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# Volatility in Equilibrium: Asymmetries and Dynamic Dependencies <sup>\*</sup>

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## Abstract

Stock market volatility clusters in time, carries a risk premium, is fractionally integrated, and exhibits asymmetric leverage effects relative to returns. This paper develops a first internally consistent equilibrium based explanation for these longstanding empirical facts. The model is cast in continuous-time and entirely self-contained, involving non-separable recursive preferences. We show that the qualitative theoretical implications from the new model match remarkably well with the distinct shapes and patterns in the sample autocorrelations of the volatility and the volatility risk premium, and the dynamic cross-correlations of the volatility measures with the returns calculated from actual high-frequency intra-day data on the S&P 500 aggregate market and VIX volatility indexes.

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**Keywords:** Equilibrium asset pricing; stochastic volatility; leverage effect; volatility feedback; option implied volatility; realized volatility; variance risk premium.

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# 1 Introduction

Modeling and forecasting of stock market return volatility has received unprecedented attention in the academic literature over the past two decades. The three most striking empirical regularities to emerge from this expanding literature arguably concern: (i) the highly persistent own dynamic dependencies in the volatility;<sup>1</sup> (ii) the existence of a typically positive volatility risk premium as manifested by the variance swap rate exceeding the corresponding expected future volatility;<sup>2</sup> (iii) the apparent asymmetry in the lead-lag relationship between returns and volatility.<sup>3</sup> Despite these now well-documented and generally accepted empirical facts, no formal model yet exists for explaining these features within a coherent economic framework. This paper fills that void by developing an entirely self-contained equilibrium based explanation for the asymmetry and volatility risk-premium that also accommodates long-run dependencies in the underlying volatility process.

Before discussing the model any further, it is instructive to illustrate anew the empirical regularities we seek to explain. To that end, the top most solid line in Figure 1 shows the sample autocorrelations for the aggregate market volatility out to a lag length of ninety days, based on daily data for the squared options-implied volatility index VIX over the past two decades; further details concerning the data and different volatility measures are given in Section 4. The autocorrelations in Figure 1 decay at a very slow rate and remain numerically large and statistically significant for all lags. Consistent with these highly persistent own dynamic dependencies in the volatility, it is now widely accepted that the typical rate of decay is so slow as to be best described by a fractionally integrated long-memory type process; for some of the earliest empirical evidence along these lines see, e.g., Robinson (1991), Ding et

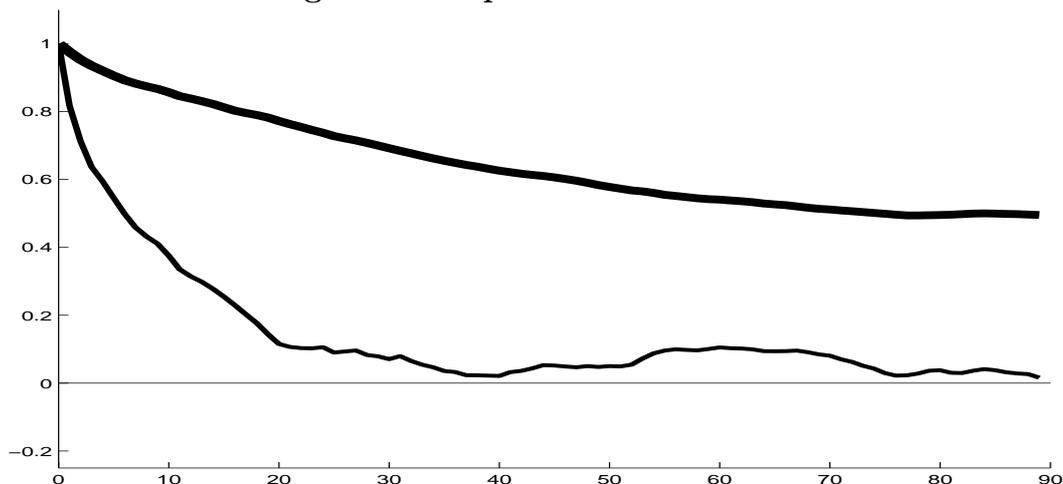
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<sup>1</sup>The historically low volatility over several years proceeding the Fall 2008 financial crises and the subsequent sustained heightened volatility provide anecdotal evidence for this idea.

<sup>2</sup>The preponderance of options traders "selling" volatility to gain the premium indirectly supports the notion of volatility carrying a risk premium.

<sup>3</sup>Again, anecdotal evidence like the heightened volatility following Russia's default and the LTCM debacle in September 1998, the relatively low volatility accompanying the rapid run-up in prices during the tech bubble, as well as the recent sharp increase in volatility accompanying the Fall 2008 financial crises and sharp market declines are all consistent with this asymmetry.

Figure 1 Sample Autocorrelations



The top most solid line shows the sample autocorrelations for the  $VIX^2$  volatility index to a lag length of 90 days. The lower line shows the sample autocorrelations for the variance risk premium. The calculations are based on daily data and variable definitions as described in more detail in Section 4.1.

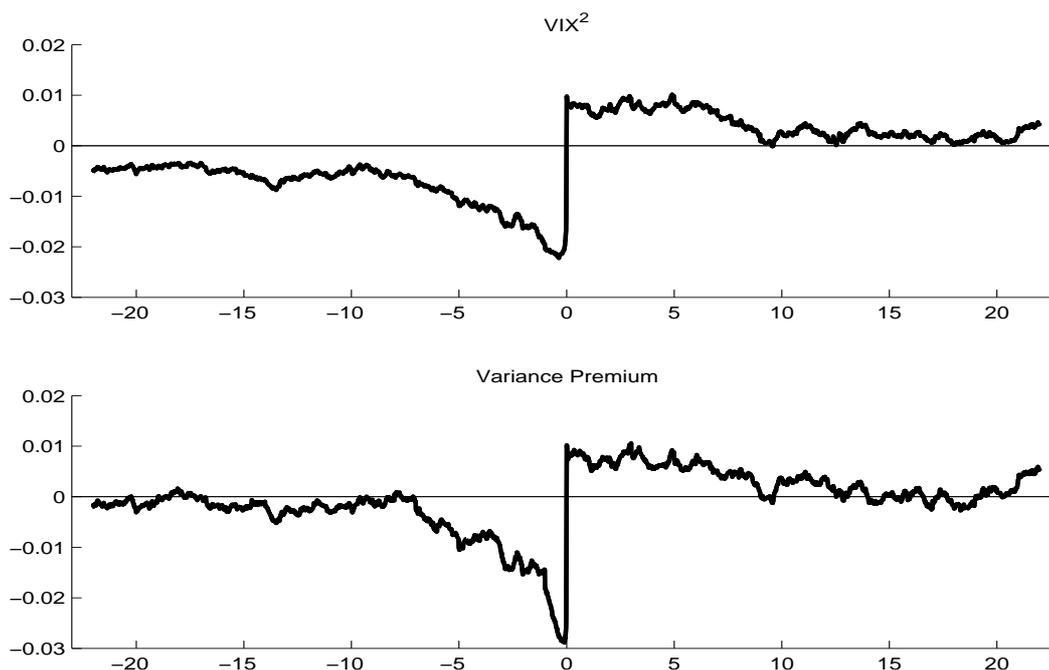
al. (1993) and Baillie et al. (1996).

The VIX index in effect represents the market's expectation of the volatility for S&P 500 index over the next month together with any premium for bearing the corresponding volatility risk.<sup>4</sup> Isolating the variance risk premium, the second line in Figure 1 shows the daily autocorrelations for the difference between the squared VIX index and the one-month-ahead forecasts from a simple reduced form time series model for the actually observed daily realized variation of the S&P 500 returns; further details concerning the high-frequency based realized volatility series and the construction of the model forecasts are again deferred to Section 4. Although the autocorrelations indicate non-trivial dependencies for up to several weeks, the premium is clearly not as persistent as the volatility process itself. Again, this is not a new empirical result per se, but one that any satisfactory economic model must be able to accommodate. For instance, the empirical analysis in Bollerslev et al. (2006a) also supports the idea of relatively fast mean reversion in the volatility risk premium, as does

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<sup>4</sup>The variance risk premium is formally defined as the difference between the expected future volatility under the risk-neutral and the actual probability measures.

Figure 2 Sample Cross-Correlations



The top panel shows the sample cross-autocorrelations between the  $VIX^2$  and lags and leads of the returns ranging from -22 to 22 days. The bottom panel shows the sample cross-autocorrelations between the variance risk premium and the returns. The calculations are based on high-frequency five-minute data and variable definitions as further detailed in Section 4.1.

the evidence of fractional co-integration between implied and realized volatility presented by, e.g., Bandi and Perron (2006) and Nielsen (2007).

Next, to highlight the aforementioned return-volatility asymmetry, the first panel in Figure 2 plots the very high frequency cross-correlations between leads and lags of the S&P 500 returns and the corresponding squared options-implied VIX volatility index. Bollerslev et al. (2006b) have previously demonstrated the advantage of using high-frequency intraday data for more effectively estimating and analyzing the lead-lag relationship between returns and volatility.<sup>5</sup> We therefore follow their lead in using five-minute observations for both the VIX and the S&P 500 returns in calculating the sample cross-correlations for leads and lags ranging up to 22 days, or 1,716 leads and lags at the five-minute sampling frequency.

<sup>5</sup>Bollerslev and Zhou (2006) have also argued that the asymmetry tend to be stronger for implied than realized volatility.

High-frequency data for the VIX have only been available since 2003, so that the cross-autocorrelations depicted in Figure 2 are necessarily based on a somewhat shorter five-year calendar-time span compared to the longer sample of daily observations used for illustrating the longer-run own volatility dependencies in the previous Figure 1. Nonetheless, this shorter high-frequency sample still reveals a striking negative pattern in the correlations between the volatility and the lagged returns, lasting for several days. On the other hand, the correlations between the volatility and the future returns are all positive, albeit close to zero. Following Black (1976), who first observed the apparent asymmetry in the return-volatility relationship, the left part of the figure and the negative correlation between lagged returns and current volatility is now commonly referred to as a “leverage effect,”<sup>6</sup> while the right part of the figure and the positive correlation between current volatility and future returns has been termed a “volatility feedback effect.” This systematic pattern in the high-frequency based cross-correlations is directly in line with the empirical evidence from numerous empirical studies based on coarser lower frequency daily data and specific parametric models, including the early influential work by French et al. (1987), Nelson (1991), Glosten et al. (1993) and Campbell and Hentschell (1992).

Taking the analysis one step further, the bottom panel in Figure 2 shows the cross-correlations between the five-minute S&P 500 returns and the variance risk premium, where as before the variance risk premium is defined as the difference between the squared VIX index and the corresponding forecast constructed from a simple reduced form time series model for the observed realized volatilities. Comparing this new picture to the return-volatility dependencies in the top panel, the signs of the cross-correlations generally coincide. However, there is a noticeable faster decay toward zero in the magnitude of the cross-correlations between the variance risk premium and the lagged returns compared to the decay in the corresponding cross-correlations between the squared VIX index and the lagged returns. This difference closely mirrors the difference in the shape and the rate of decay in the standard

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<sup>6</sup>It is now also widely agreed that the negative correlations actually have little if anything to do with changes in financial leverage; see, e.g., Figlewski and Wang (2002).

sample autocorrelations for the two daily volatility series depicted in Figure 1.

The key empirical return-volatility patterns and dynamic dependencies illustrated in the figures indicate that volatility carries a risk premium. Standard equilibrium based models build around a representative agent with time-separable utility rules out priced volatility risk; see, e.g., the discussion in Bansal and Yaron (2004). Instead, following the literature on long-run risk models pioneered by Bansal and Yaron (2004), we assume that the representative agent is equipped with Epstein-Zin-Weil preferences. Importantly, in this situation the Stochastic Discount Factor (SDF) will depend not only on the consumption growth rate, but also on the future investment opportunities. Consequently, the aggregate market return will be a function of the expected growth in the economy, as in the traditional time-separable utility case, as well as the uncertainty about the future economic growth; see, e.g., Campbell (1996). Intuitively, this also explains why investors may be willing to pay an uncertainty premium, and in turn why the VIX may differ from the corresponding actual return volatility and why the corresponding variance risk premium may serve as a separately priced risk factor.

Our model is cast in continuous time. The Epstein-Zin-Weil preference structure was first employed in a continuous-time asset pricing setting by Duffie and Epstein (1992a). However, in contrast to the expression for the SDF involving the compensator function derived in that paper, we find it more convenient to work with the discount factor expressed in terms of the consumption growth rate and the market return. As formally show below, this expression in turn results in continuous-time Euler equations analogous to the discrete-time equations originally derived by Epstein and Zin (1989), and utilizing a log-linearization allows for closed-form tractable solutions of the model for values of the Intertemporal Elasticity of Substitution (EIS) different from unity.

The model is most directly related to the equilibrium long-run risk model recently developed by Bollerslev et al. (2009), in which the volatility-of-volatility in the economy is determined by its own separate stochastic process, and the model of Drechsler and Yaron (2008), in which the expected growth rate in consumption and the volatility of consumption

growth are both allowed to “jump.” In comparison to the specific discrete-time formulations employed in those papers, however, the continuous-time formulation adopted here has the advantage of allowing for the calculation of internally consistent model implications across different sampling frequencies and return horizons, and it is able to accommodate much richer longer-run volatility dependencies including fractional integration. Importantly, the continuous-time formulation also permits a stringent definition of the VIX volatility index and avoids the inherent problem in traditional discrete-time long-run risk models with GARCH type errors that the (conditional) variance is known one period in advance and therefore formally can’t generate a variance premium. A related long-run risk type model in which the economic uncertainty, or the volatility of consumption growth, is allowed to “jump” in continuous-time has also recently been explored by Eraker (2008), in an attempt to explain the existence of a (on average) positive volatility risk premium.

