PICARD APPROXIMATION OF STOCHASTIC DIFFERENTIAL EQUATIONS AND APPLICATION TO LIBOR MODELS

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Abstract. The aim of this work is to provide fast and accurate approximation schemes for the Monte Carlo pricing of derivatives in LIBOR market models. Standard methods can be applied to solve the stochastic differential equations of the successive LIBOR rates but the methods are generally slow. Our contribution is twofold. Firstly, we propose an alternative approximation scheme based on Picard iterations. This approach is similar in accuracy to the Euler discretization, but with the feature that each rate is evolved independently of the other rates in the term structure. This enables simultaneous calculation of derivative prices of different maturities using parallel computing. Secondly, the product terms occurring in the drift of a LIBOR market model driven by a jump process grow exponentially as a function of the number of rates, quickly rendering the model intractable. We reduce this growth from exponential to quadratic using truncated expansions of the product terms. We include numerical illustrations of the accuracy and speed of our method pricing caplets, swaptions and forward rate agreements.

1. Introduction

The LIBOR market model (LMM) has become a standard model for the pricing of interest rate derivatives in recent years. The main advantage of this model in comparison to other approaches is that the evolution of discretely compounded, market-observable forward rates is modeled directly and not deduced from the evolution of unobservable factors. Moreover, the log-normal LIBOR model is consistent with the market practice of pricing caps according to Black's formula (cf. Black 1976). However, despite its apparent popularity, the LIBOR market model has certain well-known pitfalls.

On the one hand, the log-normal LIBOR model is driven by a Brownian motion, hence it cannot be calibrated adequately to the observed market data. An interest rate model is typically calibrated to the implied volatility surface from the cap market and the correlation structure of at-the-money swaptions. Several extensions of the LIBOR model have been proposed in the literature using jump-diffusions, Lévy processes or general semimartingales as the driving motion (cf. e.g. Glasserman and Kou 2003, Eberlein and

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Özkan 2005, Jamshidian 1999), or incorporating stochastic volatility effects (cf. e.g. Andersen and Brotherton-Ratcliffe 2005).

On the other hand, the dynamics of LIBOR rates are not tractable under forward measures due to the random terms that enter the dynamics of rates during the construction of the model. In particular, when the driving process has continuous paths the dynamics are tractable under their corresponding forward measure, but not under any other forward measure. When the driving process is a general semimartingale, then the dynamics of LIBOR rates are not even tractable under their very own forward measure. Consequently:

(1) if the driving process is a continuous semimartingale caplets can be priced in "closed form", but not swaptions or other multi-LIBOR derivatives;
(2) if the driving process is a general semimartingale, then even caplets cannot be priced in closed form.

The standard remedy to this problem is the so-called "frozen drift" approximation; it was first proposed by Brace et al. 1997 for the pricing of swaptions and has been used by several authors ever since. Brace et al. (2001), Dun et al. (2001) and Schlögl (2002) argue that freezing the drift is justified, since the deviation from the original equation is small in several measures.

Although the frozen drift approximation is the simplest and most popular solution, it is well-known that it does not yield acceptable results, especially for exotic derivatives and longer horizons. Therefore, several other approximations have been developed in the literature. In one line of research Daniluk and Gątarek (2005) and Kurbanmuradov et al. (2002) are looking for the best lognormal approximation of the forward LIBOR dynamics; cf. also Schoenmakers (2005). Other authors have been using linear interpolations and predictor-corrector Monte Carlo methods to get a more accurate approximation of the drift term (cf. e.g. Hunter et al. 2001 and Glasserman and Zhao 2000). We refer the reader to Joshi and Stacey (2008) for a detailed overview of that literature, some new approximation schemes and numerical experiments.

Although most of this literature focuses on the lognormal LIBOR market model, Glasserman and Merener (2003b, 2003a) have developed approximation schemes for the pricing of caps and swaptions in jump-diffusion LIBOR market models.

In this article we develop a general method for the approximation of the random terms that enter the drift of LIBOR models that is suitable for parallel computing. In particular, using Picard iterations we develop generic approximation schemes that decouple the dependence between LIBOR rates. Therefore individual rates in the tenor can be evolved independently in a Monte Carlo simulation. In addition, we treat a problem specific to LIBOR models with jumps; namely that the complexity of the drift term grows exponentially in the number of tenor dates. We expand and truncate the drift term, which yields a highly accurate approximation, while the gain in computational speed is immense. We illustrate the accuracy and efficiency of our method in an example where LIBOR rates are driven by a normal inverse Gaussian process.

