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Specification and Testing of Multiplicative Time-Varying GARCH Models with Applications

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Abstract

In this paper we develop a specification technique for building multiplicative time-varying GARCH models of Amado and Teräsvirta (2008, 2013). The variance is decomposed into an unconditional and a conditional component such that the unconditional variance component is allowed to evolve smoothly over time. This nonstationary component is defined as a linear combination of logistic transition functions with time as the transition variable. The appropriate number of transition functions is determined by a sequence of specification tests. For that purpose, a coherent modelling strategy based on statistical inference is presented. It is heavily dependent on Lagrange multiplier type misspecification tests. The tests are easily implemented as they are entirely based on auxiliary regressions. Finite-sample properties of the strategy and tests are examined by simulation. The modelling strategy is illustrated in practice with two real examples, an empirical application to daily exchange rate returns and another one to daily coffee futures returns.

JEL classification: C12; C22; C51; C52.

Key words: Conditional heteroskedasticity; Modelling volatility; Nonlinear model building; Misspecification testing; Time-varying parameter model.

1 Introduction

For more than 30 years, the ARCH model by Engle (1982) and, later, the GARCH model by Bollerslev (1986) have been successfully used to describe and forecast daily or weekly return series of financial assets and exchange rate returns. These models parameterise volatility clustering that is present in most of these series. They have been found to work well when the time series are sufficiently short. However, long return series are bound to contain slow movements or sudden breaks in variance that standard GARCH models have not been designed to describe. If these models are fitted to sufficiently long return series, the parameter estimates of these models tend to be very close to those of a (weakly nonstationary) integrated GARCH model, a fact that Diebold (1986) and Lamoureux and Lastrapes (1990) already pointed out. As Mikosch and Stărică (2004) demonstrated, this 'integrated GARCH effect' may indeed be due to nonstationarity in the variance, caused by breaks, that is, sudden shifts in the unconditional variance process.

Since many time series of asset returns contain data from periods of various degrees of turbulence, the possibility of changes in the unconditional variance should be taken seriously in modelling such series using the GARCH framework. This may be done by making the parameters, in particular the intercept, of the GARCH model time-varying. In this paper we assume that the variance of the return process is multiplicatively decomposed into two components, a stationary and a nonstationary one. The nonstationary component is typically a deterministic function of time, whereas the stationary component follows a GARCH process; see van Bellegem and von Sachs (2004), Feng (2004), Engle and Rangel (2008) and Amado and Teräsvirta (2008, 2013). For a similar approach see also Brownlees and Gallo (2010), Mazur and Pipień (2012), and Osiewalski and Pajor (2009). Mishra, Su and Ullah (2010) suggested yet another, completely stochastic, decomposition in which the second component is a slow-moving one.

In this paper, we develop a strategy of building Time-Varying GARCH (TV-GARCH) models introduced by Amado and Teräsvirta (2008) and further discussed in Amado and Teräsvirta (2013). In this model the deterministic multiplicative component is described by a linear combination of logistic transition functions of time. Our modelling strategy relies on statistical inference and proceeds from specific to general. The standard GARCH model with constant unconditional variance is first tested against the time-varying specification. If the null hypothesis is rejected, the

structure of the deterministic component is determined from the data. This is done by applying sequential hypothesis testing, using tests developed in the paper. Evaluating an estimated model is an integral part of model building, and this will be done by misspecification tests following the ideas in Eitrheim and Teräsvirta (1996) and Lundbergh and Teräsvirta (2002). Estimation of TV-GARCH models and asymptotic properties of the maximum likelihood estimators were already discussed in Amado and Teräsvirta (2013).

The outline of the paper is as follows. In Section 2 we present a short review of the TV-GARCH model and highlight some of its properties. The modelling strategy, consisting of specification, estimation and evaluation of the model, is described in Section 3. Specification and misspecification tests for the TV-GARCH model are considered in Section 4. Simulation results on the empirical performance of the tests and the specification strategy are reported in Section 5. In Section 6 we apply our modelling cycle to daily exchange rate and coffee futures price returns. Section 7 concludes.

2 The time-varying GARCH model

In this paper the tool for modelling daily or weekly return series is the time-varying GARCH model of Amado and Teräsvirta (2008, 2013) in which the unconditional variance evolves smoothly over time. Let the return series $\{y_t\}$ be given as

$$y_t = E(y_t | \mathcal{F}_{t-1}) + \varepsilon_t \quad (1)$$

where $\mathcal{F}_{t-1}$ contains the historical information available at time $t-1$. For simplicity, it is assumed that $E(y_t | \mathcal{F}_{t-1}) = 0$. The innovation sequence $\{\varepsilon_t\}$ has a conditional mean $E(\varepsilon_t | \mathcal{F}_{t-1}) = 0$, and variance σ_t^2 . Let

$$\varepsilon_t = \zeta_t \sigma_t$$

where $\{\zeta_t\} \sim \text{iid}(0, 1)$, $E\zeta_t^3 = 0$, and $E|\zeta_t^2|^{2+\phi} < \infty$, $\phi > 0$. Furthermore, σ_t^2 is assumed to have a time-varying representation measurable with respect to $\mathcal{F}_{t-1}$ with a multiplicative decomposition

$$\sigma_t^2 = h_t g_t \quad (2)$$

where h_t describes the short-run dynamics of the variance of the returns, and g_t is a positive-valued deterministic component. To allow an asymmetric response of volatility to positive and negative shocks, h_t is modelled as the GJR-GARCH(p, q) process of Glosten, Jagannathan and Runkle (1993):

$$h_t = \alpha_0 + \sum_{i=1}^q \alpha_i \varepsilon_{t-i}^{*2} + \sum_{i=1}^q \lambda_i \varepsilon_{t-i}^{*2} I(\varepsilon_{t-i}^* < 0) + \sum_{j=1}^p \beta_j h_{t-j} \quad (3)$$

where $\varepsilon_t^* = \varepsilon_t / g_t^{1/2}$ and $I(A)$ is the indicator variable, defined as $I(A) = 1$ when A is true, and zero otherwise. Equation (3) is assumed to satisfy the set of conditions for positivity and stationarity of the conditional variance of ε_t^* . This implies $\alpha_0 > 0$, $\alpha_i \geq 0$, $\alpha_i + \lambda_i \geq 0$, $i = 1, \dots, q-1$, $\alpha_q > 0$, $\alpha_q + \lambda_q > 0$, and $\beta_j \geq 0$, $j = 1, \dots, p$.

The GJR-GARCH(p, q) model is nested in (2) when $g_t \equiv 1$. The unconditional variance component g_t is smooth and time-varying, introducing nonstationarity into σ_t^2 . It is a linear combination of bounded transition functions defined as follows:

$$g_t(\boldsymbol{\theta}_1, t^*) = g_t = \delta_0 + \sum_{l=1}^r \delta_l G_l(t^*; \gamma_l, \mathbf{c}_l) \quad (4)$$

where $\boldsymbol{\theta}_1 = (\boldsymbol{\delta}', \boldsymbol{\gamma}', \mathbf{c}_1', \dots, \mathbf{c}_r')' \in \Theta_1 = (\Delta \times \Gamma \times C)$, with $\boldsymbol{\delta} = (\delta_0, \delta_1, \dots, \delta_r)'$, $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_r)'$, $\mathbf{c}_l = (c_{l1}, \dots, c_{lK_l})'$, $l = 1, \dots, r$, is an element of the parameter space of g_t . Our transition function is the general logistic transition function:

$$G_l(t^*; \gamma_l, \mathbf{c}_l) = \left(1 + \exp \left\{ -\gamma_l \prod_{k=1}^{K_l} (t^* - c_{lk}) \right\} \right)^{-1} \quad (5)$$

The transition function (5) is a continuous and non-negative function bounded between zero and one. Furthermore, the calendar time $t^* = t/T$, where T is the number of observations. We make the following assumptions about (4) and (5):

AG1. The elements of $\boldsymbol{\delta} \in \Delta$ are restricted such that $\delta_0 = \delta_0^* > 0$ (known), $\max_{j=1, \dots, q} |\delta_j| \leq M_\delta < \infty$ and $\inf_{\boldsymbol{\theta}_1 \in \Theta_1} g_t(\boldsymbol{\theta}_1, t^*) \geq g_{\min} > 0$.

AG2. The slope parameter $\gamma_l > 0$, $l = 1, \dots, r$, and the location parameters $c_{l1} < c_{l2} < \dots < c_{lK_l}$, $l = 1, \dots, r$.

AG2 and $\delta_0 = \delta_0^*$ in AG1 are identifying restrictions. That δ_0 is assumed known is due to the

fact that the decomposition is multiplicative and both h_t and g_t contain a positive constant. One of these constants has to be fixed to achieve identification. Amado and Teräsvirta (2008, 2013) set $\delta_0 = 1$, but that is not always an optimal solution in applications. This parameter has to be fixed such that $g_t(\boldsymbol{\theta}_1, t^*)$ is not too close to zero for any t^* . When $g_t \rightarrow 0$, $|\varepsilon_t^*| = |\varepsilon_t|/g_t^{1/2} \rightarrow \infty$. Thus, when g_t is very close to zero, the corresponding $|\varepsilon_t^*|$ becomes excessively high. This in turn may cause numerical problems in parameter estimation and size distortion in specification and misspecification tests. We conducted some simulations that seem to support this conclusion. How to select δ_0 to prevent g_t from becoming too close to zero is discussed in Section 3. Also note that the values of h_t , $t = 1, \dots, T$, depend on the choice of δ_0 . The conditional variance h_t in (3) is therefore a *relative* entity, not an absolute one as in stationary GARCH models. This is true for all approaches that make use of the multiplicative decomposition (2) of the variance.

The transition function allows the unconditional variance to change smoothly between regimes as a function of the transition variable t/T . The parameters $\mathbf{c}_l$ and γ_l determine the location and the speed of the transition between different regimes. When $r = K_1 = 1$, the function g_t increases (decreases) monotonically over time from 1 to $1 + \delta_1$ when $\delta_1 > 0$ ($-1 < \delta_1 < 0$), with the location centred at $t = c_1 T$. The slope parameter γ_l in (5) controls the degree of smoothness of the transition: the larger γ_l , the faster the transition between the extreme regimes. For example, when $r = K_1 = 1$ and $\gamma_1 \rightarrow \infty$, g_t collapses into a step function.

Equations (1)–(5) define the time-varying GJR-GARCH (TVGJR-GARCH) model. The unconditional variance in this model is time-varying and equals $E_t(\varepsilon_t^2) = E(\zeta_t^2 h_t g_t) = g_t E h_t$. This means that when $\delta_1 = \dots = \delta_r = 0$, the unconditional variance $E_t(\varepsilon_t^2) = \delta_0^* E h_t$ (constant). When $\delta_l \neq 0$ for $r > 1$ and $k > 1$, equations (4) and (5) form a very flexible parameterization capable of describing nonmonotonic deterministic changes in the unconditional variance. This way of introducing nonstationarity in the long run volatility component is discussed in detail by Amado and Teräsvirta (2008, 2013).

3 A strategy for building TV-GARCH models

It is seen from equations (3), (4) and (5) that there are several specification issues that have to be dealt with before the TV-GARCH or TVGJR-GARCH model can be estimated. The number of

transitions in (4) is unknown and has to be determined from the data. This is done by sequential testing. The shape of each transition function $G_l(t^*; \gamma_l, \mathbf{c}_l)$ is unknown (K_l is not assumed known) and has to be determined as well. The maximum lags p and q in h_t are also assumed unknown. They will, however, be determined only after estimating the TV-GARCH model assuming $p = q = 1$. Misspecification tests are constructed to find the appropriate lag lengths.