Other recent studies concerned with the equilibrium pricing of volatility risk include Gabaix (2009) and Wachter (2008), both of whom analyze the implications of rare disasters, and Lettau et al. (2009) who emphasize the role of low frequency movements in macroeconomic uncertainty for explaining low frequency multi-year movements in stock market valuations. Several studies more squarely rooted in the options pricing literature have also explored the equilibrium implications of allowing for richer volatility dynamics and non-standard preference structures; see, e.g., the recent papers by Benzoni et al. (2005), Eraker and Shaliastovich (2008) and Santa-Clara and Yan (2009) and the references therein. However, the empirical focus of the present paper is distinctly different from all of these other studies, and to the best of our knowledge, no other coherent economic equilibrium-based explanation for the volatility asymmetries and dynamic dependencies depicted in Figures 1 and 2 is yet available in the literature.

The plan for the rest of the paper is as follows. The new theoretical model is formally defined and solved in Section 2. The equilibrium implications from the model in regards to the return-volatility asymmetries and the own dynamic volatility dependencies are presented

in Section 3. The data used in the construction of the figures discussed above and the model's ability to reproduce these basic empirical features are the subject of Section 4. Section 5 concludes. Most of the mathematical proofs are deferred to two Appendixes.

## 2 Volatility in Equilibrium

The classic continuous-time Intertemporal CAPM of Merton (1973) is often used to justify the existence of a traditional volatility risk premium in aggregate market returns. However, the model is unable to explain the leverage effect and asymmetric return-volatility dependencies actually observed in the data. The continuous-time endowment economy developed here instead builds on the discrete-time long-run risk model pioneered by Bansal and Yaron (2004), and the extensions of that same model in Bollerslev et al. (2009) and Drechsler and Yaron (2008). We begin in the next subsection by a description of the basic continuous-time model setup and solution under short-memory dynamics. We subsequently show how the model may be extended to incorporate empirically relevant long-memory dependencies.

### 2.1 Basic Model Setup and Assumptions

Let the local geometric growth rate of consumption  $C_t$  in the economy be denoted by  $g_t \equiv \frac{dC_t}{C_t}$ . To simplify the analysis and explicitly focus on the role of time-varying volatility, we rule out any predictability in  $g_t$  by assuming that it follows the continuous martingale

$$g_t = \mu_g dt + \sigma_{g,t} dW_t^c, \quad (1)$$

where  $\mu_g$  denotes the constant mean growth rate,  $\sigma_{g,t}$  refers to the conditional volatility of the growth rate, and  $W_t^c$  is a standard Wiener process.<sup>7</sup> Further, we assume that the volatility dynamics in the economy are governed by the following continuous-time affine processes,

$$d\sigma_{g,t}^2 = \kappa_\sigma(\mu_\sigma - \sigma_t^2)dt + \sqrt{q_t}dW_t^\sigma, \quad (2)$$

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<sup>7</sup>The growth rate of consumption is identically equal to the dividend growth rate in this Lucas-tree economy.

$$dq_t = \kappa_q(\mu_q - q_t)dt + \varphi_q\sqrt{q_t}dW_t^q, \quad (3)$$

where the two Wiener processes  $W_t^\sigma$  and  $W_t^q$  are independent and jointly independent of  $W_t^c$ , and the parameters satisfy the non-negativity restrictions  $\mu_\sigma > 0$ ,  $\mu_q > 0$ ,  $\kappa_\sigma > 0$ ,  $\kappa_q > 0$ , and  $\varphi_q > 0$ .<sup>8</sup> The stochastic volatility process  $\sigma_{g,t}^2$  represents time-varying economic uncertainty in consumption growth, with the volatility-of-volatility process  $q_t$  in effect inducing an additional source of temporal variation in that same process. As discussed further below, both of these processes play an important role in generating empirically realistic time varying volatility risk premia. Note that the assumption of independent innovations across all three equations means the internal structure of the model itself must explain the return-volatility correlations, and it rules out any correlations that might otherwise arise via purely statistical channels.

We assume that the representative agent's consumption and investment decisions are based on the maximization of Epstein-Zin-Weil recursive preferences. As formally shown in Appendix A this implies that the equilibrium relationship between the inter-temporal marginal rate of substitution,  $M_t$ , consumption,  $C_t$ , and the cumulated return on the aggregate wealth portfolio,  $R_t$ , must satisfy the relation derived in equation (A.9),

$$d \log M_t + \frac{\theta}{\psi} d \log C_t + (1 - \theta) d \log R_t = -\rho \theta dt, \quad (4)$$

where  $\rho$  denotes the instantaneous subjective discount factor,  $\psi$  equals the inter-temporal elasticity of substitution, and the parameter  $\theta$  is defined by

$$\theta \equiv \frac{1 - \gamma}{1 - \frac{1}{\psi}}, \quad (5)$$

where  $\gamma$  refers to the coefficient of risk aversion. The expression in equation (4) is naturally interpreted as the continuous-time version of the widely used and celebrated discrete-time equilibrium relationship derived in Epstein and Zin (1991). In the following we will maintain

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<sup>8</sup>We also assume that  $\mu_q \kappa_q > 0.50 \varphi_q^2$ , which ensures positivity of  $q_t$ , and that  $\mu_\sigma$  is sufficiently large relative to  $\kappa_\sigma$ , so that negativity of  $\sigma_{g,t}^2$  is highly unlikely and the subsequent approximations reasonable.

the assumptions that  $\gamma > 1$  and  $\psi > 1$ , which readily implies that  $\theta < 0$ .<sup>9</sup> Consistent with the empirical regularities discussed in the introductory section, these specific parameter restrictions ensure, among other things, that asset prices fall on news of positive volatility shocks and that volatility carries a positive risk premium.

## 2.2 Basic Model Solution

Let  $\Psi_t$  denote the price-dividend ratio, or equivalently the price-consumption or the wealth-consumption ratio, of the asset that pay the consumption endowment  $\{C_{t+s}\}_{s \in [0, \infty)}$ . It is convenient to express  $\Psi_t = \Psi(x_t)$  a function of the state vector  $x_t = (\sigma_{g,t}^2, q_t)$ . The conventional linear method of solving for rational expectations models is complicated by the fact that the stochastic differential equation for  $\log(\Psi_t)$  involves the reciprocal of  $\Psi(x_t)$ . To circumvent this difficulty, we approximate  $\Psi(x)^{-1} = \exp(-\log \Psi(x))$  by the following first-order expansion,

$$\Psi(x)^{-1} \approx \exp(-\overline{\log \Psi}) - \exp(-\overline{\log \Psi})(\log \Psi(x) - \overline{\log \Psi}) = \kappa_0 - \kappa_1 \log \Psi(x),$$

where  $\kappa_1 > 0$ . This approximation plays a similar role to that of the standard Campbell-Shiller discrete-time approximation to the returns in terms of the log price-dividend ratio. Similar expressions have also previously been used in the continuous-time setting by, e.g., Campbell and Viceira (2002).

Now conjecturing a solution for  $\log(\Psi_t)$  as an affine function of the two state variables,  $\sigma_{g,t}^2$  and  $q_t$ ,

$$\log(\Psi_t) = A_0 + A_\sigma \sigma_{g,t}^2 + A_q q_t, \tag{6}$$

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<sup>9</sup>The assumption that  $\gamma > 1$  is generally agreed upon, but the assumption that  $\psi > 1$  is a matter of some debate; see, e.g., the discussion in Bansal and Yaron (2004).

and solving for the coefficients  $A_0$ ,  $A_\sigma$ , and  $A_q$ , it follows from Appendix B that

$$A_0 = \frac{A_\sigma \kappa_\sigma \mu_\sigma + A_q \kappa_q \mu_q - \kappa_0 + (1 - 1/\psi) \mu_g - \rho}{\kappa_1}, \quad (7)$$

$$A_\sigma = \frac{-\gamma}{2} \frac{1 - \frac{1}{\psi}}{\kappa_\sigma + \kappa_1}, \quad (8)$$

$$A_q = \frac{\kappa_q + \kappa_1 - \sqrt{(\kappa_q + \kappa_1)^2 - \theta^2 A_\sigma^2 \varphi_q^2}}{\theta \varphi_q^2}. \quad (9)$$

The restrictions that  $\gamma > 1$  and  $\psi > 1$ , readily imply that the impact coefficient associated with both of the volatility state variables are negative; i.e.,  $A_\sigma < 0$  and  $A_q < 0$ .<sup>10</sup> Or put differently, that the market falls on positive volatility “news.” From these explicit solutions for the three coefficients it is now also relatively straightforward to deduce the reduced form expressions for other variables of interest.

In particular, as shown in equation (B.7) in Appendix B, the equilibrium dynamics of the logarithmic cumulative return process,  $R_t$ , is given by

$$d \log(R_t) = \left[ \frac{\mu_g}{\psi} + \rho - \frac{\sigma_{g,t}^2}{2} + \frac{\gamma}{2} \left(1 - \frac{1}{\psi}\right) \sigma_{g,t}^2 - A_q q_t (\kappa_1 + \kappa_q) \right] dt + \sigma_{g,t} dW_t^c + A_\sigma \sqrt{q_t} dW_t^\sigma + A_q \varphi_q \sqrt{q_t} dW_t^q. \quad (10)$$

The directional effects of increases in the endowment volatility,  $\sigma_{g,t}^2$ , on the local expected return are generally ambiguous. However, for sufficiently high levels of risk-aversion  $\gamma$  and inter-temporal substitution  $\psi$ , endowment volatility positively affects the local expected return. Meanwhile, increases in the volatility-of-volatility,  $q_t$ , unambiguously, increase the local expected return, reflecting the compensation for bearing volatility risk. On the other hand, diffusive-type innovations in the volatility and the volatility-of-volatility,  $dW_t^\sigma$  and  $dW_t^q$ , both have a negative impact on the local returns, consistent with a leverage type effect.

To further appreciate the implications of the model and how it might help explain the empirical regularities, it is instructive to consider the model-implied equity premium,  $\pi_{r,t}$ ,

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<sup>10</sup>The solution for  $A_q$  in equation (9) represents one of a pair of roots to a quadratic equation, but it is the economically meaningful root for reasons discussed below.

as derived in equation (B.9),

$$\pi_{r,t} \equiv -\frac{1}{dt} \frac{d[R, M]_t}{R_t M_t} = \gamma \sigma_{g,t}^2 + (1 - \theta)(A_\sigma^2 + A_q^2 \varphi_q^2) q_t. \quad (11)$$

The first term,  $\gamma \sigma_{g,t}^2$ , is akin to a classic risk-return tradeoff relationship. It does not really represent a volatility risk premium *per se*, however, but rather changing prices of consumption risk induced by the presence of stochastic volatility. Instead, the second term,  $(1 - \theta) \kappa_1^2 (A_q^2 \varphi_q^2 + A_\sigma^2) q_t$ , has the interpretation of a true volatility risk premium, representing the confounding impact of the two diffusive-type innovations,  $dW_t^\sigma$  and  $dW_t^q$ .<sup>11</sup> The existence of this true volatility risk premium depends crucially on the dual assumptions of recursive utility, or  $\theta \neq 1$ , as volatility would not otherwise be a priced factor, and time varying volatility-of-volatility, in the form of the  $q_t$  process. The restrictions that  $\gamma > 1$  and  $\psi > 1$  imply that the volatility risk premium is positive.

The expression for the instantaneous risk premium in equation (11) discussed above closely mirrors the expression for the premium in the corresponding discrete-time model of Bollerslev et al. (2009). The specific formulations used in deriving these results both involve somewhat restrictive assumptions about the underlying volatility dynamics. We next discuss how the continuous-time setup and the developed model solution may be adapted to allow for more flexible and richer dynamic dependencies in the volatility, including longer-memory type effects.

## 2.3 General Model Solution

Numerous more flexible continuous-time stochastic volatility models have been proposed in the literature. We build on the framework of Comte and Renault (1996) in assuming that the evolution in  $\sigma_{g,t}^2$  may be described by

$$\sigma_{g,t}^2 = \sigma^2 + \int_{-\infty}^t a(t-s) \sqrt{q_s} dW_s^\sigma. \quad (12)$$

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<sup>11</sup>The specific root in equation (9) implies that  $A_q^2 \varphi_q^2 \rightarrow 0$  for  $\varphi_q \rightarrow 0$ , which guarantees that the premium disappears when  $q_t$  is constant, as would be required by the lack of arbitrage.

This representation is quite general and flexible. By appropriate choice of the moving average weights  $\{a(s)\}_{s \in [0, \infty)}$  it obviously includes the affine process in equation (2) as a special case. Importantly, by suitable choice of the mapping  $s \rightarrow a(s)$ , the process for  $\sigma_{g,t}^2$  may also exhibit various forms of long range dependence. In particular, setting

$$a(s) = \frac{\sigma}{\Gamma(1 + \alpha)} \left( s^\alpha - k e^{-ks} \int_0^s e^{ku} u^\alpha du \right) \quad (13)$$

results in the classic fractionally integrated process, where  $\alpha$  denotes the long-memory parameter.

To complete the specification of the model and still allow for closed form solutions, we will maintain the identical laws of motions for the consumption growth rate and the volatility-of-volatility given by equations (1) and (3), respectively. The actual solution strategy differs somewhat from that for the basic model. The full details are given in Appendix C; a precis follows.

