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On the Analysis of Endogenous Growth Models with a Balanced Growth Path ^{*}

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Abstract. This paper is the first that provides a comprehensive toolkit for analyzing deterministic, continuous-time, endogenous growth models with a balanced growth path. Since it is impossible to analyze the original dynamic system of growing variables, I suggest two different methods of transforming the model variables in order to derive a stationary system. The stationary system can then be used to test for the existence and uniqueness of transition paths and to derive transitional dynamics qualitatively (e.g. by phase diagram analysis) and quantitatively (e.g. by numerical simulation). I provide sufficient conditions under which the transformed dynamic system is autonomous, derive the functional form of the transformed system, and provide instructions of how to analyze the stability properties of the transformed system. Furthermore, I show that the class of models for which the method can be applied exhibits a specific pattern of transitional dynamics. Two identical economies that differ only with respect to the endowment of one state variable are shown to grow with the same balanced growth rate in the long-run, but at different levels.

Continuous-time Endogenous Growth Models; Transitional Dynamics; Balanced Growth; Knife-edge Conditions

JEL: C02, C60, O40

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1. INTRODUCTION

In this paper I focus on deterministic, continuous-time, endogenous growth models that exhibit a balanced growth path (BGP), i.e. in this class of models the aggregate variables grow with a constant growth rate in the long-run (see for example Lucas, 1988; Romer, 1990; Mulligan and Sala-i-Martin, 1993; Jones, 1995; Eicher and Turnovsky, 1999 and 2001; Klump and de la Grandville, 2000; Irmen 2011, Grossmann et al., 2016, or various textbook models in Barro and Sala-i-Martin, 2004; and Acemoglu, 2009). The state variables, which are usually composed of the stock of physical capital, human capital and/or other accumulable variables, determine whether an economy is on the transition path towards the BGP or grows along the BGP. In order to design an endogenous growth model exhibiting a balanced growth path, production and utility functions usually have to satisfy some kind of knife-edge conditions (see e.g. Caballe and Santos, 1993; Eicher and Turnovsky, 1999, 2001; and Christiaans, 2004). But this seems to be only a minor drawback given that models with a BGP are analytically tractable, which makes it convenient to analyze the stability and growth properties of the model. Assuming that an economy is initially in balanced growth and then hit by a shock is not only of analytical convenience, but also of logical clarity.

Unfortunately, it is difficult or even impossible to test for the existence and uniqueness of transition paths and to derive transitional dynamics qualitatively and quantitatively just by investigating the original model represented by a dynamic system of growing variables. Therefore, growth economists are used to transform the model variables such that the resulting dynamic system is stationary. The stationary model is then tested for existence and uniqueness of solutions, used to perform phase diagram analysis or to simulate transitional dynamics numerically. Conclusions derived from the stationary model are finally transferred back to the original model. However, to date – to my knowledge – no systematic procedure has been suggested by the literature to convert the growing variables into stationary variables. Furthermore, there has been no formal analysis proving that the stability analysis of existing papers is valid, i.e. showing that transition paths of the stationary dynamic system can be connected to those of the original dynamic system.

In this paper I close this gap in the literature. Instead of building on a generalized model framework with resource constraint, production technology and utility maximizing agents, the starting point of this paper is the dynamic system of growing variables, which describes the dynamics of the original, unscaled model. The advantage of this approach is that it places no restrictions on the model structure and thus the analysis is as general as possible. To this end, I extend the definition of homogeneity to

a dynamic system of differential equations and algebraic equations, and I denote it semi-homogeneity henceforth. For example, the property of semi-homogeneity is satisfied globally by models with homogenous aggregate production technology (e.g. CES production function) and isoelastic marginal utility. I show that this property is a necessary condition that must hold on a balanced growth path. Hence, the condition of semi-homogeneity can be seen as putting very little restrictions on the model.

I introduce two different methods of transforming the dynamic system into stationary variables. The first method, which I denote scale-adjustment, reduces the speed of growth of the variables by the balanced growth rate. Using this procedure, the transformed variables converge towards a constant in the long-run. I show that semi-homogeneity is a sufficient condition ensuring that the transformed system is autonomous. Furthermore, I derive the functional form of the transformed system, and provide instructions of how to analyze the stability properties of the transformed system by evaluating the Eigenvalues of the Jacobian matrix.

I show that the common method used in the literature of creating ratios of endogenous variables is equivalent to scale-adjustment. I prove that the same necessary conditions guarantee that the transformed dynamic system is stationary, I derive the functional form of the transformed system, I provide instructions of how to analyze the stability properties of the transformed system, and I show how to retrieve the time paths of the original variables. However, this way of transforming the variables suggest that the economy converges to a unique stationary point and does not reveal that initial conditions of the state variables determine at which level the economy grows in the long-run.

In order to illustrate both methods of constructing a stationary system, I apply them to the seminal Lucas-Uzawa model (Lucas, 1988; Uzawa, 1965). I show how both types of stationary systems can be derived from the original system of growing variables, and that both methods can be used for stability analysis. I also provide counterexamples of when the methods cannot be applied and the stability analysis might be flawed.

Focussing on the class of models that exhibit global homogeneity I find that they show a particular convergence pattern. Consider two economies which are both on a balanced growth path, but starting with different initial levels of accumulable assets, such that one economy has, for example, higher output. Obviously, the difference will remain forever, because both economies grow with the same balanced growth rate. Now consider two identical economies that are both endowed with the same amount of accumulable assets except for one asset, such that one economy has a lower level of output compared to the second one. I prove that these economies converge towards different balanced growth paths in the long-run. In other words, they will grow with the same rates in the long-run but at

different levels. This implies that initial differences in output, consumption, and other aggregates will persist forever – contrary to the prediction of the standard neoclassical model.

The analysis of transitional dynamics is based on the scale-adjusted dynamic system. I prove that scale adjustment transforms the balanced growth path into a Center Manifold of stationary equilibria, each of which represents an economy growing with the same balanced growth rate but at a different level. Depending on the initial conditions, an economy converges towards a particular stationary point on the manifold indicating that differences in initial conditions persist forever. Besides that, scale adjustment offers additional advantages over reduced scale-adjustment. In particular, the scale-adjusted system can be directly used for numerical analysis, and initial conditions and inequality constraints can be added without further transformations.

The paper is organized as follows. In Section 2, I collect the main definitions. In Section 3, I investigate the relation between balanced growth and semi-homogeneity. In Section 4, I propose scale-adjustment as one method to analyze endogenous growth models and I investigate transitional dynamics of this class of models. In Section 5, I introduce reduced scale-adjustment, the dominant method to analyze endogenous growth models in the literature, and compare it to scale-adjustment. In Section 6, I show how scale-adjustment and reduced scale-adjustment can be applied to a seminal growth model, the Lucas-Uzawa model. Furthermore, I provide counterexamples of when the methods cannot be applied. Section 7 concludes. In Appendix A, I collect the proofs of the theorems, and in Appendix B, I state the fundamental theorem of normally hyperbolic invariant manifolds as stated by Li et al. (2003).

2. DEFINITIONS

I start by providing some basic definitions which are necessary for the analysis. Growth models usually give rise to a dynamic system augmented with algebraic equations that describe the evolution of the economy. The starting point of the analysis is the dynamic system resulting from the maximization problem of agents. I begin with formally defining this system (see e.g. Ascher and Petzold, 1998).

DEFINITION 1 (Differential Algebraic System). *An autonomous Differential Algebraic System (DAS) is a system of which the dynamic behavior is described by a combination of ordinary differential and algebraic equations*

$$\dot{X} = F(X, Y) \tag{1a}$$

$$0 = G(X, Y) \tag{1b}$$

The dimension of X (n_X) and Y (n_Y), match the dimension of F and G , respectively, i.e. $F : \mathbb{R}^{n_X} \times \mathbb{R}^{n_Y} \rightarrow \mathbb{R}^{n_X}$ and $G : \mathbb{R}^{n_X} \times \mathbb{R}^{n_Y} \rightarrow \mathbb{R}^{n_Y}$.

In this paper I focus on a particular class of growth models, those who exhibit a balanced growth path. Whether the economy is on a balanced growth path depends on the state of the economy. This is captured by the following definition.