In order to do all this, we design a modelling strategy or model-building cycle for TV-GARCH or TVGJR-GARCH models. It is similar to the specific-to-general strategy for nonlinear models of the conditional mean considered in Teräsvirta (1998) and Teräsvirta, Tjøstheim and Granger (2010, Chapter 16), among others. The cycle or strategy consists of the following three stages:

1. Test the null hypothesis of no conditional heteroskedasticity in $\{\varepsilon_t\}$. If it is rejected, model the conditional variance h_t as defined in (3) with $p = q = 1$ *a priori*. In some applications, it may be assumed $\lambda_1 = 0$. In financial applications the test of no conditional heteroskedasticity can most often be omitted, however, because return series of sufficiently high frequency, such as daily or weekly, are likely to contain at least some conditional heteroskedasticity. The assumption $p = q = 1$ reflects the fact that a first-order GARCH model is quite often found a sufficient description of the data.
2. After estimating a first-order GARCH-type model, test the null hypothesis of constant variance against a time-varying unconditional variance with a single transition function. This implies testing $H_{10} : g_t \equiv 1$ in (4) against $H_{11} : g_t = 1 + \delta_1 G_1(t^*; \gamma_1, \mathbf{c}_1)$. Let α denote the significance level of the test. In case of a rejection, estimate a model with one transition and test $H_{20} : g_t = 1 + \delta_1 G_1(t^*; \gamma_1, \mathbf{c}_1)$ against $H_{21} : g_t = 1 + \sum_{j=1}^2 \delta_j G_j(t^*; \gamma_j, \mathbf{c}_j)$ at the significance level $\alpha\tau$, $0 < \tau < 1$. Continue until the first non-rejection of the null hypothesis. For reasons of parsimony, the significance level is reduced at each step of the testing procedure and converges to zero. After rejecting a null hypothesis, choose the type of transition, $K_l = 1, 2$, or 3 in (5). The appropriate test will be described in Section 4.
3. Estimate the model(s) using maximisation by parts as discussed in Amado and Teräsvirta (2013). The first step consists of estimating the parameters in g_t . Fix δ_0 to equal the first estimate of this parameter and keep it fixed thereafter (see the discussion in Section 2).

4. Evaluate the estimated model by means of LM or LM-type diagnostic tests to be discussed in Section 4. They are generalisations of tests in Lundbergh and Teräsvirta (2002). If the model passes all of them, tentatively accept it. Otherwise, respecify the model or consider another family of volatility models.

4 Misspecification testing of multiplicative TV-GARCH models

4.1 Three types of misspecification

In this section we shall consider misspecification tests relevant for Stage 3 of our modelling strategy. One of the tests is of particular interest for Stage 2 and can be viewed as a specification test. We begin by introducing some notation. Let $\boldsymbol{\theta}_2 = (\alpha_0, \dots, \alpha_q, \lambda_1, \dots, \lambda_q, \beta_1, \dots, \beta_p)'$ contain the parameters of h_t not included in g_t . The general idea is to construct an augmented version of the TV-GARCH model by incorporating a new component in the original model. For this purpose, we define a new function $f_t(\boldsymbol{\theta}_3)$ with the parameter vector $\boldsymbol{\theta}_3 = (\boldsymbol{\theta}'_{31}, \boldsymbol{\theta}'_{32})'$ such that $f_t(\boldsymbol{\theta}_{31}, \boldsymbol{\theta}_{32}) = 0$ if and only if $\boldsymbol{\theta}_{31} = \mathbf{0}$ or, alternatively, if and only if $\boldsymbol{\theta}_3 = \mathbf{0}$. In the former (nonstandard) case, the null hypothesis is $\boldsymbol{\theta}_{31} = \mathbf{0}$, and $\boldsymbol{\theta}_{32}$ is a vector of unidentified nuisance parameters when $\boldsymbol{\theta}_{31} = \mathbf{0}$. In the latter (standard) case, the null hypothesis is $\boldsymbol{\theta}_3 = \mathbf{0}$. Assume furthermore that $f_t(\boldsymbol{\theta}_{31}, \boldsymbol{\theta}_{32})$ is at least three times continuously differentiable in an open neighbourhood of $\boldsymbol{\theta}_{31} = \mathbf{0}$ (nonstandard case) or $\boldsymbol{\theta}_3 = \mathbf{0}$ (standard case). The three families of alternative hypothesis to be considered are as follows:

1. The deterministic component is additively misspecified:

$$\sigma_t^2 = h_t(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3)\{g_t(\boldsymbol{\theta}_1) + f_t(\boldsymbol{\theta}_3)\}.$$

2. The conditional variance component is additively misspecified:

$$\sigma_t^2 = \{h_t(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) + f_t(\boldsymbol{\theta}_3)\}g_t(\boldsymbol{\theta}_1).$$

3. The variance is multiplicatively misspecified:

$$\sigma_t^2 = h_t(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) g_t(\boldsymbol{\theta}_1) f_t(\boldsymbol{\theta}_3).$$

In the first two cases the null hypothesis is $f_t(\boldsymbol{\theta}_3) \equiv 0$, and it is assumed that this identity holds either if and only if $\boldsymbol{\theta}_{31} = \mathbf{0}$ or if and only if $\boldsymbol{\theta}_3 = \mathbf{0}$. Testing the hypothesis that the number of transitions in g_t equals r , whereas it is greater than r in the alternative, belongs to the first family. The second family contains testing a GARCH(p, q) model against either a GARCH($p+s, q$) or GARCH($p, q+s$), $s > 0$, model. Likewise, testing a linear GARCH model such as a standard GARCH but also a GJR-GARCH model against a smooth transition GARCH model fits into this framework, but in that case the identification problem (the null hypothesis is $\boldsymbol{\theta}_{31} = \mathbf{0}$) is present, see Lundbergh and Teräsvirta (2002). In the third case $f_t(\boldsymbol{\theta}_3)$ is a positive-valued function such that under the null hypothesis $f_t(\boldsymbol{\theta}_3) \equiv 1$, and this occurs if and only if $\boldsymbol{\theta}_3 = \mathbf{0}$. This design covers the case in which under the null hypothesis $\{\zeta_t\}$ is a sequence of independent identically distributed (iid) variables as in (2). Under the alternative, ζ_t has an ARCH structure. This implies $\zeta_t = z_t f_t^{1/2}$, where $\{z_t\}$ is a sequence of iid random variables with zero mean and unit variance, and

$$f_t = 1 + \sum_{j=1}^m \nu_j \zeta_{t-j}^2 \tag{6}$$

so $\boldsymbol{\theta}_3 = (\nu_1, \dots, \nu_m)'$ with $\nu_j \geq 0$, $j = 1, \dots, m$, and at least one inequality is strict. Another possibility is to assume $f_t = 1 + \sum_{j=1}^m \nu_j y_{t-j}^2$ such that y_t^2 is a positive-valued weakly stationary stochastic variable, weakly exogenous to the parameters of interest. An interesting special case may be the one in which $g_t \equiv 1$, and the multiplicative structure of the model opens up another way of introducing exogenous information to the GARCH model.

4.2 Testing the deterministic component

In order to consider the first testing problem, assume that the null model is a TVGJR-GARCH model in which the GARCH component is of order one ($p = q = 1$) in (5). The latter assumption is made for notational convenience, and a higher-order GARCH component is not excluded from consideration.

In considering Family 1, we first assume that $H_0: \boldsymbol{\theta}_3 = \mathbf{0}$. The quasi log-likelihood for observation t has the form

$$\begin{aligned} \ell(\boldsymbol{\theta}, \varepsilon_t) = & k - (1/2)\{\ln h_t(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3) + \ln\{g_t(\boldsymbol{\theta}_1) + f_t(\boldsymbol{\theta}_3)\} \\ & + \frac{\varepsilon_t^2}{h_t(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3)\{g_t(\boldsymbol{\theta}_1) + f_t(\boldsymbol{\theta}_3)\}}\}. \end{aligned} \quad (7)$$

The average score vector has the form

$$\mathbf{s}(\boldsymbol{\theta}, \varepsilon) = (\mathbf{s}_1(\boldsymbol{\theta}, \varepsilon)', \mathbf{s}_2(\boldsymbol{\theta}, \varepsilon)')' \quad (8)$$

where $\mathbf{s}_1(\boldsymbol{\theta}, \varepsilon) = (\mathbf{s}_{11}(\boldsymbol{\theta}, \varepsilon)', \mathbf{s}_{12}(\boldsymbol{\theta}, \varepsilon)')'$. In order to shorten the notation, set $h_t = h_t(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3)$, $g_t = g_t(\boldsymbol{\theta}_1)$ and $f_t = f_t(\boldsymbol{\theta}_3)$. The components of (8) are

$$\begin{aligned} \mathbf{s}_{11}(\boldsymbol{\theta}, \varepsilon) &= (2T)^{-1} \sum_{t=1}^T \frac{\partial \ell(\boldsymbol{\theta}, \varepsilon_t)}{\partial \boldsymbol{\theta}_1} = (2T)^{-1} \sum_{t=1}^T (\zeta_t^2 - 1) \left(\frac{1}{(g_t + f_t)} \frac{\partial g_t}{\partial \boldsymbol{\theta}_1} + \frac{1}{h_t} \frac{\partial h_t}{\partial \boldsymbol{\theta}_1} \right) \\ \mathbf{s}_{12}(\boldsymbol{\theta}, \varepsilon) &= (2T)^{-1} \sum_{t=1}^T \frac{\partial \ell(\boldsymbol{\theta}, \varepsilon_t)}{\partial \boldsymbol{\theta}_2} = (2T)^{-1} \sum_{t=1}^T (\zeta_t^2 - 1) \frac{1}{h_t} \frac{\partial h_t}{\partial \boldsymbol{\theta}_2} \end{aligned}$$

and

$$\mathbf{s}_2(\boldsymbol{\theta}, \varepsilon) = (2T)^{-1} \sum_{t=1}^T \frac{\partial \ell(\boldsymbol{\theta}, \varepsilon_t)}{\partial \boldsymbol{\theta}_3} = (2T)^{-1} \sum_{t=1}^T (\zeta_t^2 - 1) \left(\frac{1}{(g_t + f_t)} \frac{\partial f_t}{\partial \boldsymbol{\theta}_3} + \frac{1}{h_t} \frac{\partial h_t}{\partial \boldsymbol{\theta}_3} \right)$$

where $\zeta_t^2 = \varepsilon_t^2 / \{h_t(g_t + f_t)\}$ and $f_t = 0$ if and only if $\boldsymbol{\theta}_3 = \mathbf{0}$. The partial derivatives of h_t are as follows:

$$\begin{aligned} \frac{\partial h_t}{\partial \boldsymbol{\theta}_1} &= -\left\{ \alpha_1 \frac{\varepsilon_{t-1}^2}{(g_{t-1} + f_{t-1})^2} + \gamma_1 \frac{\varepsilon_{t-1}^2 I(\varepsilon_{t-1} < 0)}{(g_{t-1} + f_{t-1})^2} \right\} \frac{\partial g_{t-1}}{\partial \boldsymbol{\theta}_1} + \beta_1 \frac{\partial h_{t-1}}{\partial \boldsymbol{\theta}_1} \\ \frac{\partial h_t}{\partial \boldsymbol{\theta}_2} &= \frac{\partial}{\partial \boldsymbol{\theta}_2} \left\{ \alpha_0 + \alpha_1 \frac{\varepsilon_{t-1}^2}{g_{t-1} + f_{t-1}} + \gamma_1 \frac{\varepsilon_{t-1}^2 I(\varepsilon_{t-1} < 0)}{g_{t-1} + f_{t-1}} + \beta_1 h_{t-1} \right\} \\ &= \left(1, \frac{\varepsilon_{t-1}^2}{g_{t-1} + f_{t-1}}, \frac{\varepsilon_{t-1}^2 I(\varepsilon_{t-1} < 0)}{g_{t-1} + f_{t-1}}, h_{t-1} \right)' + \beta_1 \frac{\partial h_{t-1}}{\partial \boldsymbol{\theta}_2} \end{aligned}$$

and

$$\frac{\partial h_t}{\partial \boldsymbol{\theta}_3} = -\left(\alpha_1 \frac{\varepsilon_{t-1}^2}{(g_{t-1} + f_{t-1})^2} + \gamma_1 \frac{\varepsilon_{t-1}^2 I(\varepsilon_{t-1} < 0)}{(g_{t-1} + f_{t-1})^2} \right) \frac{\partial f_{t-1}}{\partial \boldsymbol{\theta}_3} + \beta_1 \frac{\partial h_{t-1}}{\partial \boldsymbol{\theta}_3}. \quad (9)$$

Furthermore, $\partial g_t / \partial \boldsymbol{\theta}_1$ is defined in Lemma A.1 (see Appendix A).