In parallel to the solution for the short-memory model discussion above, we start by conjecturing a solution for the logarithmic price-consumption ratio now of the form

$$\log(\Psi_t) = A_0 + A_q q_t + \int_{-\infty}^t A(t-s) \sqrt{q_s} dW_s^\sigma \quad (14)$$

where  $A_0$ ,  $A_q$ , and  $\{A(s)\}_{s \in [0, \infty)}$  are to be determined. Some care is needed because of subtleties related to possible arbitrage opportunities under long-memory type dependencies (Rogers, 1997). The strategy that we use relies on the fact that in the absence of arbitrage the return on a traded security must follow a semi-martingale. This allows us to split up the returns into a drift and a local martingale component. This decomposition is possible when  $A(t)$  exists and is differentiable at zero. Substituting the conjectured solution into the pricing equation (4) yields the following ordinary differential equation for  $t > s$ ,

$$A'(t-s) - \kappa_1 A(t-s) = \frac{\gamma(1 - \frac{1}{\psi})}{2} a(t-s), \quad (15)$$

and two regular equations,

$$\frac{\theta}{2} \varphi_q^2 A_q^2 - (\kappa_q + \kappa_1) A_q + \frac{\theta A(0)^2}{2} = 0 \quad (16)$$

$$A_0 = \frac{A_q \kappa_q \theta_q - \kappa_0 + (1 - \frac{1}{\psi}) \mu_g - \rho - \frac{\gamma}{2} (1 - \frac{1}{\psi}) \sigma^2}{\kappa_1}. \quad (17)$$

From the Appendix, the solutions to this system of equations are

$$A(s) = - \int_s^{+\infty} \frac{\gamma(1 - \frac{1}{\psi})}{2} e^{\kappa_1(t-\tau)} a(\tau) d\tau, \quad (18)$$

$$A_q = \frac{\kappa_q + \kappa_1 - \sqrt{(\kappa_q + \kappa_1)^2 - \theta^2 \varphi_q^2 A(0)^2}}{\theta \varphi_q^2}, \quad (19)$$

$$A_0 = \frac{A_q \kappa_q \theta_q - \kappa_0 + (1 - \frac{1}{\psi}) \mu_g - \rho - \frac{\gamma}{2} (1 - \frac{1}{\psi}) \sigma^2}{\kappa_1}, \quad (20)$$

which exists and is well defined subject to a terminal condition ruling out explosive bubble solutions and other mild regularity conditions. As before, from this set of solutions it is possible to deduce the reduced form expressions for all other variables of interest.

In particular, in parallel to the expression for the returns in the short-memory model in equation (10) above, it follows from Appendix C that the reduced form expression for the returns in the general may be expressed as,

$$d \log(R_t) = \mu_{R,t} dt + \sigma_{g,t} dW_t^c + A_q \varphi_q \sqrt{q_t} dW_t^q + A(0) \sqrt{q_t} dW_t^\sigma, \quad (21)$$

where the drift is defined by,

$$\mu_{R,t} = \rho + \frac{\mu_g}{\psi} + [-\frac{1}{2} + \frac{\gamma}{2} (1 - \frac{1}{\psi})] \sigma_{g,t}^2 - (\kappa_q + \kappa_1) A_q q_t. \quad (22)$$

Similarly, from equation (C.7) in Appendix C the equilibrium equity premium is determined by,

$$\pi_{r,t} = \gamma \sigma_t^2 + (1 - \theta) [A_q^2 \varphi_q^2 + A(0)^2] q_t = \gamma \sigma_t^2 + 2 \left( \frac{1}{\theta} - 1 \right) (\kappa_q + \kappa_1) A_q q_t. \quad (23)$$

Under the previously discussed parameter restrictions  $\gamma > 1$  and  $\psi > 1$ , implying that  $\theta < 0$ , the equity premium remains positive. So as long as  $\gamma \neq \frac{1}{\psi}$ , or  $\theta \neq 1$ , it also remains the case that stochastic volatility carries a positive risk premium. Note also that the instantaneous equity premium only depends on the  $\{a(s)\}_{s \in [0, \infty)}$  weights and the possible

long-run dependencies in the volatility through the cumulative impact determined by the integral solution for  $A(0)$  in equation (18).

We next turn to a more specific discussion of the model’s implications vis-a-vis the volatility risk premium and dynamic return-volatility dependencies.

### 3 Dynamic Equilibrium Dependencies

The equilibrium expressions discussed in the previous section characterize how the equity premium depends on the instantaneous volatility, and how the instantaneous return responds to contemporaneous volatility innovations within the model. This section further details the model’s implications in regards to the dynamic dependencies in the volatility and the volatility risk premium, and how these volatility measures co-vary with leads and lags of the returns at different horizons. We will subsequently confront these theoretical predictions with the key empirical regularities discussed in the introduction.

#### 3.1 VIX and the Volatility Risk Premium

One of the key features of the general version of the model is that consumption volatility,  $\sigma_{g,t}^2$ , may exhibit long-range dependence while the volatility of the volatility,  $q_t$ , is a short-memory process. This in turn has important implications for the serial correlation properties of the equivalent to the VIX volatility index implied by the model and the corresponding volatility risk premium, and how these measures correlate with the returns.

To begin, consider the (forward) integrated variance, or quadratic variation, of the asset price  $S_t$ ,

$$IV_{t,t+N} \equiv \int_{\tau=t}^{t+N} d[\log S, \log S]_{\tau}, \quad (24)$$

where the “brackets”  $[\ ]$  represents the standard quadratic variation process. From equation (C.8) in Appendix C the reduced form expression for the integrated variance may be

conveniently written as,

$$IV_{t,t+N} = \int_t^{t+N} \sigma_{g,\tau}^2 d\tau + (A_q^2 \varphi_q^2 + A(0)^2) \int_t^{t+N} q_\tau d\tau. \quad (25)$$

The integrated variance is, of course, random and not observed until time  $t + N$ .

The corresponding variance swap rate is defined as the time  $t$  risk-neutralized expectation of  $IV_{t,t+N}$ , say  $E_t^{\mathbb{Q}}(IV_{t,t+N})$ . This risk-neutral expectation may in theory be calculated in a completely model-free fashion from a cross-section of option prices (see, e.g., Carr and Madan, 1998; Britten-Jones and Neuberger, 2000; Jiang and Tian, 2005). This way of calculating the variance swap rate directly mirrors the definition of the (squared) VIX volatility index for the S&P 500,

$$VIX_t^2 \equiv E_t^{\mathbb{Q}}(IV_{t,t+N}), \quad (26)$$

where the horizon  $N$  is set to one month, or 22-days.<sup>12</sup>

This same risk-neutral expected variation may alternatively be calculated within the specific equilibrium model setting. In particular, it follows from equation (C.9) in Appendix C that

$$VIX_t^2 = \beta_{vx,0} + \int_{-\infty}^t h_{vx}(t-s) \sqrt{q_s} dW_s^\sigma + \beta_{vx,q} q_t, \quad (27)$$

where the dependence on  $N$  has been suppressed for notational convenience. The  $\{h_{vx}(s)\}_{s \in [0, \infty)}$  weights depend on the  $\{a(s)\}_{s \in [0, \infty)}$  moving average coefficients, and importantly inherit any long-memory decay in those coefficients. As such, an eventual slow hyperbolic decay in the autocorrelations for the  $VIX_t^2$  would therefore be entirely consistent with the implications from the general theoretical model; i.e.,

$$\text{Corr}(VIX_t^2, VIX_{t+s}^2) = c_h s^{b_h} \quad s > S, \quad (28)$$

where  $c_h > 0$  and  $b_h < 0$  are constants, and  $S$  denotes a sufficiently long lag so that the short-run dependencies have dissipated.

<sup>12</sup>A more detailed description of the mechanical calculation of the VIX index is available in the *white paper* on the CBOE website; see also the discussion in Jiang and Tian (2007).

Next, consider the variance risk premium, as formally defined by the difference between the risk-neutral and objective expectation of  $IV_{t,t+N}$ ,

$$vp_t \equiv E_t^{\mathbb{Q}}(IV_{t,t+N}) - E_t^{\mathbb{P}}(IV_{t,t+N}). \quad (29)$$

Whereas  $E_t^{\mathbb{Q}}(IV_{t,t+N})$  and  $E_t^{\mathbb{P}}(IV_{t,t+N})$  both depend on the consumption growth volatility and the volatility-of-volatility of that process, the variance risk premium is simply an affine function of the volatility-of-volatility,  $q_t$ . Specifically, from equation (C.10) in Appendix C,

$$vp_t = b_{vp,0} + b_{vp,1}q_t, \quad (30)$$

where  $b_{vp,0} >$  and  $b_{vp,1} > 0$ , reflecting the positive premium for bearing volatility risk provided that  $\theta > 0$ . Consequently, the  $vp_t$  process simply inherits the short-memory dynamic dependencies in the  $q_t$  process, and should exhibit a relatively fast exponential decay in its autocorrelation structure,

$$\text{Corr}(vp_t, vp_{t+s}) = c_q e^{-\kappa_q s} \quad s > 0, \quad (31)$$

where  $c_q$  denotes a positive constant.

The equilibrium expressions for the the variance swap rate and the premium in equations (27) and (30), respectively, also have unique implications for the dynamic cross-correlations with the returns. We review these next.

### 3.2 Return-Volatility Correlations

We are interested in calculating the dynamic cross-correlations between the two variance measures,  $VIX_t^2$  and  $vp_t$ , and leads and lags of the returns. To help elucidate the economic mechanisms, it is instructive to first review the predictions under short-memory dynamics, which allow for closed form solutions for the cross-correlations. We subsequently discuss the general case explicitly allowing for long-memory dependencies in the underlying volatility process. The cross-correlations between the variance premium and the returns are easier to calculate than those for the VIX, and we begin by considering these.

Let  $r_t \equiv d \log(R_t)$  denote the instantaneous return. We will refer to the cross-correlations between the time  $t$  premium  $vp_t$  and the future returns,  $r_{t+s}$  for  $s > 0$ , as the forward correlations. The forward correlations represent the extent to which the premium is able to forecast the returns. The correlations between the premium  $vp_t$  and the lagged returns,  $r_{t+s}$  for  $s < 0$ , on the other hand, represent the impact of movements in the past returns on the current variance premium. Given the well-known near unpredictability of returns, we would expect the forward correlations to be positive, reflecting the premium for bearing volatility risk, but small and quickly declining to zero for longer interdaily return horizons. We would expect the lagged correlations to be negative, but increasing to zero for longer daily lags, consistent with the existence of a dynamic leverage type effect. The formal theoretical predictions from the model confirm these ideas.

Specifically, from the results for the short-memory model derived in Appendix B, it follows that for  $s > 0$ ,

$$\text{Corr}(vp_t, r_{t+s}) = \beta_{R,q} \text{Var}(q_t) K_q e^{-\kappa_q s},$$

where  $\beta_{R,q}$  represents the sensitivity of the instantaneous returns to the  $q_t$  process. Since  $\beta_{R,q} > 0$  and  $K_q > 0$ , the forward correlations should all be positive. Similarly, it follows from the appendix that the cross-correlations for  $s < 0$  satisfy,

$$\text{Corr}(vp_t, r_{t-s}) = (\beta_{R,q} \text{Var}(q_t) + A_q \phi_q^2 \mu_q) K_q e^{-\kappa_q s}.$$

Since the high-frequency returns are close to unpredictable, the value for  $\beta_{R,q}$  is likely to be small. Hence, we would expect the second term involving  $A_q < 0$  to dominate the expression in parenthesis, and consequently all of the backward correlations to be negative. In summary, the model predicts,

$$\text{Corr}(vp_t, r_{t+s}) = \begin{cases} a_- e^{-\kappa_q |s|} & s < 0, \\ a_+ e^{-\kappa_q s} & s \geq 0, \end{cases} \quad (32)$$

where  $a_- < 0$  and  $a_+ > 0$ . As discussed further below, this prediction does indeed adhere very closely with the relevant empirical shapes.

The dynamic cross-correlations between the VIX<sup>2</sup> and the return are a bit more involved than those for the variance premium. However, the basic intuition is essentially the same, except that the actual formulas now also depend on the volatility process  $\sigma_{g,t}^2$  itself and its correlation with the returns. In particular, referring to Appendix B the forward correlations for  $s > 0$  takes the form,

$$\text{Corr}(\text{VIX}_t^2, r_{t+s}) = \beta_{R,\sigma} \text{Var}(\sigma_{g,t}^2) K_\sigma e^{-\kappa_\sigma s} + \beta_{R,q} \text{Var}(q_t) K_q e^{-\kappa_q s}.$$

The sign of  $\beta_{R,\sigma}$  will depend upon the preference parameters  $\psi$  and  $\gamma$ . However, it may reasonably be expected to be positive,<sup>13</sup> so that the forward cross-correlations will again be positive, with the decay toward zero ultimately determined by the dominant value of  $\kappa_\sigma$  or  $\kappa_q$ . As for the premium, the backward correlations for  $s < 0$  are slightly more complicated, taking the form,

$$\text{Corr}(\text{VIX}_t^2, r_{t-s}) = (\beta_{R,\sigma} \text{Var}(\sigma_{g,t}^2) + A_\sigma \mu_\sigma) K_\sigma e^{-\kappa_\sigma s} + (\beta_{R,q} \text{Var}(q_t) + A_q \phi_q^2 \mu_q) K_q e^{-\kappa_q s}.$$

As discussed above, given the difficulties in predicting returns, we would expect the  $\beta_{R,\sigma}$  and  $\beta_{R,q}$  terms to be relatively small and dominated by the terms involving the  $A_\sigma < 0$  and  $A_q < 0$  coefficients determining the instantaneous response of the returns to volatility innovations. Consequently, the backward correlations are naturally expected to be all negative. In summary,

$$\text{Corr}(\text{VIX}_t^2, r_{t+s}) = \begin{cases} a_{q,-} e^{-\kappa_q |s|} + a_{\sigma,-} e^{-\kappa_\sigma |s|} & s < 0, \\ a_{q,+} e^{-\kappa_q s} + a_{\sigma,+} e^{-\kappa_\sigma s} & s \geq 0, \end{cases} \quad (33)$$

where  $a_{q,-}, a_{\sigma,-} < 0$  and  $a_{q,+}, a_{\sigma,+} < 0$ . Again, these theoretical model predictions closely match what we see in the data.

