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Stochastic volatility and stochastic leverage

Almut E. D. Veraart and Luitgard A. M. Veraart

School of Economics and Management
Aarhus University
Bartholins Allé 10, Building 1322, DK-8000 Aarhus C
Denmark

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Almut E. D. Veraart [*]
CREATES
Aarhus University

Luitgard A. M. Veraart [†]
Institut für Stochastik
Universität Karlsruhe (TH)

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Abstract

This paper proposes the new concept of *stochastic leverage* in stochastic volatility models. Stochastic leverage refers to a stochastic process which replaces the classical constant correlation parameter between the asset return and the stochastic volatility process. We provide a systematic treatment of stochastic leverage and propose to model the stochastic leverage effect explicitly, e.g. by means of a linear transformation of a Jacobi process. Such models are both analytically tractable and allow for a direct economic interpretation. In particular, we propose two new stochastic volatility models which allow for a stochastic leverage effect: the generalised Heston model and the generalised Barndorff-Nielsen & Shephard model. We investigate the impact of a stochastic leverage effect in the risk neutral world by focusing on implied volatilities generated by option prices derived from our new models. Furthermore, we give a detailed account on statistical properties of the new models.

Keywords Stochastic volatility · volatility of volatility · stochastic correlation · leverage effect · Jacobi process · Ornstein–Uhlenbeck process · square root diffusion · Lévy process · Heston model · Barndorff-Nielsen & Shephard model

JEL Classification Numbers C1 · C5 · G0 · G1

1 Introduction

Stochastic volatility (SV) models for asset prices have gained great popularity in recent years. The main reason for this is that they can explain many empirical facts observed in financial markets, such as time–varying volatility and volatility clusters. In particular, they are able to reproduce the observed implied volatility smile and are therefore essential for pricing and hedging financial derivatives, see e.g. Rogers & Veraart (2008) and the references therein.

A very important empirical fact which can be modelled via a stochastic volatility model is the so–called *leverage effect*. The leverage effect refers to the relationship between asset price returns and volatility which tend to be negatively correlated. One explanation, which also led to its name, is that negative stock return might increase financial leverage which itself makes the stock riskier and therefore leads to higher volatility. This effect was initially analysed by Black (1976) and was further

[*]CREATES, School of Economics and Management, Aarhus University, DK-8000 Aarhus C, DENMARK, averaart@creates.au.dk

[†]Institut für Stochastik, Universität Karlsruhe (TH), 76128 Karlsruhe, GERMANY, veraart@stoch.uni-karlsruhe.de

supported by studies by Christie (1982), Nelson (1991) among others and, more recently, by Harvey & Shephard (1996), Bouchaud et al. (2001), Tauchen (2004, 2005), Yu (2005), Bollerslev et al. (2006). In empirical data, the leverage effect is particularly apparent, when looking at data from indices.

The leverage effect can be modelled in terms of two correlated stochastic processes which drive the asset price process and the volatility process. Well-known models incorporating the leverage effect are e.g. the Heston model (Heston 1993) and the Barndorff-Nielsen & Shephard model (Barndorff-Nielsen & Shephard 2001, 2002), (BNS model). In both models a constant correlation is assumed.

The present paper introduces the concept of a *stochastic* leverage effect in continuous time in a very general framework. We model the correlation between the asset returns and the volatility as a stochastic process. To the best of our knowledge this is the first paper that studies such a general stochastic model for the leverage.

Stochastic leverage has the advantage that it introduces an additional factor or source of randomness into a stochastic volatility model which has a natural economic interpretation. Among the stochastic volatility models, it is well known that multifactor SV models outperform single factor SV in practice, but might not necessarily be as analytically tractable. We will present how stochastic leverage can be included in stochastic volatility models such that some analytic results can still be obtained.

Another big advantage of stochastic leverage was pointed out by Carr & Wu (2007). They show that not only stochastic volatility but also stochastic skew can be observed in financial markets. They provide empirical results that stochastic skew is present in empirical currency option data and show that it is very important to account for it. They briefly mention the possibility to incorporate this feature by randomising the correlation parameter between the currency return and the stochastic volatility process, but do not further investigate this approach. This is what we do in the present paper.

Note that stochastic correlation as such has been studied in the literature before. However, the focus has mainly been on modelling the correlation between various asset prices and not between stochastic volatility and the asset price. In the context of multivariate asset price models, one stochastic process has received particular attention: the Wishart process. It has been introduced in the probability literature by Bru (1991) and has been studied extensively recently in the econometrics literature, see e.g. Gouriéroux (2006), Gouriéroux et al. (2009), since it can be used as a building block for modelling stochastic correlation between various assets. However, the shortcomings of such a model are well–known, for a discussion see Pigorsch & Stelzer (2009). In particular, the stochastic correlation generated by such models is not straightforward to interpret and, hence, we propose a different approach which leads to analytically tractable models which have a direct economic interpretation.

In the following, we will use the expression *stochastic correlation* and *stochastic leverage* interchangeably, and both refer to the stochastic process modelling the correlation between the stochastic drivers of the volatility and the asset price processes.

The outline of this article is as follows. In Section 2, we introduce the concept of stochastic leverage. We describe two general classes of stochastic volatility models (with and without jumps) which exhibit stochastic leverage. We then present specific models which can be used to model a correlation process and therefore account for the leverage effect. The most important process we study as a building block for a stochastic correlation process is the Jacobi diffusion. It is analytically tractable and can be easily extended in such a way that its support is the interval $[-1, 1]$ which makes it an ideal process to model correlation.

In Section 3, we extend the Heston model and the Barndorff-Nielsen & Shephard model by incorporating stochastic leverage using a transformed Jacobi diffusion. We develop the semimartingale characteristics corresponding to the new dynamics of the asset price.

In Section 4, we discuss the change of measure from the real world probability measure to the risk–neutral probability measure and discuss its application to pricing financial derivatives. Particular emphasis is on the structure preserving change of measure in both the generalised Heston and the generalised Barndorff-Nielsen & Shephard models.

Section 5 studies the influence of the stochastic leverage on the pricing of European plain vanilla options in the generalised Heston model. A sensitivity study shows the relationship between the various model parameters describing the stochastic leverage. In particular, we compare the results with the classical Heston model with constant correlation.

Next, we turn our attention to statistical aspects of our new stochastic volatility and stochastic leverage models. Section 6 focuses on the impact of a stochastic leverage effect on return–volatility regressions, which are widely used in econometrics for measuring both the leverage and the volatility feedback effect and, furthermore, we investigate the effect of stochastic leverage on the ability to forecast volatility based on option implied volatilities.

Section 7 is then devoted to the problem of estimating stochastic leverage non–parametrically and to parameter estimation in our new model classes.

Finally, Section 8 concludes.

All proofs are relegated to the appendix.

2 Modelling stochastic leverage

2.1 The concept of stochastic leverage

This section introduces the concept of a stochastic leverage effect, i.e. a stochastic correlation between the driving process of the asset price and the driving process of the stochastic volatility process.

Throughout the paper, we will assume that the logarithmic asset price $Y = (Y_t)_{t \geq 0}$ is given by an Itô semimartingale, which is a standard assumption, see e.g. Barndorff-Nielsen & Shephard (2002, 2007), Jacod (2008), Aït-Sahalia & Jacod (2009). An Itô semimartingale is defined as a semimartingale whose characteristics are absolutely continuous with respect to the Lebesgue measure, see e.g. Jacod (2008). In particular, its dynamics are given by

$$dY_t = \widetilde{a}_t dt + \sigma_{t-} dW_t + dJ_t, \tag{1}$$

where $\widetilde{a} = (\widetilde{a}_t)_{t \geq 0}$ is a predictable drift process, $\sigma = (\sigma_t)_{t \geq 0}$ is a predictable stochastic volatility process and $J = (J_t)_{t \geq 0}$ is a pure jump component.

For the stochastic volatility process σ , we assume that it satisfies

$$d\sigma_t^2 = \widetilde{b}_t dt + f(\sigma_t^2) dZ_t, \tag{2}$$

for a predictable process $\widetilde{b} = (\widetilde{b}_t)_{t \geq 0}$ which possibly describes the mean reversion of the squared volatility process, a deterministic function $f : \mathbb{R}_+ \to \mathbb{R}_+$ and a drift-less Lévy process $Z = (Z_t)_{t \geq 0}$. From the Lévy Khintchine formula, we know that Z consists of a Brownian motion part (denoted by $Z^c = (Z^c_t)_{t \geq 0}$) and a pure jump process denoted by $Z^d = (Z^d_t)_{t \geq 0}$. For ease of exposition, we will throughout the paper assume that Z^c is in fact a *standard* Brownian motion (if $Z^c \not\equiv 0$).

Note, that both stochastic processes $\widetilde{a}$ and $\widetilde{b}$ are assumed to be predictable stochastic processes. Particularly, they can depend on σ as well but can also have additional stochastic drivers.

In order to account for the leverage effect, most of the well–known stochastic volatility models allow for correlation between the Brownian motions, which drive the logarithmic asset price and the volatility process, i.e. $Cor(W_t, Z^c_t) \neq 0$ and/or between the jump processes which drive the asset price and the volatility, i.e. $Cor(J_t, Z_t^d) \neq 0$. In particular, these correlations are usually assumed to be constant and not time–varying or stochastic.

In this paper, we propose to model the correlation by a stochastic process. For ease of exposition, we assume throughout the paper that the leverage effect appears either in the continuous or in the jump component and not in both. However, extensions to the more general case are straightforward and can be constructed along the lines of this paper.

Definition We assume that the logarithmic asset price is given by an Itô semimartingale as defined in (1) and that the stochastic volatility process is defined as in (2).
The *stochastic correlation* or *stochastic leverage* process is defined as the predictable stochastic process $\rho = (\rho_t)_{t\geq 0}$ taking only values in $[-1,1]$ and satisfying $d[W, Z^c]_t = \rho_t dt$ or $d[J, Z^d]_t = \rho_t d[\widetilde{J}, Z^d]_t$, for a pure jump process $\widetilde{J} = \left(\widetilde{J}_t\right)_{t\geq 0}$, which is dependent of Z^d.

2.2 Stochastic leverage and stochastic volatility (of volatility)

Note that the concept of stochastic leverage is closely linked to the concept of stochastic volatility and stochastic volatility of volatility. So, before we turn to explicit models for stochastic leverage, we briefly discuss the relationship between those three quantities.

As already mentioned, one factor stochastic volatility models are not supported by empirical studies. Hence, recent research on stochastic volatility models has focused on multi–factor stochastic volatility models. However, additional sources of randomness can not only be introduced on the same level as the stochastic volatility process, but also by either stochastic leverage or stochastic volatility of volatility. The latter has recently been studied by Barndorff-Nielsen & Veraart (2009).

In particular, they have studied stochastic volatility processes of the type

$$d\sigma_t^2 = \widetilde{b}_t dt + \gamma_t f(\sigma_t^2) dZ_t,$$

where $\gamma = (\gamma_t)_{t\geq 0}$ denotes a stationary, non–negative stochastic variance of variance process, which is independent of all the other driving processes in the asset price model. If the asset price is given by (1) and if additionally Z is a Brownian motion which is correlated with W with correlation coefficient $c \in [-1,1]$ then the quadratic covariation of Y and σ^2 is given by

$$d[Y, \sigma^2]_t = c\gamma_t f(\sigma_t^2) dt, \tag{3}$$

If instead $Z \equiv J$, then

$$d[Y, \sigma^2]_t = \gamma_t f(\sigma_t^2) d[J]_t, \tag{4}$$

which is similar to the stochastic leverage mentioned above, where we have an additional source of randomness in the quadratic covariation. However, if Z is independent of W and J, then $[Y, \sigma^2] \equiv 0$.

So, we see that both stochastic leverage and stochastic volatility of volatility can lead to a similar (possibly even to the same) structure of $\left[Y, \sigma^2\right]$. However, it should be stressed that the two concepts are not identical. In particular, the existence of stochastic leverage does not necessarily imply the existence of stochastic volatility of volatility and, vice versa, the existence of stochastic volatility of volatility does not necessarily imply the existence of stochastic leverage.

Finally note that stochastic leverage is a particular case of (additional) stochastic volatility. So, we observe that there are very close links between stochastic volatility, stochastic volatility of volatility and stochastic leverage, but they are generally not the same.

2.3 Models for leverage

In this section we present different approaches to modelling a stochastic or time–varying leverage effect. The classical approach to model a constant leverage effect over time is to use a constant which models the correlation between the stochastic drivers of the asset price process and the volatility process. When looking at empirical data, however, it turns out that the leverage effect is not constant over time.

2.3.1 Time–varying, deterministic leverage

A first approach is to assume that the correlation between the two stochastic drivers is a deterministic function of time which varies between $[-1, 1]$. The advantage of such an approach is that no additional source of randomness is introduced which simplifies the calculations. In particular, for hedging purposes, one only has to deal with two sources of risk. Since this is just a special case of a stochastic leverage process, we do not go into more details.

2.3.2 Local leverage

A first step to introduce randomness into the correlation process is to assume that it is a function of the stochastic asset price and/or the stochastic volatility. Such a stochastic leverage is effectively a *local leverage* and corresponds to the concept of local volatility where volatility is modelled as a function of the underlying asset price. Models of this type have been studied by e.g. Romano & Touzi (1997) and, more recently, by Bandi & Renò (2008*b*).

2.3.3 Finite state Markov process

A next step would be to assume that the correlation process is a finite state Markov chain. If for example it is assumed that the correlation is negative, one could choose a finite number of values in $[-1, 0]$ which describe the state of the Markov chain. We will see later on that even a continuous diffusion process can be used to model a correlation process which can only visit two states.

2.3.4 Jacobi process

When modelling stochastic leverage as a stochastic process, we first need to ensure that the stochastic process only takes values in $[-1, 1]$. We could therefore use an arbitrary stochastic process such as a Brownian motion and use an appropriate function to map it onto the interval $[-1, 1]$. Such an approach, however, usually lacks economic interpretation. A natural building block for a correlation process which allows economic interpretation is the *Jacobi process*. It is a mean–reverting diffusion process which only takes values in $[0, 1]$. Applying a linear transformation to the Jacobi diffusion results in a process which only takes values in $[-1, 1]$. We will discuss more details of the Jacobi process in Section 2.4.

2.3.5 A general stochastic leverage process

Finally, we present a very general approach on how to construct a model for stochastic leverage. Similar, to the model based on the Jacobi process, we focus on constructing a stochastic process which takes only values in $[0, 1]$. Such a process can then be used as a building block for modelling stochastic correlation.

A very general framework for constructing such a process, is given as follows. Let us assume that $U^{(i)} = (U_t^{(i)})_{t\geq 0}$ for $i = 1, 2$ are independent, non–negative semimartingales. Then, we define a stochastic process $R = (R_t)_{t\geq 0}$ by

$$R_t = \frac{U_t^{(1)}}{U_t^{(1)} + U_t^{(2)}}.$$

Clearly, we always have that $0 \leq R_t \leq 1$.

Note that the Jacobi process is essentially constructed by using two independent square root diffusions (Cox et al. (1985)) for $U^{(1)}$ and $U^{(2)}$. However, we can choose various other processes for $U^{(1)}$ and $U^{(2)}$ as long as they are guaranteed to stay positive. Hence, stochastic volatility models, which satisfy the positivity requirement, are natural choices for $U^{(i)}$, where $i = 1, 2$. It is also possible, to construct a process R which is purely driven by jumps. E.g. we could model $U^{(i)}$ as a Lévy subordinator. In particular, if we choose gamma subordinators, we obtain a process R which is Beta distributed and purely jump–driven, as opposed to the Jacobi process whose stationary distribution is also given by a Beta distribution, but which is driven by a Brownian motion. Another possibility is to model the $U^{(i)}$ by non–Gaussian Ornstein Uhlenbeck processes.

In the following, we will restrict our attention to stochastic leverage models based on the Jacobi diffusion and we will study the potential of jump–driven models for stochastic correlation in future research.

2.4 The Jacobi process

The Jacobi process belongs to the class of *solvable* diffusion processes, see Albanese & Kuznetsov (2005), and is therefore analytically tractable. It is linked to the Jacobi polynomials in the sense that the eigenfunctions of the Jacobi diffusion generator can be expressed in terms of the Jacobi polynomials. Its stationary distribution is a Beta distribution. The Jacobi process has been studied by e.g Gouriéroux & Valéry (2004) in the context of how such a diffusion can be estimated from data, Gouriéroux & Jasiak (2006) studied a multivariate version as a tool for modelling smooth transitions; Larsen & Sørensen (2007) use an general class of a Jacobi-type diffusion to model the logprices of exchange rates in a target zone controlled by central banks, Forman & Sørensen (2008) study the Pearson diffusion in detail and the Jacobi process is one special case of a Pearson diffusion. Finally, Schoutens (2000) collects many properties of diffusion processes which are linked to some orthogonal polynomials.

In the following we give a brief account of the properties of a Jacobi process which will be used in our analysis. A Jacobi process satisfies the SDE

$$dV_t = (\zeta - \eta V_t)dt + \theta\sqrt{V_t(1 - V_t)}dW_t^V, \tag{5}$$

where ζ, η, θ are positive constants and W^V is a standard Brownian motion. This can be rewritten as

$$dV_t = -\eta\left(V_t - \frac{\zeta}{\eta}\right)dt + \theta\sqrt{V_t(1 - V_t)}dW_t^V.$$

This process takes values in $[0, 1]$ and is mean–reverting to $\frac{\zeta}{\eta}$ at speed η.

For a correlation coefficient ρ we want to have a process, which takes values between -1 and 1 and which is mean reverting. Hence we work with a linear transformation of the Jacobi process which is given by

$$\rho_t = 2V_t - 1. \tag{6}$$

Then from Itô's formula the dynamics of ρ are given by

$$d\rho_t = ((2\zeta - \eta) - \eta\rho_t)\, dt + \theta\sqrt{(1+\rho_t)(1-\rho_t)}dW_t^V.$$

Clearly, ρ takes values between -1 and 1 and is mean–reverting to $\frac{2\zeta-\eta}{\eta}$ at speed η.

The dynamics of the process ρ is a special case of the diffusion process studied by Larsen & Sørensen (2007). They consider a diffusion $\tilde{\rho}$ taking values in $(R_2 - R_4, R_2 + R_4)$ which satisfies

$$d\tilde{\rho}_t = -R_1(\tilde{\rho}_t - (R_2 + R_3R_4))dt + R_5\sqrt{R_4^2 - (\tilde{\rho}_t - R_2)^2}dW_t^V \tag{7}$$

where $R_1 > 0, -1 < R_3 < 1$ and W^V is a Brownian motion. This diffusion is an ergodic diffusion if and only if

$$\kappa_1 = R_1(1 - R_3)R_5^{-2} \geq 1,$$
$$\kappa_2 = R_1(1 + R_3)R_5^{-2} \geq 1.$$

Its stationary distribution is a shifted and rescaled Beta distribution and its probability density function is given by

$$f_{\tilde{\rho}}(x) = (R_4 + R_2 - x)^{\kappa_1 - 1}(R_4 - R_2 + x)^{\kappa_2 - 1}(2R_4)^{1-\kappa_1-\kappa_2}\frac{1}{B(\kappa_1, \kappa_2)},$$

for $x \in (R_2 - R_4, R_2 + R_4)$ and 0 otherwise. Here $B(\cdot, \cdot)$ denotes the Beta–function. If $\kappa_1 < 1$ then the boundary $R_2 + R_4$ can be reached in finite time, if $\kappa_2 < 1$ then the boundary $R_2 - R_4$ can be reached in finite time. The eigenfunctions of the diffusion generator can be expressed in terms of the Jacobi polynomials and can be found in Larsen & Sørensen (2007).