Setting $\widehat{h}_t = h_t(\widehat{\boldsymbol{\theta}}_{1T}, \widehat{\boldsymbol{\theta}}_{2T}, \mathbf{0})$ and $\widehat{g}_t = g_t(\widehat{\boldsymbol{\theta}}_{1T})$, and evaluating the average score under H_0 : $\boldsymbol{\theta}_3 = \mathbf{0}$ yields

$$\mathbf{s}_{11}(\widehat{\boldsymbol{\theta}}_{1T}, \widehat{\boldsymbol{\theta}}_{2T}, \mathbf{0}) = (2T)^{-1} \sum_{t=1}^T (\widehat{\zeta}_t^2 - 1) \left(\frac{1}{\widehat{g}_t} \frac{\partial g_t}{\partial \boldsymbol{\theta}_1} \Big|_{H_0} + \frac{1}{\widehat{h}_t} \frac{\partial h_t}{\partial \boldsymbol{\theta}_1} \Big|_{H_0} \right) \quad (10)$$

$$\mathbf{s}_{12}(\widehat{\boldsymbol{\theta}}_{1T}, \widehat{\boldsymbol{\theta}}_{2T}, \mathbf{0}) = (2T)^{-1} \sum_{t=1}^T (\widehat{\zeta}_t^2 - 1) \frac{1}{\widehat{h}_t} \frac{\partial h_t}{\partial \boldsymbol{\theta}_2} \Big|_{H_0} \quad (11)$$

and

$$\mathbf{s}_2(\widehat{\boldsymbol{\theta}}_{1T}, \widehat{\boldsymbol{\theta}}_{2T}, \mathbf{0}) = (2T)^{-1} \sum_{t=1}^T (\widehat{\zeta}_t^2 - 1) \left(\frac{1}{\widehat{g}_t} \frac{\partial f_t}{\partial \boldsymbol{\theta}_3} \Big|_{H_0} + \frac{1}{\widehat{h}_t} \frac{\partial h_t}{\partial \boldsymbol{\theta}_3} \Big|_{H_0} \right)$$

where $\widehat{\zeta}_t^2 = \varepsilon_t^2 / (\widehat{h}_t \widehat{g}_t)$. Let $\boldsymbol{\theta}_j^0$ be the true value of $\boldsymbol{\theta}_j$ $j = 1, 2$. It follows from Amado and Teräsvirta (2013) that under regularity conditions, $\widehat{\boldsymbol{\theta}}_{1T} \rightarrow \boldsymbol{\theta}_1^0$ and $\widehat{\boldsymbol{\theta}}_{2T} \rightarrow \boldsymbol{\theta}_2^0$ in probability as $T \rightarrow \infty$. The partial derivatives, evaluated at H_0 , are as follows:

$$\begin{aligned} \frac{\partial h_t}{\partial \boldsymbol{\theta}_1} \Big|_{H_0} &= -\{\widehat{\alpha}_1 + \widehat{\gamma}_1 I(\varepsilon_{t-1} < 0)\} \frac{\widehat{\phi}_{t-1}^2}{\widehat{g}_{t-1}} \frac{\partial g_{t-1}}{\partial \boldsymbol{\theta}_1} \Big|_{H_0} + \widehat{\beta}_1 \frac{\partial h_{t-1}}{\partial \boldsymbol{\theta}_1} \Big|_{H_0} \\ \frac{\partial h_t}{\partial \boldsymbol{\theta}_2} \Big|_{H_0} &= (1, \widehat{\phi}_{t-1}^2, \widehat{\phi}_{t-1}^2 I(\varepsilon_{t-1} < 0), \widehat{h}_{t-1})' + \widehat{\beta}_1 \frac{\partial h_{t-1}}{\partial \boldsymbol{\theta}_2} \Big|_{H_0} \end{aligned}$$

where $\widehat{\phi}_t^2 = \varepsilon_t^2 / \widehat{g}_t$, and

$$\frac{\partial h_t}{\partial \boldsymbol{\theta}_3} \Big|_{H_0} = -\{\widehat{\alpha}_1 + \widehat{\gamma}_1 I(\varepsilon_{t-1} < 0)\} \frac{\widehat{\phi}_{t-1}^2}{\widehat{g}_{t-1}} \frac{\partial f_{t-1}}{\partial \boldsymbol{\theta}_3} \Big|_{H_0} + \widehat{\beta}_1 \frac{\partial h_{t-1}}{\partial \boldsymbol{\theta}_3} \Big|_{H_0}.$$

Denoting

$$\mathbf{B}(\boldsymbol{\theta}^0) = \mathbf{E} \mathbf{s}(\boldsymbol{\theta}_1^0, \boldsymbol{\theta}_2^0, \mathbf{0}) \mathbf{s}(\boldsymbol{\theta}_1^0, \boldsymbol{\theta}_2^0, \mathbf{0})' = \begin{bmatrix} \mathbf{B}_{11}(\boldsymbol{\theta}^0) & \mathbf{B}_{12}(\boldsymbol{\theta}^0) \\ \mathbf{B}_{21}(\boldsymbol{\theta}^0) & \mathbf{B}_{22}(\boldsymbol{\theta}^0) \end{bmatrix}$$

we have that

$$T \mathbf{s}_2(\widehat{\boldsymbol{\theta}}_{1T}, \widehat{\boldsymbol{\theta}}_{2T}, \mathbf{0})' (\widehat{\mathbf{B}}_{22}(\widehat{\boldsymbol{\theta}}_T) - \widehat{\mathbf{B}}_{12}(\widehat{\boldsymbol{\theta}}_T) \widehat{\mathbf{B}}_{11}^{-1}(\widehat{\boldsymbol{\theta}}_T) \widehat{\mathbf{B}}_{21}(\widehat{\boldsymbol{\theta}}_T))^{-1} \mathbf{s}_2(\widehat{\boldsymbol{\theta}}_{1T}, \widehat{\boldsymbol{\theta}}_{2T}, \mathbf{0}) \xrightarrow{D} \chi^2(m)$$

when $\boldsymbol{\theta}_3 = \mathbf{0}$, where $\widehat{\mathbf{B}}_{ij}(\widehat{\boldsymbol{\theta}}_T)$ is a consistent plug-in estimator of $\mathbf{B}_{ij}(\boldsymbol{\theta}^0)$, $i, j = 1, 2$, and m is the dimension of $\boldsymbol{\theta}_3$. Set $\widehat{\mathbf{x}}_{jt} = \widehat{h}_t^{-1}(\partial h_t / \partial \boldsymbol{\theta}_j) \Big|_{H_0}$, $j = 1, 2, 3$, $\widehat{\mathbf{u}}_{1t} = \widehat{g}_t^{-1}(\partial g_t / \partial \boldsymbol{\theta}_1) \Big|_{H_0}$ and $\widehat{\mathbf{v}}_{3t} =$

$\widehat{g}_t^{-1}(\partial f_t / \partial \boldsymbol{\theta}_3)|_{H_0}$. Then, assuming ζ_t is normal,

$$\widehat{\mathbf{B}}_{ij}(\widehat{\boldsymbol{\theta}}_T) = \frac{1}{2T} \sum_{t=1}^T \widehat{\mathbf{r}}_{it} \widehat{\mathbf{r}}'_{jt}, \quad i, j = 1, 2 \quad (12)$$

is a consistent estimator of $\mathbf{B}_{ij}(\boldsymbol{\theta}^0)$ under H_0 , where $\widehat{\mathbf{r}}_{1t} = (\widehat{\mathbf{u}}'_{1t} + \widehat{\mathbf{x}}'_{1t}, \widehat{\mathbf{x}}'_{2t})'$ and $\widehat{\mathbf{r}}_{2t} = \widehat{\mathbf{x}}_{3t} + \widehat{\mathbf{v}}_{3t}$.

The test can be carried out in stages:

1. Estimate the TVGJR-GARCH model, save the standardised residuals $\widehat{\zeta}_t$ and construct the 'residual sum of squares' $SSR_0 = \sum_{t=1}^T (\widehat{\zeta}_t^2 - 1)^2$.
2. Regress $\widehat{\zeta}_t^2 - 1$ on $\widehat{\mathbf{r}}_{1t}$ and $\widehat{\mathbf{r}}_{2t}$ and form the residual sum of squares SSR_1 .
3. Compute the test statistic

$$LM_{\text{add-g}} = T \frac{SSR_0 - SSR_1}{SSR_0}. \quad (13)$$

Under regularity conditions, assuming that ζ_t is standard normal and that AG1-2 hold, $\sqrt{T}(\widehat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}^0) \xrightarrow{D} \mathcal{N}(\mathbf{0}, \mathbf{B}(\boldsymbol{\theta}^0))$, see Amado and Teräsvirta (2013). This implies that the statistic (13) has an asymptotic χ^2 -distribution with $\dim(\boldsymbol{\theta}_3)$ degrees of freedom when the null hypothesis is valid.

In testing r transitions against $r + 1$, which is the leading Family 1 testing situation, the aforementioned identification problem is present. Let $\boldsymbol{\theta}_3 = (\boldsymbol{\theta}'_{31}, \boldsymbol{\theta}'_{32})'$ where $\boldsymbol{\theta}_{31} = \gamma_{r+1}$ and $\boldsymbol{\theta}_{32} = (\delta_{r+1}, \mathbf{c}'_{r+1})'$. When $H_0: \gamma_{r+1} = 0$ holds,

$$f_t(\boldsymbol{\theta}_{31}, \boldsymbol{\theta}_{32}) = \delta_{r+1} \{G_{r+1}(t^*, \gamma_{r+1}, \mathbf{c}_{r+1}) - 1/2\} = 0$$

(subtracting $1/2$ from $G_{r+1}(t^*, \gamma_{r+1}, \mathbf{c}_{r+1})$ does not alter the argument), in which case $\boldsymbol{\theta}_{32} = (\delta_{r+1}, \mathbf{c}'_{r+1})'$ is a vector of unidentified nuisance parameters. We circumvent this identification problem by using the third-order Taylor expansion of $G_{r+1}(t^*, \gamma_{r+1}, \mathbf{c}_{r+1})$ around the null hypothesis as in Luukkonen, Saikkonen and Teräsvirta (1988), see also Lundbergh and Teräsvirta (2002). After reparameterisation,

$$f_t(\boldsymbol{\theta}_{31}, \boldsymbol{\theta}_{32}) = f_t^*(\boldsymbol{\delta}^*) = \delta_0^* + \delta_1^* t^* + \delta_2^* t^{*2} + \delta_3^* t^{*3} + R_{3t} = \boldsymbol{\delta}^{*'} \mathbf{t}^* + R_{3t} \quad (14)$$

where $\boldsymbol{\delta}^* = (\delta_0^*, \delta_1^*, \delta_2^*, \delta_3^*)' = (\gamma_{r+1}\bar{\delta}_0, \gamma_{r+1}\bar{\delta}_1, \gamma_{r+1}^2\bar{\delta}_2, \gamma_{r+1}^3\bar{\delta}_3)'$, $\bar{\delta}_i \neq 0, i = 0, 1, 2, 3$; $\mathbf{t}^* = (1, t^*, t^{*2}, t^{*3})'$, and R_{3t} is the remainder. Note that $\delta_0^* = 0$ only if $\mathbf{c}_{r+1} = \mathbf{0}$. The definition of $\boldsymbol{\delta}^*$ implies that the new null hypothesis is $H'_0: \boldsymbol{\delta}^* = \mathbf{0}$. Under H'_0 , $R_{3t} = 0$, and when we construct an LM-type test for this hypothesis, the remainder thus does not affect the asymptotic null distribution of the test statistic.