DEFINITION 2 (Balanced Growth). *A variable X is in balanced growth if its growth rate $\gamma_X := \frac{\dot{X}}{X}$ is constant with respect to time:*

$$\frac{d\gamma_X}{dt} = 0. \quad (2)$$

Whenever a variable satisfies this property I denote it with a star: X^* .

I assume that once the economy exhibits balanced growth, it has been on the balanced growth path forever, and will remain on the balanced growth path forever. I collect the growth rates of the variables in the vector $v \in \mathbb{R}^n$.

ASSUMPTION 1 (Balanced Growth Path). *Consider a DAS that is in balanced growth at a point (X^*, Y^*) . Then the Balanced Growth Path is defined by*

$$BGP(X^*, Y^*) = \{(X, Y) | (X, Y) = (\Lambda_X(t)X^*, \Lambda_Y(t)Y^*), t \in \mathbb{R}\}, \quad (3)$$

with $\Lambda_X \in \mathbb{R}^{n_X, n_X}$ defining the diagonal matrix of $e^{v_i t}$, $t \in \mathbb{R}$, $i = 1, \dots, n_X$ and $\Lambda_Y \in \mathbb{R}^{n_Y, n_Y}$ defining the diagonal matrix of $e^{v_i t}$, $t \in \mathbb{R}$, $i = n_X + 1, \dots, n_X + n_Y$.

To exclude special cases that are irrelevant for the analysis I assume that the BGP is one-dimensional. This means that

$$\exists v \in \mathbb{R}^{n_X + n_Y} \wedge (X^*, Y^*) \quad \text{with} \quad \left\{ (X, Y) \in BGP(X^*, Y^*) \Leftrightarrow \frac{d\gamma_{X_i}}{dt} = v_i, \frac{d\gamma_{Y_j}}{dt} = v_j, \quad i = 1, \dots, n_X, j = n_X + 1, \dots, n_X + n_Y \right\}.$$

I now turn to the DAS. The next definition extends the property of homogeneity of a function to the multi-dimensional case. For this definition I also require a vector $v \in \mathbb{R}^n$, which will turn out to be the same vector of growth rates as for the definition of balanced growth. However, for the time being I keep it unspecified.

DEFINITION 3 (Semi-Homogeneity). *A DAS is semi-homogeneous with vector $v \in \mathbb{R}^n$, $n_X + n_Y = n$ and one $v(i) \neq 0$ if for any (X, Y) satisfying the algebraic constraint $G(X, Y) = 0$*

$$F(\Lambda_X X, \Lambda_Y Y) = \Lambda_X F(X, Y) \quad (4a)$$

$$G(\Lambda_X X, \Lambda_Y Y) = 0 \quad (4b)$$

holds for all $\lambda \in \mathbb{R}^+$ with the matrices $\Lambda_X \in \mathbb{R}^{n_X, n_X}$ and $\Lambda_Y \in \mathbb{R}^{n_Y, n_Y}$ defined as the diagonal matrices of $\lambda^{v(i)}$ for all $i = 1, \dots, n_X$ and for all $i = n_X + 1, \dots, n_Y$, respectively. Note that v is not unique but from the linear space $cv, c \in \mathbb{R}$.

The property of semi-homogeneity will turn out to be of vital importance in this paper. Therefore, some intuition is warranted which type of functional forms satisfy this property. If the aggregate production technology is homogenous (for example a CES technology) and if marginal utility is isoelastic, then the DAS is semi-homogenous. Since the majority of endogenous growth models uses this type of functional forms, the class of models with semi-homogenous DAS is large. In Section (6), I show that the dynamic system of the Lucas-Uzawa model satisfies the property and I provide references to counterexamples of non semi-homogenous systems for which the results cannot be applied.

The final definition relates to the balanced growth path. It is an algebraic constraint for the DAS that has to hold on a balanced growth path. It can be seen as a knife-edge restriction for endogenous growth.

DEFINITION 4 (BGP equations). *Consider a DAS of shape (1) with n_X differential equations and n_Y static equations such that $n_X + n_Y = n$. I define the associated BGP equations as*

$$D_X F \cdot F(X, Y) - D_Y F \cdot D_Y G^{-1} \cdot D_X G \cdot F(X, Y) - \Gamma_X(X, Y) \cdot F(X, Y) = 0 \quad (5)$$

together with the static equations. The matrix $\Gamma_X(X, Y)$ is defined as the diagonal matrix of $\frac{F_i(X, Y)}{X_i}$, $i = 1 \dots n_X$. If static equations are absent, the equations reduce to

$$D_X F(X) \cdot F(X) = \Gamma(X) \cdot F(X). \quad (6)$$

3. BALANCED GROWTH AND SEMI-HOMOGENEITY

I start by investigating the relation between the BGP equations and a balanced growth path. It turns out that the BGP equations are necessary and sufficient conditions for a balanced growth path.

If the system is semi-homogenous the growth rates are equal to the vector of semi-homogeneity (up to a linear factor).

Theorem 1 (Necessary and sufficient conditions for BGP). *Consider a DAS of shape (1) with n_X differential equations and n_Y algebraic equations such that $n_X + n_Y = n$. Then, a vector X exhibits the BGP property if and only if the associated BGP equations hold. Note, that equations (5) and (6), respectively are a system of n_X equations whereas only $n_X - n_b$ are linear independent.*

Moreover, if the DAS is semi-homogeneous with vector $v \in \mathbb{R}^n$ where $v_1, \dots, v_{n_X}$ represents the vector of balanced growth rates of X , then Y also is on a balanced growth path with growth rates $v_{n_X+1}, \dots, v_n$.

The property of semi-homogeneity has to hold on the balanced growth path. Of course, the property of semi-homogeneity is only a necessary condition on the balanced growth path, but not at other points. Nonetheless, models of balanced growth usually exhibit the property of semi-homogeneity globally.

Remark 2 (Balanced growth of semi-homogenous DAS). *Consider a DAS of shape (1). The DAS exhibits a balanced growth path with vector v which is an invariant manifold*

$$M = \{(X, Y) | (X, Y) = (\Lambda_X X^*, \Lambda_Y Y^*), \lambda \in \mathbb{R}^+\}.$$

Then, the DAS is semi-homogenous on M with vector v .

4. SCALE-ADJUSTMENT AND TRANSITIONAL DYNAMICS

From now on I assume that the system is semi-homogenous with vector v . The aim of the analysis is two-fold. First, I suggest scale-adjustment as one method to transform the variables such that they are stationary. I prove that semi-homogeneity is a sufficient condition for receiving an autonomous dynamic system of scaled variables. Furthermore, I derive conditions for existence and uniqueness of the transition path.

Second, I show that the class of semi-homogenous, endogenous growth models shows a specific pattern of transitional dynamics. Two economies differing in only one of their state variables will never converge towards the same steady state level. In other words, differences in output or other aggregate variables will persist forever.

I now proceed by transforming the variables of a semi-homogenous DAS into stationary ones. The idea is to use a general method of scaling which I denote “scale-adjustment”. This method of scaling was to my knowledge first introduces in the seminal paper of Lucas (1988), however, without formal analysis. The idea of the method is to slow down the motion of variables according to their respective

balanced growth rates. This way, the variables are constant at the balanced growth path. Moreover, the dimension of the system is conserved and therefore the method allows for completely analyzing the transitional dynamics of the model.

DEFINITION 5 (Simply Scale-Adjustable). *Consider a DAS of the shape (1) and a vector $v \in \mathbb{R}^n$, $v \neq 0$ such that the transformed variables are defined as $x(i) := X(i)e^{-v(i)t}$, $i = 1, \dots, n_X$, $y(j) := Y(j)e^{-v(j)t}$, $j = n_X + 1, \dots, n$. If the resulting DAS is autonomous the system is called simply scale-adjustable:*

$$\dot{x} = f(x, y) \tag{7a}$$

$$0 = g(x, y). \tag{7b}$$

Since scale-adjustment introduces a non-autonomous component into the DAS it is not obvious that the resulting system is still autonomous. However, semi-homogeneity is a sufficient condition for the scale-adjusted system to be autonomous.

Proposition 3 (Sufficient criteria for scale-adjustability). *If a DAS is semi-homogeneous with vector v , then the scale-adjusted version is autonomous with growth rates $v(i)$, and the scaled system reads*

$$\dot{x} = F(x, y) - Vx \tag{8a}$$

$$0 = G(x, y) \tag{8b}$$

with $V \in \mathbb{R}^{n_X, n_X}$ representing the diagonal matrix of v .

