau_i}{1+\tau_k} \delta B^{-q} K_t - \frac{1+\tau_r}{1+\tau_k} R_t - \frac{1+\tau_c}{1+\tau_k} C_t \\ =\; & -\delta B^{-q} K_t - R_t - C_t + \frac{\tau_i+\tau_k}{1+\tau_k} \delta B^{-q} K_t - \frac{\tau_r-\tau_k}{1+\tau_k} R_t - \frac{\tau_c-\tau_k}{1+\tau_k} C_t. \end{aligned}$$

Therefore, the term in brackets from (A.14) can be simplified to,

$$\begin{aligned}
&\left(\frac{1-\tau_i}{1+\tau_k}\left(B^q Y_K-\delta\right)-\tau_a\right)B^{-q}K_t+\frac{1-\tau_i}{1+\tau_k}Y_L L-\frac{1+\tau_r}{1+\tau_k}R_t-\frac{1+\tau_c}{1+\tau_k}C_t\\
=\ & Y_t-\frac{\tau_i+\tau_k}{1+\tau_k}Y_t-(\tau_a+\delta)B^{-q}K_t-R_t-C_t+\frac{\tau_i+\tau_k}{1+\tau_k}\delta B^{-q}K_t-\frac{\tau_r-\tau_k}{1+\tau_k}R_t-\frac{\tau_c-\tau_k}{1+\tau_k}C_t\\
\equiv\ & Y_t-\delta B^{-q}K_t-R_t-C_t-G_t. \qquad \text{(A.15)}
\end{aligned}$$

The last equality defines total tax revenues and thus government expenditures,

$$G_t\equiv\frac{\tau_i+\tau_k}{1+\tau_k}Y_t+\left(\tau_a-\frac{\tau_i+\tau_k}{1+\tau_k}\delta\right)B^{-q}K_t+\frac{\tau_r-\tau_k}{1+\tau_k}R_t+\frac{\tau_c-\tau_k}{1+\tau_k}C_t. \qquad \text{(A.16)}$$

Inserting (A.15) into (A.14) gives

$$\begin{aligned}
dK_t &= \left\{Y_t-\delta B^{-q}K_t-R_t-C_t-G_t\right\}B^q dt+B^{q+1}\kappa_t dq_t\\
&= \left\{B^q I_t-\delta K_t\right\}dt+B^{q+1}\kappa_t dq_t,
\end{aligned}$$

which is (8). Aggregation works for G_t as defined in (A.16). Equation (A.16) is convincing, if it can be rewritten in a meaningful way. Starting at

$$\begin{aligned}
(1+\tau_k)\,G_t &\equiv \tau_k Y_t+\tau_i Y_t+(1+\tau_k)\,\tau_a B^{-q}K_t-(\tau_i+\tau_k)\,\delta B^{-q}K_t\\
&\quad+(\tau_r-\tau_k)\,R_t+(\tau_c-\tau_k)\,C_t.
\end{aligned}$$

As $\tau_k G_t=\tau_k\left(Y_t-I_t-R_t-C_t\right)$ from (4), we obtain

$$\begin{aligned}
G_t &= \tau_k I_t+\tau_i Y_t+(1+\tau_k)\,\tau_a B^{-q}K_t-(\tau_i+\tau_k)\,\delta B^{-q}K_t+\tau_r R_t+\tau_c C_t\\
&= \tau_i(Y_t-\delta B^{-q}K_t)+\tau_k(I_t-\delta B^{-q}K_t)+\tau_r R_t+\tau_c C_t+\tau_a\left(1+\tau_k\right)B^{-q}K_t,
\end{aligned}$$

i.e., government revenue in units of the consumption good before taxation.