The approximation (14) to $f_t(\boldsymbol{\theta}_{31}, \boldsymbol{\theta}_{32})$ causes a problem, however. Writing (9) using it yields

$$\frac{\partial h_t}{\partial \boldsymbol{\delta}^*}|_{H_0} = -\{\hat{\alpha}_1 + \hat{\gamma}_1 I(\varepsilon_{t-1} < 0)\} \frac{\hat{\phi}_{t-1}^2}{\hat{g}_{t-1}} \frac{\partial f_{t-1}^*}{\partial \boldsymbol{\delta}^*}|_{H_0} + \hat{\beta}_1 \frac{\partial h_{t-1}}{\partial \boldsymbol{\delta}^*}|_{H_0}. \quad (15)$$

The expression (15) cannot be used to generate estimates for the unobserved derivatives $\partial h_t / \partial \boldsymbol{\delta}^*|_{H_0}$, because $f_{t-1}^*(\boldsymbol{\delta}^*)$ does not represent the data-generating process but is only an approximation. Our solution is to ignore this component. Apart from this, the previous considerations are valid after substituting $f_t^*(\boldsymbol{\delta}^*)$ for $f_t(\boldsymbol{\theta}_{31}, \boldsymbol{\theta}_{32})$, so $\hat{\mathbf{v}}_{3t} = \hat{g}_t^{-1}(\partial f_t^* / \partial \boldsymbol{\delta}^*)|_{H_0} = \hat{g}_t^{-1} \mathbf{t}^*$, and $\hat{\mathbf{r}}_{2t} = \hat{\mathbf{v}}_{3t}$ in (12). Since $f_t^*(\boldsymbol{\delta}^*)$ is linear in parameters, the resulting test statistic has an asymptotic χ^2 -distribution with $\dim(\boldsymbol{\delta}^*)$ degrees of freedom when the null hypothesis holds.

Testing the first-order GJR-GARCH model against its TV-GARCH counterpart with a single transition constitutes an important special case. Then, as $g_t(\boldsymbol{\theta}_1) \equiv 1$, the log-likelihood function (7) simplifies to

$$\ell(\boldsymbol{\theta}, \varepsilon_t) = k - (1/2) \left\{ \ln h_t(\boldsymbol{\theta}_2, \boldsymbol{\theta}_3) + \ln f_t(\boldsymbol{\theta}_3) + \frac{\varepsilon_t^2}{h_t(\boldsymbol{\theta}_2, \boldsymbol{\theta}_3) f_t(\boldsymbol{\theta}_3)} \right\}$$

and the average score vector (8) becomes

$$\mathbf{s}(\boldsymbol{\theta}, \varepsilon_t) = (\mathbf{s}_{12}(\boldsymbol{\theta}, \varepsilon_t)', \mathbf{s}_2(\boldsymbol{\theta}, \varepsilon_t)')'$$

where $\mathbf{s}_{12}(\boldsymbol{\theta}, \varepsilon_t)$ is as before with

$$\frac{\partial h_t}{\partial \boldsymbol{\theta}_2} = \frac{\partial}{\partial \boldsymbol{\theta}_2} \left\{ \alpha_0 + \alpha_1 \frac{\varepsilon_{t-1}^2}{1 + f_{t-1}} + \gamma_1 \frac{\varepsilon_{t-1}^2 I(\varepsilon_{t-1} < 0)}{1 + f_{t-1}} + \beta_1 h_{t-1} \right\}$$

so that

$$\frac{\partial h_t}{\partial \boldsymbol{\theta}_2}|_{H_0} = (1, \varepsilon_{t-1}^2, I(\varepsilon_{t-1} < 0) \varepsilon_{t-1}^2, \hat{h}_{t-1}) + \hat{\beta}_1 \frac{\partial h_{t-1}}{\partial \boldsymbol{\theta}_2}|_{H_0}.$$

Furthermore,

$$\mathbf{s}_2(\boldsymbol{\theta}, \varepsilon) = (2T)^{-1} \sum_{t=1}^T (\zeta_t^2 - 1) \left(\frac{1}{1 + f_t} \frac{\partial f_t}{\partial \boldsymbol{\theta}_3} + \frac{1}{h_t} \frac{\partial h_t}{\partial \boldsymbol{\theta}_3} \right).$$

In testing for a single transition, f_t is replaced by f_t^* , so

$$\frac{1}{1 + f_t^*} \frac{\partial f_t^*}{\partial \boldsymbol{\delta}^*} \Big|_{H_0} = \mathbf{t}^*.$$

Even here, $\partial h_t / \partial \boldsymbol{\delta}^* \Big|_{H_0}$ is ignored, so $\hat{\mathbf{r}}_{1t} = \hat{\mathbf{x}}_{2t}$ and $\hat{\mathbf{r}}_{2t} = \hat{\mathbf{v}}_{3t} = \mathbf{t}^*$.

An interesting special case of this test is obtained by assuming $h_t = \alpha_0 > 0$. It may be relevant when the error variance of a macroeconometric model based on, say, quarterly observations, can be time-varying but without volatility clustering. The period of Great Moderation serves as a recent example. Note, however, that in this case the conditional mean is being modelled as well, whereas it was assumed away in Section 2.

When the errors are not normal, the LM-type test can be robustified to fit this situation. This can be done following Wooldridge (1991, Procedure 4.1), see also Lundbergh and Teräsvirta (2002). The test statistic can be computed as follows:

1. Estimate the TVGJR-GARCH model and save the standardised residuals $\hat{\zeta}_t$.
2. Regress $\hat{\mathbf{r}}_{2t}$ on $\hat{\mathbf{r}}_{1t}$ and compute the residual vectors $\hat{\mathbf{w}}_t$, $t = 1, \dots, T$.
3. Regress 1 on $(\hat{\zeta}_t^2 - 1)\hat{\mathbf{w}}_t$, compute the residual sum of squares SSR and test statistic

$$LM_R = T - SSR. \tag{16}$$

The statistic (16) has an asymptotic χ^2 -distribution with $\dim(\boldsymbol{\delta}^*)$ degrees of freedom when the null hypothesis is valid. When the errors are normal, (16) is asymptotically efficient, see Wooldridge (1991). This suggests that one should in practice always use the robustified statistic.

The simulations in Section 5 indicate that the empirical size of the test of zero transitions against one (linearity test) is close to the nominal one at all sample sizes simulated, that is, from $T = 1000$ upwards. In testing r transitions against $r + 1$ ones there are power issues that deserve attention. First, estimation of the parameters of the null model may affect the power of the test. As an example, consider testing one transition against two. In some cases the maximum value of

the slope parameter γ_1 in (5) has to be restricted from above for numerical reasons. Setting this maximum value too low may have an adverse effect of the power of the test. Even if the number of iterations in estimating the model grows with increases in this limit, tolerating this increase may be preferable to stopping the estimation algorithm too early. Second, the test is likely to have weak power in situations where a transition occurs at the very end of the time series. It is difficult to recognise a shift without sufficient evidence of it. This is not a problem of the test *per se* but rather a data problem. It also occurs for instance when it comes to finding breaks in a series.

Finally, one may suspect that the performance of the specification strategy based on sequential testing deteriorates when the number of transitions becomes very large. This can easily be the case when the series to be modelled is very long. A solution to this problem consists of dividing the original series into subseries that are expected to contain a relatively small number of transitions, at most two or three, say. The specification strategy outlined in Section 3 is then applied to each subsection separately, after which the results are merged and the model for the whole series estimated. This technique has recently been considered in Amado and Teräsvirta (2014) who apply it to modelling about 23000 daily returns of the Dow Jones Industrial Average index from 1920 to 2011.

4.3 Testing the conditional variance component

In Family 2, the log-likelihood for observation t has the form

$$\ell(\boldsymbol{\theta}, \varepsilon_t) = k - (1/2)[\ln\{h_t(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) + f_t(\boldsymbol{\theta}_3)\} + \ln g_t(\boldsymbol{\theta}_1) + \frac{\varepsilon_t^2}{\{h_t(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) + f_t(\boldsymbol{\theta}_3)\}g_t(\boldsymbol{\theta}_1)}]$$

and the components of the score are

$$\begin{aligned} \mathbf{s}_{11}(\boldsymbol{\theta}, \varepsilon) &= (2T)^{-1} \sum_{t=1}^T \frac{\partial \ell(\boldsymbol{\theta}, \varepsilon_t)}{\partial \boldsymbol{\theta}_1} = (2T)^{-1} \sum_{t=1}^T (\zeta_t^2 - 1) \left(\frac{1}{g_t} \frac{\partial g_t}{\partial \boldsymbol{\theta}_1} + \frac{1}{(h_t + f_t)} \frac{\partial h_t}{\partial \boldsymbol{\theta}_1} \right) \\ \mathbf{s}_{12}(\boldsymbol{\theta}, \varepsilon) &= (2T)^{-1} \sum_{t=1}^T \frac{\partial \ell(\boldsymbol{\theta}, \varepsilon_t)}{\partial \boldsymbol{\theta}_2} = (2T)^{-1} \sum_{t=1}^T (\zeta_t^2 - 1) \frac{1}{h_t + f_t} \frac{\partial h_t}{\partial \boldsymbol{\theta}_2} \end{aligned}$$

and

$$\mathbf{s}_2(\boldsymbol{\theta}, \varepsilon) = (2T)^{-1} \sum_{t=1}^T \frac{\partial \ell(\boldsymbol{\theta}, \varepsilon_t)}{\partial \boldsymbol{\theta}_3} = (2T)^{-1} \sum_{t=1}^T (\zeta_t^2 - 1) \frac{1}{(h_t + f_t)} \frac{\partial f_t}{\partial \boldsymbol{\theta}_3}$$

where

$$\begin{aligned}
\frac{\partial h_t}{\partial \boldsymbol{\theta}_1} &= \frac{\partial}{\partial \boldsymbol{\theta}_1} \left\{ \alpha_0 + \alpha_1 \frac{\varepsilon_{t-1}^2}{g_{t-1}} + \gamma_1 \frac{\varepsilon_{t-1}^2 I(\varepsilon_{t-1} < 0)}{g_{t-1}} + \beta_1 h_{t-1} \right\} \\
&= - \left\{ \alpha_1 \frac{\varepsilon_{t-1}^2}{g_{t-1}^2} + \gamma_1 \frac{\varepsilon_{t-1}^2 I(\varepsilon_{t-1} < 0)}{g_{t-1}^2} \right\} \frac{\partial g_{t-1}}{\partial \boldsymbol{\theta}_1} + \beta_1 \frac{\partial h_{t-1}}{\partial \boldsymbol{\theta}_1} \\
\frac{\partial h_t}{\partial \boldsymbol{\theta}_2} &= \left(1, \frac{\varepsilon_{t-1}^2}{g_{t-1}}, \frac{\varepsilon_{t-1}^2 I(\varepsilon_{t-1} < 0)}{g_{t-1}}, h_{t-1} \right)' + \beta_1 \frac{\partial h_{t-1}}{\partial \boldsymbol{\theta}_2}.
\end{aligned}$$