Scale-adjustment transforms a specific balanced growth path into a stationary point. Next, I investigate the properties of the scale-adjusted system and transitional dynamics in the neighborhood of a stationary point. It turns out that the balanced growth path trajectory is transformed into a Center Manifold of stationary points.¹

Theorem 4 (Properties of DAS with Balanced Growth Paths).

Consider a DAS of shape (1) that possesses a BGP solution which I will denote with an asterisk. I assume that there exists an autonomous, scale-adjusted version of (1) of the shape

$$\dot{x} = f(x, y) \tag{9a}$$

$$0 = g(x, y) \tag{9b}$$

¹For details on the concept of Center Manifolds see, for instance, Tu (1994).

with $x(i) := X(i)e^{-c_1(i)t}$, $y(j) := Y(j)e^{-c_2(j)t}$ for all components, whereby c_1, c_2 are the constant balanced growth rates. It follows that system (9) has a fixed point:

$$0 = f(x^*, y^*)$$

$$0 = g(x^*, y^*)$$

Moreover, x^* and y^* lie on a Center Manifold which consists of stationary points. This Center Manifold has the same shape in the (x, y) -space of (9) as an invariant manifold in the (X, Y) -space of (1), satisfying the BGP property.

The fact, that the scale-adjusted system exhibits a Center Manifold of stationary points demands a qualitative investigation of the dynamics in the neighborhood of a stationary point. It is not possible to apply the theorem of Hartman and Grobman, since it explicitly excludes the analysis of non-hyperbolic points (For a statement of the theorem see for instance Tu, 1994, p. 135). However, it is possible to apply the fundamental theorem of normally hyperbolic manifolds.² The Center Manifold is normally hyperbolic, since it exhibits an Eigenvalue of zero with a corresponding eigenspace tangent to the Center Manifold. Let me assume that apart from n_c Eigenvalues of zero associated to a n_c dimensional Center Manifold (CM) of stationary points there exists n_s stable and n_u unstable Eigenvalues.³ For local stability analysis I recall the following conclusions from the theorem.

- There exists a local unstable and stable manifold of the Center Manifold, W^u and W^s , of dimension $n_u + n_c$, $n_s + n_c$, respectively, tangent to the subspace spanned by the corresponding Eigenvectors. This means, a trajectory converging to the Center Manifold is locally unique if and only if $n_s + n_c$ initial conditions are given.
- Foliation: W^u and W^s are fibred by submanifolds (leaves) W_x^{uu} , W_x^{ss} , $x \in CM$ of dimension n_u and n_s , respectively, tangent to the subspace spanned by the corresponding Eigenvectors. W_x^{uu} and W_x^{ss} are characterized such that all points of W_x^{uu} and W_x^{ss} converge exponentially to x as $t \rightarrow -\infty$, $t \rightarrow \infty$, respectively. This means, all economies represented by W_x^{ss} form a class, which converges to the same stationary point in the long run. The direction of convergence is the subspace, generated by the stable Eigenvectors. Since there is no movement on the Center Manifold the qualitative behavior is the same as for the linearized system.

²For a statement of the theorem see Hirsch, Pugh and Shub (1977), or Li et al. (2003). I state the theorem and the definition of a normal hyperbolic manifold according to Li et al. in Appendix B.

³A stable or unstable Eigenvalue in this context is an Eigenvalue with a real part smaller or larger than zero, respectively.

The first point illustrates that by employing scale-adjustment it can be easily analyzed whether a specific optimal control problem yields a unique trajectory. A trajectory converging to the Center Manifold of stationary equilibria is determined uniquely by $n_s + n_c$ initial conditions transversal to the stable manifold. This corresponds to local dynamics implied by the Hartman-Grobman theorem, with the extension that zero Eigenvalues count as stable Eigenvalues. In Section 6, I show how to apply the stability analysis to the Lucas-Uzawa model.

The second point allows for conclusions about transitional dynamics. Consider two economies that grow at the balanced growth rate, but with different levels of the variables. Since both economies grow with the same growth rate, the relative difference in the variables will remain. In the scale-adjusted system, both economies are represented by two different points on the Center Manifold. Now consider two economies off the balanced growth path. The theorem states that both economies converge towards the Center Manifold of stationary equilibria. However, the initial conditions determine to which specific point of this curve each economy converges. Since the initial endowments of physical capital, human capital, or blueprints usually represent the initial conditions, these values determine the level at which the economy grows in the long-run. The results are summarized in the following remark.

Remark 5 (Path dependence of endogenous growth models). *Consider an endogenous growth model represented by a semi-homogenous DAS. I assume that the model exhibits a one-dimensional balanced growth path. Then, the long-run levels of scale-adjusted variables depend on the initial conditions, i.e. on the initial endowments of accumulable variables.*

Transitional dynamics in the original and scale-adjusted system are illustrated in Figure 1. I assume that the model has two state variables that are denoted by H and K for simplicity. The balanced growth rate of K is assumed to be μ , and that of H is assumed to be ψ such that $v = (\mu, \psi)^T$ holds. Figure 1 shows the phase diagram of the original system (1) projected on the state space in the left panel. Trajectories off the balanced growth path (dashed, dotted) converge towards the balanced growth path (solid). For the scale-adjusted system (7) (Figure 1, right panel), the balanced growth path transforms into a Center Manifold of stationary equilibria (CMSE). Each point of this curve is a stationary point of (7). Trajectories off the balanced growth path converge towards different points on the Center Manifold, depending on the initial endowment of accumulable assets.

I conclude this Section by stating a proposition which is useful for the local stability analysis. For such an analysis formally the Eigenvalues for each stationary point of the Center Manifold have to be

FIGURE 1. Phase diagram of original and scale-adjusted system

The left panel shows the phase space of the growing variables K and H . The solid black line indicates the balanced growth path (BGP). Dots indicate different initial positions from which the economy converges towards the BGP. The right panel shows the phase space of the scale-adjusted variables $Ke^{-\mu t}$ and $He^{-\psi t}$. The solid black line indicates the Center Manifold of stationary equilibria (CMSE). Dots indicate different initial positions from which the economy converges towards one steady state.

investigated. However, by applying proposition 6 it is sufficient to investigate the Eigenvalues at one single point of the Center Manifold. The reason is that the Eigenvalues are the same for all points on the Center Manifold.

Proposition 6 (Stability analysis of the scale-adjusted system). *Consider a semi-homogenous DAS that is scaled according to theorem 3:*

$$\dot{x} = F(x, y) - Vx \quad (11a)$$

$$0 = G(x, y). \quad (11b)$$

Consider one point of the Center Manifold of stationary equilibria (x^, y^*) . Then, the Jacobian matrix evaluated at any point of the Center Manifold exhibits the same Eigenvalues as the Jacobian matrix evaluated at (x^*, y^*) .*

5. REDUCED SCALE-ADJUSTMENT

I now turn to a different method of scaling, which i will denote *reduced scale-adjustment*. Different to scale-adjustment, this method of scaling is frequently used in the literature. The idea is to create ratios of variables such that the resulting artificial variables are stationary. Naturally, when ratios are used, the dimension of the system reduces by one.

DEFINITION 6 (Reduced Scale-Adjustable). *Consider a DAS of the shape (1) and a vector $v \in \mathbb{R}^n$, $v \neq 0$. The DAS is reduced scale-adjustable if there exists an index $i \leq n_d$ such that the differential algebraic system for the variables*

$$\tilde{x}_j := \frac{X_j^{v_i}}{X_i^{v_j}} \quad j = 1, \dots, n_X, \quad (12a)$$

$$\tilde{y}_j := \frac{Y_j^{v_i}}{X_i^{v_j}} \quad j = 1, \dots, n_Y \quad (12b)$$

is autonomous and depends solely on $\tilde{x}$ and $\tilde{y}$

$$\dot{\tilde{x}} = f(\tilde{x}, \tilde{y}) \quad (13a)$$

$$0 = g(\tilde{x}, \tilde{y}) . \quad (13b)$$

Note that $\tilde{x}_i \equiv 1$. I will denote X_i as the numeraire.

If the DAS is semi-homogenous the method of reduced scale-adjustment can be applied and the resulting system will no longer contain any growing variable. Note, that usually different ways of reduced scale-adjustment exists, because each variable with a non-zero balanced growth rate can be the numeraire. I provide an assumption which must hold for the transformation in order to generalize the results of the reduced scale-adjusted system.