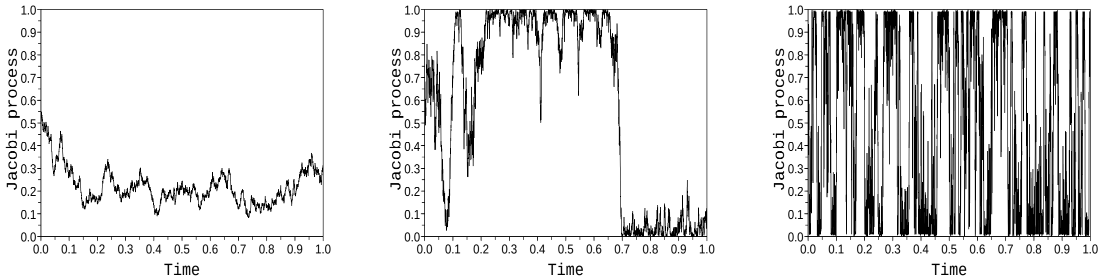
Clearly, ρ satisfies the SDE (7) if we set $R_4 = 1, R_2 = 0, R_5 = \theta, R_1 = \eta, R_3 = 2\frac{\xi}{\eta} - 1$ and require that $R_3 \in (-1, 1)$.

We have therefore found a process which stays within $[-1, 1]$ and hence can be used to model correlation. If some additional information were available, e.g. that the correlation varies within a subinterval of $[-1, 1]$, a process described by the SDE (7) with bounds chosen accordingly could be used to model the restricted correlation process. An example would be to assume that correlation is strictly negative or that it varies only within a small interval centered around 0 etc.

A very interesting characteristic of a Jacobi process is that it tends to a jump process with state space $\{0, 1\}$ and constant intensities, if θ tends to infinity. This result and further discussions can be found in Gouriéroux & Jasiak (2006). Figure 1 illustrates the sensitivity of the Jacobi process with respect to θ. Generally, the smaller the parameter θ, the smoother are the sample paths. For increasing values of θ, we observe a jump–type behaviour. In particular, the Beta distribution of the stationary Jacobi process tends in distribution to a Bernoulli distribution with parameter $\frac{\zeta}{\eta}$. It is also clear from the definition of the process that the term in front of the Brownian motion $\theta\sqrt{V_t(1 - V_t)}$ is essentially zero for V_t close to 0 or 1. It attains its maximum for $V = \frac{1}{2}$ and therefore we see that the process has a clear tendency to move away from values around $\frac{1}{2}$ and goes towards its natural bounds 0 or 1. At the boundaries the mean reversion of the process kicks in particularly strongly. If the parameter θ is large, however, the effect of a large variance in the center and a low variance at the boundaries dominates the overall behaviour of the process, i.e. the process moves towards the boundaries.

We could therefore model a rather extreme behaviour of the correlation process by increasing the parameter θ. Then the correlation process will essentially take two values. This reminds on the Markov chain approach described previously where it was assumed that the correlation process can only take a finite number of values. We therefore see that the Jacobi process can be used to model such a behaviour as well.

Generally, we find that the Jacobi diffusion and its generalisations are ideal diffusions to model stochastic correlation. We assume that stochastic correlation is mean–reverting to a long–term mean and is driven by a Brownian motion whose fluctuation can be amplified by using a higher volatility parameter for the stochastic correlation process. From an economic perspective this is perfectly sensible. We do expect that the correlation between the stock returns and the volatility process has some long term mean around which it fluctuates. If we want to model more extreme stochastic behaviour, this can be done by either playing with the volatility parameter in the Jacobi diffusion or by choosing a slower speed of mean reversion parameter.



(a) $\theta = 1$ (b) $\theta = 5$ (c) $\theta = 25$

Figure 1: **Sensitivity with respect to** θ Sample path of a Jacobi process with $V_0 = 0.5, \zeta = 0.5, \eta = 1$ and $\theta \in \{1, 5, 25\}$. Number of steps: 5000; step size 0.0002.

3 Generalised Heston and generalised BNS model with stochastic leverage

The aim of this section is to introduce two concrete models which allow for the new concept of stochastic leverage. They are extensions of stochastic volatility models which are particular popular and successful both from a practical and a theoretical point of view: The Heston model, Heston (1993), in which the stochastic volatility is modelled as a square root diffusion, see Cox et al. (1985), and the Barndorff-Nielsen & Shephard model, in which the stochastic volatility is modelled as a non–Gaussian Ornstein–Uhlenbeck process, see Barndorff-Nielsen & Shephard (2001, 2002).

3.1 Model definition

Suppose that we have a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, on which we define four independent processes: three standard Brownian motions $W = (W_t)_{t\geq 0}$, $\widetilde{W} = \left(\widetilde{W}_t\right)_{t\geq 0}$ and $W^V = \left(W_t^V\right)_{t\geq 0}$ and a Lévy subordinator $L = (L_t)_{t\geq 0}$. Throughout this paper, we denote by $Y = (Y_t)_{t\geq 0}$ the logarithmic asset price, by $S_t = S_0 \exp(Y_t)$ the asset price, where $S_0 > 0$, by $\sigma = (\sigma_t)_{t\geq 0}$ the stochastic volatility, and by $\rho = (\rho_t)_{t\geq 0}$ the stochastic correlation process.

First of all we extend the classical Heston model by allowing for stochastic correlation.

Definition The *Generalised Heston model* (GH) is defined by

$$\begin{aligned} dY_t &= \left(\mu + b\sigma_t^2\right) dt + \sigma_t dX_t, \\ dX_t &= \rho_t dW_t + \sqrt{1-\rho_t^2} d\widetilde{W}_t, \\ d\sigma_t^2 &= \alpha(\beta - \sigma_t^2)dt + \gamma\sigma_t dW_t, \end{aligned} \tag{8}$$

where $\mu, b \in \mathbb{R}$, $\alpha, \beta, \gamma > 0$, and where $\rho = (\rho_t)_{t\geq 0}$ is a stochastic correlation process. Furthermore, the processes W, $\widetilde{W}$, ρ are assumed to be independent.

If $2\alpha\beta \geq \gamma^2$, σ^2 stays almost surely positive when $\sigma_0 > 0$.

Remark It follows immediately from Lévy's Theorem, that the process X is a standard Brownian motion, see e.g. (Musiela & Rutkowski 2005, p. 232).

Definition A GH model is called *generalised Heston model with Jacobi correlation* (GHJ) if ρ satisfies

$$d\rho_t = \left((2\zeta - \eta) - \eta\rho_t\right) dt + \theta\sqrt{(1+\rho_t)(1-\rho_t)}dW_t^V,$$

where η, ζ, θ are positive constants and $W^V = \left(W_t^V\right)_{t\geq 0}$ is a standard Brownian motion.

Similarly, we can defined the generalised BNS model in the following way.

Definition The *generalised BNS model* (GBNS) is defined by

$$\begin{aligned} dY_t &= \left(\mu + b\sigma_t^2\right) dt + \sigma_t dW_t + \rho_{\lambda t} dL_{\lambda t}, \\ d\sigma_t^2 &= -\lambda\sigma_t^2 dt + dL_{\lambda t}, \end{aligned} \tag{9}$$

where $\sigma_0^2 = \int_{-\infty}^0 e^{\lambda s} dL_{\lambda s}$. Furthermore, $\mu, b \in \mathbb{R}$ and $\lambda > 0$. The stochastic correlation process ρ is assumed to be non–positive. Furthermore, we assume independence between the processes W, L and ρ.

Clearly, σ^2 is a non–Gaussian Ornstein Uhlenbeck process with stationary representation

$$\sigma_t^2 = \int_{-\infty}^t e^{-\lambda(t-s)} dL_{\lambda s}.$$

Remark By restricting ρ to be non–positive, we ensure that the jumps in the price are locally bounded, see Remark 2.1 in Hubalek & Sgarra (2007).

Remark For parameter estimation in a GBNS model, it will be necessary to fix some moments of either ρ or L to make sure that the model is uniquely identified. Otherwise, one could always multiply ρ by a constant and scale the subordinator L accordingly.

In this paper we suggest to model the correlation coefficient by a linear transformation of a Jacobi process.

Definition A GBNS model is called a *generalised BNS model with Jacobi correlation* (GBNSJ), if the stochastic leverage $\rho = (\rho_t)_{t\geq 0}$ is given by a linear transformation of a stationary Jacobi process $\rho_t = -V_t$, where

$$dV_t = (\zeta - \eta V_t)dt + \theta\sqrt{V_t(1-V_t)}dW_t^V \tag{10}$$

where W^V is a standard Brownian motion and $\zeta, \eta, \theta > 0$.

Remark Note that in models of type (9), the quadratic covariation between the price and the squared stochastic volatility is given by

$$d[Y, \sigma^2] = \rho_{\lambda t}(dL_{\lambda t})^2 = \rho_{\lambda t} d[L]_{\lambda t}. \tag{11}$$

Since the leverage effect is due to the jump component and not due to a diffusion, the process ρ does not solely describe the quadratic covariation between the log–price and the volatility, but the jumps play a direct role, too. Hence we cannot interpret ρ as a correlation coefficient as easily as in the diffusion set up. In particular, we do not have to restrict ρ to the interval $[-1, 0]$, but could allow for any negative value ρ. Hence, there is a great flexibility in how to model the stochastic process ρ. In this paper, we will focus on a process which takes values in $[-1, 0]$ and all smaller values of $[Y, \sigma^2]$ are assumed to be due to the jump size of L. Extensions to processes which can take values between $[-K, 0]$ for some $K > 0$ are straightforward. Carrying out empirical studies will help to find out which value for K is realistic and whether it should be a finite value at all or whether an unbounded process ρ describes empirical data even better.

Throughout this section, we will work with the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the triple $(W, \widetilde{W}, W^V)$ in the GH model and by (W, L_λ, W^V) in the GBNS model.

3.2 Semimartingale characteristics

First of all, we derive the semimartingale characteristics, see Jacod & Shiryaev (2003), of the new stochastic volatility and stochastic leverage models. We will need these results later, when we study how the dynamics of our new models change when we consider them under a risk–neutral probability measure.

Proposition 3.1 *The semimartingale characteristics of Y in the GH model are given by (B, C, ν_Y), where*

$$dB_t = (\mu + b\sigma_t^2)dt, \qquad dC_t = \sigma_t^2 dt, \qquad \nu_Y(dt, dx) \equiv 0,$$

Next, we study the dynamics of the asset price in the GH framework. A straightforward application of Itô's formula leads to the following result.

Proposition 3.2 *In the framework of a GH model, with logarithmic asset price $Y = (Y_t)_{t\geq 0}$, the dynamics of the asset price $S_t = \exp(Y_t)$ are given by*

$$\begin{aligned} dS_t &= S_t\left(\left(\mu + \left(b + \frac{1}{2}\right)\sigma_t^2\right)dt + \sigma_t dX_t\right) \\ &= S_t\left(\left(\mu + \left(b + \frac{1}{2}\right)\sigma_t^2\right)dt + \sigma_t\rho_t dW_t + \sigma_t\sqrt{1-\rho_t^2}d\widetilde{W}_t\right). \end{aligned}$$

Next, we turn our attention to the GBNS model. Again, we start by deriving the semimartingale characteristics of the logarithmic asset price. Since the only jump processes we deal with in the GBNS model are of finite variation, we can work with the zero truncation function, which we denote by $h(x) \equiv 0$. Furthermore, for any semimartingale X, we will denote by μ_X the Poisson random measure associated with the jumps of X and by ν_X its predictable compensator. Note that throughout this paper, we will use $\star$ to indicate integration with respect to (compensated) jump measures and $\cdot$ to indicate standard stochastic integration.

Proposition 3.3 *The semimartingale characteristics of Y in the GBNS model (defined in (9)) with respect to the truncation function $h(x) \equiv 0$ are given by (B, C, ν_Y), where*

$$dB_t = (\mu + b\sigma_t^2)dt, \qquad dC_t = \sigma_t^2 dt, \qquad \nu_Y(dt, dx) = F(t, dx)dt,$$

where $\mathbb{I}_A(-x) \star F(t, dx) = \mathbb{I}_A(\rho_{\lambda t} x) \star U_L(dx)\lambda$ for any $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ and where U_L denotes the Lévy measure of L.

From the semimartingale characteristics above, we see that the jump part of Y is not a Lévy process, since its characteristics are generally time–varying and not deterministic. Hence, our new model is a real generalisation of the BNS model and nests the BNS model when we set ρ to a constant.

Proposition 3.4 *We obtain the following representation results for the log–price and the price process in the GBNS model:*

(i) The semimartingale Y (defined in (9), (10)) can be represented as

$$Y_t = Y_0 + \left\{ \int_0^t \left(\mu + b\sigma_t^2\right) dt + \rho_{\lambda t} \star \nu_L \right\} + \left\{ \int_0^t \sigma_s dW_s + \rho_{\lambda t} \star (\mu_L - \nu_L) \right\}.$$

The term in the first bracket is of finite variation and the term in the second bracket is the sum of the continuous part plus the jump part of a local martingale.

(ii) The dynamics of the asset price $S_t = \exp(Y_t)$ are given by

$$dS_t = S_{t-} \Bigg(\left(\mu + \left(b + \frac{1}{2} \right) \sigma_t^2 + \int_0^{\lambda t} \int_0^\infty \left(e^{\rho_s x} - 1\right) U_L(dx) ds \right) dt + \sigma_{t-} dW_t + dM_t \Bigg),$$

where

$$M_t = \int_0^{\lambda t} \int_0^\infty \left(e^{\rho_s x} - 1\right) (\mu_L - \nu_L)(dx, ds),$$

which is clearly a local martingale.

Note that in this representation it becomes clear, why we assume that ρ can only take negative values, otherwise the integral above would not exist.

4 Change of measure in the GH and GBNS model and applications to option pricing

In order to use our new models for option pricing, we have to change the probability measure. So far, we have worked under the real world measure $\mathbb{P}$. We have argued that stochastic correlation between the drivers of the stochastic volatility and the asset price process is an empirical fact which can be observed when looking at empirical asset price data, i.e. using the real world probability measure $\mathbb{P}$. When we want to price options, however, we need to find an equivalent martingale measure. We will see in this section that such a measure can be constructed in such a way, that the model structure, and hence a stochastic leverage process, can be preserved under the new measure.

Let us assume that $\mathbb{P}^*$ is another probability measure which is absolutely continuous with respect to $\mathbb{P}$. Then there exists a unique, see Jacod & Shiryaev (2003, Theorem III.3.4), and up to $\mathbb{P}$-, $\mathbb{P}^*$–indistinguishability $\mathbb{P}$–martingale Z, such that, for all $t > 0$

$$Z_t = \mathbb{E}\left(\frac{d\mathbb{P}^*}{d\mathbb{P}} \middle| \mathcal{F}_t \right).$$

Z is then called the density or Radon-Nikodým derivative.

In the following, we will study three aspects: First, we study how the semimartingale characteristics of Y change under the change of measure. Next, we derive a particular representation for the density process Z. Finally, we study which class of measure changes preserves the structure of the GH and GBNS model.

Recall that a probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{A})$ is an *equivalent martingale measure* (EMM) if the discounted asset price $e^{-rt}S_t$ is a martingale under $\mathbb{P}^*$. Let $\mathcal{M}$ denote the set of equivalent martingale measures for the GH or the GBNS model and let $\mathcal{M}' \subset \mathcal{M}$ denote the set of *structure preserving* EMMs of the GH or the GBNS model. Further, we write $\mathcal{E}$ for the stochastic exponential, see (Jacod & Shiryaev 2003, I.4f).

4.1 Change of measure in the GH model

A straightforward application of the Girsanov theorem leads to the following result.

Assumption (A): Let Y denote the semimartingale defined in the GH model (8), where ρ is stochastic correlation process. Furthermore, we assume that all local martingales are representable with respect to $(W, \widetilde{W}, W^V)$.

Proposition 4.1 *Let $\mathbb{P}^* \in \mathcal{M}$ and let assumption (A) be satisfied. Then there exists a predictable and on $[0,T]$ square integrable process $\psi = \left(\psi^{(1)}, \psi^{(2)}, \psi^{(3)}\right)' = (\psi_t)_{t\geq 0}$ such that*

$$Z_t = \mathcal{E}\left(\int_0^{\cdot} \psi_s^{(1)} dW_s + \int_0^{\cdot} \psi_s^{(2)} d\widetilde{W}_s + \int_0^{\cdot} \psi_s^{(3)} dW_s^V \right)_t, \quad 0 \leq t \leq T,$$

is a density process, i.e. $Z_t = \mathbb{E}\left(\frac{d\mathbb{P}^*}{d\mathbb{P}} \middle| \mathcal{F}_t \right)$ and $\mathbb{E}(Z_T) = 1$.

The process ψ satisfies

$$\mu + \left(b + \frac{1}{2}\right)\sigma_t^2 + \sigma_t \psi_t^{(1)} - r = 0$$

$d\mathbb{P} \otimes dt$ almost surely, where $r > 0$ denotes the riskless interest rate.

The theorem above describes a general change of measure according to the Girsanov theorem for the GH model. If we specify the predictable process $\psi^{(1)}$ further, we can even define a change of measure which is structure preserving and, hence, of particular interest in applications. Such a change of measure is described in the following theorem.

Proposition 4.2 *Assume that (A) holds. Then, the process*

$$\psi_t^{(1)} = \frac{1}{\sigma_t}\left(r - \mu - \left(b + \frac{1}{2}\right)\sigma_t^2\right),$$

is a.s. square integrable on $[0,T]$ *and predictable and there exists an on* $[0,T]$ *square integrable and predictable process* $\left(\psi^{(2)}, \psi^{(3)}\right)$ *such that*

$$Z_t = \mathcal{E}\left(\int_0^{\cdot} \psi_s^{(1)} dW_s + \int_0^{\cdot} \psi_s^{(2)} d\widetilde{W}_s + \int_0^{\cdot} \psi_s^{(3)} dW_s^V\right)_t, \quad 0 \le t \le T,$$

is a density process. We obtain an EMM by defining

$$d\mathbb{P}^* = Z_T d\mathbb{P},$$

and the dynamics of the model under the probability measure $\mathbb{P}^*$ *are given by*

$$\begin{aligned} dY_t &= \left(r - \frac{1}{2}\sigma_t^2\right) dt + \sigma_t dX^*, \\ dX^* &= \rho_t dW_t^* + \sqrt{1-\rho_t^2}\, d\widetilde{W}_t^*, \\ d\sigma_t^2 &= -\alpha^*\left(\beta^* - \sigma_t^2\right) dt + \gamma\sigma_t dW_t^*, \\ \rho_t &= 2V_t - 1, \\ dV_t &= (\zeta^* - \eta^* V_t)dt + \theta\sqrt{V_t(1-V_t)}\, dW_t^{V*}, \end{aligned}$$

where $W_t^* = W_t - \int_0^t \psi_s^{(1)} ds$, $\widetilde{W}_t^* = \widetilde{W}_t - \int_0^t \psi_s^{(2)} ds$ *and* $W_t^{V*} = W_t^V - \int_0^t \psi_s^{(3)} ds$ *are independent* $\mathbb{P}^*$*–Brownian motions. Also, we have* $\alpha^* = \alpha + A$ *and* $\beta^* = \frac{\alpha\beta}{\alpha + A}$ *for some* $A \in \mathbb{R}$*. Similarly,* $\eta^* = \eta + \phi$*, and* $\zeta^* = \frac{\zeta}{\eta+\phi}$ *for a constant* $\phi \in \mathbb{R}$*. Hence* $\mathbb{P}^* \in \mathcal{M}'$*.*

4.2 Change of measure in the GBNS model

Now we focus on the change of measure in the GBNS model which is more involved than the one in the GH model due to the presence of jumps. In order to tackle this problem, we start by investigating how the the semimartingale characteristics of the logarithmic asset price process change in the GBNS model, if one performs a Girsanov change of measures.