Analogously, $\zeta_t^2 = \varepsilon_t^2 / \{(h_t + f_t)g_t\}$. The partial derivatives, evaluated at H_0 , are

$$\begin{aligned}
\frac{\partial h_t}{\partial \boldsymbol{\theta}_1}|_{H_0} &= -(\hat{\alpha}_1 \hat{\phi}_t^2 + \hat{\gamma}_1 \hat{\phi}_t^2 I(\varepsilon_{t-1} < 0)) \frac{\partial g_{t-1}}{\partial \boldsymbol{\theta}_1}|_{H_0} + \hat{\beta}_1 \frac{\partial h_{t-1}}{\partial \boldsymbol{\theta}_1}|_{H_0} \\
\frac{\partial h_t}{\partial \boldsymbol{\theta}_2}|_{H_0} &= (1, \hat{\phi}_{t-1}^2, \hat{\phi}_{t-1}^2 I(\varepsilon_{t-1} < 0), \hat{h}_{t-1})' + \hat{\beta}_1 \frac{\partial h_{t-1}}{\partial \boldsymbol{\theta}_2}|_{H_0}
\end{aligned}$$

whereas $\partial f_t / \partial \boldsymbol{\theta}_3$ depends on the alternative hypothesis. For example, if the alternative hypothesis is that $p = 2$ in the GJR-GARCH model, then $f_t = \beta_2 h_{t-2}$, so $\boldsymbol{\theta}_3 = \beta_2$. The blocks $\mathbf{s}_{11}(\hat{\boldsymbol{\theta}}_{1T}, \hat{\boldsymbol{\theta}}_{2T}, \mathbf{0})$ and $\mathbf{s}_{12}(\hat{\boldsymbol{\theta}}_{1T}, \hat{\boldsymbol{\theta}}_{2T}, \mathbf{0})$ of the average score are the same as (10) and (11), whereas

$$\mathbf{s}_2(\hat{\boldsymbol{\theta}}_{1T}, \hat{\boldsymbol{\theta}}_{2T}, \mathbf{0}) = (2T)^{-1} \sum_{t=1}^T (\hat{\zeta}_t^2 - 1) \frac{1}{\hat{h}_t} \frac{\partial f_t}{\partial \boldsymbol{\theta}_3}|_{H_0}.$$

The test is carried out in stages as in the previous case, the only difference being that $\hat{\mathbf{r}}_{2t} = \hat{h}_t^{-1}(\partial f_t / \partial \boldsymbol{\theta}_3)|_{H_0}$. To continue the example, if $f_t = \beta_2 h_{t-2}$, then $(\partial f_t / \partial \boldsymbol{\theta}_3)|_{H_0} = \hat{h}_{t-2}$. When $g_t \equiv 1$, $\mathbf{s}_{11}(\boldsymbol{\theta}, \boldsymbol{\varepsilon}) = \mathbf{0}$, and the test collapses into the corresponding misspecification test of the (GJR-) GARCH model in Lundbergh and Teräsvirta (2002). Note that the alternative model cannot have both $p \geq 2$ and $q \geq 2$, because in that case the null model with $p = q = 1$ is not identified when the alternative holds, and the standard asymptotic inference is no longer valid.

When the alternative to the TV-GARCH model is the Smooth Transition GARCH model of Hagerud (1997) and González-Rivera (1998),

$$f_t(\boldsymbol{\theta}_1, \boldsymbol{\theta}_{31}, \boldsymbol{\theta}_{32}) = (\alpha_{01} + \alpha_{11} \phi_{t-1}^2) \{G(\phi_{t-1}; \gamma, \mathbf{c}) - 1/2\}$$

where $\phi_t = \varepsilon_t / g_t^{1/2}$, $G(\phi_{t-1}; \gamma, \mathbf{c})$ is defined as in (5) and satisfies AG2 for γ and $\mathbf{c}$. Furthermore, to retain positivity of h_t under the alternative, $\alpha_i + \alpha_{i1} > 0$, $i = 0, 1$. Under H_0 : $\boldsymbol{\theta}_{31} = \gamma = 0$,

the nuisance parameter vector is $\boldsymbol{\theta}_{32} = (\alpha_{01}, \alpha_{11}, \mathbf{c}')'$. Assuming $K = 1$ in (5), approximating the transition function by the first-order Taylor expansion and merging terms gives

$$f_t^*(\boldsymbol{\theta}_1, \boldsymbol{\alpha}^*) = \alpha_1^* \phi_{t-1} + \alpha_3^* \phi_{t-1}^3$$

with $\boldsymbol{\alpha}^* = (\alpha_1^*, \alpha_3^*)'$, and the new null hypothesis becomes $H'_0: \alpha_1^* = \alpha_3^* = 0$. For the LM-type test statistic to have an asymptotic χ^2 -distribution, the moment conditions have to be strengthened (if normality is not assumed) by assuming $E\phi_t^6 < \infty$, which requires $E\zeta_t^6 < \infty$. The resulting test can also be applied to the TVGJR-GARCH model, for example to test the adequacy of the assumption that there is a threshold at $\phi_{t-1}^* = 0$ in the model.

4.4 Testing the hypothesis of no remaining ARCH

As for Family 3, the components of the average score are

$$\begin{aligned} \mathbf{s}_{11}(\boldsymbol{\theta}, \boldsymbol{\varepsilon}) &= (2T)^{-1} \sum_{t=1}^T \frac{\partial \ell(\boldsymbol{\theta}, \boldsymbol{\varepsilon}_t)}{\partial \boldsymbol{\theta}_1} = (2T)^{-1} \sum_{t=1}^T (z_t^2 - 1) \left(\frac{1}{g_t} \frac{\partial g_t}{\partial \boldsymbol{\theta}_1} + \frac{1}{h_t} \frac{\partial h_t}{\partial \boldsymbol{\theta}_1} \right) \\ \mathbf{s}_{12}(\boldsymbol{\theta}, \boldsymbol{\varepsilon}) &= (2T)^{-1} \sum_{t=1}^T \frac{\partial \ell(\boldsymbol{\theta}, \boldsymbol{\varepsilon}_t)}{\partial \boldsymbol{\theta}_2} = (2T)^{-1} \sum_{t=1}^T (z_t^2 - 1) \frac{1}{h_t} \frac{\partial h_t}{\partial \boldsymbol{\theta}_2} \end{aligned}$$

and

$$\mathbf{s}_2(\boldsymbol{\theta}, \boldsymbol{\varepsilon}) = (2T)^{-1} \sum_{t=1}^T \frac{\partial \ell(\boldsymbol{\theta}, \boldsymbol{\varepsilon}_t)}{\partial \boldsymbol{\theta}_3} = (2T)^{-1} \sum_{t=1}^T (z_t^2 - 1) \frac{1}{f_t} \frac{\partial f_t}{\partial \boldsymbol{\theta}_3}$$

where $z_t^2 = \varepsilon_t^2 / (h_t f_t g_t)$. Evaluated under H_0 , $\mathbf{s}_{11}(\boldsymbol{\theta}, \boldsymbol{\varepsilon})$ and $\mathbf{s}_{12}(\boldsymbol{\theta}, \boldsymbol{\varepsilon})$ again equal (10) and (11), whereas

$$\mathbf{s}_2(\widehat{\boldsymbol{\theta}}_{1T}, \widehat{\boldsymbol{\theta}}_{2T}, \mathbf{0}) = (2T)^{-1} \sum_{t=1}^T (\widehat{\zeta}_t^2 - 1) \frac{\partial f_t}{\partial \boldsymbol{\theta}_3} \Big|_{H_0}$$

since $z_t \equiv \zeta_t$ and $f_t \equiv 1$ under H_0 . The test can be carried out in stages as before, and now $\widehat{\mathbf{r}}_{2t} = (\partial f_t / \partial \boldsymbol{\theta}_3) \Big|_{H_0}$. When the alternative is 'ARCH in TVGJR-GARCH', mentioned in Section 4.1, $\widehat{\mathbf{r}}_{2t} = (\widehat{\zeta}_{t-1}^2, \dots, \widehat{\zeta}_{t-m}^2)'$ and under H_0 , the LM-statistic has an asymptotic χ^2 -distribution with m degrees of freedom. Even here, $g_t \equiv 1$ implies $\mathbf{s}_{11}(\boldsymbol{\theta}, \boldsymbol{\varepsilon}) = \mathbf{0}$, in which case the test collapses into the 'ARCH in GJR-GARCH' test in Lundbergh and Teräsvirta (2002), which is asymptotically equivalent to the test of Li and Mak (1994). If, in addition, $h_t \equiv \alpha_0$, we have the test of no ARCH of Engle (1982), which in turn is asymptotically equivalent to the portmanteau test of McLeod

and Li (1983).

5 Simulation study

5.1 Design of the study

In this section, we conduct a small simulation experiment to evaluate the finite-sample properties of the test against a multiplicative TV-GARCH specification. Specifically, we shall investigate the size and power properties of the LM-type test involved in the modelling strategy as well as the success rate of the specification procedures. Sample lengths of 1000, 2500 and 5000 observations are used in all simulations. To avoid the initialization effects, the first 1000 observations are discarded before generating the actual series. All the computations have been carried out using Ox, version 6.00; see Doornik (2009). The behaviour of both the non-robust and the robust test statistics is examined for several data generating processes (DGPs) that can be nested in the following TV-GARCH specification:

$$y_t = \varphi_0 + \varphi_1 y_{t-1} + \varepsilon_t, \quad \varepsilon_t = \zeta_t (h_t g_t)^{1/2}, \quad \{\zeta_t\} \sim \text{iid} \mathcal{N}(0, 1) \quad (17)$$

where

$$h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^{*2} + \beta_1 h_{t-1} + \lambda_1 I(\varepsilon_{t-1}^* < 0) \varepsilon_{t-1}^{*2}, \quad \varepsilon_t^* = \varepsilon_t / g_t^{1/2}$$

and

$$g_t = 1 + \delta_1 G_1(t^*; \gamma_1, c_1)$$

with

$$G_1(t^*; \gamma_1, c_1) = (1 + \exp(-\gamma_1(t^* - c_1)))^{-1}.$$

The data generating processes are as follows:

$$\text{DGP1: } \varphi_0 = \varphi_1 = 0, \quad h_t = 0.10 + 0.10 \varepsilon_{t-1}^2 + 0.85 h_{t-1}, \quad g_t \equiv 1.$$

$$\text{DGP2: } \varphi_0 = \varphi_1 = 0, \quad h_t = 0.05 + 0.05 \varepsilon_{t-1}^2 + 0.90 h_{t-1}, \quad g_t \equiv 1$$

$$\text{DGP3: } h_t = 0.005 + 0.05 \varepsilon_{t-1}^2 + 0.80 h_{t-1} + 0.10 I(\varepsilon_{t-1} < 0) \varepsilon_{t-1}^2, \quad g_t \equiv 1$$

DGP4: $\varphi_0 = 0.05, \varphi_1 = 0.10,$

$$h_t = 0.05 + 0.05\varepsilon_{t-1}^2 + 0.90h_{t-1}, \quad g_t \equiv 1$$

DGP5: $\varphi_0 = \varphi_1 = 0, h_t = 0.10 + 0.10\varepsilon_{t-1}^{*2} + 0.80h_{t-1},$

$$\delta_1 = \{-0.05, 0.05\}, \quad \gamma_1 = \{5, 10\}, \quad c_1 = 0.5$$

DGP6 – DGP7: $\varphi_0 = \varphi_1 = 0, h_t = 0.10 + 0.10\varepsilon_{t-1}^{*2} + 0.80h_{t-1},$

$$\delta_1 = \{-0.75, 1.5\}, \quad \gamma_1 = 5, \quad c_1 = 0.5$$

The designs DGP5–DGP7 concern testing for the multiplicative TV-GARCH model, where the midpoint of the change in g_t occurs at $c_1 = 0.5$, whereas the slope parameter γ_1 varies in the interval $\gamma_1 = \{5, 10\}$ for DGP5. For all the DGPs, except DGP4, we assume that the conditional mean of the process equals zero. Following the suggestion in Bollerslev (1986), recursive computation of h_t is initialized by using the estimated unconditional variance for the pre-sample values $t \leq 0$.