B Equilibrium dynamics

B.1 The maximized Bellman equation

The value of an optimal program of (10) is defined by

$$V(a_0,q_0)=\max_{\{c_t,i_t\}_{t=0}^{\infty}}\{U_0\},$$

which denotes the present discounted value of utility evaluated along the optimal program. Itô's formula (or change of variables, cf. Sennewald, 2007) yields

$$\begin{aligned}
dV_t &= V_a\left\{\left(\frac{1-\tau_i}{1+\tau_k}(r_t-\delta)-\tau_a\right)a_t+\frac{1-\tau_i}{1+\tau_c}w_t-c_t-\frac{1+\tau_r}{1+\tau_c}i_t\right\}dt\\
&\quad+\left\{V(a_t,q_t)-V(a_{t-},q_{t-})\right\}dq_t.
\end{aligned}$$

With $E_0(dq_t) = \lambda_t dt$, the Bellman equation reads

$$\rho V(a_0,q_0) = \max_{\{c_0,i_0\}} \left\{ u(c_0) + V_a \left\{ \left(\frac{1-\tau_i}{1+\tau_k}(r_0-\delta) - \tau_a \right) a_0 + \frac{1-\tau_i}{1+\tau_c} w_0 - c_0 - \frac{1+\tau_r}{1+\tau_c} i_0 \right\} \right.$$
$$\left. + \lambda_t \left\{ V(a_0,q_0) - V(a_{0-},q_{0-}) \right\} \right\}, \tag{A.17}$$

where the level of q_t and a_t immediately after a jump is $q_t = q_{t-} + 1$ and

$$a_t = (1-s)a_{t-} + \frac{1+\tau_k}{1+\tau_c} \frac{i_t}{R_t} \kappa_t. \tag{A.18}$$

The first-order condition for consumption reads

$$u'(c_0) = V_a(a_0,q_0), \tag{A.19}$$

whereas the first-order condition for the risky investment is

$$\frac{1+\tau_r}{1+\tau_c} V_{a_{0-}}(a_{0-},q_{0-}) = \lambda_t V_{a_0}(a_0,q_0) \frac{1+\tau_k}{1+\tau_c} \frac{\kappa_0}{R_0}. \tag{A.20}$$

Inserting (A.19) twice and rearranging gives

$$u'(c_{0-}) = \lambda_t \frac{1+\tau_k}{1+\tau_r} \frac{\kappa_0}{R_0} u'(c_0). \tag{A.21}$$

Economically, research expenditures, i_t, are chosen such that the ratio of marginal utilities from consumption an instant before and after a jump, $u'(c_{t-})/u'(c_t)$ equals to the ratio of expected payoffs of the risky research, $\lambda_t(1+\tau_k)\kappa_t i_t/R_t$, divided by its cost, $(1+\tau_r)i_t$.

The first-order conditions (A.19) and (A.20) make optimal controls a function of the state variables, $c_t = c(a_t,q_t)$ and $i_t = i(a_t,q_t)$. Hence, the maximized Bellman equation is

$$\rho V(a_t,q_t) = V_a \left\{ \left(\frac{1-\tau_i}{1+\tau_k}(r_t-\delta) - \tau_a \right) a_t + \frac{1-\tau_i}{1+\tau_c} w_t - c(a_t) - \frac{1+\tau_r}{1+\tau_c} i(a_t) \right\}$$
$$+ u(c(a_t)) + \lambda_t \left\{ V(a_t,q_t) - V(a_{t-},q_{t-}) \right\}. \tag{A.22}$$

B.2 The rule of optimal consumption

We start with the maximized Bellman equation in (A.22). Using the envelope theorem, the derivative with respect to the state variable reads[15]

$$\rho V_a = V_{aa} \left\{ \left(\frac{1-\tau_i}{1+\tau_k}(r_t-\delta) - \tau_a \right) a_t + \frac{1-\tau_i}{1+\tau_c} w_t - c(a_t) - \frac{1+\tau_r}{1+\tau_c} i(a_t) \right\}$$
$$+ \left(\frac{1-\tau_i}{1+\tau_k}(r_t-\delta) - \tau_a \right) V_a + \lambda_t \left\{ V_{a_{t-}}(a_t,q_t) - V_{a_{t-}}(a_{t-},q_{t-}) \right\}. \tag{A.23}$$

[15]The notion of V_a and $V_{a_{t-}}$ are equivalent, the term $V_{a_{t-}}$ is used if necessary for clarity.