Recall that the semimartingale characteristics of Y are given by (B, C, ν) and have been explicitly stated in Proposition 3.3. Also, let $\mathcal{P}$ denote the predictable σ–field on $\Omega \times \mathbb{R}_+$ and let $\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}$, for the Borel σ–algebra $\mathcal{B}$. From (Jacod & Shiryaev 2003, Girsanov Theorem, p.172), we know that for a probability measure $\mathbb{P}^*$ which is (locally) absolutely continuous with respect to $\mathbb{P}$, there exist a $\widetilde{\mathcal{P}}$–measurable, nonnegative function ξ and a predictable process ψ satisfying

$$\int_0^t c_s \psi_s ds < \infty \text{ and } \int_0^t \psi_s^2 c_s ds < \infty, \quad \mathbb{P}^* - a.s. \text{ for } t \in \mathbb{R}_+,$$

such that the characteristics of Y under $\mathbb{P}^*$ are given by (B', C', ν'_Y), where

$$B' = B + \int_0^t c_s \psi_s ds, \qquad C' = C, \qquad \nu_Y^{'} = \xi \cdot \nu_Y,$$

where we work – as before – with the zero truncation function. Moreover, ξ and ψ satisfy all the conditions above if and only if

$$\xi Z_- = \mathbb{E}(Z \star \mu_Y | \widetilde{\mathcal{P}})$$
$$\left\langle Z^c, \int_0^\cdot \sigma_s dW_s \right\rangle_t = \int_0^t \sigma_s^2 \psi_s Z_{s-} ds,$$

where Z denotes the corresponding density process of the measure change and $< \cdot, \cdot >$ denotes the predictable bracket.

Next, we derive an explicit representation for the Radon-Nikodým derivative Z in the change of measure. The following proposition generalises the corresponding result given in Nicolato & Venardos (2003, Theorem 3.1).

Assumption (B): Let Y denote the semimartingale defined in (9), where ρ is a stochastic correlation process satisfying $\rho \leq 0$. Furthermore, we assume that all local martingales are representable with respect to (W, L_λ, W^V) and that L is quasi–left continuous.

Proposition 4.3 *Let $\mathbb{P}^* \in \mathcal{M}$ and assume that assumption (B) is satisfied. Then there exists a predictable and on $[0,T]$ square integrable process $\psi = \left(\psi^{(1)}, \psi^{(2)}\right)' = (\psi_t)_{t \geq 0}$ and a strictly positive, predictable function $\xi = \xi(\omega, t, x)$ satisfying $(1 - \sqrt{\xi})^2 \star \nu_L < \infty$ $\mathbb{P}$–a.s. such that*

$$Z_t = \mathcal{E}\left(\int_0^\cdot \psi_s^{(1)} dW_s + \int_0^\cdot \psi_s^{(2)} dW_s^V + (\xi - 1)(\mu_L - \nu_L)\right)_t, \quad 0 \leq t \leq T,$$

is a density process, i.e. $Z_t = \mathbb{E}\left(\frac{d\mathbb{P}^*}{d\mathbb{P}} \middle| \mathcal{F}_t\right)$ and $\mathbb{E}(Z_T) = 1$.

The process ψ and the function ξ satisfy

$$\mu + \left(b + \frac{1}{2}\right)\sigma_t^2 + \lambda \int_0^\infty \left(e^{\rho \lambda_s x} - 1\right)\xi(t,x) U_L(x) dx + \sigma_t \psi_t^{(1)} - r = 0$$

$d\mathbb{P} \otimes dt$ almost surely, where $r > 0$ denotes the riskless interest rate.

Next, we restrict the class of equivalent martingale measures such that the model structure is preserved under the change of measure. Recall that we denote by $\mathcal{M}'$ the class of such structure preserving martingale measures and proceed by extending the corresponding results by Nicolato & Venardos (2003) to our more general model class. Let

$$\widetilde{\Xi}' = \left\{ \widetilde{\xi} : \mathbb{R}_+ \to \mathbb{R}_+ : \int_0^\infty \left(1 - \sqrt{\widetilde{\xi}}\right)^2 U_L(x) dx < \infty \right\},$$

denote a class of *deterministic* functions and for $\widetilde{\xi} \in \widetilde{\Xi}'$, we write

$$U_L^{\widetilde{\xi}}(x) = \widetilde{\xi}(x) U_L(x).$$

Analogously to the work by Nicolato & Venardos (2003), we deduce from $\int_0^\infty \min(1,x)U_L^{\widetilde{\xi}}(x)dx < \infty$ that

$$\int_0^{\lambda t}\int_0^\infty \left(e^{\rho_t x}-1\right)U_L^{\widetilde{\xi}}(x)dx$$

exists for negative ρ and, also, that $\mathcal{E}((\widetilde{\xi}-1)\star(\mu_L-\nu_L))$ is a true martingale for $\widetilde{\xi}\in\widetilde{\Xi}'$.

Proposition 4.4 *Let $\widetilde{\xi}\in\widetilde{\Xi}'$ and assume that (B) holds. Then, the process*

$$\psi_t^{(1)} = \frac{1}{\sigma_t}\left(r-\mu-\left(b+\frac{1}{2}\right)\sigma_t^2-\lambda\int_0^\infty\left(e^{\rho_{\lambda s}x}-1\right)\widetilde{\xi}(x)U_L(x)dx\right),$$

is predictable and a.s. square integrable on $[0,T]$ and there exists a predictable and on $[0,T]$ square integrable process $\psi^{(2)}=(\psi_t^{(2)})_{t\geq 0}$ and a strictly positive, predictable function $\xi=\xi(\omega,t,x)$ satisfying $(1-\sqrt{\xi})^2\star\nu_L<\infty$ $\mathbb{P}$–a.s. such that

$$Z_t^{\widetilde{\xi}} = \mathcal{E}\left(\int_0^{\cdot}\psi_s^{(1)}dW_s+\int_0^{\cdot}\psi_s^{(2)}dW_s^V+\left(\widetilde{\xi}-1\right)\star(\mu_L-\nu_L)\right)_t,\quad 0\leq t\leq T,$$

is a density process. We obtain an EMM by defining

$$d\mathbb{P}^{\widetilde{\xi}} = Z_T^{\widetilde{\xi}}d\mathbb{P},$$

and the dynamics of the model under the probability measure $\mathbb{P}^{\widetilde{\xi}}$ are given by

$$\begin{aligned} dY_t &= \left(r-\frac{1}{2}\sigma_t^2-\lambda\int_0^\infty\left(e^{\rho_{\lambda t}x}-1\right)\widetilde{\xi}(x)U_L(x)dx\right)dt+\sigma_t dW_t^{\widetilde{\xi}}+\rho_{\lambda t}dL_{\lambda t}^{\widetilde{\xi}},\\ d\sigma_t^2 &= -\lambda\sigma_t^2dt+dL_{\lambda t}^{\widetilde{\xi}},\\ \rho_t &= -V_t,\\ dV_t &= (\zeta^{\widetilde{\xi}}-\eta^{\widetilde{\xi}}V_t)dt+\theta\sqrt{V_t(1-V_t)}dW_t^{V,\widetilde{\xi}}, \end{aligned}$$

where $W_t^{\widetilde{\xi}}=W_t-\int_0^t\psi_s^{(1)}ds$ and $W_t^{V,\widetilde{\xi}}=W_t^V-\int_0^t\psi_s^{(2)}ds$ are $\mathbb{P}^{\widetilde{\xi}}$–Brownian motions and $L_{\lambda t}^{\widetilde{\xi}}$ is a $\mathbb{P}^{\widetilde{\xi}}$–subordinator. Also, we have $\eta^{\widetilde{\xi}}=\eta+\phi$, and $\zeta^{\widetilde{\xi}}=\frac{\zeta}{\eta+\phi}$ for a constant $\phi\in\mathbb{R}$. Furthermore $W_t^{\widetilde{\xi}}$, $W_t^{V,\widetilde{\xi}}$ and $L_{\lambda t}^{\widetilde{\xi}}$ are independent under $\mathbb{P}^{\widetilde{\xi}}$. Hence $\mathbb{P}^{\widetilde{\xi}}\in\mathcal{M}'$.

Conversely, for any $\mathbb{P}^*\in\mathcal{M}'$ there exists a deterministic function $\widetilde{\xi}\in\widetilde{\Xi}'$ such that $\mathbb{P}^*$ and $\mathbb{P}^{\widetilde{\xi}}$ coincide.

In order to conclude this section, we derive a quasi–explicit formula for the Laplace transform of the logarithmic asset price in the GBNS model, which can be used for computing option prices in a GBNS framework.

Proposition 4.5 *In the GBNS model with stochastic leverage, the Laplace transformation with $u\in\mathbb{R}$ is given by*

$$\begin{aligned} &\mathbb{E}\left(\exp(uY_T)\right|\mathcal{F}_t)\\ &=\exp\left(u(Y_t+\mu(T-t))+\left(ub+\frac{u^2}{2}\right)\frac{1}{\lambda}\left(1-e^{-\lambda(T-t)}\right)\sigma_t^2\right)\\ &\quad\mathbb{E}\left(\exp\left(\lambda\int_t^T\kappa\left(u\rho_{\lambda s}+\frac{v}{\lambda}\left(1-e^{-\lambda(T-s)}\right)\right)ds\right)\right), \end{aligned}$$

where $v=ub+\frac{u^2}{2}$ and where κ denotes the cumulant transform of the subordinator L.

So, if ρ_t is deterministic, we get a analytic expression for the Laplace transformation

$$\mathbb{E}\left(\exp(uY_T)\middle|\mathcal{F}_t\right) = \exp\left(u(Y_t + \mu(T-t)) + v\frac{1}{\lambda}\left(1 - e^{-\lambda(T-t)}\right)\sigma_t^2\right)$$
$$\exp\left(\lambda \int_t^T \kappa\left(u\rho_{\lambda s} + \frac{v}{\lambda}\left(1 - e^{-\lambda(T-s)}\right)\right) ds\right).$$

If ρ is stochastic, we have to compute the expectation

$$\mathbb{E}\left(\exp\left(\lambda \int_t^T \kappa\left(u\rho_{\lambda s} + \frac{v}{\lambda}\left(1 - e^{-\lambda(T-s)}\right)\right) ds\right)\right).$$

In general, we cannot expect to get an analytical expression for this expectation, even if we work with stochastic processes ρ whose distribution is known. In order to evaluate this expectation, one will have to use Monte Carlo methods.

5 Volatility smiles

In this section, we study the influence of stochastic leverage on European call option prices and the corresponding implied volatilities. We concentrate on the generalised Heston model (8) and assume that the stochastic correlation process ρ satisfies the SDE of a generalised Jacobi diffusion specified in (7). Before going into details on the stochastic correlation model we first briefly describe the influence of the model parameter in the classical Heston model, i.e. with constant correlation.

5.1 Influence of model parameters in the classical Heston model

The classical Heston model with constant correlation has been frequently studied in the literature. The influence of the model parameters on the shape of the implied volatility smile is therefore well understood, see e.g. (Hakala & Wystup 2002, Chapter 23). In general, the effect of changing the initial variance σ_0^2 in the classical Heston model has a very similar effect as changing the long–run variance β. The higher σ_0^2 or β the higher are the implied volatilities. The smile therefore is shifted upwards or downwards but keeps mainly the same overall shape.

In contrast, when the speed of mean reversion parameter α is changed, the overall shape of the smile changes significantly, in particular with respect to the at the money part of the smile. For higher values of α the implied volatilities corresponding to at the money strike prices are shifted upwards whereas the wings of the smile are hardly affected.

The volatility of the volatility γ is of crucial importance for the shape of the smile. If it is set equal to zero, the volatility process is deterministic and therefore the smile degenerates into a straight horizontal line. Increasing the volatility of volatility increases the convexity of the smile.

The constant correlation in the classical Heston model influences the symmetry of the volatility smile. If the correlation is 0 the smile is mainly symmetric whereas positive correlation corresponds to a shift of the minimum of the smile towards higher strikes and negative correlation towards lower strikes, i.e. positive correlation makes calls more expensive and negative correlation makes puts more expensive.

5.2 Influence of model parameters in the generalised Heston model

In order to compute European call option prices in the GH model with stochastic correlation given by (7) we use Monte Carlo methods together with a classical truncated Euler scheme with 32,000 time steps for an option with 10 years maturity, constant interest rate $r = 0$ and strikes in $\{50, 51, \ldots, 149, 150\}$. The asset price at time 0 is assumed to be 100. We simulate $100,000$ paths to compute one option price. From these prices, we derive the corresponding implied volatilities and plot them for various strikes.

We first study the influence of a stochastic correlation process in the Heston model on the shape of the implied volatility smile. The default parameters of the stochastic variance process are taken to be $\sigma_0^2 = 0.04, \alpha = 0.8, \beta = 0.04, \gamma = 0.5$. It should be noted that the CIR process with this parameterisation does not satisfy the Feller condition $2\alpha\beta \geq \gamma^2$ and therefore 0 is attainable. In empirical studies it is often observed that the stochastic volatility process does not satisfy the Feller condition, see e.g. the comments in Andersen (2007), Broadie & Kaya (2006), van Haastrecht & Pelsser (2008), and we have therefore chosen this parameterisation.

The correlation process ρ satisfies the SDE (7) with default parameters $R_0 = 0, R_1 = 1, R_2 = 0, R_3 = 0, R_4 = 1, R_5 = 1.5$. For more information on the model parameters used, see Appendix A.

In our numerical analysis we find that the overall influence of the parameters describing the dynamics of the stochastic volatility process on the shape of the smile stays the same as in the Heston model (with constant correlation). As an example we just present the influence of the volatility of volatility in Figure 2. We find that the smile is flatter for lower values of γ.

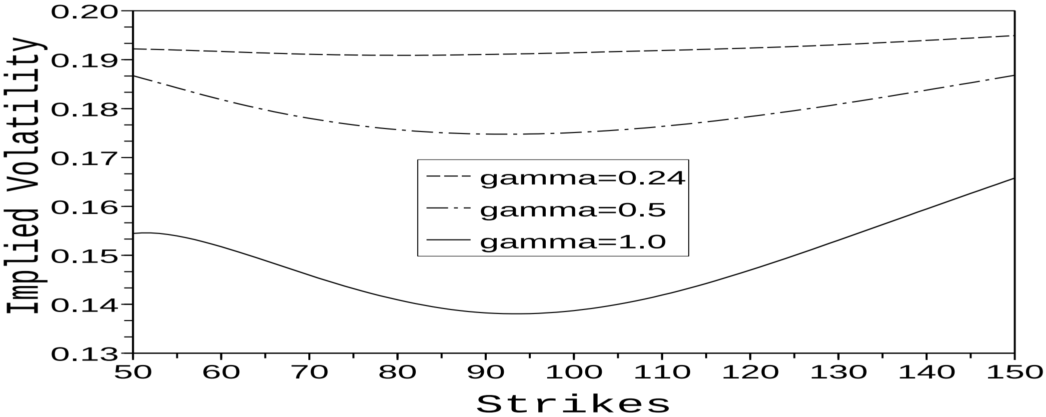


Figure 2: **Sensitivity with respect to γ for $\gamma \in \{0.24, 0.5, 1\}$.** Implied volatilities computed from simulated call prices with $S_0 = 100, T = 10, r = 0$. This Figure uses a slightly different scale for the y-axis than all other Figures due to the large variation of the implied volatilities in the present example.

Next, we consider the influence of the parameters describing the stochastic correlation process. Figure 3 shows the implied volatilities for different values of the volatility parameter R_5 of the generalised Jacobi diffusion. For small R_5, the correlation process is almost deterministic whereas, for large values of R_5, it is close to a jump–type diffusion. We find that changing the parameter R_5 results in an upwards or downwards shift of the implied volatilities. For increasing values of R_5, the implied volatilities are first shifted downwards. For $R_5 = 5$, however, we see that the implied volatilities are again almost the same as for $R_5 = 0.1$. This is due to the fact that the stochastic correlation process in that case resembles already a jump–diffusion type model and it is mostly taking extreme values $-1, 1$, which results in a very similar behaviour as just assuming an almost deterministic correlation around zero.

Figure 4 shows the sensitivity of the implied volatilities with respect to the support of the correlation process modelled in terms of the parameters R_2 and R_4. We find that these parameters have a strong influence on the shape and the location of the smile. In particular, we observe that the negative correlation process ($R_2 = -0.5, R_4 = 0.5$, i.e. support $[-1, 0]$, long–term mean -0.5) has higher implied volatilities for higher strike prices than the positive correlation process ($R_2 = 0.5, R_4 = 0.5$, i.e. support $[0, 1]$, long–term mean 0.5) and vice versa for lower strike prices. The correlation process which can take both positive and negative values ($R_2 = 0, R_4 = 1$, i.e. support $[-1, 1]$, long–term mean 0) is almost symmetric around the at the money price and exhibits a true smile shape. Indeed, we find here as well that the positive correlation process makes calls more expensive and the negative correlation process makes puts more expensive.

Figure 5 compares the correlation process varying in a smaller interval $[-0.5, 0.5]$ to those varying in the larger interval $[-1, 1]$. For both intervals we compare slow and fast mean reversion, i.e. $R_1 = 0.1$ and $R_1 = 1$ respectively. We see that the influence of the speed of mean reversion here is marginal. The support of the correlation process, however, does determine the overall location of the smile. We find that a smaller support for the correlation process results in higher overall implied volatilities.

Figure 6 compares the classical Heston model with constant correlation equals zero, to the GH were the correlation process varies in either $[-0.5, 0.5]$ with long–term mean 0 or in $[-1, 1]$ with long–term mean 0. We find that if the correlation process varies only in a small interval the resulting implied volatilities are very similar to those of the classical Heston model where a constant correlation where chosen which is the center of this interval. This is obviously not surprising. The implied volatilities for constant correlation are generally higher than those for stochastic correlation in this example. If the stochastic correlation process varies in a wider interval (here $[-1, 1]$), then the implied volatilities are generally even lower.

Figure 7 shows the influence of the parameter R_3 on the shape of the smile for a mean reversion parameter $R_1 \in \{0.1, 1\}$. We are again considering a stochastic correlation process in $[-1, 1]$ and study different values for R_3 which result in different long–term means of the stochastic correlation process. Due to the parameterisation of the model, the long term mean $R_2 + R_3 R_4$ corresponds to R_3 in this example. We find that this parameter strongly influences the shape of the smile. If the long–term mean is 0, we observe an almost symmetric smile, whereas a negative or a positive long–term mean results in a left– and a right–skew, respectively. Generally, we find that the smiles in Figure 7(b) corresponding to $R_1 = 0.1$ are flatter and exhibit a stronger smile shape rather than just skew shape compared to a fast mean reversion parameter $R_1 = 1$ in Figure 7(a). Moreover, we find that the influence of the speed of mean reversion parameter R_1 is more pronounced when the long–run mean is leading to asymmetric implied volatilities.