ASSUMPTION 2 (Identification of reduced scale-adjusted variables). *Consider a semi-homogenous DAS of form (1) and its reduced scale-adjusted version (12). I assume that there exists a time t such that the mapping $\{X_{j=1, \dots, i-1, i+1, \dots, n_X}(t) Y_{j=1, \dots, n_Y}(t)\}$ to $\{\tilde{x}(t), \tilde{y}(t)\}$ is regular.*

In other words the assumption implies that, given for example the initial value for one $X_i(0)$, the transformation must allow to recover all other $X_j(0), Y_j(0)$ from the reduced scale-adjusted variables $\tilde{x}(0), \tilde{y}(0)$. For simplicity, I will only consider reduced scale-adjusted systems with the same numeraire for all reduced scale-adjusted variables, but the results can easily be extended to the more general case.

Theorem 7 (Reduced Scale-Adjustability of semi-homogenous systems). *Consider a DAS that is semi-homogenous with vector $v \in \mathbb{R}^n$, $v \neq 0$. The DAS is reduced scale-adjustable for every $i \leq n_d$ with $v(i) > 0$. Without loss of generality I assume $v(i) = 1$. The reduced differential algebraic system reads*

$$\dot{\tilde{x}} = F(\tilde{x}, \tilde{y}) - V\tilde{x}F_i(\tilde{x}, \tilde{y}) \quad (14a)$$

$$0 = G(\tilde{x}, \tilde{y}) \quad (14b)$$

with V denoting the diagonal matrix of v . Note that the differential equation for $\tilde{x}_i$ is trivial and can be omitted.

A stationary point of (14) is a point (x^*, y^*) that satisfies the algebraic constraint and yields $\dot{\tilde{x}} = 0$ and $\dot{\tilde{y}} = 0$. However, it is not possible in general to conclude that X and Y are in balanced growth, because the dimension of the system has been reduced by one. I show that this conclusion holds for semi-homogenous systems.

Proposition 8 (Balanced growth of systems exhibiting semi-homogeneity). *Consider a DAS that is semi-homogenous with vector v , and a point (X, Y) that satisfies $G(X, Y) = 0$. I assume the system to exhibit a one-dimensional BGP. Then, if and only if X is in balanced growth it satisfies*

$$\frac{F_i(X, Y)}{X_i} \frac{1}{v_i} = \frac{F_j(X, Y)}{X_j} \frac{1}{v_j} \quad \forall \quad 1 \leq i, j \leq n_X \quad (15)$$

Moreover, if X is in balanced growth this also follows for Y .

Proposition 9 (Balanced growth of reduced systems). *Consider a semi-homogenous DAS in the reduced form of (14). If the algebraic constraint is satisfied, X and Y are in balanced growth if and only if $\dot{\tilde{x}} = 0$.*

The reduced scale-adjusted system exhibit one dimension less than the original system and the simple scale-adjusted system. This is because the Center Manifold of stationary equilibria (balanced growth path) is mapped into a single stationary point. Local stability of one point of the Center Manifold of stationary equilibria is closely related to local stability of this single fixed point of the reduced system.

Theorem 10 (Stability of reduced scale-adjusted systems). *Consider a semi-homogenous DAS in the reduced form of (14). Then, the balanced growth path is mapped onto a single hyperbolic fixed point of (14). Moreover, the Jacobian matrix of (14) evaluated at the fixed point exhibits the same number of Eigenvalues with positive and negative real part as the Jacobian matrix evaluated at one point of the Center Manifold of stationary equilibria of system (7).*

Usually, the initial conditions are stated for the original variables X and Y , and have to be translated into reduced scale-adjusted variables $\tilde{x}$ and $\tilde{y}$. I assume that the scale-adjusted system possesses

$n_s < n_X$ stable Eigenvalues and a one dimensional balanced growth path. Then, adjustment dynamics are unique, if $n_s + 1$ initial conditions are given transversal to the stable manifold:

$$I(X(0), Y(0)) = 0, \quad I : \mathbb{R}^{n_X + n_Y} \rightarrow \mathbb{R}^{n_s + 1}. \quad (16)$$

Theorem 10 states that time paths of $\tilde{x}$ and $\tilde{y}$ are unique, if (16) transforms according to

$$\tilde{i}(\tilde{x}(0), \tilde{y}(0)) = 0, \quad \tilde{i} : \mathbb{R}^{n_X + n_Y - 1} \rightarrow \mathbb{R}^{n_s} \quad (17)$$

and $\tilde{i}$ is transversal to the stable manifold of system (14). This is the case if assumption (2) holds. Furthermore, in this case it is possible to recover the original time paths of variables X and Y .

Remark 11 (retrieving time paths of original variables). *Consider a semi-homogenous DAS in the reduced form of (14). If assumption (2) holds, it is possible to retrieve the original time paths of variables X and Y .*

Remark (11) illustrates what is lost by reduced scale-adjustment, because the value for X_i from assumption (2) can be set arbitrarily. The reduced scale-adjusted system itself does not contain any information about the fibres of the stable manifold. Economies converging towards the balanced growth path (i.e. the Center Manifold of the scale-adjusted system) on different fibres are summarized in one single trajectory in the reduced scale-adjusted system. Therefore, the dependence of the long-run level of variables on initial conditions is no longer visible for the reduced scale-adjusted system.

6. APPLICATIONS OF THE THEORY

I illustrate the method of scale adjustment and reduced scale adjustment by applying both procedures on a seminal endogenous growth model, the Lucas-Uzawa model. I also provide counterexamples of cases when the theory cannot be applied and the stability analysis could be flawed.

6.1. The Lucas-Uzawa model. The model was introduced by Uzawa (1965) and Lucas (1988), and has since then been analyzed in different versions and using different methods (see e.g. Mulligan and Sala-I-Martin, 1993; Benhabib and Perli, 1993; Caballe and Santos, 1993; Ortigueira and Santos, 1997; Gomez, 2003; Boucekkine et al., 2008; Boucekkine and Ruiz-Tamarit, 2008; Hiraguchi, 2009, and Ben-Gad, 2012). I rely mainly on the exposition in Caballe and Santos (1993), who analyze the scale-adjusted version of the model, and Benhabib and Perli (1993), who investigate the reduced scale-adjusted version.

To keep the analysis simple I focus on the model without external effect of human capital, constant population, and no depreciation of physical capital and human capital. To be consistent with the notation used in this paper, uppercase letters denote the original, growing variables, and lowercase letters denote scale-adjusted, stationary variables.

The representative household maximizes

$$V = \int_0^\infty \frac{C^{1-\sigma}}{1-\sigma} e^{-\rho t} dt \quad (18)$$

subject to the accumulation equations for physical and human capital, respectively,

$$\dot{K} = K^\alpha (Hu)^{1-\alpha} - C \quad (19)$$

$$\dot{H} = \delta H(1 - u), \quad (20)$$

and initial conditions $K(0) = K_0$, and $H(0) = H_0$. It is straightforward to derive a four-dimension system of differential equations governing the evolution of the economy together with the initial conditions and two transversality conditions:

$$\dot{K} = K^\alpha (Hu)^{1-\alpha} - C \quad (21a)$$

$$\dot{H} = \delta H(1 - u) \quad (21b)$$

$$\dot{C} = \frac{C}{\sigma} (\alpha K^{\alpha-1} (Hu)^{1-\alpha} - \rho) \quad (21c)$$

$$\dot{u} = u \left(\frac{\delta(1-\alpha)}{\alpha} + \delta u - \frac{C}{K} \right). \quad (21d)$$

It can easily be verified that the system (21) is semi-homogenous with vector $v = (1, 1, 1, 0)$, i.e. multiplying K , H , and C on the right hand side with a factor λ multiplies $\dot{K}$, $\dot{H}$, and $\dot{C}$ with λ and leaves $\dot{u}$ unaffected.