Now observe that from (A.18), $V_{a_{t-}}(a_t, q_t) = V_{a_t}(1-s)$. Rearranging gives

$$\begin{aligned} &\left(\rho - \frac{1-\tau_i}{1+\tau_k}(r_t - \delta) + \tau_a + \lambda_t\right) V_a - (1-s)\,\lambda_t V_{a_t} = \\ &V_{aa}\left\{\left(\frac{1-\tau_i}{1+\tau_k}(r_t-\delta) - \tau_a\right) a_t + \frac{1-\tau_i}{1+\tau_c} w_t - c_t - \frac{1+\tau_r}{1+\tau_c} i_t\right\}. \end{aligned} \tag{A.24}$$

Computing the differential of the costate variable, using the budget constraint (12) and Itô's formula (change of variables) gives

$$\begin{aligned} dV_a = \; & V_{aa}\left\{\left(\frac{1-\tau_i}{1+\tau_k}(r_t-\delta) - \tau_a\right) a_t + \frac{1-\tau_i}{1+\tau_c} w_t - c_t - \frac{1+\tau_r}{1+\tau_c} i_t\right\} dt \\ & + \left\{V_{a_t}(a_t, q_t) - V_{a_{t-}}(a_{t-}, q_{t-})\right\} dq_t. \end{aligned}$$

Inserting equation (A.24) yields the evolution of the costate variable as

$$dV_a = \left(\rho - \frac{1-\tau_i}{1+\tau_k}(r_t - \delta) + \tau_a + \lambda_t\right) V_a - (1-s)\,\lambda_t V_{a_t} dt + \left\{V_{a_t}(a_t, q_t) - V_{a_{t-}}(a_{t-}, q_{t-})\right\} dq_t.$$

Dividing by the costate an instant before the jump, V_a, we obtain

$$\frac{dV_a}{V_a} = \left\{\rho - \frac{1-\tau_i}{1+\tau_k}(r_t - \delta) + \tau_a + \lambda_t - (1-s)\,\lambda_t \frac{V_{a_t}(a_t, q_t)}{V_{a_{t-}}(a_{t-}, q_{t-})}\right\} dt + \left\{\frac{V_{a_t}(a_t, q_t)}{V_{a_{t-}}(a_{t-}, q_{t-})} - 1\right\} dq_t.$$

The first-order condition for consumption (A.19) yields $dV_a = du'(c_t)$. Inserting this term as well as using the first order condition (A.19) gives the optimal rule as

$$\frac{du'(c_{t-})}{u'(c_{t-})} = \left(\rho - \frac{1-\tau_i}{1+\tau_k}(r_t - \delta) + \tau_a + \lambda_t - (1-s)\,\lambda_t \frac{u'(c_t)}{u'(c_{t-})}\right) dt + \left(\frac{u'(c_t)}{u'(c_{t-})} - 1\right) dq_t. \tag{A.25}$$

Note that the ratio of marginal utilities before and after a jump is determined by the first-order condition of risky investment (A.21),

$$\frac{u'(c_t)}{u'(c_{t-})} = \frac{R_t}{\kappa_t}\frac{1+\tau_r}{1+\tau_k}\lambda_t^{-1} = \frac{1+\tau_r}{1+\tau_k}\lambda_t^{\frac{\gamma}{1-\gamma}} D/\kappa_0 = \xi^{-\sigma}, \tag{A.26}$$

where we inserted $R_t = \lambda_t^{\frac{1}{1-\gamma}} D_t$ from (1), and used the assumptions of (14) and (15).

Using (11) and Itô's formula, we may write the rule for optimal consumption (A.25) as

$$\begin{aligned} dC_t = & -\frac{1}{\sigma}\left(\rho - \frac{1-\tau_i}{1+\tau_k}(r_t - \delta) + \tau_a + \lambda_t - (1-s)\,\lambda_t \xi^{-\sigma}\right) C_t^{-\sigma}\left(C_t^{-\sigma}\right)^{-\frac{1}{\sigma}-1} dt \\ & + \left(\left(C_{t-}^{-\sigma} + \left\{\frac{u'(c_t)}{u'(c_{t-})} - 1\right\} C_{t-}^{-\sigma}\right)^{-\frac{1}{\sigma}} - C_{t-}\right) dq_t \\ = & -\frac{1}{\sigma}\left(\rho - \frac{1-\tau_i}{1+\tau_k}(r_t - \delta) + \tau_a + \lambda_t - (1-s)\,\lambda_t \xi^{-\sigma}\right) C_t dt + \left(C_t - C_{t-}\right) dq_t. \end{aligned}$$

In terms of cyclical consumption, we obtain the Euler equation

$$d\hat{C}_t = -\frac{1}{\sigma}\left(\rho - \frac{1-\tau_i}{1+\tau_k}(r_t - \delta) + \tau_a + \lambda_t - (1-s)\,\lambda_t \xi^{-\sigma}\right) \hat{C}_t dt + \left(\hat{C}_t - \hat{C}_{t-}\right) dq_t. \tag{A.27}$$

B.3 Proof of Theorem 1

The idea of this proof is to first assume that $\hat{C}_t = \Psi\hat{K}_t$ and then derive conditions under which this holds. We first derive the evolution of the cyclical capital index, and split up the proof by sequentially considering the jump and the deterministic component of the SDE.