In general, we find that for negative support of the correlation function or for a negative long–term mean, the European call prices are higher for strike prices smaller than S_0 and are smaller for strike prices greater than S_0 than the prices for positive support or positive long–term mean or symmetric correlation processes which can take both negative and positive values.

In principal, the support of the correlation process together with its long–term mean can be used to model the symmetry of the smile. For asymmetric smile the speed of mean reversion parameter can be chosen to increase or decrease the convexity of the smile. For symmetric smiles the influence of this parameter seems to be marginal. The length of the support of the correlation process can be used to shift the implied volatilities upwards or downwards. In particular the more values the correlation process can take, the lower are usually the implied volatilities.

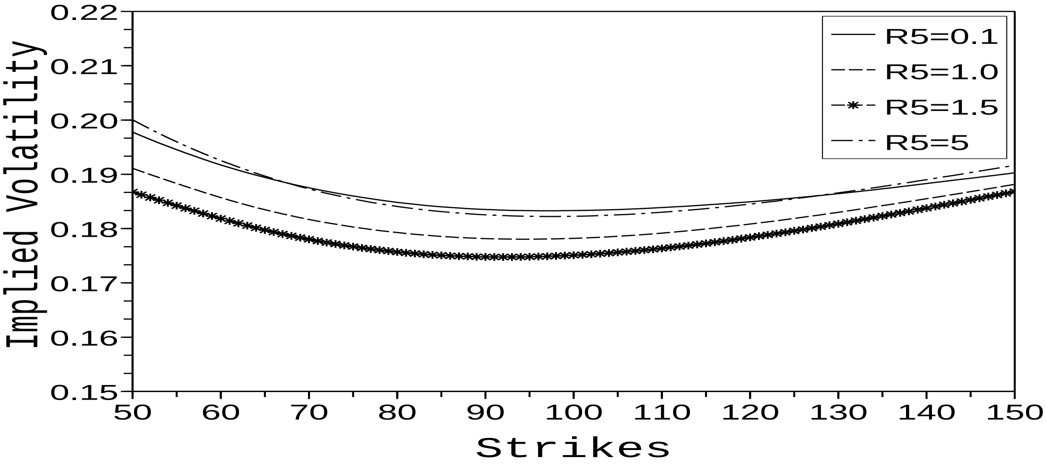


Figure 3: **Sensitivity with respect to R_5 for $R_5 \in \{0.1, 1.5, 1\}$.**

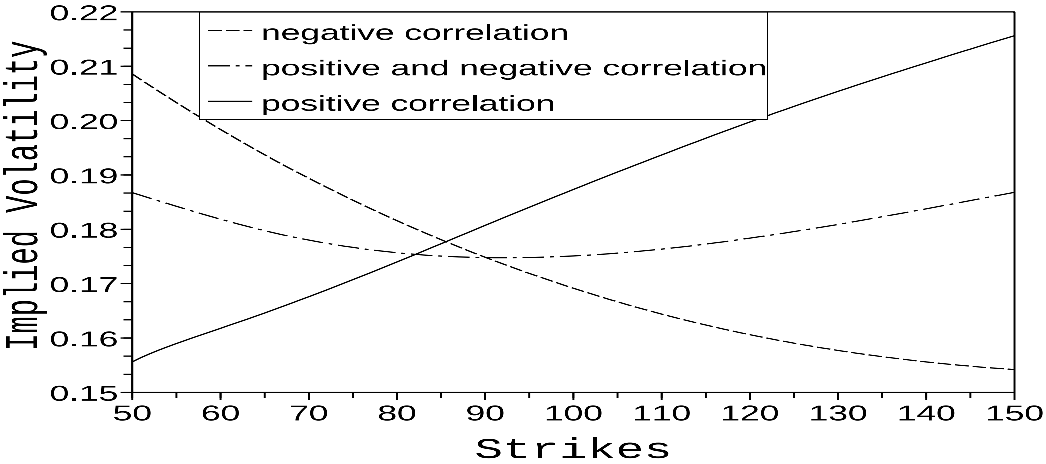


Figure 4: **Sensitivity with respect to R_2, R_4.**

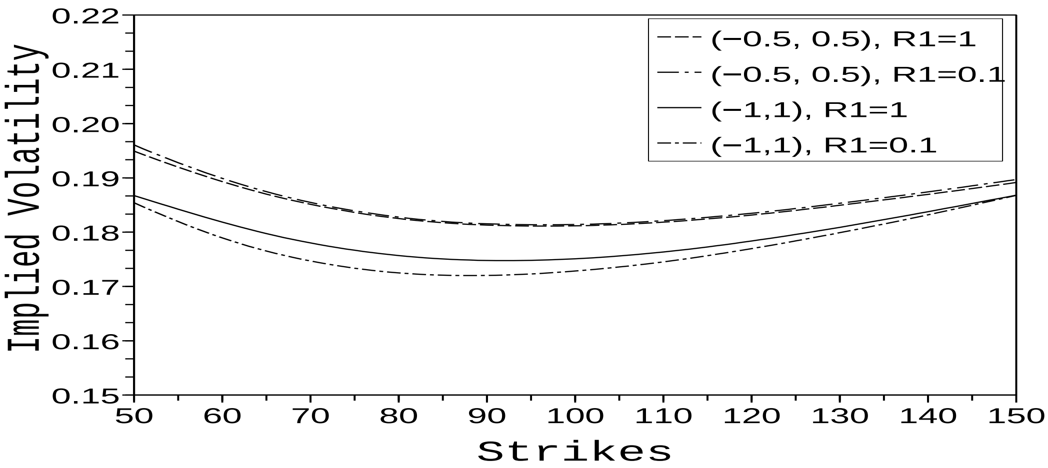


Figure 5: **Sensitivity with respect to R_1, R_2, R_4.**

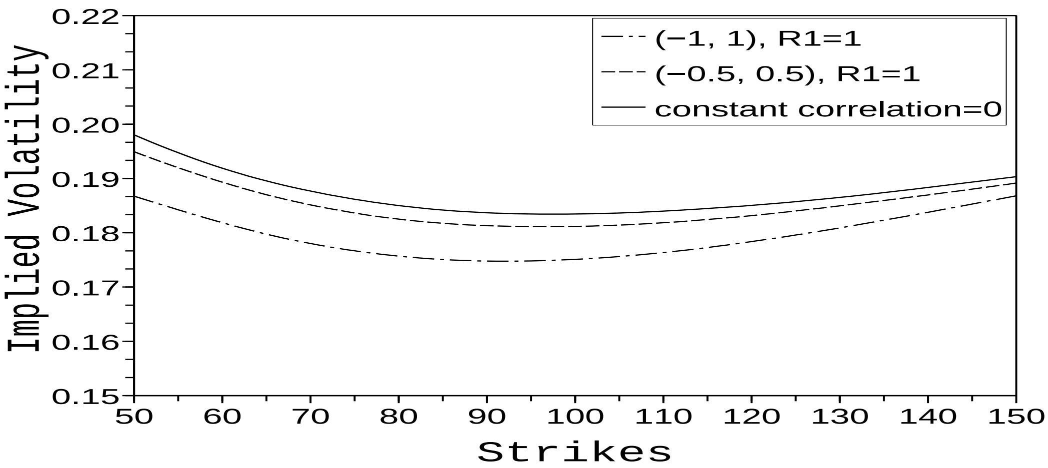
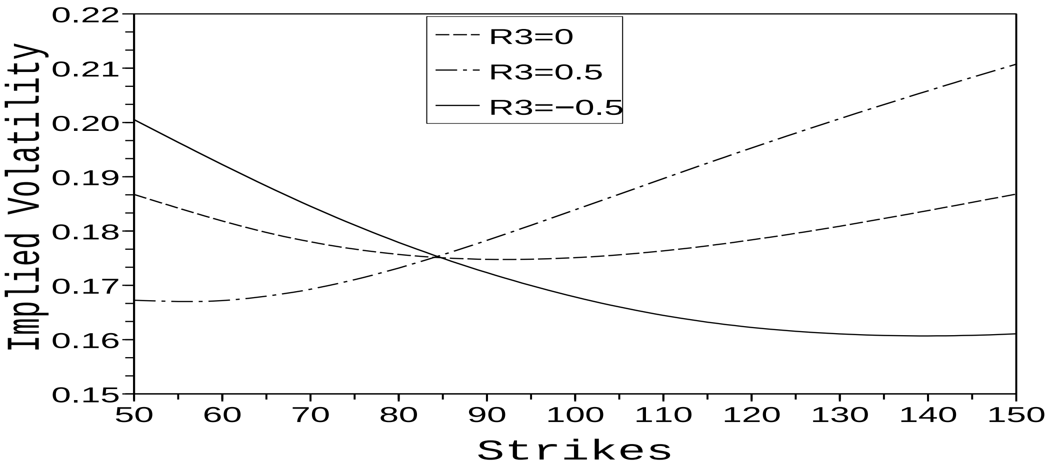
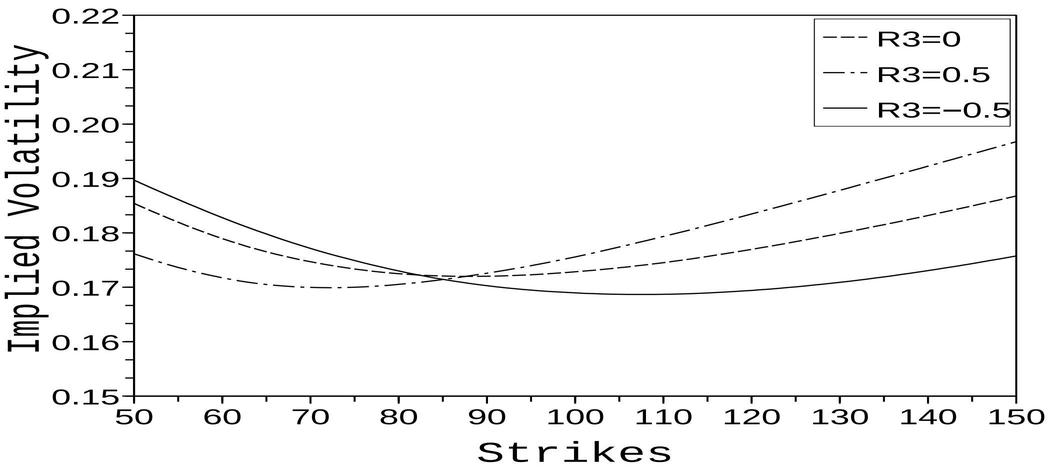


Figure 6: **Influence of stochastic correlation.** Comparison of the classical Heston model with constant correlation $= 0$ and the Heston model with stochastic correlation process in $(-0.5, 0.5)$ with long–term mean $= 0$ and in $(-1, 1)$ with long–term mean $= 0$.



(a) Sensitivity with respect to R_3 for $R_1 = 1$.



(b) Sensitivity with respect to R_3 for $R_1 = 0.1$.

Figure 7: **Sensitivity with respect to R_3 for different values of R_1.**

6 Influence of leverage effect and volatility feedback effect on return–volatility regressions

So far, we have studied the concept of stochastic leverage and have investigated its impact in the context of equivalent martingale measures and option pricing. This section and the following one are now devoted to more statistical aspects of stochastic leverage, which arise when one is interested in *measuring* the leverage effect.

We start with a very simple set up: A standard tool in econometrics for quantifying the empirical leverage effect is by means of return–volatility regressions of various types. The aim of this section is now to study the impact of stochastic leverage on such return–volatility regressions and to investigate further how well we can measure both the leverage effect and the volatility feedback effect in the GH and GBNSJ model based on a simple regression framework. A short note on forecasting volatility based on option implied volatility will round off this section and will link it to the topics of change of measure and applications to option pricing we studied before.

Our work extends recent work by Bollerslev & Zhou (2006), Bollerslev et al. (2006) to the GH model with stochastic leverage and furthermore derives the corresponding results for the GBNSJ which allows for stochastic leverage in terms of a linearly transformed Jacobi process. Note in particular that, throughout this section, we work under the assumption that the instantaneous volatility feedback effect, which is described by the parameter b in the drift of the log–price process is *positive* in order to obtain results which are comparable to the ones in Bollerslev & Zhou (2006).

6.1 Leverage effect

First of all, we focus on the classical leverage effect, which is referred to as the – usually negative – correlation between lagged returns and current volatility. In order to measure this effect, econometricians usually study the population regressions for integrated volatility, given by

$$\int_t^{t+h} \sigma_u^2 du = G + D \int_{t-h}^{t} dY_u + \epsilon_{t,t+h}, \tag{12}$$

and for the option implied volatility, we write

$$\mathbb{E}_t^* \left(\int_t^{t+h} \sigma_u^2 du \right) = G^* + D^* \int_{t-h}^{t} dY_u + \epsilon_{t,t+h}^*, \tag{13}$$

for constants $G, D, G^*, D^* \in \mathbb{R}$ and for white noise processes ϵ, ϵ^*. Note that we denote by $\mathbb{E}_t^*$ the conditional expectation under the equivalent, risk–neutral martingale probability measure $\mathbb{P}^*$, conditional on $\mathcal{F}_t$.

In the literature, we find some empirical studies which find D to be positive and others which indicate that D is negative (for a discussion on these results and further references to the corresponding empirical studies see Bollerslev & Zhou (2006)). In this section, we will shed some light on how both a positive and a negative leverage effect can be explained both in a GH model and in a GBNS model. Motivated by the empirical findings, we will assume throughout this section that $\mathbb{E}(\rho) < 0$, which is in line with the assumption of Bollerslev & Zhou (2006), who assumed that the deterministic, instantaneous correlation coefficient is negative.

Recall that the GH model is given by equation (8), i.e.

$$dY_t = (\mu + b\sigma_t^2)dt + \sigma_t \left(\rho_t dW_t + \sqrt{1-\rho_t^2}\widetilde{W}_t \right),$$
$$d\sigma_t^2 = \alpha(\beta - \sigma_t^2)dt + \gamma\sigma_t dW_t,$$

for positive constants $\mu, b, \alpha, \beta, \gamma$, and where $W, \widetilde{W}$ are independent Brownian motions and ρ is a stochastic process taking values in $[-1, 1]$, which is independent of W and $\widetilde{W}$. For ease of exposition, we will additionally assume that ρ is *stationary*, e.g. ρ is a linearly transformed stationary Jacobi process. Furthermore, when we apply a structure preserving change of measure as discussed in Section 4, the memory parameter of the volatility process under the risk neutral measure is given by $\alpha^* = \alpha + A$, where the constant A is usually assumed to be negative, i.e. $\alpha^* < \alpha$, and the long term mean under the risk neutral measure is denoted by β^*.

Proposition 6.1 *In the GH model, with standard parameter restrictions $\alpha, \alpha^*, \beta, \beta^*, \gamma, b > 0$, $A < 0$ and with a stationary ρ with $\mathbb{E}(\rho) < 0$, the population slope parameters and intercepts in the population regressions (12) and (13) are given by*

$$D = \frac{\left(\frac{b\beta\gamma^2}{2\alpha} + \beta\mathbb{E}(\rho)\gamma \right) \frac{1}{\alpha^2} \left(1 - e^{-\alpha h}\right)^2}{\beta h + \frac{b\beta\gamma}{\alpha^2}\left(\frac{b\gamma}{\alpha} + 2\mathbb{E}(\rho) \right)\left(-1 + h\alpha + e^{-\alpha h}\right)},$$
$$G = h\beta - D(\mu + b\beta)h,$$

and

$$D^* = \frac{\left(\frac{b\beta\gamma^2}{2\alpha} + \beta\mathbb{E}(\rho)\gamma \right) \frac{1}{\alpha} \left(1 - e^{-\alpha h}\right) \frac{1}{\alpha^*}\left(1 - e^{-\alpha^* h}\right)}{\beta h + \frac{b\beta\gamma}{\alpha^2}\left(\frac{b\gamma}{\alpha} + 2\mathbb{E}(\rho) \right)\left(-1 + h\alpha + e^{-\alpha h}\right)},$$
$$G^* = (\beta - \beta^*)\frac{1}{\alpha^*}\left(1 - e^{-\alpha^* h}\right) + h\beta^* - D^*(\mu + b\beta)h.$$

Also, if $0 < b < -2\frac{\mathbb{E}(\rho)\alpha}{\gamma}$, then $D^* < D < 0$ and if $0 < -2\frac{\mathbb{E}(\rho)\alpha}{\gamma} < b$, we get $0 < D < D^*$.

As in Bollerslev & Zhou (2006), we also find for the GH model which allows for a stochastic leverage effect that the *empirical* leverage coefficient, given by D and D^*, respectively, depends positively on the *instantaneous* volatility feedback parameter b and negatively on the *instantaneous* leverage parameter which is given by the mean of the stochastic leverage effect $\mathbb{E}(\rho)$. Depending on which effect is more pronounced we might find both a negative or a positive empirical leverage coefficient in empirical studies.

Next, we study the population regressions in the GBNSJ model. Recall that the GBNSJ model is defined by equations (9) and (10). Furthermore, we will denote by λ^* the memory parameter of σ^2 under the risk neutral measure.

Proposition 6.2 *In the GBNSJ model, with standard parameter restrictions $\mu, b, \lambda, \lambda^*, \zeta, \eta, \theta, \kappa_1 = \mathbb{E}(L_1), \kappa_2 = Var(L_1), \kappa_1^* = \mathbb{E}^*(L_1) > 0$ and $\mathbb{E}(\rho) < 0$, the population slope parameters and intercepts in the population regressions (12) and (13) are given by*

$$D = \frac{\kappa_2\left(\frac{b}{2} + \frac{\mathbb{E}(\rho)}{\lambda}\right)\frac{1}{\lambda^2}\left(1 - e^{-\lambda h}\right)^2 - \kappa_1^2\mathbb{E}(\rho)h\left(-1 + \lambda h + e^{-\lambda h}\right)}{Var\left(\int_{t-h}^t dY_s\right)},$$

where

$$Var\left(\int_{t-h}^{t} dY_s\right) = b\kappa_2\left\{2\lambda\mathbb{E}(\rho)+b\right\}\frac{1}{\lambda^2}\left(-1+\lambda h+e^{-\lambda h}\right) + 2\lambda^2\kappa_1^2\mathbb{E}\left(\rho^2\right)\frac{1}{\lambda^2\eta^2}\left(-1-\lambda\eta h+e^{\lambda\eta h}\right)+\kappa_1\left(1+\lambda\mathbb{E}(\rho)\right)h.$$

The intercept in the regression is given by

$$G = h\kappa_1 - D(\mu + b\kappa_1 + \lambda\kappa_1\mathbb{E}(\rho))h.$$

For the implied volatility, we get

$$D^* = \frac{\left(\frac{b\kappa_2}{2}+\mathbb{E}(\rho)\kappa_2\lambda\right)\frac{1}{\lambda}\left(1-e^{-\lambda h}\right)\frac{1}{\lambda^*}\left(1-e^{-\lambda^* h}\right)}{Var\left(\int_{t-h}^{t} dY_s\right)},$$

$$G^* = (\kappa_1-\kappa_1^*)\frac{1}{\lambda^*}\left(1-e^{-\lambda^* h}\right)+h\kappa_1^*-D^*(\mu+b\kappa_1+\lambda\kappa_1\mathbb{E}(\rho))h.$$

The findings in the GBNSJ model are similar to the ones we obtain in the GH model. The first part of the numerator of the empirical leverage coefficient is basically equivalent to the coefficient we obtain in the GH model. However, in addition, we obtain a term which depends positively on the instantaneous mean of the stochastic leverage. That means that, even if the volatility feedback parameter is small or even zero, we could nevertheless find a positive empirical leverage coefficient. Note that in the population regression based on the option implied volatility, the extra term, which depends positively on the instantaneous leverage effect, is no longer present.