5.2 Results of the study

In this section we shall first report results on the size and power properties of the non-robust and the robust parameter constancy tests. Then we turn to the results on the performance of our misspecification tests.

5.2.1 Size and power simulations

Results of the size simulations are presented graphically in Figure 1. The graphs show for each test the discrepancies in size (the actual minus the nominal size) plotted against the nominal sizes from 0.1%, 0.3%, 0.5%, ..., 10%. For each DGP, the total of number of replications equals 5000. In each subgraph we present the size discrepancies for DGP1–DGP4 and the three sample sizes. In the left panels, one can observe that the non-robust parameter constancy test is severely size-distorted at all sample sizes. Furthermore, we also investigate the size results of the test statistics when the errors are not identically distributed. Results on the robust version of the parameter constancy test are presented in the right panel of Figure 1. As can be seen, the robust test is reasonably well-sized for all sample sizes. The only exception is a slight positive size-distortion at the sample size $T = 1000$ for DGP3, but the empirical size becomes more accurate as the sample size increases. The robust test is also well-sized for the three sample sizes when the conditional mean is misspecified. Our

main conclusion is that for $T \geq 2500$, the robust version of our test statistic has a null distribution that is a rather good approximation to the finite-sample distributions.

Although there exist several parameter constancy tests in the GARCH literature, none of them can be considered a direct benchmark for our constant unconditional variance test. In what follows, we only show results of the power simulations for the robust version of the test. Rejection frequencies for the robust LM-type test against a multiplicative alternative are shown in Figure 2. On each graph, the actual rejection frequencies are plotted against the nominal significance levels 0.1%, 0.3%, 0.5%, ..., 10%. Instead of the size-adjusted power curves suggested by Davidson and MacKinnon (1998), we simply report power curves as the robust tests have good size properties. The power results have been obtained by generating artificial data from DGP5. As before, we consider sample sizes of 1000, 2500 and 5000 observations. The rejection frequencies of the LM test statistics shown in the left panel are moderate when $T = 1000$ and increase with the sample size. As expected, the rejection frequencies are an increasing function of the sample size and of the slope parameter γ_1 . It is interesting to note that the robust LM-type test statistic turns out to be quite powerful even for small values of the parameter δ_1 and short time series.

5.2.2 Simulating the misspecification tests

In this section we consider the performance of the misspecification tests for the multiplicative TV-GARCH models discussed in Section 4. This is done by studying the size properties of the three types of misspecification tests. As before, the first 1000 observations of each generated series are discarded to minimise the initialization effects. All experiments have been performed with a total number of replications equal to 2000. The results concern the case where the simulation design follows a TV-GARCH model with one transition and where the centre of the change is located halfway through the sample.

First, we consider the case of testing the deterministic component g_t . This test can also be seen as a specification test for the unconditional variance component. The size results of the non-robust and robust versions of the test of a single transition against a double transition for designs DGP6–DGP7 are reported in Table 1. Because the simulation results are highly sensitive to the estimates of the slope parameter γ_1 , we estimate the model by fixing the parameters of the unconditional component to their true values. This helps us to focus on the properties of the test.

Under non-normal errors, the size distortion in the non-robustified version of the test is rather severe for the three sample sizes. On the other hand, the robust tests are well-sized for samples larger than 2500, but positively size-distorted for $T = 1000$.

Results of the misspecification tests for the conditional variance component are reported in Table 2. For each experiment, we evaluate the size properties using three different tests: the tests of GARCH(1,1) against GARCH(1,2) and GARCH(2,1) models, and the linearity test of no smooth transition GARCH(1,1) model of order 1. As before, we compute both the ordinary and the robust versions for each test. The most striking result is that the non-robust tests are strongly oversized at all nominal levels and sample sizes. The robust tests have an empirical size close to the nominal size for most cases. In some occasions, however, the robust tests against higher-order GARCH models may be conservative when the nominal size is set at 5% and 10%.

Finally, the results of the test of no remaining ARCH are reported in Table 3. For each design, the robustified version of the tests correct the strong size distortions of the ordinary tests. The robust test seems, however, to be undersized at the 5% and 10% nominal level when the sample size equals 5000. Overall, the main conclusion is that in applied work one should apply the robustified version of the three types of misspecification tests.

6 Applications

In this section we shall consider two empirical examples, each of them involving a financial time series. The series in the first example is the spot exchange rate of the Singapore dollar versus the U.S. dollar (SPD/USD), while the second application concerns the futures price of coffee (Coffee). Both series are observed at a daily frequency and transformed into continuously compounded rates of return.

6.1 Exchange rate returns

The time series of this section consists of daily returns of the spot SPD/USD exchange rate provided by the Federal Reserve Bank of New York. A graph of the series is shown in Figure 3. It covers the period from May 1, 1997 until July 11, 2005, yielding a total of 2060 observations. At first sight it appears that one can distinguish two different regimes in the series. A period of high volatility

occurs during the East Asian financial crisis due to the significant depreciation of the Singapore dollar relative to the U.S. dollar. After the crisis, the volatility of the currency returns descends to a low level.

Descriptive statistics for the SPD/USD exchange rate returns are reported in the first row of Table 4. There is plenty of excess kurtosis, and the estimated skewness is strongly negative. These values are mainly due to a couple of large negative returns early in the series. Indeed, the robust measure of skewness suggests that the marginal distribution of the returns is not skewed. The robust centred kurtosis is substantially smaller than the standard measure, but there is excess kurtosis as may be expected. For robust measures, see Kim and White (2004) and Teräsvirta and Zhao (2011). The estimate of the first-order autocorrelation for the returns equals -0.048 . An AR(1) model was fitted to the series to account for serial correlation. The GARCH(1,1) model fitted to the residual series shows high persistence of volatility. The misspecification tests in Table 7 that does not contain the test of constant unconditional variance, do not generally indicate misspecification. The 'no ARCH in GARCH' test when $r = 5$ is the only exception. However, the p -value 0.022 is not particularly small, given that the number of observations exceeds 2000.

The robust specification test of constant unconditional variance against a time-varying one in Table 6 has the p -value equal to 0.001. The short test sequence indicates that one should choose $K_1 = 2$, as H_{02} is rejected more strongly than the other hypotheses H_{01} and H_{03} . This may at first appear a bit surprising, given Figure 3. A TV-GARCH model with a single transition appears adequate in the sense that the test for another transition has $p = 0.78$. These values are from robust statistics.

The diagnostic tests of this model in Table 7 do not indicate misspecification. The robust linearity test against the smooth transition GARCH has a p -value slightly below 0.05 (the corresponding value for the non-robust statistic is 0.048), but given the large number of observations that is not strong evidence against linearity. The model is thus accepted as our final model for the SPD/USD daily return series.

Turning to the properties of the estimated model, it is seen from Table 4 that the excess kurtosis of the rescaled residuals $\{\hat{\varepsilon}_t/\hat{g}_t^{1/2}\}$ is substantially smaller than that of $\{\hat{\varepsilon}_t\}$. Furthermore, negative skewness has been reduced from -0.91 to less than -0.17 . This is a consequence of the fact that the original skewness estimate was strongly affected by a couple of very large negative returns

during the Asian crisis, and rescaling has reduced their relative magnitude. As may be expected, the robust skewness estimate is not affected by rescaling.

The estimated g_t has the following form:

$$\hat{g}_t = \{3.958 - 3.4905 G_1(t^*; \hat{\gamma}_1, \hat{\mathbf{c}}_1)\}_{(0.0161)}, \quad (18)$$

where

$$G_1(t^*; \hat{\gamma}_1, \hat{\mathbf{c}}_1) = (1 + \exp\{-1000(t^* - 0.0287)_{(-)}(t^* - 0.1891)_{(0.0021)}\})^{-1}_{(0.0017)} \quad (19)$$

Since the sequence of estimates of the slope parameter γ converges towards a very large value, γ has been fixed to 1000. This is an adequate approximation since the estimates for the remaining parameters do not change beyond $\gamma = 1000$ and numerical problems are avoided when calculating standard errors. The standard errors of the parameter estimates are calculated using numerical second derivatives conditionally on $\gamma = 1000$. The graph of the transition function (18)-(19) can be found in Figure 4. It shows how the tranquil period just before the crisis in the cause for selecting $K_1 = 2$, which then gives $\hat{g}_t$ its hump shape. For comparison, Figure 5 shows the corresponding graph obtained from the fitted Spline-GARCH model of Engle and Rangel (2008). It is clearly smoother than the one in Figure 5 but has the same general shape.

The parameter estimates of the GARCH component of the TV-GARCH model appear in Table 4. While $\hat{\alpha}_1 + \hat{\beta}_1 = 0.994$ in the GARCH model, after introducing g_t the same sum equals 0.933, which suggests a remarkable drop in persistence. The characteristics of the model are further illustrated by two figures. Figure 6 shows how the autocorrelations of $|\hat{\varepsilon}_t|/\hat{g}_t^{1/2}$ in Panel (a) are considerably lower than those of $|\hat{\varepsilon}_t|$ in Panel (b). The first-order autocorrelation that equals 0.304 for $|\hat{\varepsilon}_t|$ is only equal to 0.079 for $|\hat{\varepsilon}_t|/\hat{g}_t^{1/2}$. The decay rate of the autocorrelations of $|\hat{\varepsilon}_t|/\hat{g}_t^{1/2}$ is quite rapid. The unconditional component clearly has a dominating influence on the variance, and the amount of conditional heteroskedasticity left to be modelled after the changing unconditional variance has been accounted for is relatively small in comparison.

The graph of the conditional standard deviation $h_t^{1/2}$ in Panel (a) of Figure 7 clearly shows a long period of high volatility that is more than a simple cluster. Panel (b) shows that in the final model this period is explained by the deterministic component g_t , and that the graph of $h_t^{1/2}$ does not show signs of nonstationarity. This is precisely what one would expect after noting the

GARCH parameter estimates in Table 5.

Figure 8 contains the estimated news impact curves, see Engle and Ng (1993), of the traditional GARCH(1,1) model and the ones of the TV-GARCH(1,1) model for three regimes. The news impact curve of the TVGJR-GARCH model is time-varying because it depends on g_{t-1} . The news impact curve of the GJR-GARCH model is time-invariant, whereas this is not the case for the TVGJR-GARCH model. Symmetry in the response of volatility to news is common to both models. But then, time-varying news impact curves distinguish between responses of volatility to news in calm and turbulent times. When there is plenty of turbulence, a piece of news of a given size may drown in the general uncertainty, while the same news can have a large impact during a tranquil period.

6.2 Coffee futures price returns

Our second financial time series consists of commodity futures price quotes for coffee traded at the Coffee, Sugar and Cocoa Exchange. To avoid price jumps at the expiration date of the contract, we use the continuous back-adjusted futures of closing prices of coffee with 3-month maturities calculated by Commodity Systems, Inc. (CSI) on February 17, 2012. The period extends from 3 January 2000 until 23 March 2011, consisting of 2801 observations. The return series is plotted in Figure 9. By looking at the graph one may distinguish two regimes in volatility. There is a period of relatively low volatility in the beginning of the sample but during the second half of the series volatility increases to a rather high level. Volatility clustering is observed throughout the period.