Given the capital accumulation constraint in (8) with the market clearing condition (4) and using Itô's formula (change of variables) for $\hat{K}_t$ defined in (16), we obtain

$$\begin{aligned} d\hat{K}_t &= ((Y_t - C_t - R_t - G_t)B^q - \delta K_t)\, A^{-\frac{q}{\alpha}} dt + \left(\left(K_{t-} + B^{q+1}\kappa_t\right) A^{-\frac{q+1}{\alpha}} - A^{-\frac{q}{\alpha}} K_{t-}\right) dq_t \\ &= \left(\hat{Y}_t - \hat{C}_t - \hat{R}_t - \hat{G}_t - \delta\hat{K}_t\right) dt + \left(A^{-\frac{1}{\alpha}} + A^{-1}\kappa - 1\right) \hat{K}_{t-} dq_t, \end{aligned}$$

where we inserted (15) and used the definition of cyclical variables from (16), $\hat{X}_t = A^{-q}X_t$ for $X_t \in (Y_t, C_t, R_t, G_t)$. With the definition of the parameter ξ in Theorem 1, we obtain

$$d\hat{K}_t = \left(\hat{Y}_t - \hat{C}_t - \hat{R}_t - \hat{G}_t - \delta\hat{K}_t\right) dt + \left(A^{-1}\xi - 1\right) \hat{K}_{t-} dq_t. \tag{A.28}$$

Considering the jump term of the SDE in (A.28), an instant after successful research, the level of cyclical capital is $\hat{K}_t = A^{-1}\xi\hat{K}_{t-}$. This implies that in the linear case of $\hat{C}_t = \Psi\hat{K}_t$, consumption jumps by $\hat{C}_t/\hat{C}_{t-} = A^{-1}\xi$ as well. ¿From (A.26), we obtain another expression for the jump of cyclical consumption using $u'(c_t) = c_t^{-\sigma}$ from (11) and $C_t = Lc_t$,

$$C_t/C_{t-} = A\hat{C}_t/\hat{C}_{t-} = \left(\frac{1+\tau_r}{1+\tau_k}\lambda_t^{\frac{\gamma}{1-\gamma}} D/\kappa\right)^{-\frac{1}{\sigma}}.$$

Thus, we obtain a constant arrival rate by solving for $\lambda = \lambda_t$ as

$$\xi = \left(\frac{1+\tau_r}{1+\tau_k}\lambda^{\frac{\gamma}{1-\gamma}} D/\kappa\right)^{-\frac{1}{\sigma}} \;\Leftrightarrow\; \lambda = \left(\frac{1+\tau_k}{1+\tau_r}\frac{\kappa}{D}\xi^{-\sigma}\right)^{\frac{1-\gamma}{\gamma}},$$

which is (21). Using (1) and $R_t = A^q\hat{R}_t$ gives

$$\hat{R}_t = \lambda^{\frac{1}{1-\gamma}}\hat{K}_t D = \frac{1+\tau_k}{1+\tau_r}\kappa\lambda\xi^{-\sigma}\hat{K}_t \;\equiv\; \Gamma\hat{K}_t. \tag{A.29}$$

Optimal consumption $\hat{C}_t = \Psi\hat{K}_t$ also requires that consumption and capital grow at the same rates over the cycle. In other words, the SDEs or growth rates $d\hat{C}_t/\hat{C}_t$ and $d\hat{K}_t/\hat{K}_t$ resulting from (A.27) and (A.28), respectively, must share the same deterministic part,

$$-\frac{1}{\sigma}\left(\rho - \frac{1-\tau_i}{1+\tau_k}(r_t - \delta) + \tau_a + \lambda - (1-s)\,\lambda\xi^{-\sigma}\right) = \frac{\hat{Y}_t - \hat{C}_t - \hat{R}_t - \hat{G}_t - \delta\hat{K}_t}{\hat{K}_t}.$$