Evaluating the BGP equations (5) for the system (21) yields four equations with one linear dependency. Simplifying gives three equations characterizing the BGP

$$\frac{C^*}{K^*} = \frac{\delta}{\alpha} - \frac{\delta - \rho}{\sigma} \quad (22a)$$

$$\left(\frac{H^* u^*}{K^*} \right)^{1-\alpha} = \frac{\delta}{\alpha} \quad (22b)$$

$$u^* = 1 - \frac{\delta - \rho}{\sigma \delta}. \quad (22c)$$

Substituting these relations back into the system of differential equations (21) yields the balanced growth rate of K , H , and C as $\eta := \frac{\delta - \rho}{\sigma}$ and reveals that u is stationary. Since $0 < u < 1$ must hold, I get $\delta > \rho$ as a parameter restriction which I assume to hold henceforth.

6.2. The scale-adjusted system. I now proceed with deriving the scale-adjusted system. The scale adjusted variables are

$$k := Ke^{-\eta t}, \quad h := He^{-\eta t}, \quad c := Ce^{-\eta t}, \quad \text{and} \quad u. \quad (23)$$

According to theorem (3) the scale-adjusted system reads

$$\dot{k} = k^\alpha (hu)^{1-\alpha} - c - \nu k \quad (24a)$$

$$\dot{h} = \delta h(1 - u) - \nu h \quad (24b)$$

$$\dot{c} = \frac{c}{\sigma} (\alpha k^{\alpha-1} (hu)^{1-\alpha} - \rho) - \nu c \quad (24c)$$

$$\dot{u} = u \left(\frac{\delta(1-\alpha)}{\alpha} + \delta u - \frac{c}{k} \right). \quad (24d)$$

It is straightforward to verify that equations (22) define stationary points also of the scale-adjusted system (24). Hence, scale-adjustment transformed the balanced growth path into a Center Manifold of stationary equilibria.

I now turn to the stability analysis of the BGP. Calculating the Eigenvalues of the Jacobian matrix at one of the stationary points gives⁴

$$0 \quad (25a)$$

$$-\frac{(1-\alpha)\delta}{\alpha} < 0 \quad (25b)$$

$$\frac{(1+\sigma)\delta - \rho}{\sigma} > 0 \quad (25c)$$

$$\frac{\alpha(\rho - \delta) + \delta\sigma}{\alpha\sigma} > 0 \quad (25d)$$

The zero Eigenvalue is associated with the manifold of stationary equilibria and the corresponding Eigenvector is tangent to the manifold. For the parameter restriction $\delta > \rho$ imposed above one Eigenvalue is negative and two Eigenvalues are positive. The dimension of the stable manifold associated with the Center Manifold of stationary equilibria is two, because it equals the sum of zero Eigenvalues and Eigenvalues with negative real part. Together with two initial conditions this implies a unique

⁴According to proposition 6, the Eigenvalues are the same for each stationary point of the Center Manifold.

adjustment path. Finding one negative Eigenvalue implies that, for each stationary point from the Center Manifold, there exists a one-dimensional stable fibre. This can be exploited to derive further insights about transitional dynamics: In the scale-adjusted system the convergence path of economies converging towards the Center Manifold is tangent to the Eigenvector associated with the negative Eigenvalue.

The results from the stability analysis confirm the findings of previous local stability analyses, albeit using a different method.⁵ Previous studies found that when there is no external effect of human capital on production the adjustment paths towards the balanced growth path exists and are unique, given initial conditions for physical and human capital (see e.g. Caballe and Santos, 1993; Benhabib and Perli, 1994).

Besides the fact that the scale adjusted system can be directly used for stability analysis, there are further advantages from using scale-adjustment compared to reduced scale adjustment. Most importantly, the scale-adjusted system can be employed for numerical analysis without any further transformations (see Trimborn et al., 2008). In particular, initial conditions and inequality constraints can directly be considered for the numerical analysis.⁶ Furthermore, the transition paths of the scale-adjusted variables directly reveal how economies starting with different initial endowments of physical and human capital develop over time and how the initial values affect the level at which the economy grows in the long-run (sf. Figure 1).

6.3. The reduced scale-adjusted system. Now I turn to deriving the reduced scale adjusted system. For defining scale-adjusted variables there are different possibilities to define stationary ratios of growing variables. I follow Benhabib and Perli (1994) and define the following stationary variables

$$x := \frac{K}{H}, \quad q := \frac{C}{K}, \quad \text{and} \quad u. \quad (26)$$

Applying Theorem (7) and simplifying gives the reduced scale-adjusted system

$$\dot{x} = x^\alpha u^{1-\alpha} - qx - \delta(1-u)x \quad (27a)$$

$$\dot{q} = \left(\frac{\alpha}{\sigma} - 1\right) x^{\alpha-1} u^{1-\alpha} q + q^2 - \frac{\rho}{\sigma} q \quad (27b)$$

$$\dot{u} = u \left(\frac{\delta(1-\alpha)}{\alpha} + \delta u - q \right). \quad (27c)$$

⁵It must be acknowledged that Caballe and Santos (1993) also derive a scale-adjusted dynamic system of a variant of this model and use the Eigenvalues of the Jacobian matrix evaluated at a steady state for the stability analysis. However, they do not calculate the Eigenvalues, but instead conclude from the structure of the optimization problem that one Eigenvalue must be negative and two Eigenvalues must be positive.

⁶Trimborn (2013) shows how to solve this kind of models numerically when inequality constraints are present.

It is straightforward to verify that the stationary points of system (27) are given by

$$x^* = \left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}} u^* \quad (28a)$$

$$q^* = \frac{\delta}{\alpha} - \frac{\delta - \rho}{\sigma} \quad (28b)$$

$$u^* = 1 + \frac{\rho - \delta}{\sigma\delta}. \quad (28c)$$

Benhabib and Perli (1994) show that the Jacobian matrix of this system evaluated at the steady state exhibits two Eigenvalues with positive real part and one negative Eigenvalue. Hence, the steady state is saddle-point stable and the stable manifold is one-dimensional. This confirms theorem (10), i.e. the reduced scale-adjusted system preserves the sign of all Eigenvalues compared to scale-adjustment, except for the zero Eigenvalue which is associated to the Center Manifold. The zero Eigenvalue disappears because of the dimension is reduced by one. Two initial conditions for the variables K and H transform into one initial condition for variable x for the reduced scale-adjusted system. Therefore, the analysis of system (27) shows that the transition paths are determined uniquely, just as the analysis of system (24).

6.4. Counterexamples. The stability analysis of the reduced scale-adjusted system is tied to two necessary conditions. First, the original system has to be semi-homogenous, and second, the reduction in dimension must not exceed the dimension of the balanced growth path, which is usually equal to one. I will give counterexamples when one of each condition is violated in order to demonstrate what is not covered by theorem (10).

First, I consider the case in which it is possible to reduce the dimension of the system by more than the dimension of the balanced growth path. Ben-Gad (2012) analyzes a variant of the Lucas-Uzawa model. By introducing the variables $\tilde{c} := CHu$ and $\tilde{k} := KHu$ he is able to derive a reduced form, two-dimensional dynamic system in $\tilde{c}$ and $\tilde{k}$ and conducts a stability analysis for this system. Any conclusion from the sign of the Eigenvalues of the Jacobian matrix, however, is not covered by theorem (10), because the dimension was reduced from four to two, by more than that of the balanced growth path.⁷

Second, I consider the case in which the system is not semi-homogenous. Ruiz-Tamarit and Ventura-Marco (2001) focus on the seminal Ramsey-Cass-Koopmans model. They start with the conventional

⁷It should be noted that it is nonetheless possible to apply the theorem. The two-dimensional dynamic system can be easily augmented by a third differential equation for u . Analyzing the augmented dynamic system would result in exactly the same Eigenvalues as before, augmented by an additional positive Eigenvalue. All conditions of theorem (10) are satisfied such that the analysis of the extended system is covered by the theorem.

two-dimensional system of per-capita consumption, c , and per-capita capital, k , describing the evolution of the economy. Note, that this system is not semi-homogenous. By applying a parameter restriction they are able to reduce the original two-dimensional system into a one-dimensional differential equation for the ratio $z := c/k$. They convincingly argue that the stability analysis of the one-dimensional equation is incorrect. This constitutes an interesting counterexample, since reducing the dimension of a non semi-homogenous system might result in an incorrect stability analysis of the reduced system.