The general model allowing for long-memory in the volatility essentially give rise to the same basic patterns and predictions. However, the formal derivations are somewhat more complicated and the actual values of the cross-correlations will ultimately depend upon the

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<sup>13</sup>The prototypical values  $\psi = 1.5$  and  $\gamma = 10$  from Bansal and Yaron (2004) implies that  $\beta_{R,\sigma} = 7/6$ .

specific process for  $\sigma_{g,t}^2$  and the corresponding moving average coefficients  $\{a(s)\}_{s \in [0, \infty)}$ . We briefly sketch the relevant tools and ideas needed to determine the correlations.

The economics of the problem remain exactly the same. The main interactions between the return and volatility are twofold: one consists in the forward effect of volatility innovations on future expected returns, the other involves the feedback effect of lagged return innovations, or the diffusive part of the returns, on current volatility. To elucidate these separate effects within the general model setting, it is useful to define the auxiliary variable

$$r_{t,s} \equiv \begin{cases} \sigma_{g,t+s} dW_{t+s}^c + A_q \varphi_q \sqrt{q_{t+s}} dW_{t+s}^q + A(0) \sqrt{q_{t+s}} dW_{t+s}^\sigma & s < 0, \\ \mu_{R,t+s} & s \geq 0. \end{cases}$$

The variable  $r_{t,s}$  is equal to the local diffusive part of the equilibrium return process for  $s < 0$ , and it equals the local mean of the equilibrium return process for  $s \geq 0$ , as defined in equation (22). As such, the basic shape of the cross-covariances between the variance risk premium  $vp_t$  and  $r_{t,s}$  directly mirrors that of the cross-covariances with the returns,  $r_{t+s}$ . In particular, it follows directly from the expression for  $vp_t$  in equation (30) that the forward correlations with  $r_{t,s}$  must be proportional to the autocovariances of the  $q_t$  process. That is for  $s > 0$ ,

$$\text{Cov}(vp_t, r_{t,s}) = K_r e^{-\kappa_q s},$$

where  $K_r > 0$  denotes a positive constant of proportionality. To derive the backward correlations, write  $q_t$  in integral form,

$$q_t = \varphi_q \int_{u=-\infty}^t e^{\kappa_q(u-t)} \sqrt{q_u} dW_u^q.$$

From this expression it follows that for  $s < 0$ ,

$$\text{Cov}(vp_t, r_{t,s}) = e^{-\kappa_q |s|} \mathbb{E}(\varphi_q b_{vp,1} \sqrt{q_{t+s}} dW_{t+s}^q, A_q \varphi_q \sqrt{q_{t+s}} dW_{t+s}^q) = A_q b_{vp,1} \varphi_q^2 \mathbb{E}(q_t) e^{-\kappa_q |s|},$$

so that all of the backward autocovariances are again negative decaying at an exponential rate under the maintained assumption that the preference parameters  $\gamma > 1$  and  $\psi > 1$

and therefore  $A_q < 0$ . The dynamic patterns in the cross-correlations for  $r_{t,s}$ , of course, mimic those of the cross-covariances. While those for the returns  $r_{t-s}$  will be given by an attenuated version of the cross-correlations for  $r_{t,s}$ , consistent with the implications from the short-memory model summarized in equation (32) above.

The theoretical predictions for the dynamic cross-correlations between the VIX and the returns within the general model setting are not quite as clear-cut as those for the variance premium. The  $q_t$  process determining the variance risk premium essentially gets confounded with the classical consumption risk premium, and there are also potential side effects from long-range dependence. However, the underlying economic mechanisms remain the same as for the short-memory model, and we would expect a similar pattern of mixed negative backward correlations and positive forward correlations to hold true.

## 4 Empirical Results

The equilibrium framework developed above completely characterizes the dynamic dependencies in the returns and the volatility. Of course, the specific solution of the model will invariably depend upon the choice of preference parameters and the values of the parameters for the underlying consumption growth rate and volatility dynamics. However, the model is obviously somewhat stylized and direct estimation based on actual consumption data would be challenging at best. Instead, we will evaluate the model's direct implications in regards to the autocorrelations and cross-correlations derived in the previous section, and in particular how well the basic patterns implied by the model match those of the actual data depicted in Figures 1 and 2. We begin in the next subsection with a discussion of the data and pertinent summary statistics underlying the figures.

### 4.1 Data Description

Our tick-by-tick data for the S&P 500 futures contract was obtained from Tick Data Inc. To alleviate the impact of market microstructure "noise" in the calculation of the cross-

correlations, and the realized volatility measures discussed further below, we follow the dominant approach in the literature and convert the tick-by-tick prices to equally spaced five-minute observations.<sup>14</sup> With 77 five-minute intervals per trading day and one overnight return, this leaves us with a total of 78 “high-frequency” return observations per day. Standard summary statistics for the corresponding daily returns over the January 2, 1990, through October 31, 2007, sample period and the five-minute returns over the shorter September 23, 2003, to August 31, 2007, sample are reported in the first column in Table 1.

The autocorrelations for the VIX in the top panel in Figure 1 are based on daily data from January 2, 1990, through October 31, 2007. These data are freely available from the Chicago Board Options Exchange (CBOE).<sup>15</sup> From the summary statistics reported in Table 1, the average value of the  $VIX^2$  over the sample equals 32.81, or 16.41 in standard deviation form, but it varies quite considerably over the sample, as indicated by the standard deviation of 23.70. This variation is quite persistent, however, as suggested by the slow decay in the aforementioned autocorrelations depicted in Figure 1. This strong persistence is also immediately evident from the actual time series plot of the data in Figure 5 in the appendix.

The tick-by-tick data for the VIX used in the construction of the cross-correlations shown in the top panel in Figure 2 was again obtained from Tick Data Inc. High-frequency data for the VIX has only been available since the introduction of the “new” model-free VIX index on September 22, 2003. The relevant summary statistics for the  $VIX^2$  over the shorter September 22, 2003, to August, 2007, high-frequency sample, reported in the bottom part of Table 1, are broadly consistent with those over the longer daily sample, and the time series plots in Figures 5 and 6 in the appendix also reveal the same basic features. Most notably, the average value is somewhat lower over the more recent sample, and not surprisingly, the

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<sup>14</sup>The specific choice of a five-minute sampling frequency strike a reasonable balance between confounding market microstructure effects when sampling too frequently and the loss of important information concerning fundamental price movements when sampling more coarsely; see, e.g., the discussion and references in Andersen et al. (2007b), where the same futures data and five-minute sampling frequency have been used from a different perspective.

<sup>15</sup>The VIX index is reported in annualized units by the CBOE. We convert the series to monthly units using the transformation  $VIX_t^2 = 30/365 VIX_{CBOE,t}^2$ .

**Table 1 Summary Statistics**

	$r_t$	$VIX_t^2$	$RV_{t,t+22}$	$\hat{v}p_t$
Daily Sampling (1990-2007)				
Mean	11.14	32.81	23.74	8.96
Standard Deviation	15.67	23.70	24.12	12.62
Skewness	-0.05	1.88	2.59	-1.86
Excess Kurtosis	3.75	4.66	8.01	17.53
5-Minute Sampling (2003-2007)				
Mean	8.96	17.48	11.28	6.17
Standard Deviation	11.31	8.40	7.68	5.25
Skewness	0.70	2.90	3.81	1.94
Excess Kurtosis	42.12	15.12	17.90	9.40

The table reports summary statistics for continuously-compounded returns  $r_t$ , implied variances  $VIX_t^2$ , monthly realized variances  $RV_{t,t+22}$ , and the variance risk premium  $\hat{v}p_t = VIX_t^2 - \hat{E}RV_{t,t+22}$ . The realized variances are constructed from the summation of high-frequency five-minute squared returns. The expectations for the future variances  $\hat{E}_tRV_{t,t+22}$  are based on the HAR-RV forecasting model discussed in the text. All the variables are in percentage form. The daily data extend from January 2, 1990 to October 31, 2007. The five-minute sample spans September 22, 2003 to August 31, 2007.

kurtosis is substantially higher when the data is sampled at the five-minute frequency.

The integrated variance  $IV_{t,t+N}$  defined within the theoretical model is, of course, not directly observable. However, it may be consistently estimated in a completely model-free manner by the corresponding realized variation based on an increasing number of observations over the fixed time-interval  $[t, t + N]$  (see, e.g., Andersen, Bollerslev and Diebold, 2009). As previously noted, to guard against the adverse impact of market microstructure effects when sampling too frequently, we follow the common approach in the literature and rely on the summation of equally spaced five-minute squared returns.<sup>16</sup> With 77 five-minute intervals per trading day and the overnight return, the  $N$ -day-ahead realized variation is then simply given by,

$$RV_{t,t+N} = \sum_{i=1}^{78N} (\log S_{t+i/78} - \log S_{t+(i-1)/78})^2.$$

<sup>16</sup>Recent studies, e.g., Zhang et al. (2005) and Barndorff-Nielsen et al. (2008), have proposed more efficient and complicated ways by which to annihilated the impact of market microstructure effects. However, the simple-to-implement estimator that we use here remains dominant, and importantly allows for easy verification and replication of the results.

With the notable exception of a lower mean, the summary statistics for the resulting one-month realized volatility measures  $RV_{t,t+22}$  reported in Table 1 are generally close to those for the VIX. The corresponding time series plots in Figures 5 and 6 are also very similar to those for the VIX.

The higher average value of the VIX compared to the actual realized volatility reflects a premium for bearing volatility risk. Meanwhile, the variance risk premium is formally defined as the difference between the objective and risk-neutralized expectation of the forward integrated variance. While the risk-neutral expectation and the actually observed values of  $IV_{t,t+N}$  may both be estimated in a completely model-free fashion by the VIX and the realized volatilities, respectively, the calculation of the objective expectation  $E_t(IV_{t,t+N})$  necessitates some mild auxiliary modeling assumptions. Motivated by the results in Andersen et al. (2003) that simple autoregressive type models estimated directly for the realized volatility typically perform on par with, and often better, than specific parametric modeling approaches designed to forecast the integrated volatility,<sup>17</sup> we will here rely on the HAR-RV model structure first proposed by Corsi (2004), and subsequently used by Andersen et al. (2007a) among many others, in approximating the objective expectation. Specifically, define the one-day-ahead expectation by the linear projection of the realized volatility on the lagged daily, weekly and monthly realized volatilities,

$$E_t(RV_{t,t+1}) = \beta_{rv,0} + \beta_{rv,1}RV_{t-1,t} + \beta_{rv,2}RV_{t-5,t} + \beta_{rv,3}RV_{t-22,t}.$$

The one-month expectation  $E_t(IV_{t,t+22}) = E_t(RV_{t,t+22})$  is then simply obtained by iterating the projection forward.<sup>18</sup>

The summary statistics for the resulting variance risk premium  $\hat{v}p_t = VIX_t^2 - \hat{E}RV_{t,t+22}$ , reported in the last column in Table 1, confirm the positive expected return for selling volatility, but also show that the magnitude of the premium varies substantially over time.

<sup>17</sup>Andersen et al. (2004) have formally shown that for the stochastic volatility models most commonly applied, the loss in efficiency from the use of reduced form autoregressive models for the realized volatility is typically small; see also Sizova (2008).

<sup>18</sup>The actual estimates for the  $\beta$ 's are directly in line with the results reported in the extant literature and available upon request.

At the same time, the plots in Figures 5 and 6 indicate much less persistent dependencies in the premium than for the VIX and the realized volatilities. Of course, the previously discussed autocorrelations in Figure 1 already confirm this important difference between the premium and the VIX. We next turn to our discussion of the model’s ability to match this basic feature along with the other key dynamic dependencies observed in the data.

## 4.2 Model Implied Auto- and Cross-Correlations

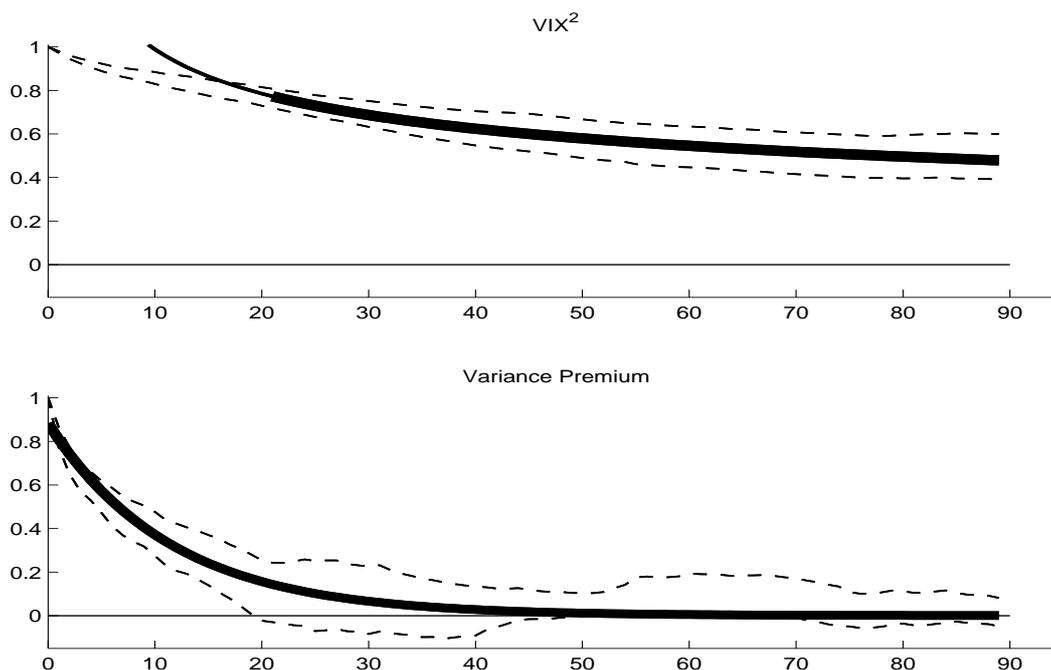
A full characterization of the model-implied autocorrelations for the integrated volatility would require that all of the moving average weights  $\{a(s)\}_{s \in [0, \infty)}$  in equation (12) be completely specified. Importantly, however, as discussed in Section 3.1, any long-run dependencies in these coefficients are directly translated to similar long-run dependencies in the moving average weights  $\{h_{vx}(s)\}_{s \in [0, \infty)}$  describing the equilibrium process for the  $VIX_t^2$  in equation (27). The top panel in Figure 3 shows the best fitting model-implied autocorrelations from estimating the slowly hyperbolic decaying autocorrelation structure in (28) to the actual daily sample autocorrelations for the VIX starting at a lag length of  $S = 22$ .<sup>19</sup> The figure also shows the conventional ninety-five percent confidence intervals for the sample autocorrelations. The model-implied autocorrelations do a remarkable job at describing the long-run dependencies inherent in the VIX, always falling well inside the confidence bands. Of course, the fit does *not* match at all well if extrapolated to the 1-22 day interval, which is to be expected.