The method we develop is universal and can be applied to any LIBOR model driven by a general semimartingale. However, we focus on the Lévy LIBOR model as a characteristic example of a LIBOR model driven by a general semimartingale.

The article is structured as follows: in section 2 we review time-inhomogeneous Lévy process, and in section 3 we revisit the Lévy LIBOR model and explain in detail the computational problems. In section 4 we derive the Picard approximation scheme and the drift expansions. In section 5 we briefly describe the main derivatives on LIBOR rates. Finally, section 6 contains a numerical illustration.

2. Lévy processes

Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a complete stochastic basis, where $\mathcal{F} = \mathcal{F}_{T_*}$ and the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T_*]}$ satisfies the usual conditions; we assume that $T_* \in \mathbb{R}_{\geqslant 0}$ is a finite time horizon. The driving process $H = (H_t)_{0\leq t\leq T_*}$ is a *process with independent increments* and *absolutely continuous* characteristics; this is also called a *time-inhomogeneous Lévy process*. That is, H is an adapted, càdlàg, real-valued stochastic process with independent increments, starting from zero, where the law of H_t, $t \in [0, T_*]$, is described by the characteristic function

$$\mathbb{E}\big[\mathrm{e}^{iuH_t}\big] = \exp\left(\int_0^t \Big[ib_s u - \frac{c_s}{2}u^2 + \int_{\mathbb{R}} (\mathrm{e}^{iux} - 1 - iux)F_s(\mathrm{d}x)\Big]\mathrm{d}s\right); \quad (2.1)$$

here $b_t \in \mathbb{R}$, $c_t \in \mathbb{R}_{\geqslant 0}$ and F_t is a Lévy measure, i.e. satisfies $F_t(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)F_t(\mathrm{d}x) < \infty$, for all $t \in [0, T_*]$. In addition, the process H satisfies Assumptions $(\mathbb{AC})$ and $(\mathbb{EM})$ given below.

Assumption $(\mathbb{AC})$. The triplets (b_t, c_t, F_t) satisfy

$$\int_0^{T_*} \left(|b_t| + c_t + \int_{\mathbb{R}} (1 \wedge |x|^2)F_t(\mathrm{d}x)\right)\mathrm{d}t < \infty. \quad (2.2)$$

Assumption $(\mathbb{EM})$. There exist constants $M, \varepsilon > 0$ such that for every $u \in [-(1 + \varepsilon)M, (1 + \varepsilon)M] =: \mathbb{M}$

$$\int_0^{T_*} \int_{\{|x|>1\}} \mathrm{e}^{ux} F_t(\mathrm{d}x)\mathrm{d}t < \infty. \quad (2.3)$$

Moreover, without loss of generality, we assume that $\int_{\{|x|>1\}} \mathrm{e}^{ux}F_t(\mathrm{d}x) < \infty$ for all $t \in [0, T_*]$ and $u \in \mathbb{M}$.

These assumptions render the process $H = (H_t)_{0\leq t\leq T_*}$ a *special* semimartingale, therefore it has the canonical decomposition (cf. Jacod and Shiryaev 2003, II.2.38, and Eberlein et al. 2005)

$$H = \int_0^{\cdot} b_s\mathrm{d}s + \int_0^{\cdot} \sqrt{c_s}\mathrm{d}W_s + \int_0^{\cdot}\int_{\mathbb{R}} x(\mu^H - \nu)(\mathrm{d}s, \mathrm{d}x), \quad (2.4)$$

where μ^H is the random measure of jumps of the process H, ν is the $\mathbb{P}$-compensator of μ^H, and $W = (W_t)_{0\leq t\leq T_*}$ is a $\mathbb{P}$-standard Brownian motion. The *triplet of predictable characteristics* of H with respect to the measure $\mathbb{P}$, $\mathbb{T}(H|\mathbb{P}) = (B, C, \nu)$, is

$$B = \int_0^{\cdot} b_s \mathrm{d}s, \qquad C = \int_0^{\cdot} c_s \mathrm{d}s, \qquad \nu([0,\cdot]\times A) = \int_0^{\cdot}\int_A F_s(\mathrm{d}x)\mathrm{d}s, \tag{2.5}$$

where $A \in \mathcal{B}(\mathbb{R})$; the triplet (b, c, F) represents the *local characteristics* of H. In addition, the triplet of predictable characteristics (B, C, ν) determines the distribution of H, as the Lévy–Khintchine formula (2.1) obviously dictates.