7. CONCLUDING REMARKS

In this paper I provided a toolkit for analyzing deterministic, continuous-time, endogenous growth models with a balanced growth path. Different to other papers, the starting point of the analysis is the dynamic system of growing variables. To this end, I extended the property of homogeneity to a dynamic system and denote it semi-homogeneity. I showed that this is a necessary condition which the dynamic system has to satisfy on the balanced growth path. Dynamic systems that are globally semi-homogenous can easily be transformed into stationary dynamic systems. I derived the functional form of the transformed system, and provided instructions of how to analyze the stability properties of the transformed system.

Results from this paper can be used as a toolkit for designing and analyzing endogenous growth models. Of course, it is known in the literature that production and utility functions have to satisfy certain conditions in order to allow for a balanced growth path, which has led to a small literature on the knife-edge conditions of endogenous growth models (see e.g. Caballe and Santos, 1993; Eicher and Turnovsky, 1999, 2001; and Christiaans, 2004;). In this paper I provided a more general analysis by showing that these knife-edge conditions allow not only for a balanced growth path, but at the same time for transforming the model into stationary variables. Hence, the insights can be used to “engineer” models which are analytically tractable.

I also proved that the class of models yielding globally semi-homogenous dynamic systems show a specific convergence pattern: While all economies grow with the same balanced growth rate in the long-run, they may converge towards different levels. This implies that, in contrast to predictions of the neoclassical model, initial differences in output and other aggregates persist in the long-run.

APPENDIX A: PROOFS

Proof of theorem 1.

Proof. Differentiating the static equation with respect to time yields

$$\begin{aligned} D_Y G \cdot \dot{Y} + D_X G \cdot F(X, Y) &= 0 \\ \Leftrightarrow \dot{Y} &= -D_Y G^{-1} \cdot D_X G \cdot F(X, Y) \end{aligned} \quad (29)$$

I assume, that the inverse exists.⁸ Now, differentiating the equation $F(X, Y) = \Gamma_X(X, Y) \cdot X$ with respect to time and applying the fact that the growth rates are constant on the balanced growth path yields

$$D_X F \cdot F(X, Y) + D_Y F \cdot \dot{Y} = \Gamma_X(X, Y) \cdot F(X, Y)$$

Substituting for $\dot{Y}$ yields the n_X dimensional equation

$$D_X F \cdot F(X, Y) - D_Y F \cdot D_Y G^{-1} \cdot D_X G \cdot F(X, Y) - \Gamma_X(X, Y) \cdot F(X, Y) = 0$$

These equations do not require the static equations to hold. Therefore, together with the n_Y static equations the BGP manifold is well defined.

To proof sufficiency of the equations for balanced growth assume that the balanced growth assumption for X does not hold necessarily. Reconsidering the calculations above the equations have to be augmented by an additional term embodying the change of the growth rates with respect to time.

$$\begin{aligned} D_X F \cdot F(X, Y) - D_Y F \cdot D_Y G^{-1} \cdot D_X G \cdot F(X, Y) \\ - \Gamma_X(X, Y) \cdot F(X, Y) - \frac{d\Gamma_X(X, Y)}{dt} X = 0 \end{aligned}$$

The last summand is zero if and only if X is on a BGP.

It remains to proof that if X is on a BGP and the system is semi-homogeneous, Y will also fulfill the BGP property. Semi-homogeneity was not exploited until now.

Differentiating the algebraic equation of semi-homogeneity with respect to λ and evaluating at $\lambda = 1$ and for an (X, Y) with $G(X, Y) = 0$ yields

$$D_X G(X, Y) \text{diag}(v_X) X + D_Y G(X, Y) \text{diag}(v_Y) Y = 0$$

⁸This is a sufficient condition for the DAS to be of differential index one. It is not necessary, however, it is only slightly stronger.

whereby $\text{diag}(v_X)$ and $\text{diag}(v_Y)$ are diagonal matrices with X and Y components of the vector v of semi-homogeneity in the diagonal. Subtracting equation (29) yields

$$D_X G(X, Y)(\text{diag}(v_X)X - F(X, Y)) + D_Y G(X, Y)(\text{diag}(v_Y)Y - \dot{Y}) = 0$$

Since the first bracket is zero and $D_Y G$ is regular, it follows directly that Y is in balanced growth with the growth rates being equal to the vector of semi-homogeneity, because balanced growth rates and the vector of semi-homogeneity are the same for the dynamic variables. $\square$

Proof of remark 2.

Proof. Assume that X and Y are in balanced growth with vector v , i.e. $(X, Y) \in M$. In this case it is straightforward to see that the definition of semi-homogeneity with vector v holds on M . $\square$

Proof of proposition 3.

Proof. From the definition $x = \Lambda_X X$, $y = \Lambda_Y Y$ with Λ_X , Λ_Y representing diagonal matrices of $e^{-v(i)t}$, $i = 1, \dots, n_X$ and $e^{-v(j)t}$, $j = n_X+1, \dots, n$, respectively, it follows that:

$$\begin{aligned} \dot{x} &= -V\Lambda_X X + \Lambda_X \dot{X} \\ &= -V\Lambda_X X + \Lambda_X F(X, Y) \\ &= -V\Lambda_X X + F(\Lambda_X X, \Lambda_Y Y) \\ &= -Vx + F(x, y) \end{aligned}$$

where V denotes the diagonal matrix of v . Analogous for G . $\square$

Proof of theorem 4.

Proof. Note first, that if a BGP of (1) is a stationary point of (9) in one point of time, then this holds for the maximum interval of definition.⁹ Therefore let us assume this holds for time zero, i.e. the positive orbit of the flow ϕ , $\{\phi_i(X_0^*, Y_0^*)\}_{i \in [0, \infty)}$ is a BGP and the corresponding vector (x_0^*, y_0^*) is a stationary point. Note that the latter will be constant for all t . Now consider the positive orbit of the flow ϕ , $\{\phi_i(\phi_t(X_0^*, Y_0^*))\}_{i \in [0, \infty)}$ for some $t > 0$ which lies on the same invariant manifold and therefore exhibits the same BGP property and the same growth rates. This will also define a stationary point

⁹For convenience, the upper case variables will now always refer to (1) and the lower case variables to (9).

(x_1^*, y_1^*) , since the limit of the ratio is constant:

$$\lim_{t \rightarrow \infty} \frac{X_0^*(i)e^{c_1(i)t}}{X_1^*(i)e^{c_1(i)t}} = \frac{X_0(i)}{X_1(i)} = \lim_{t \rightarrow \infty} \frac{x_0^*(i)}{x_1^*(i)} \quad \forall i, 1 \leq i \leq n_X$$

But $(x_0^*, y_0^*) \neq (x_1^*, y_1^*)$ because for time zero $(X, Y) = (x, y)$ and $\phi_0(X_0^*, Y_0^*) \neq \phi_0(\phi_t(X_0^*, Y_0^*))$ since (X, Y) is growing in at least one component. Since the flow is continuous and differentiable in t , this mapping in (x, y) will define a manifold of stationary points which in turn implies the existence of a Center Manifold. Furthermore, since the mapping $(X, Y) \rightarrow (x, y)$ is the identity for $t = 0$ the shape of this Center Manifold is the same as that of the invariant manifold. $\square$

Proof of remark 5.

Proof. A Center Manifold of stationary equilibria is a ∞ -normally hyperbolic invariant manifold. The proof follows by applying the fundamental theorem of normally hyperbolic invariant manifolds to the scale-adjusted system. $\square$

Proof of proposition 6.

Proof. Consider a stationary point (x^*, y^*) of the scale-adjusted system

$$\dot{x} = F(x, y) - Vx \tag{30a}$$

$$0 = G(x, y) \tag{30b}$$

From the property of semi-homogeneity it follows that the (one-dimensional) Center Manifold of equilibria is defined by

$$M = \{(x, y) | (x, y) = (\Lambda_X x, \Lambda_Y y), \lambda \in \mathbb{R}^+\} \tag{31}$$

The aim is to show that an Eigenvalue of (30) at (x^*, y^*) is also an Eigenvalue of (30) at $(\Lambda_X x^*, \Lambda_Y y^*)$ for some $\lambda \neq 1$.