One of the key predictions of the theoretical model is that the equilibrium volatility risk premium is an affine function of the volatility-of-volatility, and thereby is short memory, despite the fact that the integrated volatility and its risk neutral expectation may both display long-memory dependencies. Intuitively, as discussed above, everything except for the volatility-of-volatility gets risk neutralized out in equation (30). This, of course, is consistent with the shape of the autocorrelation function displayed in Figure 1, which in sharp contrast

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<sup>19</sup>The  $R^2$  from estimating the functional relationship is an impressive 0.993, although this value should be carefully interpreted because of the strong serial correlation in the residuals from the fit.

**Figure 3 Model Implied Autocorrelations**



The top panel shows the autocorrelations for the  $VIX^2$  volatility index to a lag length of 90 days. The solid line gives the model implied autocorrelations under the assumption of long-memory in the underlying fundamental volatility process. The bottom panel shows the autocorrelations for the variance risk premium  $vp$ . The solid line gives the model implied autocorrelations. The pair of dashed lines included in both panels represent ninety-five percent confidence intervals for the corresponding sample autocorrelations based on daily data from 1990 through 2007.

to the one for the VIX dies out relatively quickly. As a more formal verification of this distinct implication from the model, we fitted the functional form in equation (31) to the actual sample autocorrelations for the premium. The fit shown in the bottom panel of Figure 3 is again excellent, and the model-implied autocorrelations easily fall within the ninety-five percent confidence intervals over the entire 1-90 day range.<sup>20</sup>

The observed volatility feedback and leverage effects evident in the dynamic cross-correlations in Figure 2 arguably present the more challenging and difficult to explain empirical dependencies. Consider first the cross-correlations for the variance risk premium. The negative backward correlations start out at a slightly larger absolute value than the forward correla-

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<sup>20</sup>The fitted  $R^2$  equals 0.934.

tions, and both decay toward zero at what appears to be an exponential rate. This apparent pattern in the cross-correlations between the premium and leads and lags of the returns is entirely consistent with the model-implied correlations summarized in (32). The bottom panel in Figure 4 shows the resulting fit along with the ninety-five percent confidence intervals for the sample cross-correlations.<sup>21</sup> The theoretical model obviously delivers very accurate predictions for the actually observed dynamic dependencies between the returns and the variance risk premium.

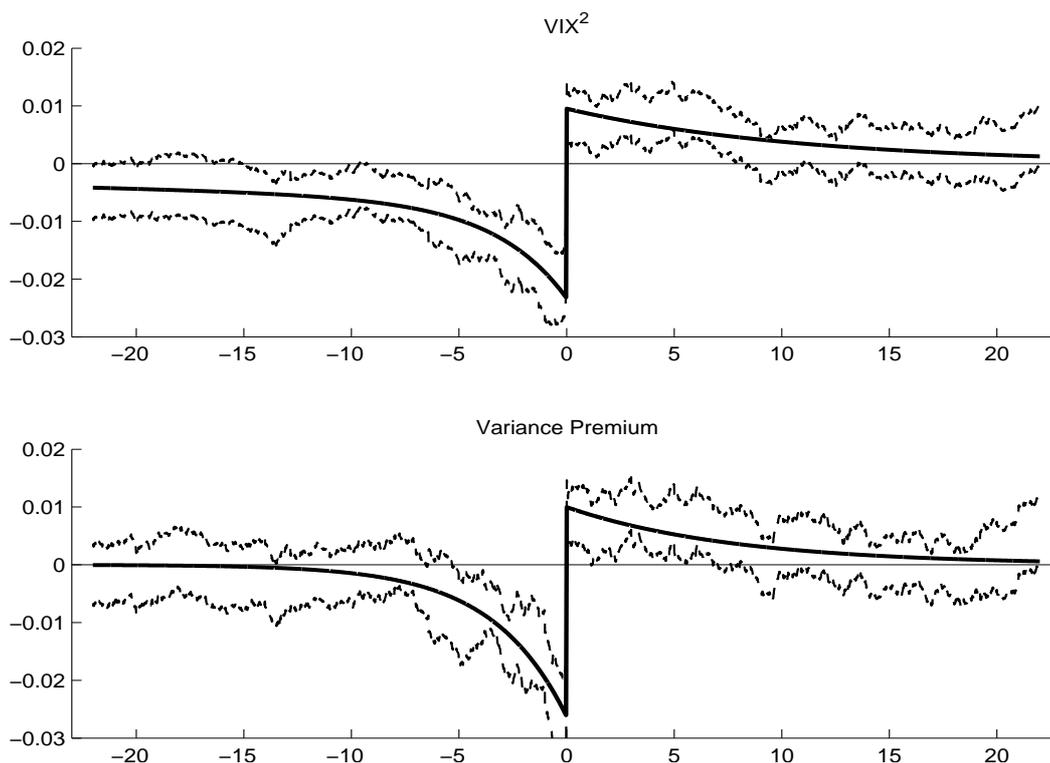
The theoretical predictions for the VIX-return cross-correlations are not quite as clear-cut as those for the premium. As discussed in Section 3.2 above, the volatility risk premium in effect gets confounded with the classical consumption risk premium, and within the general theoretical model setting there may also be potential side effects from long-range dependence. In parallel to the model-implied autocorrelations for the VIX, a complete characterization of these separate effects would require that the underlying fundamental consumption growth rate volatility process  $\sigma_{g,t}^2$  and the corresponding moving average weights  $\{a(s)\}_{s \in [0, \infty)}$  be fully specified. Short of such a specification, the basic pattern and decay in the cross-correlations may naturally be expected to adhere to the functional form in (33). The top panel in Figure 4 shows the resulting fit to the sample cross-correlations.

Comparing the observed backward correlations for the VIX in the left part of Figure 2 to those for the variance premium in the bottom panel, the dynamic leverage effect is clearly more prolonged for the VIX. This slower decay is very well described by the mixed exponential functions shown in Figure 4. At the same time, the differences between the forward correlations for the VIX and the premium, and in turn the impact on future returns attributable to the classical consumption risk premium and mean-variance tradeoff, appear less pronounced. In fact, the relatively fast decay rates in the empirically observed forward correlations are well described by a single exponential function for both the premium and

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<sup>21</sup>The fitted  $R^2$ 's for the backward and forward correlations equal 0.907 and 0.678, respectively. Of course, some of the sample cross-correlations used in the fit are not statistically different from zero.

Figure 4 Model Implied Cross-Correlations



The top panel shows the sample cross-autocorrelations between the  $VIX^2$  volatility index and lags and leads of the returns ranging from -22 to 22 days. The bottom panel shows the cross-correlations between the variance risk premium  $vp$  and the returns. The solid lines give the cross-correlations implied by the theoretical model. The pair of dashed lines represent ninety-five percent confidence intervals for the corresponding sample cross-correlations based on high-frequency 5-minute observations from 2003 through 2007.

the VIX.<sup>22</sup>

Summing up the empirical results, the qualitative implications from the new theoretical model do an admirable job in terms of matching the key dynamic dependencies in the aggregate market returns and volatilities. The previously documented autocorrelations for the volatility and volatility risk premium and the puzzling high-frequency based cross-correlation patterns in Figures 1 and 2, may all be explained by the model, with the model predictions well within conventional statistical confidence intervals.

<sup>22</sup>The fitted  $R^2$ 's for a double exponential for the VIX backward correlations and a single exponential for the VIX forward correlations equal 0.941 and 0.709, respectively.

## 5 Conclusion

The aggregate stock market volatility exhibits long-range dependence, while the variance risk premium, defined as the difference between the objective and risk-neutral expectation of the forward variance, shows much less persistence. Consistent with the well documented leverage effect, there is also a distinct and prolonged asymmetry in the relationship between volatility and past and future returns. We provide a first self-contained equilibrium based explanation for these empirical facts. The return on the aggregate market defined within the new model depends not only on the prospects of future economy growth, but also on the current uncertainty about the future economic conditions, and importantly explains the presence of a separate premium for bearing variance risk.

The dynamic return-volatility dependencies that motivate our theoretical model were all well-established prior to the advent of the current financial crises. Although the same basic dependencies have been shown to hold true during other market downturns and recessions, the scope of the current crises is different. We look forward to the opportunity of analyzing more recent data and formally test the model's ability to explain the dependencies observed therein.

Our explanation of the empirical facts is entirely risk-based, and depends critically on the temporal variation in the variance risk premium defined within the model. The wedge between the objective and risk-neutral expectation of the forward variance may alternatively be interpreted as a proxy for the aggregate degree of risk aversion in the market, and any temporal variation in the empirically observed variance risk premium thus indicative of changes in the way in which systematic risk is valued (see, e.g., Aït-Sahalia and Lo, 2000; Bollerslev, Gibson and Zhou, 2006a; Gordon and St-Amour, 2004). Although it might be difficult to contemplate systematic changes in the level of risk aversion at the frequencies emphasized here, time-varying volatility risk *and* time-varying attitudes toward risk likely both play a role in explaining the temporal variation in expected returns and risk premia (e.g., Bekaert, Engstrom and Xing, 2009). It would be interesting to extend the model developed

here to somehow allow for changes in the underlying preference parameters and risk-attitudes to more clearly delineate these effects. We conjecture that this may be especially important in fully understanding the return-volatility dependencies observed during the current financial crises.

## A Continuous-Time Equilibrium and SDF

The generalized preferences that we use are a continuous-time version of the Epstein-Zin-Weil discrete-time utility:

$$\tilde{V}_t^{1-\frac{1}{\psi}} = (1 - e^{-\rho h}) C_t^{1-\frac{1}{\psi}} + e^{-\rho h} \left[ E_t \tilde{V}_{t+h}^{1-\gamma} \right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}}. \quad (\text{A.1})$$

Assuming that  $\tilde{V}_t^{1-\gamma}$  is a semi-martingale, we approximate its conditional expectation over a short time-interval  $h$  by the linear function:

$$E_t \tilde{V}_{t+h}^{1-\gamma} \approx D(\tilde{V}_t^{1-\gamma})h + \tilde{V}_t^{1-\gamma}, \quad (\text{A.2})$$

where  $D(\cdot)$  denotes the drift of the argument. Plugging the conditional expectation in (A.2) into the definition (A.1), and taking limits around  $h = 0$ , the drift for the utility may be expressed as a function of consumption and utility:

$$D(\tilde{V}_t^{1-\gamma}) = \theta \rho \left[ 1 - \frac{C_t^{1-\frac{1}{\psi}}}{\tilde{V}_t^{1-\frac{1}{\psi}}} \right] \tilde{V}_t^{1-\gamma}.$$

The original utility  $\tilde{V}_t$  in (A.1) can be replaced by any ordinally equivalent utility  $V_t = \varphi(\tilde{V}_t)$ , where the transformation  $\varphi(\cdot)$  is strictly increasing. Following Duffie and Epstein (1992b), we apply the transformation  $\varphi(\cdot)$  that is linear in  $\tilde{V}_t^{1-\gamma}$ :

$$V_t = \frac{1}{1-\gamma} \tilde{V}_t^{1-\gamma}.$$

Given this choice of  $\varphi(\cdot)$ , the preferences may be simply defined through the recursive condition:

$$DV_t + f(C_t, V_t) = 0, \quad (\text{A.3})$$

where the normalized drift equals

$$f(c, v) = \frac{\rho}{1-\frac{1}{\psi}} \frac{c^{1-\frac{1}{\psi}} - [(1-\gamma)v]^{1/\theta}}{[(1-\gamma)v]^{1/\theta-1}}. \quad (\text{A.4})$$

This in effect constitutes the formal definition of the Epstein-Zin-Weil preferences in continuous time; see also Duffie and Epstein (1992a).

## A.1 SDF in Continuous Time

In this section we derive the exact formula for the Stochastic Discount Factor (SDF) under the Epstein-Zin-Weil preferences in equation (A.4) as a function of the return on the consumption asset and the consumption growth rate.