Inserting capital rewards from (13), $r_t = B^qY_K = \alpha\hat{Y}_t/\hat{K}_t$ and multiply by $\sigma\hat{K}_t$ to obtain

$$\frac{1-\tau_i}{1+\tau_k}\alpha\hat{Y}_t = \left(\frac{1-\tau_i}{1+\tau_k}\delta + \tau_a + \omega\right)\hat{K}_t + \sigma\left(\hat{Y}_t - \hat{C}_t - \hat{R}_t - \hat{G}_t - \delta\hat{K}_t\right),$$

where we defined $\omega \equiv \rho + \lambda - (1-s)\,\lambda\xi^{-\sigma}$. Inserting $\hat{C}_t = \Psi \hat{K}_t$ associated with $\hat{R}_t = \Gamma \hat{K}_t$ from (A.29), and using $\hat{G}_t = A^{-q} G_t$ from (A.16) the condition can be simplified to

$$\frac{1-\tau_i}{1+\tau_k}(\alpha-\sigma)\hat{Y}_t = \left\{\frac{1-\tau_i}{1+\tau_k}\delta + \tau_a + \omega - \sigma\left(\tau_a + \frac{1+\tau_r}{1+\tau_k}\Gamma + \frac{1+\tau_c}{1+\tau_k}\Psi + \frac{1-\tau_i}{1+\tau_k}\delta\right)\right\}\hat{K}_t. \tag{A.30}$$

Hence, (A.30) holds if the expression in front of $\hat{Y}$ holds, requiring $\alpha = \sigma$ and if the expression in front of $\hat{K}$ in (A.30) holds, requiring

$$\frac{1-\tau_i}{1+\tau_k}\delta + \tau_a + \omega = \sigma\left(\tau_a + \frac{1+\tau_r}{1+\tau_k}\Gamma + \frac{1+\tau_c}{1+\tau_k}\Psi + \frac{1-\tau_i}{1+\tau_k}\delta\right).$$

Solving for Ψ and inserting $\omega = \rho + \lambda - (1-s)\,\lambda\xi^{-\sigma}$ yields the expression for Ψ in (18).

C Volatility measures and endogeneity

C.1 The second moment in the long run

We show that the second moment of cyclical utility, $\hat{m}_t^2$, is constant in the long run. From (47), the ODE for the second moment is $d\hat{m}_t^2 = \left(2\vartheta_0 \hat{m}_t^1 - \left(2\vartheta_1 + \lambda\left(1-(1-\vartheta_2)^2\right)\right)\hat{m}_t^2\right)dt$. Inserting the solution of the first moment from (45) gives

$$d\hat{m}_t^2 = \left(2\vartheta_0\left(e^{-(\vartheta_1+\lambda\vartheta_2)t}\left(\hat{m}_0^1 - \hat{m}^1\right) + \hat{m}^1\right) - \left(2\vartheta_1 + \lambda\left(1-(1-\vartheta_2)^2\right)\right)\hat{m}_t^2\right)dt.$$

Solving this deterministic differential equation gives

$$\hat{m}_t^2 = e^{-\left(2\vartheta_1+\lambda\left(1-(1-\vartheta_2)^2\right)\right)t}\hat{m}_0^2 + \int_0^t 2\vartheta_0\left(e^{-(\vartheta_1+\lambda\vartheta_2)s}\left(\hat{m}_0^1 - \hat{m}^1\right) + \hat{m}^1\right)e^{\left(2\vartheta_1+\lambda\left(1-(1-\vartheta_2)^2\right)\right)(s-t)}ds.$$

Computing the integral yields

$$\begin{aligned}\hat{m}_t^2 &= e^{-\left(2\vartheta_1+\lambda\left(1-(1-\vartheta_2)^2\right)\right)t}m_0^2 + 2\vartheta_0\left(\hat{m}_0^1 - \hat{m}^1\right)\frac{e^{-(\vartheta_1+\lambda\vartheta_2)t} - e^{-\left(2\vartheta_1+\lambda\left(1-(1-\vartheta_2)^2\right)\right)t}}{\vartheta_1 + \lambda(1-\vartheta_2)\vartheta_2} \\ &\quad + 2\vartheta_0\hat{m}^1\frac{1-e^{-\left(2\vartheta_1+\lambda\left(1-(1-\vartheta_2)^2\right)\right)t}}{2\vartheta_1 + \lambda\left(1-(1-\vartheta_2)^2\right)}.\end{aligned}$$