Summary statistics for the return series can be found in the third row of Table 4. Both robust and standard non-robust measures of skewness and kurtosis are provided. There is no evidence of skewness but, as may be expected, there is excess kurtosis in the series. The return series has very little serial correlation as the first-order autocorrelation coefficient is very close to zero. Nevertheless, we proceed as before and estimate an autoregressive process of order 1 for filtering out any linear dependence in the return series.

To account for asymmetric effects on volatility, we begin by fitting a GJR-GARCH(1,1) model to the residual series. The results can be found in the 6 and 7 columns of Table 5. The conditional variance is strongly persistent as $\hat{\alpha}_1 + \hat{\beta}_1 + \hat{\lambda}_1/2 = 0.996$. Results of misspecification tests of this model, other than tests of the deterministic component, can be found in Table 8. They do not

indicate any misspecification.

We now turn our attention to the specification tests of g_t . The test results can be found in Table 6. Outcomes for both non-robust and robust statistics are reported. The null hypothesis of constant unconditional variance is rejected very strongly as the p -value of the robust test equals 2×10^{-4} . The test sequence for specifying the structure of the deterministic function g_t points towards $K_1 = 1$, as H_{01} and H_{03} are strongly rejected, whereas H_{02} is not. Fitting the TVGJR-GARCH model with a single transition function and $K_1 = 1$ to the series and testing for another transition yields the p -value 0.011 for the robust statistic. We take this to signify that another transition is needed to adequately characterise the deterministic component of the TVGJR-GARCH(1,1) model. The test sequence points at $K_2 = 1$. The robust test of two transitions against (at least) three has a p -value 0.20, so two transitions suffice.

The estimated g_t with two transitions has the following form:

$$\hat{g}_t = \{0.5638 + \underset{(0.0598)}{0.7330}G_1(t^*; \hat{\gamma}_1, \hat{\mathbf{c}}_1) + \underset{(0.1611)}{1.6448}G_2(t^*; \hat{\gamma}_1, \hat{\mathbf{c}}_1)\} \quad (20)$$

where

$$G_1(t^*; \hat{\gamma}_1, \hat{\mathbf{c}}_1) = (1 + \exp\{-\underset{(-)}{1000}(t^* - \underset{(0.0007)}{0.3866})\})^{-1} \quad (21)$$

and

$$G_2(t^*; \hat{\gamma}_1, \hat{\mathbf{c}}_1) = (1 + \exp\{-\underset{(-)}{1000}(t^* - \underset{(0.0006)}{0.7257})\})^{-1}. \quad (22)$$

The graph of the deterministic component (20)-(22) is depicted in Figure 10. The two transitions are clearly visible and illustrate how volatility increases over time, first around 40% of the observations and then somewhat after two thirds of the sample period. The deterministic component from the estimated spline-GARCH(1,1) model in Figure 11 is quite different from the one in Figure 10. It is smooth, and the difference between the two graphs is largest at the end, where the spline function points straight up, whereas $\hat{g}_t$ stabilises to a certain level well before the end of the observation period.

The results of the misspecification tests of the estimated TVGJR-GARCH model in Table 8 do not indicate any model misspecification. This is not surprising as the GJR-GARCH model already passed these tests. Taken together, the results suggest that the main source of misspecification in

the GJR-GARCH model is lack of a proper deterministic component.

It is of interest even here to compare the TVGJR-GARCH model with the GJR-GARCH specification; see Table 5. The log-likelihood of the former model is considerably higher than that of the latter. The persistence of volatility has decreased remarkably: in the TVGJR-GARCH model equals $\hat{\alpha}_1 + \hat{\beta}_1 + \hat{\lambda}_1/2 = 0.940$. The differences between the two specifications are further illustrated by two figures. Figure 12 contains the autocorrelations of $|\hat{\varepsilon}_t|$ in Panel (a) and those of $|\hat{\varepsilon}_t|/\hat{g}_t^{1/2}$ in Panel (b). In Panel (a), the autocorrelations retain the 'long-memory property', a very slow decay as a function of the lag, which translates into the high persistence in the GJR-GARCH model. Panel (b) shows that the autocorrelations of $|\hat{\varepsilon}_t|/\hat{g}_t^{1/2}$ have decreased considerably, and only a few of them exceed two standard deviations of $|\hat{\varepsilon}_t|$ under the iid normality assumption, marked by the horizontal line in the figure.

The fourth row of Table 4 contains the skewness and kurtosis estimates for the standardised residuals. The non-robust kurtosis shows that kurtosis has increased from that of the raw residuals. This is due to the fact that rescaling has shrunk most returns towards zero, which affects the denominator of the kurtosis, but as a result some outliers now stick out, which in turn has an effect on the numerator. This explanation is supported by the robust centred kurtosis that suggests some excess kurtosis but less than the standard measure.

Figure 13 contains the estimated conditional standard deviations $h_t^{1/2}$ of $\{\hat{\varepsilon}_t\}$ for the GJR-GARCH(1,1) model and the ones of $\{\hat{\varepsilon}_t/\hat{g}_t^{1/2}\}$. For the GJR-GARCH model, see Panel (a), the graph looks rather 'nonstationary' in that there appears to be a nonlinear 'trend'. From the graph in Panel (b) it is seen that volatility (the conditional standard deviation of $\{\hat{\varepsilon}_t/\hat{g}_t^{1/2}\}$) is still changing over time, but the persistent changes have been absorbed by the deterministic component. As in the previous example, a major part of the variation in the variance of the daily coffee futures return series can be ascribed to the deterministic component g_t , and only a minor portion of the variation is left to be parameterised by the stationary GJR-GARCH component h_t . This component is still needed because of the presence of volatility clustering, but its contribution to the autocorrelation function of $|\hat{\varepsilon}_t|$ is much smaller than that of g_t .

The estimated news impact curve of the standard GJR-GARCH(1,1) model in Figure 14 is compared with corresponding curves of the TVGJR-GARCH(1,1) model. The effect of 'good news' (or positive shocks) on volatility is larger than that of 'bad news' (negative shocks). In fact,

negative shocks have almost no impact on volatility. As in the exchange rate example, the impact of news, in this case 'good news', depends on the degree of turbulence in the coffee futures market.

7 Concluding remarks

In this paper we develop a modelling strategy for building the TV-GARCH model of Amado and Teräsvirta (2013) based on statistical inference. Time-variation of the unconditional variance is incorporated in the model in a multiplicative form. The nonstationary component is described by a linear combination of smooth transition functions whereas the stationary component is of a GARCH-type model. The problem of selecting the appropriate number of transitions is solved by sequentially applying Lagrange multiplier tests. The test statistics are robustified against deviations from the iid assumption. The model is first specified, which includes determining the structure of the deterministic component by a test sequence and then it is estimated. Misspecification tests are derived for evaluating the estimated TV-GARCH model. Our simulation experiments suggest that robust versions of the tests have reasonable good properties already in samples of moderate size.

We put our TV-GARCH models to test by applying the modelling-cycle strategy to commodity price returns and exchange rate returns. We find that a constant unconditional variance against a time-varying one is strongly rejected for both return series. Fitting a traditional GARCH or GJR-GARCH model to these series yields results that are quite different from the ones obtained by our approach and point at the presence of long memory in volatility. Once the model accounts for the time-variation in the unconditional variance, the fit of the models substantially improves and evidence for long memory is considerably weakened in both applications.

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Appendix A

Lemma A.1. The function $\partial\ell(\boldsymbol{\theta}_1, \varepsilon_t)/\partial\boldsymbol{\theta}_1$ has the form

$$\begin{aligned}\partial\ell(\boldsymbol{\theta}_1, \varepsilon_t)/\partial\boldsymbol{\theta}_1 &= -(1/2)\frac{\partial}{\partial\boldsymbol{\theta}_1}\left\{\ln g_t(\boldsymbol{\theta}_1) + \frac{\varepsilon_t^2}{g_t(\boldsymbol{\theta}_1)}\right\} \\ &= (1/2)\left(\frac{\varepsilon_t^2}{g_t(\boldsymbol{\theta}_1)} - 1\right)\frac{1}{g_t(\boldsymbol{\theta}_1)}\frac{\partial g_t(\boldsymbol{\theta}_1)}{\partial\boldsymbol{\theta}_1}\end{aligned}$$

where

$$\frac{\partial g_t(\boldsymbol{\theta}_1)}{\partial\boldsymbol{\theta}_1} = \left(\frac{\partial g_t(\boldsymbol{\theta}_1)}{\partial\alpha_0} \frac{\partial g_t(\boldsymbol{\theta}_1)}{\partial\boldsymbol{\delta}^{*'}} \frac{\partial g_t(\boldsymbol{\theta}_1)}{\partial\boldsymbol{\gamma}'} \frac{\partial g_t(\boldsymbol{\theta}_1)}{\partial\boldsymbol{c}'}\right)'.$$

with

$$\partial g_t/\partial\alpha_0 = 1$$

$$\partial g_t/\partial\boldsymbol{\delta}^* = (G_{1t}, \dots, G_{rt})' \quad (23)$$

$$\partial g_t/\partial\boldsymbol{\gamma} = (g_{\gamma 1t}, \dots, g_{\gamma rt})' \quad (24)$$

$$\partial g_t/\partial\boldsymbol{c} = (g_{c1t}, \dots, g_{crt})'. \quad (25)$$

In (24),

$$g_{\gamma jt} = \delta_j^* G_{jt}(1 - G_{jt})(t^* - c_j)$$

and in (25),

$$g_{cjt} = -\gamma_j \delta_j^* G_{jt}(1 - G_{jt})$$

for $j = 1, \dots, r$.

Appendix B: Figures and Tables

Figure 1: Size discrepancy plots of the tests against a time-varying unconditional variance. The size discrepancies of the non-robust (left panel) and robust (right panel) LM tests are plotted against the nominal size. The lines in each plot correspond to sample sizes 1000 ($\times$), 2500 ($+$), and 5000 ($\diamond$). The total number of replications equals 5000.

Figure 2: Power curves of the robust test against a time-varying unconditional variance. The actual rejection frequency is plotted against the nominal size. The lines in each plot correspond to sample sizes 1000 ($\times$), 2500 (+), and 5000 ($\diamond$). The total number of replications equals 5000.

Figure 3: Daily returns of the Singapore Dollar versus US dollar exchange rate from May 1, 1997 until July 11, 2005 (2060 observations).

Figure 4: Graph of the final estimated function g_t for the SPD/USD returns model as a smooth function of the rescaled time variable t^* as given in (18)-(19).

Figure 5: Graph of the final estimated function g_t for the SPD/USD returns model as a smooth function of the rescaled time variable t^* as given by the Spline-GARCH model.

Figure 6: Sample autocorrelations of absolute log returns of the SPD/USD residuals and for the standardised variable $|\hat{\varepsilon}_t|/\hat{g}_{t_{SPD/USD}}^{1/2}$ with the 95% confidence bounds.

Figure 7: Conditional standard deviation of the GARCH(1,1) model for the SPD/USD residuals and for the standardised variable $\hat{\varepsilon}_t/\hat{g}_{t_{SPD/USD}}^{1/2}$.

Figure 8: News impact curves of the GARCH(1,1) (solid line in boldface) and the TV-GARCH(1,1) models for several regimes in the exchange rate returns application. The time-varying news impact curves are plotted for the lower regime, i.e. $G_1(t^*) = 0$ (dotted line), for an intermediate regime, i.e. $G_1(t^*) = 0.5$ (dashed line) and for the higher regime, i.e. $G_1(t^*) = 1$ (solid line).

Figure 9: Commodity adjusted futures price returns for coffee from January 3, 2000 until March 23, 2011 (2801 observations).

Figure 10: Graph of the final estimated function g_t for the commodity coffee returns model as a smooth function of the rescaled time variable t^* as given in (20)-(22).