6.2 Volatility feedback effect

Now we focus on the so–called *volatility feedback effect* which is regarded as the usually positive correlation between current volatility and future returns. In order to measure it empirically, econometricians usually focus on the following two population regressions:

$$\int_t^{t+h} dY_u = \widetilde{G} + \widetilde{D}\int_t^{t+h}\sigma_u^2 du + \widetilde{\epsilon}_{t,t+h}, \tag{14}$$

and

$$\int_t^{t+h} dY_u = \widetilde{G}^* + \widetilde{D}^*\mathbb{E}_t^*\left(\int_t^{t+h}\sigma_u^2 du\right) + \widetilde{\epsilon}^*_{t,t+h}, \tag{15}$$

for constants $\widetilde{G}, \widetilde{D}, \widetilde{G}^*, \widetilde{D}^* \in \mathbb{R}$ and where $\widetilde{\epsilon}, \widetilde{\epsilon}^*$ denote white noise processes.

Note that the main difference between the leverage and the volatility feedback effect, from an empirical point of view, lies in the *causality* (as discussed by Bollerslev et al. (2006)): While the leverage effect describes how a negative price increment leads to an increase in subsequent volatility, the volatility feedback effect explains how an increase in volatility can give rise to negative price increments.

The corresponding results for the volatility feedback effect in the GH and GBNSJ model are now given as follows.

Proposition 6.3 *In the GH model, with standard parameter restrictions $\alpha, \alpha^*, \beta, \beta^*, \gamma, b > 0$, $A < 0$ and with a stationary ρ with $\mathbb{E}(\rho) < 0$ and $\mu \neq 0$, the population slope parameters and intercepts in the regressions (14) and (15) are given by*

$$\widetilde{D} = b + \frac{\alpha \mathbb{E}(\rho)}{\gamma} < b, \qquad \widetilde{G} = \left(\mu - \frac{\beta\alpha\mathbb{E}(\rho)}{\gamma}\right) h,$$

and

$$\widetilde{D}^* = b \frac{\frac{1}{\alpha}\left(1 - e^{-\alpha h}\right)}{\frac{1}{\alpha^*}\left(1 - e^{-\alpha^* h}\right)} < b,$$
$$\widetilde{G}^* = (\mu + b\beta)h - \widetilde{D}^* \left(\beta^* h + (\beta - \beta^*)\frac{1}{\alpha^*}\left(1 - e^{-\alpha h}\right)\right).$$

Also, we get for $0 < b < -\frac{\mathbb{E}(\rho)\alpha}{\gamma}$ that

$$\widetilde{D} < 0 < \widetilde{D}^* < b,$$

and for $0 < -\frac{\mathbb{E}(\rho)\alpha}{\gamma} < b < \frac{\alpha\mathbb{E}(\rho)}{\gamma} \frac{\frac{1}{\alpha^*}\left(1-e^{-\alpha^* h}\right)}{\frac{1}{\alpha}\left(1-e^{-\alpha h}\right) - \frac{1}{\alpha^*}\left(1-e^{-\alpha^* h}\right)}$, we have

$$0 < \widetilde{D} < \widetilde{D}^* < b.$$

The results above show clearly, that the instantaneous volatility feedback effect which is given by b is systematically underestimated by the empirical volatility feedback effect $\widetilde{D}$ in the presence of stochastic leverage with negative mean. The same result also holds when we look at the option implied return–volatility regression.

In the GBNSJ model, the findings are very similar again. An additional nice feature in this modelling framework is that the intercept of the population regression equals the *true* drift, which was not the case in the generalised Heston model.

Proposition 6.4 *In the GBNSJ model, with standard parameter restrictions $\mu, b, \lambda, \lambda^*, \zeta, \eta, \theta, \kappa_1 = \mathbb{E}(L_1), \kappa_2 = Var(L_1), \kappa_1^* = \mathbb{E}^*(L_1) > 0$, $\mathbb{E}(\rho) < 0$ and $\mu \neq 0$ and $\lambda^* \leq \lambda$, the population slope parameters and intercepts are given by*

$$\widetilde{D} = b + \lambda\mathbb{E}(\rho) < b, \qquad \widetilde{G} = \mu h,$$

and

$$\widetilde{D}^* = b \frac{\frac{1}{\lambda}\left(1 - e^{-\lambda h}\right)}{\frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right)} \leq b,$$
$$\widetilde{G}^* = (\mu + b\kappa_1 + \lambda\kappa_1\mathbb{E}(\rho))h - \widetilde{D}^* \left(\kappa_1^* h + (\kappa_1 - \kappa_1^*)\frac{1}{\lambda^*}\left(1 - e^{-\lambda h}\right)\right).$$

Also, if additionally $\lambda^* < \lambda^*$, we get for $0 < b < -\mathbb{E}(\rho)\lambda$ that

$$\widetilde{D} < 0 < \widetilde{D}^* < b,$$

and for $0 < -\mathbb{E}(\rho)\lambda < b < \lambda\mathbb{E}(\rho)\frac{\frac{1}{\lambda^*}\left(1-e^{-\lambda^* h}\right)}{\frac{1}{\lambda}\left(1-e^{-\lambda h}\right) - \frac{1}{\lambda^*}\left(1-e^{-\lambda^* h}\right)}$, we have

$$0 < \widetilde{D} < \widetilde{D}^* < b.$$

6.3 Implied volatility forecasting bias

The final part of this section focuses on the impact of the leverage effect on creating a bias when forecasting volatility based on option implied volatility. The corresponding population regression is given by

$$\int_t^{t+h} \sigma_u^2 du = \overline{G} + \overline{D}\mathbb{E}_t^* \left(\int_t^{t+h} \sigma_u^2 du \right) + \overline{\epsilon}_{t,t+h}, \tag{16}$$

where $\overline{G}, \overline{D} \in \mathbb{R}$ and $\overline{\epsilon}$ denotes white noise. Ideally, we would like to have that $\overline{G} = 0$ and that $\overline{D} = 1$. This would imply that the implied volatility generates unbiased volatility forecasts. However, we obtain the following results for the GH and GBNSJ model.

Proposition 6.5 *Under the same assumptions as in Proposition 6.3 and Proposition 6.4 the slope and the intercept of the population regression (16) in the GH model is given by*

$$\overline{D} = \frac{\frac{1}{\alpha}\left(1 - e^{-\alpha h}\right)}{\frac{1}{\alpha^*}\left(1 - e^{-\alpha^* h}\right)} < 1,$$
$$\overline{G} = \beta \left(h - \frac{1}{\alpha}\left(1 - e^{-\alpha h}\right)\right) + \beta^* \left(h - \frac{1}{\alpha^*}\left(1 - e^{-\alpha^* h}\right)\right) \frac{\frac{1}{\alpha}\left(1 - e^{-\alpha h}\right)}{\frac{1}{\alpha^*}\left(1 - e^{-\alpha^* h}\right)},$$

and in the GBNSJ model, we get

$$\overline{D} = \frac{\frac{1}{\lambda}\left(1 - e^{-\lambda h}\right)}{\frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right)} \leq 1,$$
$$\overline{G} = \kappa_1 \left(h - \frac{1}{\lambda}\left(1 - e^{-\lambda h}\right)\right) + \kappa_1^* \left(h - \frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right)\right) \frac{\frac{1}{\lambda}\left(1 - e^{-\lambda h}\right)}{\frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right)}.$$

So we observe that while we underestimate the slope coefficient in the GH model, this is not necessarily true in the GBNSJ model. However, under the additional assumption that $\lambda^* < \lambda$, we get that $\overline{D} < 1$ in both models.

After we have studied a very simple method for quantifying leverage–type effects in form of return–volatility regressions and after we have found out what impact the *stochastic* leverage has on such regressions, we will next turn to more advanced methods for measuring the leverage effect.

7 A short note on estimation and inference

Last but not least, we address the problem of estimation and inference in our two new models in the presence of stochastic leverage. In fact, there are two types of estimation problems which should be discussed: First of all, there is the question of how we can estimate stochastic leverage non–parametrically and how we can make inference on it. Second, it is interesting to investigate how the model parameters in both the GHJ and the GBNSJ model can be estimated.

Throughout this section, we will work under the physical probability measure and we will assume that we observe the asset price at high frequencies. Furthermore, we ignore any sort of microstructure effect and just refer to Bandi & Russell (2008), Zhang et al. (2005), Hansen & Lunde (2006), Jacod et al. (2008), Barndorff-Nielsen et al. (2008*a*,*b*) and the references therein for a detailed account on this matter.

7.1 Non–parametric estimation of the leverage effect

We start with the question of how to estimate the stochastic leverage non–parametrically. This question has recently been addressed by Bandi & Renò (2008*b*) in a modelling framework which allows for *local* stochastic leverage. However, here we are interested in even more general models for stochastic leverage. So, we will proceed differently. Since, stochastic leverage is a special case of stochastic volatility it is natural to use similar methods for estimating the leverage than the ones which are very successful in estimating stochastic volatility.

Throughout the section, we assume that we observe the logarithmic asset price $Y = (Y_t)_{t\geq 0}$ over a time interval $[0, T]$ for some $T > 0$ at times $i\Delta_n$ for $i = 0, 1, \ldots, \lfloor T/\Delta_n \rfloor$ for some $\Delta_n > 0$ such that $\Delta_n \to 0$ as $n \to \infty$. Then we write $\Delta_i^n Y = Y_{i\Delta_n} - Y_{(i-1)\Delta_n}$ for the ith return of Y. In the following, we always assume that $0 \leq t \leq T$.

A key quantity in estimating the stochastic volatility/leverage is the quadratic variation process, see e.g. Protter (2004), denoted by the square bracket $[\cdot]$ and its empirical counter part, the *realised variance* (RV) (Andersen & Bollerslev (1998), Barndorff-Nielsen & Shephard (2001, 2002)), which is defined by

$$RV_t^n = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Delta_i^n Y)^2 .$$

Clearly, $RV_t^n \to [Y]_t$ as $n \to \infty$, where the convergence is uniform on compacts in probability (ucp), see Protter (2004).

Note that in the GH model, the quadratic variation $[Y]_t$ is given by $\int_0^t \sigma_s^2 ds$, whereas in the GBNS model we obtain $\int_0^t \sigma_s^2 ds + \sum_{0\leq s\leq t} \rho_{\lambda s}^2 \left(L_{\lambda s}\right)^2$.

The concept of realised variance has been generalised to realised multipower variation (Barndorff-Nielsen & Shephard (2006*a*), Barndorff-Nielsen et al. (2006), Jacod (2008),Veraart (2009)) and truncated realised variance (Mancini (2001, 2006), Jacod (2008)) in order to estimate the continuous and the discontinuous part of the quadratic variation separately. These methods seem to be promising tools when we look at the GBNS model, since the stochastic correlation coefficient appears in the jump part of the quadratic variation and can be estimated separately based on the difference of realised variance and realised multipower variation. However, we observe that, when we are in the GH model, the stochastic leverage has no contribution to the quadratic variation. In order to estimate it we should hence focus on the quadratic covariation instead. In the GH model, we have

$$d[Y, \sigma^2]_t = \gamma \sigma_t^2 \rho_t dt, \tag{17}$$

and in the GBNS model, we get

$$d[Y, \sigma^2]_t = \rho_{\lambda t} d[L]_{\lambda t}. \tag{18}$$

In order to estimate $[Y, \sigma^2]$ non–parametrically, one can use an estimator proposed by Mykland & Zhang (2009), which is based on the sum of the products of the high frequency increments of the logarithmic asset price and suitably normalised estimated spot variances, see e.g. Lee & Mykland (2006), Bandi & Renò (2008*a*). Note that Mykland & Zhang (2009) focus on continuous Itô processes for the asset prices which, clearly, do not include the GBNS model. However, extensions of their work to the jump case (using similar reasoning as in Jacod (2008)) are likely to be straightforward to derive and will be studied elsewhere in future research.

7.2 Parameter estimation in models which allow for stochastic leverage

Next, we turn our attention to the question of how we can estimate the model parameters which specify the GH and the GBNS model. Parameter estimation in stochastic volatility models has been studied intensively in the last decade. Popular methods include (quasi-) maximum likelihood estimation, see Barndorff-Nielsen & Shephard (2006*b*), Gallant (1997), generalised methods of moments, see e.g. Bollerslev & Zhou (2002), and simulation methods, see e.g. Roberts et al. (2004), Frühwirth-Schnatter & Sögner (2001), Griffin & Steel (2006), and Aït-Sahalia & Kimmel (2007) and the references therein.

Following the work by Barndorff-Nielsen & Shephard (2006*b*), Veraart (2008), Todorov (2009*b*,*a*), we can use the time series of realised variances/multipower variation for estimating the model parameters of the generalised Heston and BNS model, since we can compute all moments of interest in explicit form (see the appendix) or can use suitable approximations as in Todorov (2009*a*), and hence, can use quasi–maximum likelihood methods or general methods of moments for estimating the model parameters and for making inference on them. An implementation of such estimation methods will be left for future research.

8 Concluding remarks

This paper contains a systematic treatment of the new concept of a stochastic leverage effect in stochastic volatility models. By modelling the stochastic leverage effect explicitly, e.g. by means of a linear transformation of a Jacobi process, we have found an analytically tractable asset price model which allows for an easy economic interpretation of both stochastic volatility and stochastic leverage.

In order to get a better understanding of such models, we have proposed two new stochastic volatility models which allow for stochastic leverage: the generalised Heston model and the generalised Barndorff–Nielsen & Shephard model.

We have studied in detail how such models behave under both the empirical and the risk neutral probability measure and we have investigated the implied volatility patterns of such more general stochastic volatility and stochastic leverage models. Finally, we have addressed statistical aspects, in such a new model class. In particular, we have given explicit results on how a stochastic leverage effect affects return–volatility regression and the ability to forecast volatility based on option implied volatilities. Furthermore, we have indicated how one can estimate stochastic leverage and the parameters in these new model classes.

In future research, it will be interesting to study multivariate extensions of our new models and to carry out an empirical study by implementing the estimation techniques described in this paper.

Acknowledgement

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APPENDIX

A Choice of parameters for the simulation study

The following table contains all the parameter values used in the simulations for Figure 2 - 7.

We considered European call options with asset price at time 0 given by $S_0 = 100$, maturity $T = 10$ and interest rate $r = 0$.

R_0	R_1	R_2	R_3	R_4	R_5	σ_0^2	α	β	γ
Figure 2: Influence of the volatility of volatility γ:									
0.0	1.0	0.0	0.0	1.0	1.5	0.04	0.8	0.04	**0.24**
0.0	1.0	0.0	0.0	1.0	1.5	0.04	0.8	0.04	**0.5**
0.0	1.0	0.0	0.0	1.0	1.5	0.04	0.8	0.04	**1.0**
Figure 3: Influence of the volatility of the correlation process R_5:									
0.0	1.0	0.0	0.0	1.0	**0.1**	0.04	0.8	0.04	0.5
0.0	1.0	0.0	0.0	1.0	**1.0**	0.04	0.8	0.04	0.5
0.0	1.0	0.0	0.0	1.0	**1.5**	0.04	0.8	0.04	0.5
0.0	1.0	0.0	0.0	1.0	**5.0**	0.04	0.8	0.04	0.5
Figure 4: Influence of the range of the correlation process modelled by R_2, R_4:									
-0.5	1.0	**-0.5**	0.0	**0.5**	1.5	0.04	0.8	0.04	0.5
0.0	1.0	**0.0**	0.0	**1.0**	1.5	0.04	0.8	0.04	0.5
0.5	1.0	**0.5**	0.0	**0.5**	1.5	0.04	0.8	0.04	0.5
Figure 5: Influence of the range of the correlation process modelled by R_4 and the speed of mean reversion R_1:									
0.0	**1.0**	0.0	0.0	**0.5**	1.5	0.04	0.8	0.04	0.5
0.0	**0.1**	0.0	0.0	**0.5**	1.5	0.04	0.8	0.04	0.5
0.0	**1.0**	0.0	0.0	**1.0**	1.5	0.04	0.8	0.04	0.5
0.0	**0.1**	0.0	0.0	**1.0**	1.5	0.04	0.8	0.04	0.5
Figure 6: Influence of the stochastic correlation versus constant correlation:									
0.0	1.0	0.0	0.0	**1.0**	1.5	0.04	0.8	0.04	0.5
0.0	1.0	0.0	0.0	**0.5**	1.5	0.04	0.8	0.04	0.5
0.0	**0.0**	0.0	0.0	**0.0**	**0.0**	0.04	0.8	0.04	0.5
Figure 7(a): Influence of the long–term mean influenced by R_3 for speed of mean reversion $R_1 = 1$:									
0.0	1.0	0.0	**0**	1.0	1.5	0.04	0.8	0.04	0.5
0.5	1.0	0.0	**0.5**	1.0	1.5	0.04	0.8	0.04	0.5
-0.5	1.0	0.0	**-0.5**	1.0	1.5	0.04	0.8	0.04	0.5
Figure 7(b): Influence of the long–term mean influenced by R_3 for speed of mean reversion $R_1 = 0.1$:									
0.0	0.1	0.0	**0.0**	1.0	1.5	0.04	0.8	0.04	0.5
0.5	0.1	0.0	**0.5**	1.0	1.5	0.04	0.8	0.04	0.5
-0.5	0.1	0.0	**-0.5**	1.0	1.5	0.04	0.8	0.04	0.5

Table 1: Parameter values used in the simulations for Figure 2 - 7.