Since $1-\vartheta_2 = \left(A^{-1}\xi\right)^{1-\sigma} < 1$, the exponential terms decrease in t. With t sufficiently large, those terms vanish and the expression becomes

$$\hat{m}^2 = \frac{2\vartheta_0\hat{m}^1}{2\vartheta_1 + \lambda\left(1-(1-\vartheta_2)^2\right)}.$$

C.2 Proof of Lemma 1

Observe that the covariance is

$$\begin{aligned}
&\lim_{t\to\infty} Cov_0\left(\ln\hat{K}_t - \ln\hat{K}_{t-\Delta}, q_t - q_{t-\Delta}\right)\\
&= \lim_{t\to\infty} E_0\left(\int_{t-\Delta}^{t} d(\ln\hat{K}_s)\left(\int_{t-\Delta}^{t} dq_s - \lambda\Delta\right)\right)\\
&= \lim_{t\to\infty} E_0\left(\int_{t-\Delta}^{t}\left(\left(\theta_0\hat{K}_s^{\alpha-1} - \theta_1\right)ds + \ln\left(1-\theta_2\right)dq_s\right)\int_{t-\Delta}^{t} dq_s\right)\\
&= \lim_{t\to\infty} E_0\left(\int_{t-\Delta}^{t}\left(\left(\frac{1-\tau_i}{1+\tau_k}r_t/\alpha - \theta_1\right)ds + \ln\left(1-\theta_2\right)dq_s\right)\int_{t-\Delta}^{t} dq_s\right)\\
&= \lim_{t\to\infty} E_0\left(\int_{t-\Delta}^{t}\left(\frac{1-\tau_i}{1+\tau_k}r_t/\alpha - \theta_1\right)ds\int_{t-\Delta}^{t} dq_s + \ln\left(1-\theta_2\right)\left(q_\Delta\right)^2\right)\\
&= \lim_{t\to\infty} E_0\left(\int_{t-\Delta}^{t}\frac{1-\tau_i}{1+\tau_k}r_t/\alpha ds\int_{t-\Delta}^{t} dq_s - \theta_1\Delta q_\Delta + \ln\left(1-\theta_2\right)\left(q_\Delta\right)^2\right)\\
&= \lim_{t\to\infty} E_0\left(\int_{t-\Delta}^{t}\frac{1-\tau_i}{1+\tau_k}r_t/\alpha ds\int_{t-\Delta}^{t} dq_s\right) - \lim_{t\to\infty} E_0\left(\theta_1\Delta q_\Delta - \ln\left(1-\theta_2\right)\left(q_\Delta\right)^2\right)\\
&= \frac{(1-\alpha)\theta_1 - (1-\alpha)\ln(1-\theta_2)\lambda}{1-\alpha}\Delta\lambda\Delta - \theta_1\Delta\lambda\Delta + \ln\left(1-\theta_2\right)\lambda\Delta(1+\lambda\Delta)\\
&= \theta_1\lambda\Delta^2 - \ln(1-\theta_2)(\lambda\Delta)^2 - \theta_1\Delta\lambda\Delta + \ln\left(1-\theta_2\right)\lambda\Delta(1+\lambda\Delta)\\
&= \ln\left(1-\theta_2\right)\lambda\Delta,
\end{aligned}$$

where we used the property that $E(q_\Delta^2) = \lambda\Delta(1+\lambda\Delta)$. Moreover, we used that

$$\int_{t-\Delta}^{t}\int_{t-\Delta}^{t}\left(\frac{1-\tau_i}{1+\tau_k}r_s/\alpha\right)dsd(q_u - \lambda u)$$

is a martingale (cf. García and Griego, 1994, Theorem 5.3).