In a first step I apply the implicit function theorem to calculate the DAS' Jacobian matrix, which Eigenvalues give information about the dynamic behavior in the neighborhood of the stationary point.¹⁰ This Jacobian matrix is represented by

$$\begin{aligned} & D_x \left(F(x, -D_y G(x, y)^{-1} D_x G(x, y)x) - Vx \right) \\ &= D_X \left(F(X, -D_Y G(X, Y)^{-1} D_X G(X, Y)X) - VX \right) \end{aligned}$$

¹⁰The fundamental theorem of normally hyperbolic invariant manifolds is stated for a pure system of differential equations. By applying the implicit function theorem I formally reduce the DAS to a pure system of differential equations.

$$= D_X F(X, Y) - D_Y F(X, Y) \cdot D_Y G(X, Y)^{-1} D_X G(X, Y) - V \quad (32)$$

Applying the chain rule yields

$$D_{\Lambda_X X} F(\Lambda_X X, \Lambda_Y Y) \Lambda_X = D_X (F(\Lambda_X X, \Lambda_Y Y)) = \Lambda_X D_X F(X, Y)$$

$$D_{\Lambda_Y Y} F(\Lambda_X X, \Lambda_Y Y) \Lambda_Y = D_Y (F(\Lambda_X X, \Lambda_Y Y)) = \Lambda_X D_Y F(X, Y)$$

$$D_{\Lambda_X X} G(\Lambda_X X, \Lambda_Y Y) \Lambda_X = D_X (G(\Lambda_X X, \Lambda_Y Y)) = D_X G(X, Y)$$

$$D_{\Lambda_Y Y} G(\Lambda_X X, \Lambda_Y Y) \Lambda_Y = D_Y (G(\Lambda_X X, \Lambda_Y Y)) = D_Y G(X, Y)$$

and hence

$$\Lambda_X^{-1} D_{\Lambda_X X} F(\Lambda_X X, \Lambda_Y Y) \Lambda_X = D_X F(X, Y)$$

$$\Lambda_X^{-1} D_{\Lambda_Y Y} F(\Lambda_X X, \Lambda_Y Y) \Lambda_Y = D_Y F(X, Y)$$

$$D_{\Lambda_X X} G(\Lambda_X X, \Lambda_Y Y) \Lambda_X = D_X G(X, Y)$$

$$D_{\Lambda_Y Y} G(\Lambda_X X, \Lambda_Y Y) \Lambda_Y = D_Y G(X, Y)$$

Substituting into (32) yields

$$\begin{aligned} & D_x (F(x, -D_y G(x, y)^{-1} D_x G(x, y)x) - Vx) \\ = & \Lambda_X^{-1} D_{\Lambda_X X} F(\Lambda_X X, \Lambda_Y Y) \Lambda_X \\ & - \Lambda_X^{-1} D_{\Lambda_Y Y} F(\Lambda_X X, \Lambda_Y Y) \Lambda_Y \cdot \Lambda_Y^{-1} D_{\Lambda_Y Y} G(\Lambda_X X, \Lambda_Y Y)^{-1} D_{\Lambda_X X} G(\Lambda_X X, \Lambda_Y Y) \Lambda_X - V \\ = & \Lambda_X^{-1} \left[D_{\Lambda_X X} F(\Lambda_X X, \Lambda_Y Y) \right. \\ & \left. - D_{\Lambda_Y Y} F(\Lambda_X X, \Lambda_Y Y) D_{\Lambda_Y Y} G(\Lambda_X X, \Lambda_Y Y)^{-1} D_{\Lambda_X X} G(\Lambda_X X, \Lambda_Y Y) \right] \Lambda_X - V \\ = & \Lambda_X^{-1} \left[D_{\Lambda_X X} F(\Lambda_X X, D_{\Lambda_Y Y} G(\Lambda_X X, \Lambda_Y Y)^{-1} D_{\Lambda_X X} G(\Lambda_X X, \Lambda_Y Y) \Lambda_X X) \right] \Lambda_X - V \\ = & \Lambda_X^{-1} \left[D_{\Lambda_X X} F(\Lambda_X X, D_{\Lambda_Y Y} G(\Lambda_X X, \Lambda_Y Y)^{-1} D_{\Lambda_X X} G(\Lambda_X X, \Lambda_Y Y) \Lambda_X X - V \Lambda_X X) \right] \Lambda_X \end{aligned}$$

The last step follows because Λ_X and V are diagonal matrices and thus commutative.

Therefore, the Jacobian matrices of the dynamic system at (X, Y) and $(\Lambda_X X, \Lambda_Y Y)$ are similar and, thus, have the same Eigenvalues. $\square$

Proof of theorem 7.

Proof. From the definition $\tilde{x}_j = \frac{X_j}{X_i^{v_j}}$ it follows that

$$\begin{aligned}\frac{\dot{\tilde{x}}_j}{\tilde{x}_j} &= \frac{\dot{X}_j}{X_j} - v(j) \frac{\dot{X}_i}{X_i} \\ &= \frac{F_j(X, Y)}{X_j} - v(j) \frac{F_i(X, Y)}{X_i}\end{aligned}$$

Multiplying the equation with $\tilde{x}_j$ yields

$$\dot{\tilde{x}}_j = \frac{F_j(X, Y)}{X_i^{v_j}} - v(j) \tilde{x}_j \frac{F_i(X, Y)}{X_i}$$

Applying the defining equation of semi-homogeneity component wise for F_j and F_i with $\lambda := X_j^{-1}$ and $\lambda := X_i^{-1}$, respectively, yields

$$\dot{\tilde{x}}_j = F_j(\tilde{x}, \tilde{y}) - v(j) \tilde{x}_j F_i(\tilde{x}, \tilde{y})$$

The form of G also follows from the defining equation of semi-homogeneity component wise for G_j with $\lambda := X_i^{-1}$. $\square$

Proof of proposition 8.

Proof. The claim can be restated as

$$F(X, Y) = s(X, Y) \text{diag}(v_X)X, \quad s : (\mathbb{R}^{n_X}, \mathbb{R}^{n_Y}) \rightarrow \mathbb{R} \quad \Leftrightarrow \quad X \in BGP \quad (33)$$

I assume X to exhibit balanced growth. From the defining equation of semi-homogeneity it follows by differentiating with respect to λ and evaluating for $\lambda = 1$ that

$$\begin{aligned}D_X F(X, Y) \text{diag}(v_X)X + D_Y F(X, Y) \text{diag}(v_Y)Y &= \text{diag}(v_X)F(X, Y) \\ D_X G(X, Y) \text{diag}(v_X)X + D_Y G(X, Y) \text{diag}(v_Y)Y &= 0\end{aligned}$$

for an (X, Y) that satisfies $G(X, Y) = 0$.

The second equation inserted into the first one yields

$$\begin{aligned}D_X F(X, Y) \text{diag}(v_X)X - D_Y F(X, Y) D_Y G(X, Y)^{-1} D_X G(X, Y) \text{diag}(v_X)X \\ = \text{diag}(v_X)F(X, Y)\end{aligned} \quad (34)$$

From the BGP equation it follows that $F(X, Y)$ is in the kernel of

$$D_X F(X, Y) - D_Y F(X, Y) D_Y G(X, Y)^{-1} D_X G(X, Y) - \Gamma_X(X, Y)$$

Rewriting equation (34) yields

$$(D_X F(X, Y) - D_Y F(X, Y) D_Y G(X, Y)^{-1} D_X G(X, Y) - \Gamma_X(X, Y)) \text{diag}(v_X) X = 0$$

Therefore

$$F(X, Y) = s \cdot \text{diag}(v_X) X \quad \Leftrightarrow \quad \Gamma_X = s \cdot \text{diag}(v_X)$$

with some $s \in \mathbb{R}$, if the BGP is one-dimensional.

Now assume the left hand side of (33) holds. Differentiating the equation with respect to time and employing the algebraic equation yields

$$\begin{aligned} & D_X F F(X, Y) - D_Y F(X, Y) D_Y G(X, Y)^{-1} D_X G(X, Y) F(X, Y) \\ &= s(X, Y) \text{diag}(v_X) F(X, Y) + \frac{ds(X, Y)}{dt} \text{diag}(v_X) X \end{aligned} \quad (35)$$

Multiplying (34) with $s(X, Y)$ and substituting the assumption yields

$$\begin{aligned} & D_X F F(X, Y) - D_Y F(X, Y) D_Y G(X, Y)^{-1} D_X G(X, Y) F(X, Y) \\ &= s(X, Y) \text{diag}(v_X) F(X, Y) \end{aligned} \quad (36)$$

Subtracting the equation from (35) yields

$$\frac{ds(X, Y)}{dt} = 0 \quad (37)$$

Hence, X is in balanced growth.