B Proofs

B.1 Proofs for Section 3

Proof of Proposition 3.3: The drift and the continuous martingale part are straightforward to derive and, for the jump part, we get $\Delta Y_t = \rho_{\lambda t-}\Delta L_{\lambda t}$, and

$$\begin{aligned} Y_t^d &= \int_0^t \int_{-\infty}^0 z\mu_Y(dz, ds) = \int_0^t \rho_{\lambda s-} dL_{\lambda s} = \sum_{0\le s\le t} \rho_{\lambda s-}\Delta L_{\lambda s} \\ &= \int_0^t \int_0^\infty x\rho_{\lambda s-}\mu_L(dx, \lambda ds), \end{aligned}$$

where μ_L denotes the Poisson random measure associated with L with predictable compensator ν_L. In particular, for any $A \in \mathcal{B}(\mathbb{R}\setminus\{0\})$, we get

$$\mathbb{I}_A(-x) \star \mu_Y = \mathbb{I}_A(\rho_\lambda x) \star \mu_L,$$

and, hence,

$$\mathbb{I}_A(-x) \star \nu_Y = \mathbb{I}_A(\rho_\lambda x) \star \nu_L.$$

Note that ν_L is factorisable and homogeneous with $\nu_L(dx, ds) = U_L(dx)ds$, where U_L is the Lévy measure of L. Hence, we get for any $A \in \mathcal{B}(\mathbb{R}\setminus\{0\})$

$$\mathbb{I}_A(-x) \star \nu_Y(dx, dt) = \mathbb{I}_A(\rho_\lambda x) \star U_L(dx)\lambda dt.$$

□

B.2 Proofs for Section 4

Proof of Proposition 4.2: Proposition 4.2 is a straightforward application of the Girsanov theorem. Note that the independence of the Brownian motions under the risk neutral measure follows along the lines of Musiela & Rutkowski (2005, p. 233). □

Proof of Proposition 4.3: We start this proof by giving a very general outline on how to construct the density process Z. It turns out that Z is given by the Doléans–Dade exponential of the local martingale N, where N is constructed in the following. Note that throughout this proof, we follow closely Jacod & Shiryaev (2003, III.5). Let

$$\begin{aligned} a_t &= \nu_Y(\{t\}, \mathbb{R}), \\ \widehat{\xi}_t &= \begin{cases} \int \xi(t,x)\nu_Y(\{t\}, dx), & \text{if the integral exists} \\ \infty, & \text{otherwise}. \end{cases} \end{aligned}$$

We define a predictable time by

$$\Sigma = \inf\{t : \text{either } \widehat{\xi}_t > 1, \text{or } a_t = 1 \text{and } \widehat{\xi}_t < 1\},$$

which is always positive and we define

$$H_t = \int_0^t \psi_s \sigma_s^2 \mathbb{I}_{[0,\Sigma)}(s)ds + \int_0^t \int_0^\infty \left(1 - \sqrt{\xi(s,x)}\right)^2 \mathbb{I}_{[0,\Sigma)}(s)\nu_Y(ds,dx) + \sum_{s \le t} \left(\sqrt{1-a_s} - \sqrt{1-\widehat{\xi}_s}\right)^2 \mathbb{I}_{[0,\Sigma)}(s).$$

Furthermore, let

$$T_n = \inf(t, H_t \ge n), \qquad \Delta = [0,\Sigma) \cap (\cup_n [0,T_n]).$$

From Jacod & Shiryaev (2003, Proposition III.5.10), we get that there is a unique process N on the set Δ such that for every stopping time τ which satisfies $[0,\tau] \subset \Delta$, the stopped process N^τ is a $\mathbb{P}$–local martingale given by

$$N^\tau = (\psi \mathbb{I}_{\{[0,\tau]\}}) \cdot Y^c + \widetilde{V}\mathbb{I}_{\{[0,\tau]\}} \star (\mu_Y - \nu_Y),$$
$$\widetilde{V} = \left(\xi - 1 + \frac{\widehat{\xi} - a}{1-a}\mathbb{I}_{\{a<1\}}\right)\mathbb{I}_{\{[0,\Sigma)\}},$$

under suitable integrability conditions on the processes such that the integrals above exist.

The density process has then the form

$$Z = Z_0 + (Z_- \psi) \cdot Y^c + \left(Z_- \widetilde{V}\right) \star (\mu_Y - \nu_Y) + Z',$$

where all quantities are as defined above and Z' is a $\mathbb{P}$–local martingale with $Z'_0 = 0$, $\left\langle Z'^c, Y^c \right\rangle = 0$ and $\mathbb{E}\left(Z \star \mu_Y | \widetilde{\mathcal{P}}\right) = 0$, see Jacod & Shiryaev (2003, Lemma III.5.17).

If we additionally assume that all local martingales under $\mathbb{P}$ are representable relative to Y, see Jacod & Shiryaev (2003, III.4c), and that $a_t = 0$ (which implies that Y is *quasi–left continuous*), then the density process is, according to Jacod & Shiryaev (2003, Theorem III.5.19), given by

$$Z = Z_0 + (Z_- \psi) \cdot Y^c + (Z_-(\xi - 1)) \star (\mu_Y - \nu_Y),$$

which is the density mentioned by Nicolato & Venardos (2003). This result can also be written as

$$Z_t = \begin{cases} Z_0 \exp\left(N_t - \frac{1}{2}\int_0^t \psi_s^2 \sigma_s^2 ds\right) \prod_{0 \le s \le t} (1 + \Delta N_s) e^{-\Delta N_s}, & \text{if } t \in \Delta, \\ 0, & \text{if } t \notin \Delta \end{cases}.$$

Furthermore, we deduce from Jacod & Shiryaev (2003, II.8) that the formula above is equivalent to

$$Z_t = \begin{cases} Z_0 \mathcal{E}(N_t), & \text{if } t \in \Delta, \\ 0, & \text{if } t \notin \Delta \end{cases}.$$

Now, we have all the means to carry out the proof of the proposition: Using the above results, we apply the generalised Girsanov theorem to the triple (W, W^V, L). In particular, we get that

$$W_t^* = W_t - \int_0^t \psi_s^{(1)} ds, \qquad W_t^{V*} = W_t^V - \int_0^t \psi_s^{(2)} ds,$$

are Brownian motions with respect to $\mathbb{P}^*$, and

$$\nu_L^*(dt, dx) = \lambda \xi(t, x) U_L(x) dx dt$$

is the compensator of μ_{L_λ} under $\mathbb{P}^*$. Under the new measure $\mathbb{P}^*$, the asset price S satisfies the following SDE:

$$dS_t = S_{t-}(b_t^* dt + \sigma_t dW_t^* + dM_t^*),$$

where

$$\int_0^{\lambda t} \int_0^\infty (e^{\rho_t x} - 1) (\mu_L - \nu_L^*)(dx, ds),$$

and

$$b_t^* = \mu + \left(b + \frac{1}{2}\right) \sigma_t^2 + \lambda \int_0^\infty (e^{\rho_{\lambda s} x} - 1) \xi(t, x) U_L(dx) + \sigma_t \psi_t^{(1)}.$$

In order to ensure that $e^{-rt} S_t$ is a (local) martingale under $\mathbb{P}$, we set $b_t^* = r$. □

Proof of Proposition 4.4: The first part of the proof follows from the general Girsanov theorem and the arguments given in the proof of Nicolato & Venardos (2003, Theorem 3.2). Additionally, we deduce the independence of $L^{\widetilde{\xi}}$ and $W^{V,\widetilde{\xi}}$ under $\mathbb{P}^*$ from Sato (1999, Theorem 19.3). Finally, a straightforward computation along the lines of Musiela & Rutkowski (2005, p. 233) shows that $W^{V,\widetilde{\xi}}$ and $W^{\widetilde{\xi}}$ are independent under $\mathbb{P}^*$. □

Proof of Proposition 4.5: Let $u \in \mathbb{R}$. Then

$$\begin{aligned}
&\mathbb{E} \left(\exp(u(Y_T - Y_t)) \middle| \mathcal{F}_t\right) \\
&= \mathbb{E} \left(\exp \left(u \left(\mu(T-t) + b \int_t^T \sigma_s^2 ds \right) + u \int_t^T \sigma_s dW_s + u \int_t^T \rho_{\lambda s} dL_{\lambda s} \right) \middle| \mathcal{F}_t \right) \\
&= \mathbb{E} \left(\mathbb{E} \left(\exp \left(u \left(\mu(T-t) + b \int_t^T \sigma_s^2 ds \right) + u \int_t^T \sigma_s dW_s + u \int_t^T \rho_{\lambda s} dL_{\lambda s} \right) \right.\right. \\
&\qquad \left.\left. | \mathcal{F}^L, \mathcal{F}^{W^V} \right) \middle| \mathcal{F}_t \right) \\
&= \mathbb{E} \left(\exp \left(u \left(\mu(T-t) + b \int_t^T \sigma_s^2 ds \right) + u \int_t^T \rho_{\lambda s} dL_{\lambda s} \right) \right. \\
&\qquad \left. \mathbb{E} \left(\exp \left(u \int_t^T \sigma_s dW_s \right) \middle| \mathcal{F}^L, \mathcal{F}^{W^V} \right) \middle| \mathcal{F}_t \right).
\end{aligned}$$

Note that

$$\mathbb{E} \left(\exp \left(u \int_t^T \sigma_s dW_s \right) \middle| \mathcal{F}^L, \mathcal{F}^{W^V} \right) = \exp \left(\frac{u^2}{2} \int_t^T \sigma_s^2 ds \right),$$

hence, we have

$$\begin{aligned}
&\mathbb{E}\left(\exp(u(Y_T - Y_t)) \middle| \mathcal{F}_t\right) \\
&= \mathbb{E}\left(\exp\left(u\mu(T-t) + u\int_t^T \rho_{\lambda s} dL_{\lambda s} + \left(ub + \frac{u^2}{2}\right)\int_t^T \sigma_s^2 ds\right) \middle| \mathcal{F}_t\right) \\
&= \exp(u\mu(T-t))\mathbb{E}\left(\exp\left(u\int_t^T \rho_{\lambda s} dL_{\lambda s} + \left(ub + \frac{u^2}{2}\right)\int_t^T \sigma_s^2 ds\right) \middle| \mathcal{F}_t\right).
\end{aligned}$$

Note that

$$\int_t^T \sigma_s^2 ds = \frac{1}{\lambda}\left(1 - e^{-\lambda(T-t)}\right)\sigma_t^2 + \frac{1}{\lambda}\int_t^T \left(1 - e^{-\lambda(T-s)}\right) dL_{\lambda s}.$$

Hence, we get for $v = \left(ub + \frac{u^2}{2}\right)$ that

$$\begin{aligned}
\mathbb{E}\left(\exp(u(Y_T - Y_t)) \middle| \mathcal{F}_t\right) = \exp&\left(u\mu(T-t) + v\frac{1}{\lambda}\left(1 - e^{-\lambda(T-t)}\right)\sigma_t^2\right) \\
&\mathbb{E}\left(\exp\left(\int_t^T \left(u\rho_{\lambda s} + \frac{v}{\lambda}\left(1 - e^{-\lambda(T-s)}\right)\right) dL_{\lambda s}\right) \middle| \mathcal{F}_t\right).
\end{aligned}$$

Now we apply Cont & Tankov (2004, Lemma 15.1) and obtain

$$\begin{aligned}
&\mathbb{E}\left(\exp\left(\int_t^T \left(u\rho_{\lambda s} + \frac{v}{\lambda}\left(1 - e^{-\lambda(T-s)}\right)\right) dL_{\lambda s}\right) \middle| \mathcal{F}^{W^V}, \mathcal{F}_t\right) \\
&= \exp\left(\lambda\int_t^T \kappa\left(u\rho_{\lambda s} + \frac{v}{\lambda}\left(1 - e^{-\lambda(T-s)}\right)\right) ds\right),
\end{aligned}$$

where κ denotes the cumulant transform of the subordinator L. □

B.3 Proofs for Section 6

Proof of Proposition 6.1: Note that

$$\mathbb{E}\left(\sigma_0^2\right) = \beta, \qquad \mathbb{E}\left(\sigma_0^4\right) = \beta^2 + \frac{\beta\gamma^2}{2\alpha}, \qquad Var\left(\sigma_0^2\right) = \frac{\beta\gamma^2}{2\alpha},$$
$$\mathbb{E}\left(\sigma_u^2, \sigma_s^2\right) = \beta^2 + \frac{\beta\gamma^2}{2\alpha}e^{-\alpha|u-s|}, \qquad Cor\left(\sigma_u^2, \sigma_s^2\right) = e^{-\alpha|u-s|}.$$

First, we compute $Var\left(\int_{t-h}^t dY_s\right)$. Note that

$$\int_{t-h}^t dY_s = Y_t - Y_{t-h} = \mu h + b\int_{t-h}^t \sigma_s^2 ds + \int_{t-h}^t \sigma_s dX_s.$$

Clearly,

$$\mathbb{E}\left(\int_{t-h}^t dY_s\right) = \mu h + bh\mathbb{E}\left(\sigma_0^2\right) = (\mu + b\beta)\, h.$$

Also,

$$\begin{aligned}\mathbb{E}\left(Y_t^2\right) &= \mathbb{E}\left(2\int_0^t Y_d dY_s + [Y]_t\right)\\ &= 2\mathbb{E}\left(\int_0^t\int_0^s \left((\mu + b\sigma_u^2)du + \sigma_u dX_u\right)\left(\mu + b\sigma_s^2\right)ds\right) + t\mathbb{E}\left(\sigma_0^2\right)\\ &= \mu^2t^2 + 2\mu b\beta t^2 + b^2\frac{\beta\gamma^2}{\alpha^3}\left(-1 + \alpha t + e^{-\alpha t}\right) + b^2\beta^2t^2 + t\beta.\end{aligned}$$

Note that

$$\begin{aligned}\mathbb{E}\left(\int_0^t\int_0^s \sigma_u dX_u\sigma_s^2 ds\right) &= \mathbb{E}\left(\int_0^t\int_0^s \sigma_u\rho_u dW_u\sigma_s^2 ds\right)\\ &= -\alpha\mathbb{E}\left(\int_0^t\int_0^s\int_0^x \sigma_u\rho_u dW_u\sigma_x^2 dxds + \beta\mathbb{E}(\rho)\gamma\frac{t^2}{2}\right),\end{aligned}$$

and when we solve this integral equation, we get

$$\mathbb{E}\left(\int_0^t\int_0^s \sigma_u dX_u\sigma_s^2 ds\right) = \frac{\beta\mathbb{E}(\rho)\gamma}{\alpha^2}\left(-1 + \alpha t + e^{-\alpha t}\right).$$

Since Y has stationary increments, we conclude that

$$\begin{aligned}\mathbb{E}\left((Y_t - Y_{t-h})^2\right) = \mu^2h^2 + 2\mu b\beta h^2 + b^2\left(-\frac{\beta\gamma^2}{\alpha^3} + \frac{\beta\gamma^2}{\alpha^2}h + \beta^2h^2 + \frac{\beta\gamma^2}{\alpha^3}e^{-\alpha h}\right) + h\beta\\ + 2b\frac{\beta\mathbb{E}(\rho)\gamma}{\alpha^2}\left(-1 + \alpha h + e^{-\alpha h}\right).\end{aligned}$$

Hence

$$\begin{aligned}Var\left(Y_t - Y_{t-h}\right) &= \beta h + \left(b^2\frac{\beta\gamma^2}{\alpha^3} + 2b\frac{\beta\mathbb{E}(\rho)\gamma}{\alpha^2}\right)\left(-1 + h\alpha + e^{-\alpha h}\right)\\ &= \beta h + \frac{b\beta\gamma}{\alpha^2}\left(\frac{b\gamma}{\alpha} + 2\mathbb{E}(\rho)\right)\left(-1 + h\alpha + e^{-\alpha h}\right).\end{aligned}$$

Next, we compute the covariance. Clearly,

$$\begin{aligned}&Cov\left(\int_t^{t+h}\sigma_u^2 du, \int_{t-h}^t dY_s\right)\\ &= bCov\left(\int_t^{t+h}\sigma_u^2 du, \int_{t-h}^t \sigma_s^2 ds\right) + Cov\left(\int_t^{t+h}\sigma_u^2 du, \int_{t-h}^t \sigma_s dX_s\right)\\ &= \left(\frac{b\beta\gamma^2}{2\alpha} + \beta\mathbb{E}(\rho)\gamma\right)\frac{1}{\alpha^2}\left(1 - e^{-\alpha h}\right)^2,\end{aligned}$$

since

$$Cov\left(\int_t^{t+h}\sigma_u^2 du, \int_{t-h}^t \sigma_s^2 ds\right) = \frac{\beta\gamma^2}{2\alpha}\int_t^{t+h}\int_{t-h}^t e^{-\alpha|u-s|}duds = \frac{\beta\gamma^2}{2\alpha}\frac{1}{\alpha^2}\left(1 - e^{-\alpha h}\right)^2,$$

and

$$Cov\left(\int_t^{t+h}\sigma_u^2du, \int_{t-h}^t \sigma_s dX_s\right) = \mathbb{E}\left(\int_t^{t+h}\int_{t-h}^t \sigma_s\rho_s dW_s\sigma_u^2du\right)$$
$$= \frac{\beta\mathbb{E}(\rho)\gamma}{\alpha^2}\left(1-e^{-\alpha h}\right)^2.$$

Hence, we get

$$D := \frac{Cov\left(\int_t^{t+h}\sigma_u^2du, \int_{t-h}^t dY_s\right)}{Var\left(\int_{t-h}^t dY_s\right)} = \frac{\left(\frac{b\beta\gamma^2}{2\alpha}+\beta\mathbb{E}(\rho)\gamma\right)\frac{1}{\alpha^2}\left(1-e^{-\alpha h}\right)^2}{\beta h + \frac{b\beta\gamma}{\alpha^2}\left(\frac{b\gamma}{\alpha}+2\mathbb{E}(\rho)\right)\left(-1+h\alpha+e^{-\alpha h}\right)}.$$

Hence, the intercept in the regression is given by

$$G := \mathbb{E}\left(\int_t^{t+h}\sigma_s^2ds\right) - \frac{Cov\left(\int_t^{t+h}\sigma_u^2du, \int_{t-h}^t dY_s\right)}{Var\left(\int_{t-h}^t dY_s\right)}\mathbb{E}\left(Y_t - Y_{t-h}\right)$$
$$= h\beta - D(\mu + b\beta)h.$$

The coefficient of the implied volatility asymmetry is defined by

$$D^* = \frac{Cov\left(\mathbb{E}_t^*\left(\int_t^{t+h}\sigma_u^2du\right), \int_{t-h}^t dY_s\right)}{Var\left(\int_{t-h}^t dY_s\right)}.$$

Note that $\mathbb{E}_t^*\left(\int_t^{t+h}\sigma_u^2du\right) = \frac{1}{\alpha^*}\left(1-e^{-\alpha^* h}\right)\left(\sigma_t^2-\beta^*\right)+\beta^*h$, and, hence,

$$Cov\left(\mathbb{E}_t^*\left(\int_t^{t+h}\sigma_u^2du\right), \int_{t-h}^t dY_s\right)$$
$$= Cov\left(\frac{1}{\alpha^*}\left(1-e^{-\alpha^* h}\right)\sigma_t^2 + \beta^*\left(h - \frac{1}{\alpha^*}\left(1-e^{-\alpha^* h}\right)\right), \int_{t-h}^t dY_s\right)$$
$$= b\frac{1}{\alpha^*}\left(1-e^{-\alpha^* h}\right)Cov\left(\sigma_t^2, \int_{t-h}^t\sigma_s^2ds\right) + \frac{1}{\alpha^*}\left(1-e^{-\alpha^* h}\right)Cov\left(\sigma_t^2, \int_{t-h}^t\sigma_s dX_s\right)$$
$$= b\frac{\beta\gamma^2}{2\alpha}\frac{1}{\alpha^*}\left(1-e^{-\alpha^* h}\right)\frac{1}{\alpha}\left(1-e^{-\alpha h}\right) + \mathbb{E}(\rho)\beta\gamma\frac{1}{\alpha^*}\left(1-e^{-\alpha^* h}\right)\frac{1}{\alpha}\left(1-e^{-\alpha h}\right).$$

Hence, we get

$$D^* = \frac{\left(\frac{b\beta\gamma^2}{2\alpha}+\beta\mathbb{E}(\rho)\gamma\right)\frac{1}{\alpha}\left(1-e^{-\alpha h}\right)\frac{1}{\alpha^*}\left(1-e^{-\alpha^* h}\right)}{\beta h + \frac{b\beta\gamma}{\alpha^2}\left(\frac{b\gamma}{\alpha}+2\mathbb{E}(\rho)\right)\left(-1+h\alpha+e^{-\alpha h}\right)},$$

and, for the intercept, we get

$$G^* = \left(\beta-\beta^*\right)\frac{1}{\alpha^*}\left(1-e^{-\alpha^* h}\right) + h\beta^* - D^*(\mu+b\beta)h.$$