C.3 Volatility and taxation

For reading convenience, we compute the derivative of the speed of convergence, ϑ_1, with respect to tax rates for later use. When we insert (21) into the definition from (25), we find

$$\vartheta_1 = \frac{1-\sigma}{\sigma}\left(\rho + \left(\frac{1+\tau_k}{1+\tau_r}\frac{\kappa}{D}\xi^{-\sigma}\right)^{\frac{1-\gamma}{\gamma}}\left(1-(1-s)\,\xi^{-\sigma}\right) + \frac{1-\tau_i}{1+\tau_k}\delta + \tau_a\right). \tag{A.31}$$

Hence, higher tax rates τ_r and τ_i reduce ϑ_1, while a higher τ_a coincides with a higher value ϑ_1. The effect of τ_k is ambiguous, it increases λ but decreases the effective rate of depreciation.

Starting from (A.31) and computing the derivative explicitly, however, shows that the partial derivative $\partial\vartheta_1/\partial\tau_k = \frac{1-\sigma}{\sigma}\left(\left(1-(1-s)\,\xi^{-\sigma}\right)\partial\lambda/\partial\tau_k - \frac{1-\tau_i}{(1+\tau_k)^2}\delta\right)$ is positive as long as $\left(1-(1-s)\,\xi^{-\sigma}\right)\partial\lambda/\partial\tau_k \geq \frac{1-\tau_i}{(1+\tau_k)^2}\delta$. When computing $\partial\lambda/\partial\tau_k = \frac{1-\gamma}{\gamma}\frac{1}{1+\tau_k}\lambda$ from (21), one obtains $\partial\vartheta_1/\partial\tau_k \geq 0 \Leftrightarrow \left(1-(1-s)\,\xi^{-\sigma}\right)\frac{1-\gamma}{\gamma}\lambda \geq \frac{1-\tau_i}{1+\tau_k}\delta$. This holds if the rate of physical depreciation δ is sufficiently small.

C.3.1 Coefficient of variation

From (30), volatility increases through higher taxation when ϑ_1/λ decreases, that means $\frac{\partial}{\partial \tau_j} cv^2 \geq 0 \Leftrightarrow \frac{\partial}{\partial \tau_j}(\vartheta_1/\lambda) \leq 0$. It holds for the income tax, τ_i, and for the tax on wealth, τ_a. From (A.31) and (21),

$$\vartheta_1/\lambda = \frac{1-\sigma}{\sigma}\left(\rho + \lambda\left(1-(1-s)\xi^{-\sigma}\right) + \frac{1-\tau_i}{1+\tau_k}\delta + \tau_a\right)/\lambda. \tag{A.32}$$

The only partial derivatives for which the sign is ambiguous is for τ_k and τ_r. As τ_r affects only the arrival rate, the sign of $\partial(\vartheta_1/\lambda)/\partial\tau_r$ is

$$\text{sgn}\left[\frac{\partial(\vartheta_1/\lambda)}{\partial \tau_r}\right] = \text{sgn}\left[\frac{\partial(\vartheta_1/\lambda)}{\partial\lambda}\frac{\partial\lambda}{\partial\tau_r}\right].$$

As $\partial\lambda/\partial\tau_r < 0$, the cv increases when $\partial(\vartheta_1/\lambda)/\partial\lambda$ is positive. Computing this gives

$$\begin{aligned}\frac{\partial(\vartheta_1/\lambda)}{\partial\lambda} &= \left(\lambda\frac{\partial}{\partial\lambda}\vartheta_1 - \vartheta_1\right)/\lambda^2 > 0 \;\Leftrightarrow\; \lambda\frac{1-\sigma}{\sigma}\left(1-(1-s)\xi^{-\sigma}\right) > \vartheta_1 \\ &\Leftrightarrow\; 0 > \rho + \frac{1-\tau_i}{1+\tau_k}\delta + \tau_a.\end{aligned}$$

As this does not hold for a positive tax of wealth, the cv decreases when τ_r increases.