Finally, if X is in balanced growth the same holds for Y as can be seen at the proof of theorem (1). □

Proof of proposition 9.

Proof. The proof follows directly from proposition 8. □

Proof of theorem 10.

Proof. I define $\tilde{h} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ according to

$$\tilde{h}(x, y) = \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_{i-1} \\ x_i \\ \tilde{x}_{i+1} \\ \tilde{x}_{n_X} \\ \tilde{y}_1 \\ \vdots \\ \tilde{y}_{n_Y} \end{pmatrix} \quad h(x, y) = \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_{i-1} \\ 1 \\ \tilde{x}_{i+1} \\ \tilde{x}_{n_X} \\ \tilde{y}_1 \\ \vdots \\ \tilde{y}_{n_Y} \end{pmatrix} \quad (38)$$

If I assume $i = 1$, the Jacobian matrices of $\tilde{h}$ and h read

$$D\tilde{h} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -\frac{v_2 x_2}{x_1^{v_2+1}} & x_1^{-v_2} & 0 & \dots & 0 \\ -\frac{v_3 x_3}{x_1^{v_3+1}} & 0 & x_1^{-v_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{v_n y_{n_Y}}{x_1^{v_n+1}} & 0 & 0 & \dots & x_1^{-v_n} \end{bmatrix} \quad Dh = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ -\frac{v_2 x_2}{x_1^{v_2+1}} & x_1^{-v_2} & 0 & \dots & 0 \\ -\frac{v_3 x_3}{x_1^{v_3+1}} & 0 & x_1^{-v_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{v_n y_{n_Y}}{x_1^{v_n+1}} & 0 & 0 & \dots & x_1^{-v_n} \end{bmatrix}$$

and, hence, $D\tilde{h}$ is regular while Dh is not.

The manifold

$$M = \{(x, y) | (x, y) = (\Lambda_X x, \Lambda_Y y), \lambda \in \mathbb{R}^+\}$$

is mapped on a single point $(\tilde{x}, \tilde{y})$ by h and, hence, the Center Manifold of stationary equilibria is mapped on a single point that is a stationary point of system (14).

Let ϕ denote the flow implied by the scale-adjusted DAS, (7), and $\tilde{\phi}$ denote the flow implied by the reduced scale-adjusted DAS, (14), but the trivial differential equation for $\tilde{x}_i$ be replaced by the differential equation for x_i . Let $\hat{\phi}$ denote the flow implied by the reduced scale-adjusted system alone. Both DAS are constructed from the same original DAS and, hence, $\tilde{h}$ is a conjugacy.

$$\phi = \tilde{h}^{-1} \circ \tilde{\phi} \circ \tilde{h}$$

Since a conjugacy preserves the real parts of the Eigenvalues of the underlying dynamic system by sign (+, −, 0) at a stationary point (see Li et al., 2003), ϕ and $\tilde{\phi}$ imply the same topology.

However, h is only a semi-conjugacy since it is not invertible but fulfills

$$h \circ \phi = \hat{\phi} \circ h.$$

Since the rank of h is $n - 1$, h preserves all the real parts of the Eigenvalues of ϕ by sign but one. Since the Eigenvector of Eigenvalue zero is tangent to the Center Manifold of stationary equilibria and this Center Manifold is mapped on a single point, I identified the Eigenvalue that is not preserved by h . Since this is the Eigenvalue zero, all real parts of Eigenvalues not equal to zero are preserved by sign by h . $\square$

Proof of remark 11.

Proof. The solution trajectories of the reduced scale-adjusted system comprise $\tilde{x}(0)$ and $\tilde{y}(0)$. If $X_i(0)$ is given, the initial conditions $X(0)$ and $Y(0)$ for the original system can be obtained. Integrating yields the time paths of variables X and Y . $\square$

APPENDIX B: FUNDAMENTAL THEOREM OF NORMALLY HYPERBOLIC INVARIANT MANIFOLDS

DEFINITION 7. (*r*-normally hyperbolic invariant manifold, Hirsch et al. (1977), Li et al. (2003))
An invariant manifold $V \subset M$ is called *r*-normally hyperbolic ($1 \leq r \leq \infty$) with respect to a flow ϕ^t , if

- (1) ϕ^t is C^r
- (2) $T_V M$, the tangent bundle of M restricted on V , splits into 3 continuous subbundles N^u , TV , and N^s such that

$$T_V M = N^u \oplus TV \oplus N^s$$

Thereby, TV is tangent to the invariant manifold V .

- (3) There exists a vector norm $\|\cdot\|$ such that for every $x \in \mathbb{R}$ and $0 \leq k \leq r$

$$\begin{aligned} \|D\phi^t|_{N_x^u}\|_{inf} &> \|D\phi^t|_{T_x V}\|_{sup}^k \\ \|D\phi^t|_{N_x^s}\|_{sup} &< \|D\phi^t|_{T_x V}\|_{inf}^k. \end{aligned}$$

Thereby, given a matrix A , $\|A\|_{inf}$ and $\|A\|_{sup}$ are defined as

$$\begin{aligned}\|A\|_{inf} &:= \inf_{\|x\|=1} \{\|Ax\|\} \\ \|A\|_{sup} &:= \sup_{\|x\|=1} \{\|Ax\|\}\end{aligned}$$

Theorem 12. (Fundamental theorem of normally hyperbolic invariant manifolds, Li et al. (2003))

Let $\phi^t : M \rightarrow M$ be a C^r flow of a C^∞ manifold M with $r \geq 1$ leaving the C^1 submanifold $V \subset M$ invariant, where V is assumed either being compact or being a leaf or a union of leaves of a foliation of M . Assume that ϕ^t is r -normally hyperbolic at V respective to the tangent bundle splitting $T_V M = N^u \oplus TV \oplus N^s$, where $D\phi^t$ exponentially expands and contracts the vectors in N^u and N^s , respectively. Then

- (1) *Existence:* There exists locally ϕ^t -invariant submanifolds $W^u(\phi^t)$ and $W^s(\phi^t)$, called a local unstable manifold and a local stable manifold at V , respectively, tangent at V to $N^u \oplus TV$, and $TV \oplus N^s$, respectively.
- (2) *Uniqueness:* Any locally invariant set near V lies in $W^u \cup W^s$.
- (3) *Characterization:* W^s consists of all points whose forward ϕ^t -orbits never strays far from V , and W^u of all points whose reverse ϕ^t -orbits never stray far from V .
- (4) *Smoothness:* W^u , W^s and V are class C^r .
- (5) *Foliation:* W^u and W^s are invariantly fibered by C^r submanifolds W_x^{uu} , W_x^{ss} , $x \in V$, tangent at x to N_x^u and N_x^s , respectively. W^{uu} and W^{ss} are invariant in the sense that $\phi^t(W_x^{ss}) \subset W_{\phi^t(x)}^{ss}$ for $t < 0$ and $\phi^t(W_x^{uu}) \subset W_{\phi^t(x)}^{uu}$ for $t > 0$. W_x^{ss} is characterized by $\|\phi(y) - \phi(x)\| \rightarrow 0$ exponentially as $t \rightarrow \infty$ for any $y \in W_x^{ss}$, and W_x^{uu} is characterized by $\|\phi(y) - \phi(x)\| \rightarrow 0$ exponentially as $t \rightarrow -\infty$ for any $y \in W_x^{uu}$.
- (6) *Continuity:* The leaves of foliation W^{uu} and W^{ss} are continuous on parameter $x \in V$ in C^1 -topology.
- (7) *Permanence:* If $\tilde{\phi}^t$ is another C^r flow on M and is C^r close to ϕ^t (i.e., close in C^r -norm). Then $\tilde{\phi}^t$ is r -normally hyperbolic at some unique submanifold $\tilde{V}$, which is C^r close to V . The invariant manifolds $W^u(\tilde{\phi}^t)$, $W^s(\tilde{\phi}^t)$, and the leaves $W_x^{uu}(\tilde{\phi}^t)$, $W_x^{ss}(\tilde{\phi}^t)$, are C^r close to those of ϕ^t .
- (8) *Linearization:* Near V , ϕ^t is topologically conjugate (i.e. C^0 conjugate) to $D\phi^t|_{N^u \oplus N^s}$, the restriction of the Jacobian of the flow to the subspace $N^u \oplus N^s$.

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