For notational convenience, denote the logarithmic welfare-consumption ratio:

$$vc_t \equiv \log \frac{\tilde{V}_t}{C_t} = \frac{1}{1-\gamma} \log([1-\gamma]V_t) - \log C_t. \quad (\text{A.5})$$

The normalized drift in equation (A.4) may then alternatively be represented as:

$$f(C_t, V_t) = \rho\theta V_t \left( e^{-(1-\frac{1}{\psi})vc_t} - 1 \right).$$

Duffie and Epstein(1992) have previously derive the SDF for recursive preferences with an arbitrary normalized drift  $f(C_t, V_t)$ :

$$M_t = \exp^{\int_0^t f_v(C_s, V_s) ds} f_c(C_t, V_t). \quad (\text{A.6})$$

The dynamics for the SDF thus follows from the dynamics of the two partial derivative,

$$f_c \equiv \frac{\partial f}{\partial C_t} = (1-\gamma)\rho C_t^{-\gamma} e^{-\log(1-\gamma) + (\frac{1}{\psi}-\gamma)vc_t},$$

and

$$f_v \equiv \frac{\partial f}{\partial V_t} = \theta\rho(e^{-(1-\frac{1}{\psi})vc_t} - 1)\left(1 - \frac{1}{\theta}\right) - \rho.$$

Now, note that

$$V_t^{\frac{1}{\theta}} M_t C_t^{\frac{1}{\psi}} = \rho e^{\frac{\log(1-\gamma)}{\theta}} V_t e^{\int_0^t f_v(C_s, V_s) ds}.$$

Extracting the drift of the process on the left-hand-side of the above equation, taking into account that the drift of the  $V_t$  process equals  $-f(C_t, V_t)$ , and rearranging the terms, we obtain the following relation:

$$D \left[ \frac{e^{\frac{\log(1-\gamma)}{\theta}}}{\rho} \left( \frac{V_t^{\frac{1}{1-\gamma}}}{C_t} \right)^{1-\frac{1}{\psi}} M_t C_t \right] = -M_t C_t.$$

In other words,

$$\frac{e^{\frac{\log(1-\gamma)}{\theta}}}{\rho} \left( \frac{V_t^{1-\gamma}}{C_t} \right)^{1-\frac{1}{\psi}}$$

satisfies the Euler condition for the price-dividend ratio of the consumption asset. Together with appropriate terminal conditions, this implies that this expression must be equal to the price-dividend ratio. In logarithm terms, there is a one-to-one correspondence between the price-dividend ratio of the consumption asset and the welfare-consumption ratio:

$$\log \Psi_t = -\log \rho + \left(1 - \frac{1}{\psi}\right) v c_t. \quad (\text{A.7})$$

The formula for the price-dividend ratio implies that the total return satisfies:

$$\frac{dR_t}{R_t} \equiv \frac{dS_t + C_t dt}{S_t} = \frac{d[\Psi_t C_t]}{\Psi_t C_t} + \frac{dt}{\Psi_t}. \quad (\text{A.8})$$

Combining the definitions of the return in (A.8), the SDF in (A.6), and the price-dividend ratio in (A.7), we obtain the following relation:

$$\begin{aligned} d \log M_t + (1 - \theta) d \log R_t + \frac{\theta}{\psi} d \log C_t \\ &= [f_v(C_t, V_t) dt + d \log f_c(C_t, V_t)] + (1 - \theta) [d \log C_t + d \log \Psi_t + \frac{dt}{\Psi_t}] + \frac{\theta}{\psi} d \log C_t \\ &= \rho \theta dt, \end{aligned} \quad (\text{A.9})$$

where the last equality follows from two identities:

$$f_v(C_t, V_t) + (1 - \theta) \frac{1}{\Psi_t} = \rho \theta,$$

$$d \log f_c(C_t, V_t) + (1 - \theta) [d \log C_t + d \log \Psi_t] + \frac{\theta}{\psi} d \log C_t = 0.$$

The expression for the SDF in equation (A.9) as a function of the return on aggregate consumption and consumption growth may naturally be seen as the continuous-time version of the similar discrete-time relationship in Bansal and Yaron (2004). We next proceed to study the asset pricing implications of the model and this SDF, including expressions for the risk-free rate, the return on the consumption asset, and the variance risk premium.

## B Model Solution Under Short Memory Dynamics

### B.1 Pricing of the Consumption Asset

Suppose, that the dynamics of the consumption growth  $g_t \equiv \frac{dC_t}{C_t}$  is determined by the following system of equations:

$$\frac{dC_t}{C_t} = \mu_g dt + \sigma_{g,t} dW_t^c, \quad (\text{B.1})$$

$$d\sigma_{g,t}^2 = \kappa_\sigma(\mu_\sigma - \sigma_{g,t}^2)dt + \sqrt{q_t} dW_t^\sigma, \quad (\text{B.2})$$

$$dq_t = \kappa_q(\mu_q - q_t)dt + \varphi_q \sqrt{q_t} dW_t^q, \quad (\text{B.3})$$

where all of the shocks are uncorrelated. Under the risk-neutral measure the asset return must be a martingale with respect to information at time  $t$ , i.e.,

$$D(M_t R_t) = 0.$$

It follows from the formula for the SDF in (A.9) and the definition of the return in (A.8) that

$$d \log(M_t R_t) = \theta(d \log \Psi_t + d \log C_t + \frac{dt}{\Psi_t}) - \frac{\theta}{\psi} d \log C_t - \rho \theta dt$$

Substituting the price-dividend ratio  $\Psi_t \equiv \Psi(x_t)$  into the above condition for the drift  $D(M_t R_t)$  yields the following pricing relation:

$$\begin{aligned} \theta D \log \Psi(x_t) + \frac{\theta}{\Psi(x_t)} + (1 - \gamma)(\mu_g - \frac{\sigma_{g,t}^2}{2}) - \rho \theta + \\ + \frac{\theta^2}{2} \frac{d[\log \Psi(x), \log \Psi(x)]_t}{dt} + \frac{1}{2}(1 - \gamma)^2 \sigma_{g,t}^2 = 0, \end{aligned} \quad (\text{B.4})$$

where  $D \log \Psi(x_t)$  denotes the drift of  $\log \Psi(x_t)$ , and  $[\log \Psi(x), \log \Psi(x)]_t$  refers to the quadratic variation, whose increment characterizes the variance of shocks to  $\log \Psi(x_t)$ . The pricing relation in (B.4) may now be solved using the first-order approximations similar to Campbell and Viceira (2002),

$$\frac{1}{\Psi(x_t)} = \exp(-\log \Psi(x_t)) \approx \exp(-\overline{\log \Psi}) - \exp(-\overline{\log \Psi})(\log \Psi(x_t) - \overline{\log \Psi}).$$

In particular, under this linearization it is natural to conjecture that log price-dividend ratio is linear in the states:

$$\log \Psi(x_t) = A_0 + A_\sigma \sigma_{g,t}^2 + A_q q_t, \quad (\text{B.5})$$

and therefore

$$\frac{1}{\Psi(x_t)} \approx -\kappa_0 - \kappa_1(A_0 + A_\sigma \sigma_{g,t}^2 + A_q q_t). \quad (\text{B.6})$$

Substituting the conjectured solution for  $\Psi(x_t)$  in (B.5) and its inverse value in (B.6) into the pricing condition (B.4), we find the coefficients:

$$\begin{aligned} A_0 &= \frac{A_\sigma \kappa_\sigma \mu_\sigma + A_q \kappa_q \mu_q - \kappa_0 + (1 - 1/\psi)\mu_g - \rho}{\kappa_1}, \\ A_\sigma &= \frac{-\gamma}{2} \frac{1 - \frac{1}{\psi}}{\kappa_\sigma + \kappa_1}, \\ A_q &= \frac{\kappa_q + \kappa_1 - \sqrt{(\kappa_q + \kappa_1)^2 - \theta^2 A_\sigma^2 \varphi_q^2}}{\theta \varphi_q^2}, \end{aligned}$$

where the value of  $A_q$  is the root of a quadratic equation that is bounded away from  $\infty$  as  $\phi_q$  goes to zero. Note that similar to the discrete-time case, the loadings  $A_\sigma$  and  $A_q$  are negative for values of the inter-temporal elasticity of substitution  $\psi > 1$ .

Combining the dynamics of dividends  $C_t$  and the dynamics of the price-dividend ratio, we obtain the dynamics for the total log-return under the objective measure:

$$\begin{aligned} d \log R_t &= \left( \frac{\mu_g}{\psi} + \rho + \left( -\frac{1}{2} + \frac{\gamma}{2} \left( 1 - \frac{1}{\psi} \right) \right) \sigma_{g,t}^2 - A_q (\kappa_1 + \kappa_q) q_t \right) dt \\ &\quad + \sigma_{g,t} dW_t^c + A_\sigma \sqrt{q_t} dW_t^\sigma + A_q \varphi_q \sqrt{q_t} dW_t^q. \end{aligned} \quad (\text{B.7})$$

The dynamics for the stochastic discount factor follows from (A.9):

$$\begin{aligned} \frac{dM_t}{M_t} &= \left[ \frac{\gamma}{2} \left( 1 + \frac{1}{\psi} \right) \sigma_{g,t}^2 - \left( 1 - \frac{1}{\theta} \right) (\kappa_q + \kappa_1) A_q q_t - \frac{1}{\psi} \mu_g - \rho \right] dt \\ &\quad - \gamma \sigma_{g,t} dW_t^g + (\theta - 1) [A_\sigma \sqrt{q_t} dW_t^\sigma + A_q \varphi_q \sqrt{q_t} dW_t^q] \end{aligned} \quad (\text{B.8})$$

The dynamics of SDF readily defines the risk-free rate, the equity premium, and the risk-neutral probability measure. For example, the risk-free rate is simply given by the drift of the SDF:

$$r_{f,t} \equiv -E_t \frac{dM_t}{M_t} = \frac{1}{\psi} \mu_g + \rho - \frac{\gamma}{2} \left(1 + \frac{1}{\psi}\right) \sigma_{g,t}^2 + \left(1 - \frac{1}{\theta}\right) (\kappa_q + \kappa_1) A_q q_t.$$

The equity premium is obtained as the ‘‘covariance’’ between the total return and the SDF:

$$\pi_{r,t} \equiv \frac{1}{dt} \frac{d[R, M]_t}{R_t M_t} = \gamma \sigma_{g,t}^2 - (\theta - 1) (A_\sigma^2 + A_q^2 \varphi_q^2) q_t. \quad (\text{B.9})$$

Lastly, the diffusion part of the SDF (B.8) defines the transition from the processes under the objective measure to the risk-neutral measure:

$$\begin{aligned} \frac{dC_t}{C_t} &= (\mu_g - \gamma \sigma_{g,t}^2) dt + \sigma_{g,t} d\tilde{W}_t^c \\ d\sigma_{g,t}^2 &= (\kappa_\sigma (\mu_\sigma - \sigma_{g,t}^2) + (\theta - 1) A_\sigma q_t) dt + \sqrt{q_t} d\tilde{W}_t^\sigma \end{aligned} \quad (\text{B.10})$$

$$dq_t = (\kappa_q (\mu_q - q_t) + (\theta - 1) A_q \varphi_q^2 q_t) dt + \varphi_q \sqrt{q_t} d\tilde{W}_t^q, \quad (\text{B.11})$$

where  $d\tilde{W}_t^c$ ,  $d\tilde{W}_t^\sigma$ , and  $d\tilde{W}_t^q$  are all uncorrelated Brownian motions under the risk-neutral probability measure.

## B.2 Variance Premium

The variability of the future asset price is determined by the integrated variance:

$$IV_{t,t+N} \equiv \int_{\tau=t}^{t+N} d[\log S, \log S]_\tau = \int_{\tau=t}^{t+N} \sigma_{g\tau}^2 d\tau + (A_\sigma^2 + A_q^2 \varphi_q^2) \int_{\tau=t}^{t+N} q_\tau d\tau. \quad (\text{B.12})$$

The variance risk premium by definition is given by the difference between the expected values of the integrated variance under the objective and risk-neutral measures:

$$vp_t \equiv E_t^Q IV_{t,t+N} - E_t^P IV_{t,t+N}.$$

Under the objective measure, the consumption variance  $\sigma_{g,t}^2$  and the volatility-of-volatility  $q_t$  are both affine processes with expectations:

$$E_t^P q_{t+\Delta t} = [q_t - \mu_q] e^{-\kappa_q \Delta t} + \mu_q,$$

$$E_t^P \sigma_{g,t+\Delta t}^2 = [\sigma_{g,t}^2 - \mu_\sigma] e^{-\kappa_\sigma \Delta t} + \mu_\sigma.$$

Under the risk-neutral measure, the volatility-of-volatility  $q_t$  remains an affine process, with the mean-reversion  $\tilde{\kappa}_q = \kappa_q - (\theta - 1)A_q \varphi_q^2$  and the mean  $\tilde{\mu}_q = \frac{\kappa_q \mu_q}{\tilde{\kappa}_q}$  given by equation (B.11). Thus, the expectation of  $q_{t+\Delta t}$  under the risk-neutral measure simply equals:

$$E_t^Q q_{t+\Delta t} = [q_t - \tilde{\mu}_q] e^{-\tilde{\kappa}_q \Delta t} + \tilde{\mu}_q.$$

The process for the variance  $\sigma_{g,t}^2$  under the risk-neutral measure in equation (B.10) is qualitatively different. The conditional mean now depends not only on its own value, but also on the current realization of  $q_t$ :

$$E_t^Q \sigma_{g,t+\Delta t} = \tilde{\mu}_\sigma + (\sigma_{g,t}^2 - \tilde{\mu}_\sigma - \Delta_q) e^{-\kappa_\sigma \Delta t} + \Delta_q e^{-\tilde{\kappa}_q \Delta t},$$

where,

$$\Delta_q = \frac{(\theta - 1)A_\sigma}{\kappa_\sigma - \tilde{\kappa}_q} [q_t - \tilde{\mu}_q],$$

and  $\tilde{\mu}_\sigma = \mu_\sigma + \frac{(\theta-1)A_\sigma}{\kappa_\sigma} \tilde{\mu}_q$  is equal to the risk-neutral unconditional mean of the variance.