We now compute the partial derivative of (A.32) for τ_k,

$$\begin{aligned}\frac{\partial(\vartheta_1/\lambda)}{\partial\tau_k} &= \left(\frac{\partial\vartheta_1}{\partial\tau_k}\lambda - \frac{\partial\lambda}{\partial\tau_k}\vartheta_1\right)/\lambda^2 = \left[\left(\frac{\partial\vartheta_1}{\partial\lambda}\frac{\partial\lambda}{\partial\tau_k} - \frac{1-\tau_i}{(1+\tau_k)^2}\delta\right)\lambda - \frac{\partial\lambda}{\partial\tau_k}\vartheta_1\right]/\lambda^2 < 0 \\ &\Leftrightarrow\; \frac{1-\sigma}{\sigma}\lambda\left(1-(1-s)\xi^{-\sigma}\right)\frac{\partial\lambda}{\partial\tau_k} < \frac{1-\tau_i}{(1+\tau_k)^2}\delta\lambda \\ &+ \frac{\partial\lambda}{\partial\tau_k}\left[\frac{1-\sigma}{\sigma}\left(\rho + \lambda\left(1-(1-s)\xi^{-\sigma}\right) + \frac{1-\tau_i}{1+\tau_k}\delta + \tau_a\right)\right] \\ &\Leftrightarrow\; 0 < \frac{1-\tau_i}{(1+\tau_k)^2}\delta\lambda + \frac{\partial\lambda}{\partial\tau_k}\left[\frac{1-\sigma}{\sigma}\left(\rho + \frac{1-\tau_i}{1+\tau_k}\delta + \tau_a\right)\right],\end{aligned}$$

which holds for any positive tax on wealth.

C.3.2 Measures based on growth rates

From (36), we can study the effect on the variance of output growth rates mainly through the tax effects on the variance of the rental rate of capital, r_t.

$$\frac{\partial Var\left(\frac{1-\tau_i}{1+\tau_k}r\right)}{\partial\tau_j} \geq 0 \;\Leftrightarrow\; \frac{\partial\left(\frac{1-\tau_i}{1+\tau_k}E(r)\lambda\right)}{\partial\tau_j} \geq 0 \;\Leftrightarrow\; \frac{\partial(\vartheta_1\lambda + \ln\left(\frac{1}{1-\vartheta_2}\right)\lambda^2)}{\partial\tau_j} \geq 0$$

where

$$\vartheta_1\lambda = \frac{1-\sigma}{\sigma}\left(\rho + \lambda\left(1-(1-s)\xi^{-\sigma}\right) + \frac{1-\tau_i}{1+\tau_k}\delta + \tau_a\right)\lambda. \tag{A.33}$$

The only partial derivatives for which the sign is ambiguous is for τ_k,

$$\frac{\partial(\vartheta_1\lambda)}{\partial\tau_k} = \frac{1-\sigma}{\sigma}\left(\partial\lambda/\partial\tau_k(1-(1-s)\xi^{-\sigma}) - (1-\tau_i)/(1+\tau_k)^2\delta\right)\lambda + \vartheta_1$$

which remains ambiguous, but for δ sufficiently small is positive.

Economics Working Paper

2008-12: Maria Knoth Humlum: Timing of Family Income, Borrowing Constraints and Child Achievement

2008-13: Erich Gundlach and Martin Paldam: Income and Democracy: A Comment on Acemoglu, Johnson, Robinson, and Yared (2008)

2008-14: Helena Skyt Nielsen, Torben Sørensen and Christopher Taber: Estimating the Effect of Student Aid on College Enrollment: Evidence from a Government Grant Policy Reform

2008-15: Erich Gundlach and Martin Paldam: The Democratic Transition. A study of the causality between income and the Gastil democracy index

2008-16: Alexander K. Koch and Eloïc Peyrache: Aligning Ambition and Incentives

2009-01: Alexander K. Koch, Albrecht Morgenstern and Philippe Raab: Career concerns incentives: An experimental test

2009-02: Maria Knoth Humlum and Rune Majlund Vejlin: The Effects of Financial Aid in High School on Academic and Labor Market Outcomes: A Quasi-Experimental Study

2009-03: Philipp J.H. Schröder and Allan Sørensen: Second Thoughts on Exporter Productivity

2009-04: Lene Kromann: Does Employee Body Weight Affect Employers' Behavior?

2009-05: Martin Paldam: The welfare state and Baumol’s law

2009-06: Erich Gundlach and Martin Paldam: The agricultural and the democratic transitions - Causality and the Roundup model

2009-07: Torben M. Andersen: Fiscal policy and the global financial crisis

2009-08: Helena Skyt Nielsen: Causes and Consequences of a Father’s Child Leave: Evidence from a Reform of Leave Schemes

2009-09: Marianne Simonsen and Lars Skipper: The Family Gap Reconsidered: What Wombmates Reveal

2009-10: Olaf Posch and Klaus Wälde: On the non-causal link between volatility and growth