Figure 11: Graph of the final estimated function g_t for the commodity coffee returns model as a smooth function of the rescaled time variable t^* as given by the Spline-GARCH model.

Figure 12: Sample autocorrelations of absolute log returns of the coffee residuals and the standardised variable $|\hat{\varepsilon}_t|/\hat{g}_{tCoffee}^{1/2}$ with the 95% confidence bounds.

Figure 13: Conditional standard deviation of the GJR-GARCH(1,1) model for the coffee residuals and the standardised variable $\hat{\varepsilon}_t/\hat{g}_{tCoffee}^{1/2}$.

Figure 14: News impact curves of the GJR-GARCH(1,1) (thick black curve) and the TVGJR-GARCH(1,1) for different regimes in the commodity coffee returns application. The time-varying news impact curves are plotted for the lower regime, i.e. $G_1(t^*) = G_2(t^*) = 0$ (dotted grey curve), for the intermediate regimes, i.e. $G_1(t^*) = G_2(t^*) = 0.5$ (dashed grey curve); $G_1(t^*) = 1$ and $G_2(t^*) = 0$ (grey curve); $G_1(t^*) = 0$ and $G_2(t^*) = 1$ (dashed black curve), and for the higher regime, i.e. $G_1(t^*) = G_2(t^*) = 1$ (dotted black curve).

Table 1: Empirical size of the misspecification test of type 1: Testing the deterministic component g_t . Each cell represents the proportion of rejections at the given nominal significance level.

	$T = 1000$			$T = 2500$			$T = 5000$		
	1%	5%	10%	1%	5%	10%	1%	5%	10%
DGP6									
LM	0.118	0.339	0.517	0.092	0.290	0.439	0.072	0.265	0.430
LM _R	0.018	0.087	0.156	0.015	0.070	0.131	0.012	0.051	0.109
DGP7									
LM	0.124	0.338	0.505	0.090	0.280	0.444	0.074	0.264	0.425
LM _R	0.018	0.081	0.156	0.017	0.067	0.131	0.016	0.063	0.117

Table 2: Empirical size of the misspecification test of type 2: Testing the conditional variance component h_t . Each cell represents the proportion of rejections at the given nominal significance level.

	$T = 1000$			$T = 2500$			$T = 5000$		
	1%	5%	10%	1%	5%	10%	1%	5%	10%
Test of GARCH(1,1) vs. GARCH(1,2)									
DGP6									
LM	0.113	0.402	0.637	0.110	0.387	0.611	0.101	0.348	0.556
LM _R	0.015	0.057	0.107	0.014	0.050	0.093	0.008	0.053	0.103
DGP7									
LM	0.131	0.445	0.677	0.260	0.602	0.799	0.401	0.729	0.855
LM _R	0.012	0.055	0.100	0.009	0.048	0.094	0.008	0.039	0.077
Test of GARCH(1,1) vs. GARCH(2,1)									
DGP6									
LM	0.098	0.385	0.638	0.102	0.362	0.595	0.081	0.296	0.548
LM _R	0.007	0.042	0.103	0.010	0.042	0.092	0.006	0.039	0.086
DGP7									
LM	0.102	0.422	0.658	0.254	0.596	0.788	0.437	0.757	0.870
LM _R	0.006	0.039	0.089	0.008	0.053	0.103	0.018	0.082	0.144
Test of no smooth transition GARCH(1,1) of order 1									
DGP6									
LM	0.069	0.220	0.408	0.058	0.211	0.372	0.057	0.195	0.332
LM _R	0.010	0.053	0.111	0.009	0.052	0.096	0.009	0.050	0.100
DGP7									
LM	0.070	0.258	0.450	0.146	0.418	0.599	0.225	0.545	0.707
LM _R	0.011	0.066	0.120	0.011	0.061	0.114	0.014	0.052	0.098

Table 3: Empirical size of the misspecification test of type 3: Testing the hypothesis of no remaining ARCH. Each cell represents the proportion of rejections at the given nominal significance level.

	$T = 1000$			$T = 2500$			$T = 5000$		
	1%	5%	10%	1%	5%	10%	1%	5%	10%
DGP6									
LM	0.255	0.518	0.709	0.297	0.531	0.695	0.234	0.434	0.600
LM_R	0.012	0.063	0.117	0.012	0.042	0.088	0.011	0.052	0.091
DGP7									
LM	0.396	0.681	0.831	0.430	0.700	0.836	0.520	0.778	0.871
LM_R	0.020	0.070	0.124	0.012	0.041	0.078	0.006	0.032	0.061

Table 4: Descriptive statistics for the daily returns

	Min	Max	Mean	Std.Dev.	Skewness	Rob. SK	Ex. Kurtosis	Rob. KR	$\hat{\rho}_1$
SPD/USD	-4.144	2.762	0.008	0.415	-0.905	-0.004	14.59	0.166	-0.048
$\hat{\varepsilon}_t/\hat{g}_{tSPD/USD}^{1/2}$	-4.198	4.220	-0.011	0.802	-0.170	-0.041	2.131	0.133	0.022
Coffee	-8.209	6.322	-0.002	1.144	-0.075	0.018	4.131	0.333	-0.019
$\hat{\varepsilon}_t/\hat{g}_{tCoffee}^{1/2}$	-5.056	6.393	-0.008	0.736	0.113	-0.036	4.940	0.220	-0.006

Notes: Robust SK denotes the robust measure for skewness based on quantiles proposed by Bowley (see Kim and White (2004)) and the robust KR denotes the robust centred coefficient for kurtosis proposed by Moors (see Kim and White (2004)). The coefficient $\hat{\rho}_1$ denotes the estimate of the first-order autocorrelation parameter.

Table 5: Estimation results in the two applications

	SPD/USD		$\hat{\varepsilon}_t/\hat{g}_{tSPD/USD}^{1/2}$		Coffee		$\hat{\varepsilon}_t/\hat{g}_{tCoffee}^{1/2}$	
	Estimate	(Std. error)	Estimate	(Std. error)	Estimate	(Std. error)	Estimate	(Std. error)
$\hat{\varphi}_0$	0.008	(0.009)	0.008	(0.009)	-0.002	(0.022)	-0.002	(0.022)
$\hat{\varphi}_1$	-0.048	(0.022)	-0.048	(0.022)	0.019	(0.019)	0.019	(0.019)
$\hat{\alpha}_0$	0.001	(0.001)	0.011	(0.005)	0.009	(0.004)	0.055	(0.017)
$\hat{\alpha}_1$	0.056	(0.016)	0.065	(0.020)	0.104	(0.025)	0.110	(0.023)
$\hat{\beta}_1$	0.938	(0.019)	0.868	(0.042)	0.938	(0.014)	0.892	(0.026)
$\hat{\lambda}_1$	-	(-)	-	(-)	-0.093	(0.026)	-0.123	(0.023)
$\hat{\alpha}_1 + \hat{\beta}_1 + \frac{\hat{\lambda}_1}{2}$	0.994		0.933		0.996		0.940	
Log-Lik	-637.6		-605.9		-4019.0		-3965.9	

Notes: The table contains the parameter estimates from the AR(1)-GJR-GARCH(1,1) and AR(1)-TVGJR-GARCH(1,1) models for both applications. The estimated AR(1)-TVGJR-GARCH(1,1) model has the form $y_t = \varphi_0 + \varphi_1 y_{t-1} + \varepsilon_t$ where $\varepsilon_t = \zeta_t h_t^{1/2} g_t^{1/2}$, $\{\zeta_t\} \sim \text{iid}(0, 1)$, and $h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1} + \lambda_1 I(\varepsilon_{t-1}^2 < 0) \varepsilon_{t-1}^2$, where $\varepsilon_t^* = \varepsilon_t / g_t^{1/2}$, $I(A)$ is the indicator variable, defined as $I(A) = 1$ when A is true, and 0 otherwise, and the estimated component g_t has the form (18)-(19) in the exchange series application and the form (20)-(22) in the coffee returns application. When the function $g_t = 1$, we get the AR(1)-GJR-GARCH(1,1) model.

Table 6: Results of the sequence of tests of constant unconditional variance against a time-varying GARCH model

Transitions in the		Test		H ₀₃		H ₀₂		H ₀₁	
alternative model		LM	p-value	LM ₃	p-value	LM ₂	p-value	LM ₁	p-value
Single transition									
Coffee		19.33	7×10^{-4}	9.170	0.002	0.110	0.740	10.08	0.001
Coffee - Robust		21.85	2×10^{-4}	10.75	0.001	0.114	0.736	10.73	0.001
SPD/USD		20.79	3×10^{-4}	5.710	0.017	9.555	0.002	5.591	0.018
SPD/USD - Robust		17.66	0.001	5.609	0.018	8.857	0.003	3.648	0.056
Double transition									
Coffee		11.26	0.024	3.521	0.061	0.490	0.484	7.262	0.007
Coffee - Robust		13.12	0.011	3.994	0.046	0.463	0.496	3.849	0.050
SPD/USD		5.721	0.221	0.050	0.823	0.043	0.836	5.629	0.018
SPD/USD - Robust		1.777	0.777	0.065	0.799	0.046	0.830	1.481	0.224
Triple transition									
Coffee		8.004	0.091	2.464	0.117	0.011	0.917	5.534	0.019
Coffee - Robust		5.936	0.204	2.193	0.139	0.009	0.925	1.420	0.233
SPD/USD		—	—	—	—	—	—	—	—
SPD/USD - Robust		—	—	—	—	—	—	—	—

Table 7: Misspecification tests for the GARCH model in the exchange rate returns application

SPD/USD	No ARCH-in-GARCH			GARCH(1,1) vs.		GARCH(1,1) vs		No ST-GARCH
	$r = 1$	$r = 5$	$r = 10$	GARCH(1,2)		GARCH(2,1)		
<u>Returns</u>								
LM test	0.334	13.10	15.73	0.550		2.786		2.244
p -value	0.563	0.022	0.108	0.458		0.095		0.326
LM Rob test	0.551	7.366	13.05	0.692		4.446		3.845
p -value	0.458	0.195	0.221	0.406		0.035		0.146
<u>Std>Returns</u>								
LM test	0.020	5.274	7.828	0.009		1.105		1.686
p -value	0.888	0.383	0.646	0.924		0.293		0.430
LM Rob test	0.024	5.209	9.456	0.008		1.885		6.090
p -value	0.877	0.391	0.489	0.929		0.170		0.048

Notes: The tests are those against remaining ARCH in the standardised residuals, GARCH(1,2) and GARCH(2,1) models, additional transition in the function g_t , and ST-GARCH(1,1) model.

Table 8: Misspecification tests for the GJR-GARCH model in the commodity coffee returns application

Coffee	No ARCH-in-GARCH			GARCH(1,1) vs.		GARCH(1,1) vs		No ST-GARCH
	$r = 1$	$r = 5$	$r = 10$	GARCH(1,2)	GARCH(1,1)	GARCH(2,1)		
<u>Returns</u>								
LM test	1.354	3.812	6.100	0.732	0.845	0.845	1.489	
p -value	0.245	0.577	0.807	0.392	0.358	0.358	0.475	
LM Rob test	0.858	4.659	9.733	0.054	0.166	0.166	2.042	
p -value	0.354	0.459	0.464	0.817	0.683	0.683	0.360	
<u>Std>Returns</u>								
LM test	2.882	11.08	12.36	2.079	2.573	2.573	2.129	
p -value	0.090	0.050	0.262	0.149	0.109	0.109	0.345	
LM Rob test	0.965	5.868	8.76	0.370	0.897	0.897	0.440	
p -value	0.326	0.319	0.555	0.543	0.344	0.344	0.803	

Notes: The tests are those against remaining ARCH in the standardised residuals, GARCH(1,2) and GARCH(2,1) models, additional transition in the function g_t , and ST-GARCH(1,1) model.