The differences in the conditional expectations of the state variables under the risk-neutral and the objective measures can be represented as:

$$E_t^Q q_{t+\Delta t} - E_t^P q_t = q_t [e^{-\tilde{\kappa}_q \Delta t} - e^{-\kappa_q \Delta t}] + \kappa_q \mu_q \left[ \frac{1 - e^{-\tilde{\kappa}_q \Delta t}}{\tilde{\kappa}_q} - \frac{1 - e^{-\kappa_q \Delta t}}{\kappa_q} \right],$$

$$\frac{E_t^Q \sigma_{g,t+\Delta t}^2 - E_t^P \sigma_{g,t+\Delta t}^2}{(\theta - 1)A_\sigma} = q_t \left[ \frac{e^{-\tilde{\kappa}_q \Delta t} - e^{-\kappa_\sigma \Delta t}}{\kappa_\sigma - \tilde{\kappa}_q} \right] + \tilde{\mu}_q \left[ \frac{1 - e^{-\kappa_\sigma \Delta t}}{\kappa_\sigma} - \frac{e^{-\tilde{\kappa}_q \Delta t} - e^{-\kappa_\sigma \Delta t}}{\kappa_\sigma - \tilde{\kappa}_q} \right].$$

Since  $\exp(-x)$  and  $(1 - \exp(-x))/x$  are both decreasing functions in  $x$ , it follows that  $E_t^Q q_{t+\Delta t} > E_t^P q_t$  for  $\tilde{\kappa}_q < \kappa_q$  ( $A_q < 0$ ). Similarly, it is possible to show that for any positive  $\tilde{\kappa}_q$ ,  $\kappa_\sigma$ , and  $\Delta t$ , the expressions in square brackets in the second equation above are both greater than zero. Thus, the variance and volatility-of-volatility are both expected to be higher under the risk-neutral measure. Since the integrated variance in (B.12) depends on future values of  $q_t$  and  $\sigma_{g,t}^2$ , the variance premium  $vp_t \equiv E_t^Q IV_{t,t+N} - E_t^P IV_{t,t+N}$  must be positive.

Going one step further, the variance premium may be expressed as:

$$vp_t = \beta_{pr,0} + \beta_{pr,1}q_t, \quad (\text{B.13})$$

where

$$\begin{aligned} \beta_{pr,0} &= (\tilde{\mu}_\sigma - \mu_\sigma) \left[ N - \frac{1 - e^{-\kappa_\sigma N}}{\kappa_\sigma} \right] + (\tilde{\mu}_q - \mu_q)(A_\sigma^2 + A_q^2\varphi_q^2) \left[ N - \frac{1 - e^{-\kappa_q N}}{\kappa_q} \right] - \beta_{pr,1}\tilde{\mu}_q, \\ \beta_{pr,1} &= \left[ \frac{1 - e^{-\tilde{\kappa}_q N}}{\tilde{\kappa}_q} - \frac{1 - e^{-\kappa_\sigma N}}{\kappa_\sigma} \right] \frac{(\theta - 1)A_\sigma}{\kappa_\sigma - \tilde{\kappa}_q} + (A_\sigma^2 + A_q^2\varphi_q^2) \left[ \frac{1 - e^{-\tilde{\kappa}_q N}}{\tilde{\kappa}_q} - \frac{1 - e^{-\kappa_q N}}{\kappa_q} \right]. \end{aligned}$$

The expression in (B.13) is obtained by taking the difference between the expectations of the integrated variance under the objective measure,

$$E_t^P IV_{t,t+N} = \mu_\sigma N + \frac{1 - e^{-\kappa_\sigma N}}{\kappa_\sigma} (\sigma_{g,t}^2 - \mu_\sigma) + (A_\sigma^2 + A_q^2\varphi_q^2) \left( \mu_q N + \frac{1 - e^{-\kappa_q N}}{\kappa_q} (q_t - \mu_q) \right),$$

and under the risk-neutral measure,

$$E_t^Q IV_{t,t+N} = \beta_{VIX,0} + \beta_{VIX,\sigma}\sigma_{g,t}^2 + \beta_{VIX,q}q_t^2, \quad (\text{B.14})$$

where

$$\begin{aligned} \beta_{VIX,0} &= \tilde{\mu}_\sigma \left[ T - \frac{1 - e^{-\kappa_\sigma T}}{\kappa_\sigma} \right] + \\ &+ \left[ \frac{(\theta - 1)A_\sigma}{\kappa_\sigma - \tilde{\kappa}_q} \tilde{\mu}_q \right] \left[ \frac{1 - e^{-\kappa_\sigma T}}{\kappa_\sigma} - \frac{1 - e^{-\tilde{\kappa}_q T}}{\tilde{\kappa}_q} \right] + (A_\sigma^2 + A_q^2\varphi_q^2) \left[ T - \frac{1 - e^{-\tilde{\kappa}_q T}}{\tilde{\kappa}_q} \right] \tilde{\mu}_q, \\ \beta_{VIX,\sigma} &= \frac{1 - e^{-\kappa_\sigma T}}{\kappa_\sigma}, \\ \beta_{VIX,q} &= \frac{(\theta - 1)A_\sigma}{\kappa_\sigma - \tilde{\kappa}_q} \left[ \frac{1 - e^{-\tilde{\kappa}_q T}}{\tilde{\kappa}_q} - \frac{1 - e^{-\kappa_\sigma T}}{\kappa_\sigma} \right] + (A_\sigma^2 + A_q^2\varphi_q^2) \frac{1 - e^{-\tilde{\kappa}_q T}}{\tilde{\kappa}_q}. \end{aligned}$$

As discussed further in the main text, the expectation under the risk-neutral measure corresponds directly to the VIX<sup>2</sup> volatility index, hence the subscript notation for the  $\beta$ 's.

### B.3 Return-Volatility Cross-Correlations

The return over a short time-interval  $\Delta t$  is approximately equal to:

$$\begin{aligned} \Delta \log(R_t) &\approx \left( \frac{\mu_g}{\psi} + \rho - \frac{\sigma_{g,t}^2}{2} + \frac{\gamma}{2} \left( 1 - \frac{1}{\psi} \right) \sigma_{g,t}^2 - A_q(\kappa_1 + \kappa_q)q_t \right) \Delta t + \\ &+ \sigma_{g,t} \Delta W_t^c + A_\sigma \sqrt{q_t} \Delta W_t^\sigma + A_q \varphi_q \sqrt{q_t} \Delta W_t^q, \end{aligned}$$

where the operator  $\Delta$  denotes the increment to the process over the  $[t, t + \Delta]$  time-interval. The variance of the return equals:

$$\text{Var}(\Delta \log R_t) = [\beta_{R,\sigma}^2 \text{Var}(\sigma_{g,t}^2) + \beta_{R,q}^2 \text{Var}q_t] (\Delta t)^2 + \mu_\sigma \Delta t + [A_\sigma^2 + A_q^2 \varphi_q^2] \mu_q \Delta t,$$

where  $\beta_{R,\sigma} = -0.5 + \frac{\gamma}{2}(1 - \frac{1}{\psi})$  and  $\beta_{R,q} = -A_q(\kappa_1 + \kappa_q)$ .

From equation (B.13), the variance premium is directly proportional to  $q_t$ . Hence, the correlation between the premium and the return is solely determined by the correlation of the return with the  $q_t$  process. The covariance between  $q_t$  and a future return,  $\Delta \log R_{t+l}$  and  $l > 0$ , is equal to the covariance of  $q_t$  with the drift part of the return:

$$\text{cov}(q_t, \Delta \log R_{t+l}) = e^{-\kappa_q l} \text{Var}q_t \beta_{R,q} \Delta t, \quad l \geq 0.$$

The covariance of  $q_t$  with the past return,  $\Delta \log R_{t-l}$  and  $l < 0$ , consists of two parts. The covariance with the drift  $e^{-\kappa_q l} \text{Var}q_t \beta_{R,q} \Delta t$ , and the covariance with the diffusive part  $A_q \varphi_q \text{cov}(q_t, \sqrt{q_{t-l}} \Delta W_{t-l}^q) = A_q \varphi_q^2 \mu_q e^{-\kappa_q l} \Delta t$ . Combining these effects, the cross-correlation function for the variance risk premium and the returns may be conveniently expressed as:

$$\text{corr}(q_t, \Delta \log R_{t+l}) = \frac{(\text{Var}q_t \beta_{R,q} + I_{l < 0} A_q \varphi_q^2 \mu_q) e^{-\kappa_q |l|} \Delta t}{\sqrt{\text{Var}q_t} \sqrt{\text{Var}(\Delta \log R_t)}},$$

for any value of  $l$ .

The expression for the  $\text{VIX}_t^2 \equiv E_t^Q IV_{t,t+N}$  in (B.14) involves a linear function of  $q_t$  and the variance  $\sigma_{g,t}$ , with loadings  $\beta_{VIX,q}$  and  $\beta_{VIX,\sigma}$ , respectively. The covariance of the  $\text{VIX}_t^2$  with any future return depends solely on the covariance with the drift of the return:

$$\text{cov}(\text{VIX}_t^2, \Delta \log R_{t+l}) = (\beta_{R,\sigma} \beta_{VIX,\sigma} \text{Var}\sigma_{g,t}^2 e^{-\kappa_\sigma l} + \beta_{R,q} \beta_{VIX,q} \text{Var}q_t e^{-\kappa_q l}) \Delta t.$$

The covariance of the  $\text{VIX}_t^2$  with past returns includes the covariances with the drift and the diffusion:

$$\begin{aligned} \text{cov}(\text{VIX}_t^2, \Delta \log R_{t-l}) &= \beta_{VIX,\sigma} \beta_{R,\sigma} e^{-\kappa_\sigma l} \text{Var}\sigma_{g,t}^2 \Delta t + \beta_{VIX,\sigma} A_\sigma \text{cov}(\sigma_{g,t}^2, \sqrt{q_{t-l}} dW_{t-l}^\sigma) + \\ &\quad + \beta_{VIX,q} \beta_{R,q} e^{-\kappa_q l} \text{Var}q_t \Delta t + \beta_{VIX,q} A_q \varphi_q \text{cov}(q_t, \sqrt{q_{t-l}} dW_{t-l}^q) \\ &= ([\beta_{R,\sigma} \text{Var}\sigma_{g,t}^2 + A_\sigma \mu_\sigma] \beta_{VIX,\sigma} e^{-\kappa_\sigma l} + [\beta_{R,q} \text{Var}q_t + A_q \varphi_q^2 \mu_q] \beta_{VIX,q} e^{-\kappa_q l}) \Delta t. \end{aligned}$$

Combining these expressions, the cross-correlations between the  $VIX_t^2$  and the returns may be succinctly written:

$$\text{corr}(VIX_t^2, \Delta \log R_{t+l}) = \frac{(\beta_{R,\sigma} \text{Var} \sigma_{g,t}^2 + I_{l < 0} A_\sigma \mu_q) \beta_{VIX,\sigma} e^{-\kappa_\sigma |l|} \Delta t + (\beta_{R,q} \text{Var} q_t + I_{l < 0} A_q \varphi_q^2 \mu_q) \beta_{VIX,q} e^{-\kappa_q |l|} \Delta t}{\sqrt{(\beta_{VIX,\sigma}^2 \text{Var} \sigma_{g,t}^2 + \beta_{VIX,q}^2 \text{Var} q_t) \text{Var}(\Delta \log R_t)}},$$

for any value of  $l$ .

## C General Model Solution

To allow for more flexible volatility dynamics, we assume that the process for the consumption variance has a general MA-representation:

$$\sigma_{g,t}^2 = \sigma^2 + \int_{-\infty}^t a(t-s) \sqrt{q_s} dW_s^\sigma \quad (\text{C.1})$$

This specification includes the short-memory model in equation (B.2) above as a special case, but importantly allows for much richer dynamic dependencies, including long-memory in which the  $a(t-s)$  coefficients decrease at a slow hyperbolic rate. We maintain the identical short-memory process for the volatility-of-volatility in equation (B.3).

The pricing relation in (B.4) remains the same. In parallel to the solution method for the short-memory model used above, the linearization of the price-dividend ratio reduces the problem to a system of linear equations. In general, all the shocks in (C.1) need to be included in the conjectured solution for the dividend-price ratio:

$$\log \Psi_t = A_0 + A_q q_t + \int_{-\infty}^t A(t-s) \sqrt{q_s} dW_s^\sigma. \quad (\text{C.2})$$

Following Rogers (1997), if the price is a semi-martingale, as it must be to prevent arbitrage, and  $A(t)$  exists and is differentiable at zero, the dynamics of the price-dividend ratio may be decomposed into separate drift and diffusion terms:

$$d \log \Psi_t = \left[ A_q \kappa_q (\mu_q - q_t) + \int_{-\infty}^t A'(t-s) \sqrt{q_s} dW_s^\sigma \right] dt + A_q \varphi_q \sqrt{q_t} dW_t^q + A(0) \sqrt{q_t} dW_t^\sigma.$$

Now, substituting the conjectured solution into the pricing equation imply that the loadings on the variance shocks must satisfy:

$$A'(t-s) - \kappa_1 A(t-s) = \frac{\gamma(1 - \frac{1}{\psi})}{2} a(t-s),$$

for all  $t \geq s$ . Solving this system along with the constant and the loading on  $q_t$  in equation (C.2) we obtain:

$$A(t) = - \int_t^{+\infty} \frac{\gamma(1 - \frac{1}{\psi})}{2} e^{\kappa_1(t-\tau)} a(\tau) d\tau, \quad (\text{C.3})$$

$$A_q = \frac{\kappa_q + \kappa_1 - \sqrt{(\kappa_q + \kappa_1)^2 - \theta^2 \varphi_q^2 A(0)^2}}{\theta \varphi_q^2}, \quad (\text{C.4})$$

$$A_0 = \frac{A_q \kappa_q \mu_q - \kappa_0 + (1 - \frac{1}{\psi}) \mu_g - \rho - \frac{\gamma}{2} (1 - \frac{1}{\psi}) \sigma^2}{\kappa_1}. \quad (\text{C.5})$$

If absolute values of the coefficients  $a(t)$  are decreasing in  $t$  (or grow at a rate less than exponential) and  $|a(t)| < \infty$ , then  $A(t)$  is well-defined and  $A'(0) = \kappa_1 A(0) + \frac{\gamma(1-\frac{1}{\psi})}{2} a(0)$  is finite.