Clearly,

$$Var\left(Y_t - Y_{t-h}\right) = \beta h + \frac{b\beta\gamma}{\alpha^2}\left(\frac{b\gamma}{\alpha} + 2\mathbb{E}(\rho)\right)\left(-1 + h\alpha + e^{-\alpha h}\right) > 0,$$

hence, if $0 < b < -2\frac{\mathbb{E}(\rho)\alpha}{\gamma}$, then $D^* < D < 0$ and if $0 < -2\frac{\mathbb{E}(\rho)\alpha}{\gamma} < b$, we get $0 < D < D^*$. Recall that $\alpha^* = \alpha + A$. So, the assumption that $A < 0$ leads to $0 < \frac{1}{\alpha}\left(1 - e^{-\alpha h}\right) < \frac{1}{\alpha^*}\left(1 - e^{-\alpha^* h}\right)$. □

Proof of Proposition 6.2: Note that for

$$\begin{aligned} dY_t &= (\mu + b\sigma_t^2)dt + \sigma_t dW_s + \rho_{\lambda t} dL_{\lambda t}, \\ d\sigma_t^2 &= -\lambda\sigma_t^2 dt + dL_{\lambda t}, \\ d\rho_t &= (\zeta + \eta\rho_t)dt + \theta\sqrt{-\rho_t(1+\rho_t)}dW_t^V, \end{aligned}$$

we have

$$\mathbb{E}\left(\sigma_0^2\right) = \kappa_1, \qquad \mathbb{E}\left(\sigma_0^4\right) = \kappa_1^2 + \frac{\kappa_2}{2}, \qquad Var\left(\sigma_0^2\right) = \frac{\kappa_2}{2},$$
$$\mathbb{E}\left(\sigma_u^2, \sigma_s^2\right) = \kappa_1^2 + \frac{\kappa_2}{2}e^{(-\lambda|u-s|)}, \qquad Cor\left(\sigma_u^2, \sigma_s^2\right) = e^{-\lambda|u-s|}.$$

First, we compute $Var\left(\int_{t-h}^t dY_s\right)$. Note that

$$\int_{t-h}^t dY_s = Y_t - Y_{t-h} = \mu h + b\int_{t-h}^t \sigma_s^2 ds + \int_{t-h}^t \sigma_s dW_s + \int_t^{t+h} \rho_{\lambda s} dL_{\lambda s}.$$

Clearly,

$$\mathbb{E}\left(\int_{t-h}^t dY_s\right) = \mu h + bh\mathbb{E}\left(\sigma_0^2\right) + \lambda\kappa_1\mathbb{E}(\rho)h = \left(\mu + b\kappa_1 + \lambda\kappa_1\mathbb{E}(\rho)\right)h.$$

Note that $\mathbb{E}(\rho) = -\frac{\zeta}{\eta}$. Also,

$$\begin{aligned} \mathbb{E}\left(Y_t^2\right) &= \mathbb{E}\left(2\int_0^t Y_d dY_s + [Y]_t\right) \\ &= 2\mathbb{E}\left(\int_0^t\int_0^s \left((\mu + b\sigma_u^2)du + \sigma_u dW_u + \rho_{\lambda u} dL_{\lambda u}\right)\left(\left(\mu + b\sigma_s^2\right)ds + \rho_{\lambda s} dL_{\lambda s}\right)\right) \\ &\quad + t\mathbb{E}\left(\sigma_0^2\right) + t\lambda\kappa_1\mathbb{E}\left(\rho\right). \end{aligned}$$

Note that

$$\begin{aligned} &2\mathbb{E}\left(\int_0^t\int_0^s \left(\mu + b\sigma_u^2\right)du\left(\mu + b\sigma_s^2\right)ds\right) \\ &\qquad = \mu^2 t^2 + 2\mu b\kappa_1 t^2 + b^2\kappa_2\frac{1}{\lambda^2}\left(-1 + e^{-\lambda t} + \lambda t\right) + b^2\kappa_1^2 t^2, \end{aligned}$$

and

$$\mathbb{E}\left(2\int_0^t\int_0^s \left(\mu + b\sigma_u^2\right)du\rho_{\lambda s} dL_{\lambda s}\right) = \mu\lambda\kappa_1\mathbb{E}\left(\rho\right)t^2 + \lambda\kappa_1^2 b\mathbb{E}\left(\rho\right)t^2 = \left(\mu + b\kappa_1\right)\lambda\kappa_1\mathbb{E}\left(\rho\right)t^2.$$

Furthermore, we have

$$\mathbb{E}\left(\int_0^t \int_0^s \sigma_u dW_u \sigma_s^2 ds\right) = -\lambda \mathbb{E}\left(\int_0^t \int_0^s \int_0^x \sigma_u dW_u \sigma_x^2 dx ds\right),$$

and when we solve this integral equation with initial value 0, we get $\mathbb{E}\left(\int_0^t \int_0^s \sigma_u dW_u \sigma_s^2 ds\right) = 0$, and, hence,

$$\mathbb{E}\left(2\int_0^t \int_0^s \sigma_u dW_u \left(\mu + b\sigma_s^2\right) ds\right) = 0.$$

Also, we have

$$2\mathbb{E}\left(\int_0^t \int_0^s \sigma_u dW_u \rho_{\lambda s} dL_{\lambda s}\right) = 2\lambda\kappa_1 \int_0^t \mathbb{E}\left(\int_0^s \sigma_u dW_u \rho_{\lambda s} ds\right)$$
$$= 2\lambda\kappa_1 \int_0^t \mathbb{E}\left(\int_0^s \sigma_u dW_u\right) \mathbb{E}\left(\rho_{\lambda s}\right) ds = 0,$$

and

$$2\mathbb{E}\left(\int_0^t \int_0^s \rho_{\lambda u} dL_{\lambda u} \left(\mu + b\sigma_s^2\right) ds\right) = \mu\lambda\kappa_1 \mathbb{E}(\rho) t^2 + 2b\mathbb{E}\left(\int_0^t \int_0^s \rho_{\lambda u} dL_{\lambda u} \sigma_s^2 ds\right),$$

where

$$\begin{aligned}
&\mathbb{E}\left(\int_0^s \rho_{\lambda u} dL_{\lambda u} \sigma_s^2\right) \\
&= \mathbb{E}\left(\int_0^s \int_0^u \rho_{\lambda x} dL_{\lambda x} d\sigma_u^2\right) + \mathbb{E}\left(\int_0^s \sigma_u^2 \rho_{\lambda u} dL_{\lambda u}\right) + \mathbb{E}\left(\int_0^s \rho_{\lambda u} d[L]_{\lambda u}\right) \\
&= -\lambda \mathbb{E}\left(\int_0^s \int_0^u \rho_{\lambda x} dL_{\lambda x} \sigma_u^2 du\right) + \mathbb{E}\left(\int_0^s \int_0^u \rho_{\lambda x} dL_{\lambda x} dL_{\lambda u}\right) + \lambda\kappa_1^2 \mathbb{E}(\rho) s + \lambda\kappa_2 \mathbb{E}(\rho) s \\
&= -\lambda \mathbb{E}\left(\int_0^s \int_0^u \rho_{\lambda x} dL_{\lambda x} \sigma_u^2 du\right) + \lambda^2\kappa_1^2 \mathbb{E}(\rho) \frac{s^2}{2} + \lambda\left(\kappa_1^2 + \kappa_2\right) \mathbb{E}(\rho) s.
\end{aligned}$$

So, for $y(t) = \mathbb{E}\left(\int_0^t \rho_{\lambda u} dL_{\lambda u} \sigma_t^2\right)$ we obtain an ordinary differential equation (ODE) of type

$$y'(t) + \lambda y(t) = \lambda^2 \kappa_1^2 \mathbb{E}(\rho) t + \lambda(\kappa_1^2 + \kappa_2),$$

with $y(0) = 0$. Hence, we get

$$y(t) = \mathbb{E}\left(\int_0^t \rho_{\lambda u} dL_{\lambda u} \sigma_t^2\right) = \left(1 - e^{-\lambda t}\right) \kappa_2 \mathbb{E}(\rho) + \lambda\kappa_1^2 \mathbb{E}(\rho) t,$$

and

$$\mathbb{E}\left(\int_0^t \int_0^s \rho_{\lambda u} dL_{\lambda u} \sigma_s^2 ds\right) = \kappa_2 \mathbb{E}(\rho) \frac{1}{\lambda}\left(-1 + \lambda t + e^{-\lambda t}\right) + \frac{1}{2}\lambda\kappa_1^2 \mathbb{E}(\rho) t^2.$$

Note that using the same reasoning as above, we get for any $0 \leq x \leq t$:

$$\mathbb{E}\left(\int_x^t \rho_{\lambda s} dL_{\lambda s} \sigma_t^2\right) = \left(1 - e^{-\lambda(t-x)}\right) \kappa_2 \mathbb{E}(\rho) + \lambda \kappa_1^2 \mathbb{E}(\rho)(t-x).$$

Altogether, we have

$$2\mathbb{E}\left(\int_0^t \int_0^s \rho_{\lambda u} dL_{\lambda u} \left(\mu + b\sigma_s^2\right) ds\right)$$
$$= \lambda \kappa_1 \mathbb{E}(\rho) \left(\mu + b\kappa_1\right) t^2 + 2b\lambda\kappa_2 \mathbb{E}(\rho) \frac{1}{\lambda^2} \left(-1 + \lambda t + e^{-\lambda t}\right),$$

and

$$\mathbb{E}\left(2\int_0^t \int_0^s \rho_{\lambda u} dL_{\lambda u} \rho_{\lambda s} dL_{\lambda s}\right) = 2\lambda\kappa_1 \int_0^t \mathbb{E}\left(\int_0^s \rho_{\lambda u} dL_{\lambda u} \rho_{\lambda s}\right) ds$$
$$= 2\lambda\kappa_1 \int_0^t \left(\lambda^2 \zeta \kappa_1 \mathbb{E}\left(\rho\right) \frac{s^2}{2} + \lambda\eta \mathbb{E}\left(\int_0^s \int_0^u \rho_{\lambda x} dL_{\lambda x}\right) \rho_{\lambda u} du + \lambda \kappa_1 \mathbb{E}\left(\rho^2\right) s\right) ds,$$

where

$$\mathbb{E}\left(\int_0^s \rho_{\lambda u} dL_{\lambda u} \rho_{\lambda s}\right) = \mathbb{E}\left(\int_0^s \int_0^u \rho_{\lambda x} dL_{\lambda x}\right) d\rho_{\lambda u} + \mathbb{E}\left(\int_0^s \rho_{\lambda u}^2 dL_{\lambda u}\right)$$
$$= \lambda\zeta \mathbb{E}\left(\int_0^s \int_0^u \rho_{\lambda x} dL_{\lambda x}\right) du + \lambda\eta \mathbb{E}\left(\int_0^s \int_0^u \rho_{\lambda x} dL_{\lambda x}\right) \rho_{\lambda u} du + \lambda\kappa_1 \mathbb{E}\left(\rho^2\right) s$$
$$= \lambda^2 \zeta \kappa_1 \mathbb{E}\left(\rho\right) \frac{s^2}{2} + \lambda\eta \mathbb{E}\left(\int_0^s \int_0^u \rho_{\lambda x} dL_{\lambda x}\right) \rho_{\lambda u} du + \lambda\kappa_1 \mathbb{E}\left(\rho^2\right) s.$$

Hence, we get

$$\mathbb{E}\left(2\int_0^t \int_0^s \rho_{\lambda u} dL_{\lambda u} \rho_{\lambda s} dL_{\lambda s}\right)$$
$$= 2\lambda\kappa_1 \left(\lambda\kappa_1 \mathbb{E}\left(\rho^2\right) + \lambda \frac{\zeta}{\eta} \kappa_1 \mathbb{E}(\rho)\right) \frac{1}{\lambda^2 \eta^2} \left(-1 - \lambda\eta t + e^{\lambda\eta t}\right) - \lambda^2 \kappa_1^2 \frac{\zeta}{\eta} \mathbb{E}(\rho) t^2.$$

Altogether, we get

$$\mathbb{E}\left((Y_t)^2\right) = \mu^2 t^2 + 2\mu b\kappa_1 t^2 + b^2 \kappa_2 \frac{1}{\lambda^2} \left(-1 + e^{-\lambda t} + \lambda t\right) + b^2 \kappa_1^2 t^2$$
$$+ 2\left(\mu + b\kappa_1\right) \lambda\kappa_1 \mathbb{E}\left(\rho\right) t^2 + 2b\lambda\kappa_2 \mathbb{E}(\rho) \frac{1}{\lambda^2} \left(-1 + \lambda t + e^{-\lambda t}\right)$$
$$+ 2\lambda\kappa_1 \left(\lambda\kappa_1 \mathbb{E}\left(\rho^2\right) + \lambda \frac{\zeta}{\eta} \kappa_1 \mathbb{E}(\rho)\right) \frac{1}{\lambda^2 \eta^2} \left(-1 - \lambda\eta t + e^{\lambda\eta t}\right) - \lambda^2 \kappa_1^2 \frac{\zeta}{\eta} \mathbb{E}(\rho) t^2$$
$$+ \kappa_1 \left(1 + \lambda \mathbb{E}(\rho)\right) t.$$

Since Y has stationary increments, we conclude that $\mathbb{E}\left((Y_t - Y_{t-h})^2\right) = \mathbb{E}\left((Y_h)^2\right)$. Hence

$$Var\left(Y_t - Y_{t-h}\right) = b\kappa_2 \left\{2\lambda \mathbb{E}(\rho) + b\right\} \frac{1}{\lambda^2} \left(-1 + \lambda h + e^{-\lambda h}\right)$$
$$+ 2\lambda^2 \kappa_1^2 \mathbb{E}\left(\rho^2\right) \frac{1}{\lambda^2 \eta^2} \left(-1 - \lambda\eta h + e^{\lambda\eta h}\right) + \kappa_1 \left(1 + \lambda \mathbb{E}(\rho)\right) h.$$

Next, we compute the covariance. Clearly,

$$\begin{aligned}
&Cov\left(\int_t^{t+h}\sigma_u^2du, \int_{t-h}^t dY_s\right)\\
&= bCov\left(\int_t^{t+h}\sigma_u^2du, \int_{t-h}^t \sigma_s^2ds\right) + Cov\left(\int_t^{t+h}\sigma_u^2du, \int_{t-h}^t \rho_{\lambda s}dL_{\lambda s}\right),
\end{aligned}$$

where

$$Cov\left(\int_t^{t+h}\sigma_u^2du, \int_{t-h}^t \sigma_s^2ds\right) = \frac{\kappa_2}{2}\int_t^{t+h}\int_{t-h}^t e^{-\lambda|u-s|}duds = \frac{\kappa_2}{2}\frac{1}{\lambda^2}\left(1-e^{-\lambda h}\right)^2,$$

and

$$\begin{aligned}
Cov\left(\int_t^{t+h}\sigma_u^2du, \int_{t-h}^t \rho_{\lambda s}dL_{\lambda s}\right) &= \mathbb{E}\left(\int_t^{t+h}\int_{t-h}^t \rho_{\lambda s}dL_{\lambda s}\sigma_u^2du\right) - \lambda\kappa_1^2\mathbb{E}(\rho)h^2\\
&= \int_t^{t+h}\mathbb{E}\left(\int_{t-h}^t \rho_{\lambda s}dL_{\lambda s}\sigma_u^2\right)du - \lambda\kappa_1^2\mathbb{E}(\rho)h^2,
\end{aligned}$$

where for $u \geq t$:

$$\begin{aligned}
&\mathbb{E}\left(\int_{t-h}^t \rho_{\lambda s}dL_{\lambda s}\sigma_u^2\right) = \mathbb{E}\left(\int_{t-h}^t \rho_{\lambda t}dL_{\lambda s}\sigma_t^2\right) + \mathbb{E}\left(\int_{t-h}^t \rho_{\lambda s}dL_{\lambda s}\int_t^u d\sigma_x^2\right)\\
&= \mathbb{E}\left(\int_{t-h}^t \rho_{\lambda t}dL_{\lambda s}\sigma_t^2\right) - \lambda\mathbb{E}\left(\int_{t-h}^t \rho_{\lambda s}dL_{\lambda s}\int_t^u \sigma_x^2dx\right) + \mathbb{E}\left(\int_{t-h}^t \rho_{\lambda s}dL_{\lambda s}\int_t^u dL_{\lambda x}\right)\\
&= \lambda\kappa_1^2\mathbb{E}(\rho)h + \kappa_2\mathbb{E}(\rho)\left(1-e^{-\lambda h}\right) - \lambda\mathbb{E}\left(\int_{t-h}^t \rho_{\lambda s}dL_{\lambda s}\int_t^u \sigma_x^2dx\right).
\end{aligned}$$

So, we obtain a differential equation of the type

$$\begin{aligned}
&y'(u) + \lambda y(u) = 0,\\
&\qquad y(t) = \lambda\kappa_1^2\mathbb{E}(\rho)h + \kappa_2\mathbb{E}(\rho)\left(1-e^{-\lambda h}\right) =: A(h).
\end{aligned}$$

From solving the above ODE, we get

$$\mathbb{E}\left(\int_{t-h}^t \rho_{\lambda s}dL_{\lambda s}\sigma_u^2\right) = A(h)e^{-\lambda(u-t)},$$

and

$$\begin{aligned}
\int_t^{t+h}\mathbb{E}\left(\int_{t-h}^t \rho_{\lambda s}dL_{\lambda s}\sigma_u^2\right)du &= A(h)\frac{1}{\lambda}\left(1-e^{-\lambda h}\right)\\
&= \kappa_1^2\mathbb{E}(\rho)h\left(1-e^{-\lambda h}\right) + \lambda\kappa_2\mathbb{E}(\rho)\frac{1}{\lambda^2}\left(1-e^{-\lambda h}\right)^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
Cov\left(\int_t^{t+h}\sigma_u^2du, \int_{t-h}^t \rho_{\lambda s}dL_{\lambda s}\right) &= \kappa_1^2\mathbb{E}(\rho)h\left(1-\lambda h-e^{-\lambda h}\right)\\
&\quad + \lambda\kappa_2\mathbb{E}(\rho)\frac{1}{\lambda^2}\left(1-e^{-\lambda h}\right)^2.
\end{aligned}$$

Altogether, we have

$$Cov\left(\int_t^{t+h}\sigma_u^2du,\int_{t-h}^t dY_s\right)=\kappa_2\left(\frac{b}{2}+\mathbb{E}(\rho)\lambda\right)\frac{1}{\lambda^2}\left(1-e^{-\lambda h}\right)^2 - \kappa_1^2\mathbb{E}(\rho)h\left(-1+\lambda h+e^{-\lambda h}\right).$$

Hence, we get

$$D:=\frac{Cov\left(\int_t^{t+h}\sigma_u^2du,\int_{t-h}^t dY_s\right)}{Var\left(\int_{t-h}^t dY_s\right)} =\frac{\kappa_2\left(\frac{b}{2}+\mathbb{E}(\rho)\lambda\right)\frac{1}{\lambda^2}\left(1-e^{-\lambda h}\right)^2-\kappa_1^2\mathbb{E}(\rho)h\left(-1+\lambda h+e^{-\lambda h}\right)}{Var\left(\int_{t-h}^t dY_s\right)},$$

$$Var\left(\int_{t-h}^t dY_s\right)=b\kappa_2\left\{2\lambda\mathbb{E}(\rho)+b\right\}\frac{1}{\lambda^2}\left(-1+\lambda h+e^{-\lambda h}\right) + 2\lambda^2\kappa_1^2\mathbb{E}\left(\rho^2\right)\frac{1}{\lambda^2\eta^2}\left(-1-\lambda\eta h+e^{\lambda\eta h}\right)+\kappa_1\left(1+\lambda\mathbb{E}(\rho)\right)h$$

Hence, the intercept in the regression is given by

$$G:=\mathbb{E}\left(\int_t^{t+h}\sigma_s^2ds\right)-\frac{Cov\left(\int_t^{t+h}\sigma_u^2du,\int_{t-h}^t dY_s\right)}{Var\left(\int_{t-h}^t dY_s\right)}\mathbb{E}\left(Y_t-Y_{t-h}\right) = h\kappa_1-D(\mu+b\kappa_1+\lambda\kappa_1\mathbb{E}(\rho))h.$$

The coefficient of the implied volatility asymmetry is defined by

$$D^*=\frac{Cov\left(\mathbb{E}_t^*\left(\int_t^{t+h}\sigma_u^2du\right),\int_{t-h}^t dY_s\right)}{Var\left(\int_{t-h}^t dY_s\right)}.$$

Here, we work with structure preserving changes of measure as discussed earlier. Under the new measure, the volatility process satisfies

$$d\sigma_t^2=-\lambda^*\sigma_t^2dt+dL^*_{\lambda^*t},$$

where L^* is a Lévy subordinator under the new measure with $\kappa_1^*=\mathbb{E}\left(L_1^*\right)$. Then

$$\mathbb{E}_t^*\left(\int_t^{t+h}\sigma_u^2du\right)=\int_t^{t+h}\mathbb{E}_t^*\left(\sigma_u^2\right)du,$$

where

$$\mathbb{E}_t^*\left(\sigma_u^2\right)=\sigma_t^2+\mathbb{E}_t^*\left(\int_t^u d\sigma_x^2\right)=\sigma_t^2-\lambda^*\mathbb{E}_t^*\left(\int_t^u\sigma_x^2dx\right)+\mathbb{E}_t^*\left(\int_t^u dL^*_{\lambda x}\right) =\sigma_t^2-\lambda^*\int_t^u\mathbb{E}_t^*\left(\sigma_x^2\right)dx+\lambda^*\kappa_1^*(u-t),$$

which leads to

$$\mathbb{E}_t^*\left(\sigma_u^2\right) = \sigma_t^2 e^{-\lambda^*(u-t)} + \kappa_1^*\left(1 - e^{-\lambda^*(u-t)}\right),$$

and, hence,

$$\mathbb{E}_t^*\left(\int_t^{t+h} \sigma_u^2 du\right) = \sigma_t^2 \frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right) + \kappa_1^*\left(h - \frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right)\right).$$

Hence,

$$\begin{aligned}
&Cov\left(\mathbb{E}_t^*\left(\int_t^{t+h} \sigma_u^2 du\right), \int_{t-h}^t dY_s\right) \\
&= Cov\left(\frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right)\sigma_t^2, \int_{t-h}^t dY_s\right) \\
&= b\frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right) Cov\left(\sigma_t^2, \int_{t-h}^t \sigma_s^2 ds\right) + \frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right) Cov\left(\sigma_t^2, \int_{t-h}^t \rho_{\lambda s} dL_{\lambda s}\right) \\
&= b\frac{\kappa_2}{2}\frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right)\frac{1}{\lambda}\left(1 - e^{-\lambda h}\right) + \mathbb{E}(\rho)\kappa_2 \frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right)\left(1 - e^{-\lambda h}\right).
\end{aligned}$$

Hence, we get

$$D^* = \frac{\left(\frac{b\kappa_2}{2} + \mathbb{E}(\rho)\kappa_2\lambda\right)\frac{1}{\lambda}\left(1 - e^{-\lambda h}\right)\frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right)}{Var\left(\int_{t-h}^t dY_s\right)},$$

and, for the intercept, we get

$$G^* = (\kappa_1 - \kappa_1^*)\frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right) + h\kappa_1^* - D^*(\mu + b\kappa_1 + \lambda\kappa_1\mathbb{E}(\rho))h.$$

Clearly,

$$Var\left(Y_t - Y_{t-h}\right) > 0,$$

and

$$-\kappa_1^2\mathbb{E}(\rho)h\left(-1 + \lambda h + e^{-\lambda h}\right) > 0,$$

hence,

$$D > \frac{\kappa_2\left(\frac{b}{2} + \mathbb{E}(\rho)\lambda\right)\frac{1}{\lambda^2}\left(1 - e^{-\lambda h}\right)^2}{Var\left(\int_{t-h}^t dY_s\right)}.$$

So, if $b < -2\lambda\mathbb{E}(\rho)$ and $\lambda^* < \lambda$, hence $0 < \frac{1}{\lambda}\left(1 - e^{-\lambda h}\right) < \frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right)$, we get $D^* < D$.

□

Proof of Proposition 6.3: We have to compute

$$\frac{Cov\left(\int_t^{t+h} dY_s, \int_t^{t+h} \sigma_u^2 du\right)}{Var\left(\int_t^{t+h} \sigma_u^2 du\right)}.$$

Clearly,

$$Var\left(\int_t^{t+h} \sigma_u^2 du\right) = \frac{\beta\gamma^2}{\alpha}\frac{1}{\alpha^2}\left(-1 + \alpha h + e^{-\alpha h}\right).$$

For the covariance, we get

$$\begin{aligned}
&Cov\left(\int_t^{t+h} dY_s, \int_t^{t+h} \sigma_u^2 du\right)\\
&= bCov\left(\int_t^{t+h} \sigma_s^2 ds, \int_t^{t+h} \sigma_u^2 du\right) + Cov\left(\int_t^{t+h} \sigma_s dX_s, \int_t^{t+h} \sigma_u^2 du\right),
\end{aligned}$$

where

$$\begin{aligned}
Cov\left(\int_t^{t+h} \sigma_s dX_s, \int_t^{t+h} \sigma_u^2 du\right) &= \mathbb{E}\left(\int_t^{t+h} \sigma_s \rho_s dW_s \int_t^{t+h} \sigma_u^2 du\right)\\
= \mathbb{E}\left(\int_t^{t+h}\int_t^{u} \sigma_s \rho_s dW_s \sigma_u^2 du\right) &= \beta\mathbb{E}(\rho)\gamma\frac{1}{\alpha^2}\left(-1 + \alpha h + e^{-\alpha h}\right).
\end{aligned}$$

Hence, we get

$$\widetilde{D} = b + \frac{\alpha\mathbb{E}(\rho)}{\gamma}.$$

Clearly, $\widetilde{D} < 0$ if $0 < b < -\mathbb{E}(\rho)\alpha/\gamma$ and $\widetilde{D} > 0$ if $0 < -\mathbb{E}(\rho)\alpha/\gamma < b$. For the intercept, we get

$$\widetilde{G} = \mathbb{E}(Y_{t+h} - Y_t) - \widetilde{D}\mathbb{E}\left(\int_t^{t+h} \sigma_u^2 du\right) = \left(\mu - \frac{\beta\alpha\mathbb{E}(\rho)}{\gamma}\right)h.$$

For the implied volatility, we get

$$Var\left(\mathbb{E}_t^*\left(\int_t^{t+h} \sigma_u^2 du\right)\right) = \frac{\beta\gamma^2}{2\alpha}\frac{1}{\alpha^{*2}}\left(1 - e^{-\alpha^* h}\right)^2,$$

and, similarly as before, using $\mathbb{E}_t^*\left(\int_t^{t+h}\sigma_u^2 du\right) = \frac{1}{\alpha^*}\left(1 - e^{-\alpha^* h}\right)\left(\sigma_t^2 - \beta^*\right) + \beta^* h$,

$$Cov\left(Y_{t+h} - Y_t, \mathbb{E}_t^*\left(\int_t^{t+h} \sigma_u^2 du\right)\right) = b\frac{\beta\gamma^2}{2\alpha}\frac{1}{\alpha}\left(1 - e^{-\alpha h}\right)\frac{1}{\alpha^*}\left(1 - e^{-\alpha^* h}\right).$$

Combining the above results, we get

$$\widetilde{D}^* = b\frac{\frac{1}{\alpha}\left(1 - e^{-\alpha h}\right)}{\frac{1}{\alpha^*}\left(1 - e^{-\alpha^* h}\right)}.$$

Clearly, $0 < \widetilde{D}^* < b$ given that $b > 0$ and $A < 0$. For the intercept, we get

$$\widetilde{G}^* = \mathbb{E}(Y_{t+h} - Y_t) - \widetilde{D}^* \mathbb{E}\left(\mathbb{E}_t^*\left(\int_t^{t+h} \sigma_u^2 du\right)\right)$$
$$= (\mu + b\beta)h - \widetilde{D}^*\left(\beta^* h + (\beta - \beta^*)\frac{1}{\alpha^*}\left(1 - e^{-\alpha h}\right)\right).$$

Altogether, we get for $0 < b < -\mathbb{E}(\rho)\alpha/\gamma$ that $\widetilde{D} < 0 < \widetilde{D}^*$, and for $0 < -\mathbb{E}(\rho)\alpha/\gamma < b < \frac{\alpha\mathbb{E}(\rho)}{\gamma}\frac{\frac{1}{\alpha^*}\left(1-e^{-\alpha^* h}\right)}{\frac{1}{\alpha}\left(1-e^{-\alpha h}\right)-\frac{1}{\alpha^*}\left(1-e^{-\alpha^* h}\right)}$, we have $0 < \widetilde{D} < \widetilde{D}^*$. Finally, similarly as before, we get

$$Cov\left(Y_{t+h} - Y_t, \mathbb{E}_t\left(\int_t^{t+h} \sigma_u^2 du\right)\right) = b\frac{\beta\gamma^2}{2\alpha}\frac{1}{\alpha^2}\left(1 - e^{-\alpha h}\right)^2.$$

□

Proof of Proposition 6.4: We have to compute

$$\frac{Cov\left(\int_t^{t+h} dY_s, \int_t^{t+h} \sigma_u^2 du\right)}{Var\left(\int_t^{t+h} \sigma_u^2 du\right)}.$$

Clearly,

$$Var\left(\int_t^{t+h} \sigma_u^2 du\right) = \kappa_2 \frac{1}{\lambda^2}\left(-1 + \lambda h + e^{-\lambda h}\right).$$

For the covariance, we get

$$Cov\left(\int_t^{t+h} dY_s, \int_t^{t+h} \sigma_u^2 du\right) = bCov\left(\int_t^{t+h} \sigma_s^2 ds, \int_t^{t+h} \sigma_u^2 du\right)$$
$$+ Cov\left(\int_t^{t+h} \sigma_s dW_s, \int_t^{t+h} \sigma_u^2 du\right) + Cov\left(\int_t^{t+h} \rho_{\lambda s} dL_{\lambda s}, \int_t^{t+h} \sigma_u^2 du\right),$$

where

$$Cov\left(\int_t^{t+h} \sigma_s dW_s, \int_t^{t+h} \sigma_u^2 du\right) = \mathbb{E}\left(\int_t^{t+h}\int_t^u \sigma_s dW_s \sigma_u^2 du\right) = 0,$$

and

$$Cov\left(\int_t^{t+h} \rho_{\lambda s} dL_{\lambda s}, \int_t^{t+h} \sigma_u^2 du\right) = \mathbb{E}\left(\int_t^{t+h} \rho_{\lambda s} dL_{\lambda s} \int_t^{t+h} \sigma_u^2 du\right) - \lambda\kappa_1^2 \mathbb{E}(\rho) h^2$$
$$= \mathbb{E}\left(\int_t^{t+h}\int_t^u \rho_{\lambda s} dL_{\lambda s} \sigma_u^2 du\right) + \mathbb{E}\left(\int_t^{t+h}\int_t^u \sigma_s^2 ds \rho_{\lambda u} dL_{\lambda u}\right) - \lambda\kappa_1^2 \mathbb{E}(\rho) h^2$$
$$= \int_t^{t+h}\left(\kappa_2 \mathbb{E}(\rho)\left(1 - e^{-\lambda(u-t)}\right) + \lambda\kappa_1^2 \mathbb{E}(\rho)(u - t)\right) du + \frac{1}{2}\lambda\kappa_1^2 \mathbb{E}(\rho) h^2 - \lambda\kappa_1^2 \mathbb{E}(\rho) h^2$$
$$= \kappa_2 \mathbb{E}(\rho)\frac{1}{\lambda}\left(-1 + \lambda h + e^{-\lambda h}\right).$$

Hence, we get

$$\widetilde{D} = b + \lambda \mathbb{E}(\rho).$$

Clearly, $\widetilde{D} < 0$ if $0 < b < -\mathbb{E}(\rho)\lambda$ and $\widetilde{D} > 0$ if $0 < -\mathbb{E}(\rho)\lambda < b$. For the intercept, we get

$$\widetilde{G} = \mathbb{E}(Y_{t+h} - Y_t) - \widetilde{D}\mathbb{E}\left(\int_t^{t+h} \sigma_u^2 du\right) = \mu h.$$

For the implied volatility, we get

$$Var\left(\mathbb{E}_t^*\left(\int_t^{t+h} \sigma_u^2 du\right)\right) = \frac{\kappa_2}{2}\frac{1}{\lambda^{*2}}\left(1 - e^{-\lambda^* h}\right)^2,$$

and, similarly as before, using $\mathbb{E}_t^*\left(\int_t^{t+h}\sigma_u^2 du\right) = \frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right)\left(\sigma_t^2 - \kappa_1^*\right) + \kappa_1^* h$,

$$\begin{aligned} Cov\left(Y_{t+h} - Y_t, \mathbb{E}_t^*\left(\int_t^{t+h} \sigma_u^2 du\right)\right) &= Cov\left(\int_t^{t+h} dY_s, \sigma_t^2 \frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right)\right) \\ &= b\frac{\kappa_2}{2}\frac{1}{\lambda}\left(1 - e^{-\lambda h}\right)\frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right). \end{aligned}$$

Combining the above results, we get

$$\widetilde{D}^* = b\frac{\frac{1}{\lambda}\left(1 - e^{-\lambda h}\right)}{\frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right)}.$$

Clearly, $0 < \widetilde{D}^* \leq b$ given that $b > 0$ and $\lambda^* \leq \lambda$. For the intercept, we get

$$\begin{aligned} \widetilde{G}^* &= \mathbb{E}(Y_{t+h} - Y_t) - \widetilde{D}^*\mathbb{E}\left(\mathbb{E}_t^*\left(\int_t^{t+h} \sigma_u^2 du\right)\right) \\ &= (\mu + b\kappa_1 + \lambda\kappa_1\mathbb{E}(\rho))h - \widetilde{D}^*\left(\kappa_1^* h + (\kappa_1 - \kappa_1^*)\frac{1}{\lambda^*}\left(1 - e^{-\lambda^* h}\right)\right). \end{aligned}$$

Altogether, we get for $0 < b < -\mathbb{E}(\rho)\lambda$ that $\widetilde{D} < 0 < \widetilde{D}^*$, and for $0 < -\mathbb{E}(\rho)\lambda < b < \lambda\mathbb{E}(\rho)\frac{\frac{1}{\lambda^*}\left(1-e^{-\lambda^* h}\right)}{\frac{1}{\lambda}\left(1-e^{-\lambda h}\right)-\frac{1}{\lambda^*}\left(1-e^{-\lambda^* h}\right)}$, we have $0 < \widetilde{D} < \widetilde{D}^*$. Finally, similarly as before, we get

$$Cov\left(Y_{t+h} - Y_t, \mathbb{E}_t\left(\int_t^{t+h} \sigma_u^2 du\right)\right) = (b/2 + \mathbb{E}(\rho)\lambda)\,\kappa_2\frac{1}{\lambda^2}\left(1 - e^{-\lambda h}\right)^2.$$

□

Proof of Proposition 6.5: Note that

$$\begin{aligned} \overline{D} &= \frac{Cov\left(\int_t^{t+h} \sigma_u^2 du, \mathbb{E}_t^*\left(\int_t^{t+h} \sigma_u^2 du\right)\right)}{Var\left(\mathbb{E}_t^*\left(\int_t^{t+h} \sigma_u^2 du\right)\right)} = \frac{\frac{\beta\gamma^2}{2\alpha}\frac{1}{\alpha}\left(1 - e^{-\alpha h}\right)\frac{1}{\alpha^*}\left(1 - e^{-\alpha^* h}\right)}{\frac{\beta\gamma^2}{2\alpha}\left(\frac{1}{\alpha^*}\left(1 - e^{-\alpha^* h}\right)\right)^2} \\ &= \frac{\frac{1}{\alpha}\left(1 - e^{-\alpha h}\right)}{\frac{1}{\alpha^*}\left(1 - e^{-\alpha^* h}\right)} < 1, \end{aligned}$$

where we used the results from the proof of the previous propositions. For the intercept, we get

$$\begin{aligned}\overline{G} &= \mathbb{E}\left(\int_t^{t+h} \sigma_u^2 du\right) - \overline{D}\mathbb{E}\left(\mathbb{E}_t^*\left(\int_t^{t+h} \sigma_u^2 du\right)\right)\\ &= \beta h - \overline{D}\mathbb{E}\left(\frac{1}{\alpha^*}\left(1-e^{-\alpha^* h}\right)\sigma_t^2 + \beta^*\left(h - \frac{1}{\alpha^*}\left(1-e^{-\alpha^* h}\right)\right)\right)\\ &= \beta\left(h - \frac{1}{\alpha}\left(1-e^{-\alpha h}\right)\right) + \beta^*\left(h - \frac{1}{\alpha^*}\left(1-e^{-\alpha^* h}\right)\right)\frac{\frac{1}{\alpha}\left(1-e^{-\alpha h}\right)}{\frac{1}{\alpha^*}\left(1-e^{-\alpha^* h}\right)}.\end{aligned}$$

Note that for the GBNSJ model, we get

$$\overline{D} = \frac{Cov\left(\int_t^{t+h} \sigma_u^2 du, \mathbb{E}_t^*\left(\int_t^{t+h} \sigma_u^2 du\right)\right)}{Var\left(\mathbb{E}_t^*\left(\int_t^{t+h} \sigma_u^2 du\right)\right)} = \frac{\frac{1}{\lambda}\left(1-e^{-\lambda h}\right)}{\frac{1}{\lambda^*}\left(1-e^{-\lambda^* h}\right)} \leq 1,$$

where we used the results from the proof of the previous propositions. For the intercept, we get

$$\begin{aligned}\overline{G} &= \mathbb{E}\left(\int_t^{t+h} \sigma_u^2 du\right) - \overline{D}\mathbb{E}\left(\mathbb{E}_t^*\left(\int_t^{t+h} \sigma_u^2 du\right)\right)\\ &= \kappa_1 h - \overline{D}\mathbb{E}\left(\frac{1}{\lambda^*}\left(1-e^{-\lambda^* h}\right)\sigma_t^2 + \kappa_1^*\left(h - \frac{1}{\lambda^*}\left(1-e^{-\lambda^* h}\right)\right)\right)\\ &= \kappa_1\left(h - \frac{1}{\lambda}\left(1-e^{-\lambda h}\right)\right) + \kappa_1^*\left(h - \frac{1}{\lambda^*}\left(1-e^{-\lambda^* h}\right)\right)\frac{\frac{1}{\lambda}\left(1-e^{-\lambda h}\right)}{\frac{1}{\lambda^*}\left(1-e^{-\lambda^* h}\right)}.\end{aligned}$$

□

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