Substituting the solution for the price-dividend ratio into the expression for the SDF, it follows that

$$\begin{aligned} \frac{dM_t}{M_t} = & \left[ -\frac{\mu_g}{\psi} - \rho + \frac{\gamma}{2} \left(1 + \frac{1}{\psi}\right) \sigma_{g,t}^2 + \left(\frac{1}{\theta} - 1\right) (\kappa_1 + \kappa_q) A_q q_t \right] dt \\ & - \gamma \sigma_{g,t} dW_t^c + (\theta - 1) A_q \varphi_q \sqrt{q_t} dW_t^q + (\theta - 1) A(0) \sqrt{q_t} dW_t^\sigma. \end{aligned}$$

As before, the risk-free rate is simply defined by the drift of  $M_t$ :

$$r_t^{rf} = - E_t \frac{dM_t}{M_t dt} = \frac{\mu_g}{\psi} + \rho - \frac{\gamma}{2} \left(1 + \frac{1}{\psi}\right) \sigma_{g,t}^2 + \left(1 - \frac{1}{\theta}\right) (\kappa_1 + \kappa_q) A_q q_t.$$

Since  $d \log R_t = d \log C_t + d \log \Psi_t + \Psi_t^{-1} dt$ , and therefore

$$d \log R_t = D \log R_t dt + \sigma_{g,t} dW_t^c + A_q \varphi_q \sqrt{q_t} dW_t^q + A(0) \sqrt{q_t} dW_t^\sigma, \quad (\text{C.6})$$

the equity premium equals:

$$\pi_{r,t} = \gamma \sigma_{g,t}^2 + (1 - \theta) [A_q^2 \varphi_q^2 + A(0)^2] q_t = \gamma \sigma_{g,t}^2 + 2 \left( \frac{1}{\theta} - 1 \right) (\kappa_q + \kappa_1) A_q q_t. \quad (\text{C.7})$$

And, the dynamics of the return is determined by:

$$d \log R_t = \left[ \rho + \frac{\mu_g}{\psi} + \left[ -\frac{1}{2} + \frac{\gamma}{2} \left( 1 - \frac{1}{\psi} \right) \right] \sigma_{g,t}^2 - (\kappa_q + \kappa_1) A_q q_t \right] dt \\ + \sigma_{g,t} dW_t^c + A_q \varphi_q \sqrt{q_t} dW_t^q + A(0) \sqrt{q_t} dW_t^\sigma.$$

The integrated variance may generally be expressed as:

$$IV_{t,t+N} = \int_t^{t+N} \sigma_{g,\tau}^2 d\tau + (A_q^2 \varphi_q^2 + A(0)^2) \int_t^{t+N} q_\tau d\tau. \quad (\text{C.8})$$

The expected value of the integrated variance under the objective measure equals:

$$E_t^P IV_{t,t+N} = \sigma^2 N + \int_{-\infty}^t \left[ \int_t^{t+N} A(\tau - s) d\tau \right] \sqrt{q_s} dW_s^\sigma \\ + (A_q^2 \varphi_q^2 + A(0)^2) \left[ \mu_q N + \frac{1 - e^{-\kappa_q N}}{\kappa_q} (q_t - \mu_q) \right].$$

The expectation under the risk-neutral measure is:

$$E_t^Q IV_{t,t+N} = \sigma^2 N + \int_{-\infty}^t \left[ \int_t^{t+N} A(\tau - s) d\tau \right] \sqrt{q_s} dW_s^\sigma \\ + E_t^Q \int_t^{t+N} \int_t^\tau A(\tau - s) \sqrt{q_s} dW_s^\sigma d\tau + (A_q^2 \varphi_q^2 + A(0)^2) \left[ \tilde{\mu}_q N + \frac{1 - e^{-\tilde{\kappa}_q N}}{\tilde{\kappa}_q} (q_t - \tilde{\mu}_q) \right], \quad (\text{C.9})$$

where  $\tilde{\kappa}_q = \kappa_q - (\theta - 1) A_q \varphi_q^2$  refers to the mean-reversion of  $q_t$  under the risk-neutral probability, and  $\tilde{\mu}_q = \kappa_q / \tilde{\kappa}_q \mu_q$  denotes the corresponding expectation. Moreover, under the risk-neutral measure:

$$E_t^Q \sqrt{q_s} dW_s^\sigma = E_t^Q [\sqrt{q_s} d\tilde{W}_s^\sigma + (\theta - 1) A(0) q_s ds] = (\theta - 1) A(0) (\tilde{\mu}_q + e^{-\tilde{\kappa}_q (s-t)} (q_t - \tilde{\mu}_q)) ds.$$

Consequently, the variance risk premium defined by the difference between  $E_t^Q IV_{t,t+N}$  and  $E_t^P IV_{t,t+N}$  is again a linear function of  $q_t$ :

$$vp_t = \beta_{pr,0} + \beta_{pr,1} q_t, \quad (\text{C.10})$$

with the two coefficients now defined by:

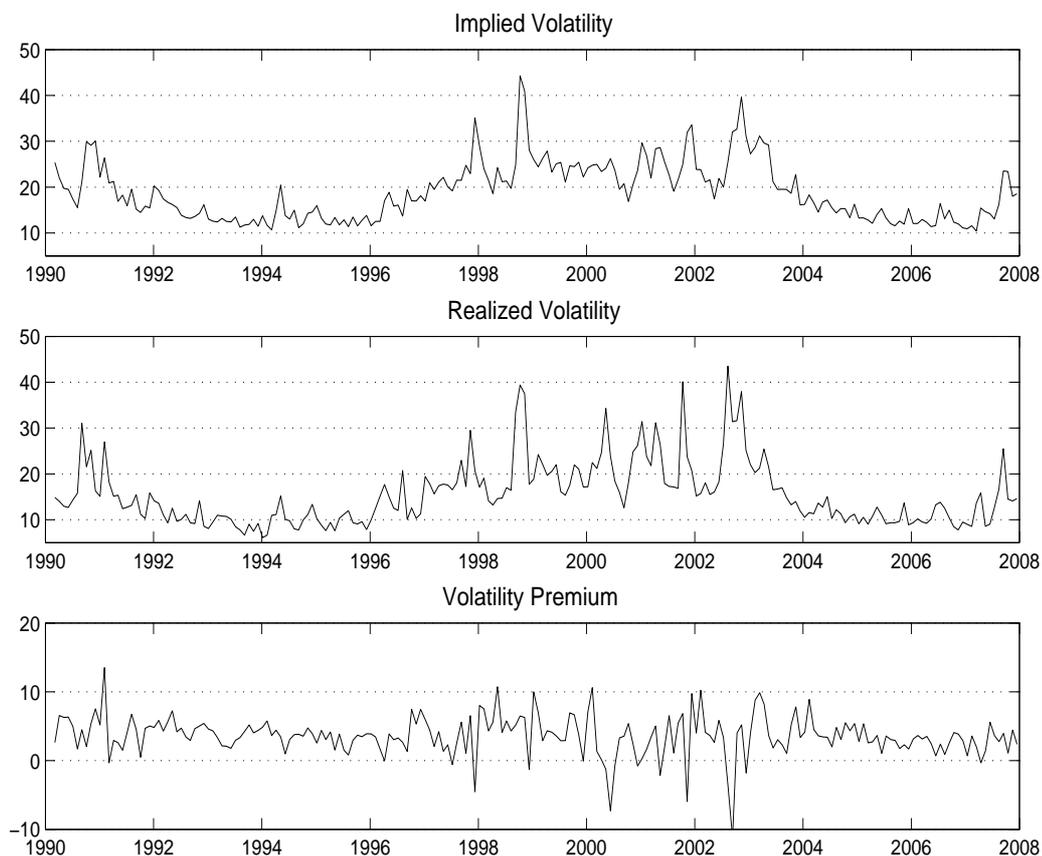
$$\beta_{pr,0} = (\theta - 1) A(0) \tilde{\mu}_q \int_t^{t+N} \int_t^\tau A(\tau - s) (1 - e^{-\tilde{\kappa}_q (s-t)}) ds d\tau + \\ + (A_q^2 \varphi_q^2 + A(0)^2) \left[ \tilde{\mu}_q \left( N - \frac{1 - e^{-\tilde{\kappa}_q N}}{\tilde{\kappa}_q} \right) - \mu_q \left( N - \frac{1 - e^{-\kappa_q N}}{\kappa_q} \right) \right],$$

$$\beta_{pr,1} = (\theta - 1)A(0) \int_t^{t+N} \int_t^\tau A(\tau - s)e^{-\tilde{\kappa}_q(s-t)} ds d\tau +$$

$$+ (A_q^2 \varphi_q^2 + A(0)^2) \left[ \frac{1 - e^{-\tilde{\kappa}_q N}}{\tilde{\kappa}_q} - \frac{1 - e^{-\kappa_q N}}{\kappa_q} \right].$$

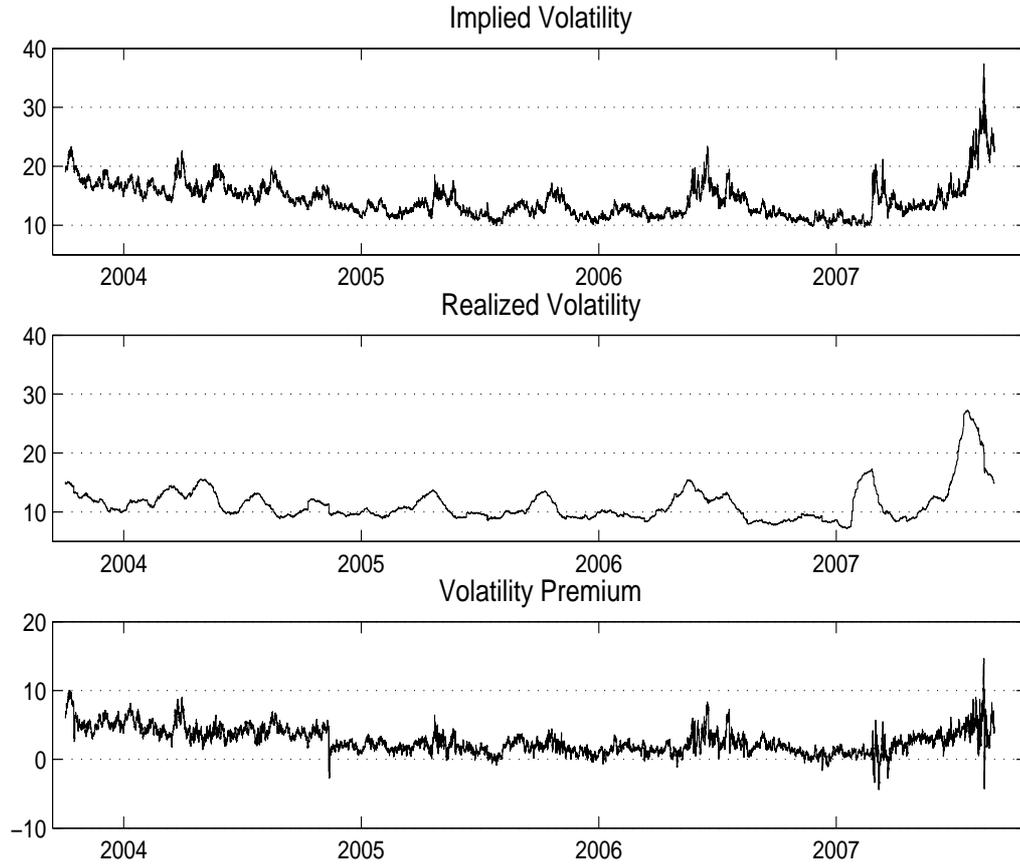
## D Time Series Plots

Figure 5 Volatility Measures, Daily Sample



The figure shows the  $VIX_t$  implied volatility index, the realized volatility  $RV_{t,t+22}^{1/2}$ , and the volatility risk premium  $\hat{v}p_t = VIX_t - E_t(RV_{t,t+22})^{1/2}$  over the January 2, 1990 to October 31, 2007 sample period. All of the volatility measures are plotted at the monthly frequency in annualized percentage units. The realized volatilities are constructed from the summation of high-frequency five-minute squared returns. The expectations for the future variances  $\hat{E}_t RV_{t,t+22}$  are based on the HAR-RV forecasting model discussed in the text.

Figure 6 Volatility Measures, 5-Minute Sample



The figure shows the  $VIX_t$  implied volatility index, the realized volatility  $RV_{t,t+22}^{1/2}$ , and the volatility risk premium  $\hat{v}p_t = VIX_t - E_t(RV_{t,t+22})^{1/2}$  over the September 23, 2003 to August 31, 2007 sample period. All of the volatility measures are plotted at the daily frequency in annualized percentage units. The realized volatilities are constructed from the summation of high-frequency five-minute squared returns. The expectations for the future variances  $\hat{E}_t RV_{t,t+22}$  are based on the HAR-RV forecasting model discussed in the text.

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