We denote by κ_s the *cumulant generating function* associated to the infinitely divisible distribution with Lévy triplet (b_s, c_s, F_s), i.e. for $z \in \mathbb{M}$ and $s \in [0, T_*]$

$$\kappa_s(z) := b_s z + \frac{c_s}{2} z^2 + \int_{\mathbb{R}} (\mathrm{e}^{zx} - 1 - zx) F_s(\mathrm{d}x). \tag{2.6}$$

Using Assumption ($\mathbb{EM}$) we can extend κ_s to the complex domain $\mathbb{C}$, for $z \in \mathbb{C}$ with $\Re z \in \mathbb{M}$, and the characteristic function of H_t can be written as

$$\mathbb{E}\big[\mathrm{e}^{iuH_t}\big] = \exp\bigg(\int_0^t \kappa_s(iu)\mathrm{d}s\bigg). \tag{2.7}$$

If H is a Lévy process, i.e. time-homogeneous, then (b_s, c_s, F_s) – and thus also κ_s – do not depend on s. In that case, κ equals the cumulant (log-moment) generating function of H_1.

3. The Lévy LIBOR model

3.1. Model description. The Lévy LIBOR model was developed by Eberlein and Özkan (2005) following the seminal articles on LIBOR market models driven by Brownian motion by Sandmann et al. (1995), Miltersen et al. (1997) and Brace et al. (1997); see also Glasserman and Kou (2003) and Jamshidian (1999) for LIBOR market models driven by jump processes and general semimartingales respectively. The Lévy LIBOR model is a *market model* where the forward LIBOR rate is modeled directly and is driven by a time-inhomogeneous Lévy process.

Let $0 = T_0 < T_1 < \cdots < T_N < T_{N+1} = T_*$ denote a discrete tenor structure where $\delta_i = T_{i+1} - T_i$, $i \in \{0, 1, \ldots, N\}$. Consider a complete stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P}_{T_*})$ and a time-inhomogeneous Lévy process $H = (H_t)_{0\leq t\leq T_*}$ satisfying Assumptions ($\mathbb{AC}$) and ($\mathbb{EM}$). The process H has predictable characteristics $(0, C, \nu^{T_*})$ or local characteristics $(0, c, F^{T_*})$, and its canonical decomposition is

$$H = \int_0^{\cdot} \sqrt{c_s} \mathrm{d}W_s^{T_*} + \int_0^{\cdot}\int_{\mathbb{R}} x(\mu^H - \nu^{T_*})(\mathrm{d}s, \mathrm{d}x), \tag{3.1}$$

where W^{T_*} is a $\mathbb{P}_{T_*}$-standard Brownian motion, μ^H is the random measure associated with the jumps of H and ν^{T_*} is the $\mathbb{P}_{T_*}$-compensator of μ^H. We further assume that the following conditions are in force.

(LR1): For any maturity T_i there exists a bounded, continuous, deterministic function $\lambda(\cdot, T_i) : [0, T_i] \to \mathbb{R}$, which represents the volatility of the forward LIBOR rate process $L(\cdot, T_i)$. Moreover,

$$\sum_{i=1}^{N} \left|\lambda(s, T_i)\right| \leq M,$$

for all $s \in [0, T_*]$, where M is the constant from Assumption ($\mathbb{EM}$), and $\lambda(s, T_i) = 0$ for all $s > T_i$.

(LR2): The initial term structure $B(0, T_i)$, $1 \leq i \leq N + 1$, is strictly positive and strictly decreasing. Consequently, the initial term structure of forward LIBOR rates is given, for $1 \leq i \leq N$, by

$$L(0, T_i) = \frac{1}{\delta_i}\left(\frac{B(0, T_i)}{B(0, T_i + \delta_i)} - 1\right) > 0. \tag{3.2}$$

The construction of the Lévy LIBOR model starts by postulating that the dynamics of the forward LIBOR rate with the longest maturity $L(\cdot, T_N)$ is driven by the time-inhomogeneous Lévy process H and evolve as a martingale under the terminal forward measure $\mathbb{P}_{T_*}$. Then, the dynamics of the LIBOR rates for the preceding maturities are constructed by backward induction; they are driven by the same process H and evolve as martingales under their corresponding forward measures.

Let us denote by $\mathbb{P}_{T_{i+1}}$ the forward measure associated to the settlement date T_{i+1}, $i \in \{0, \dots, N\}$. The dynamics of the forward LIBOR rate $L(\cdot, T_i)$, for an arbitrary T_i, is given by

$$L(t, T_i) = L(0, T_i) \exp\left(\int_0^t b^L(s, T_i)\mathrm{d}s + \int_0^t \lambda(s, T_i)\mathrm{d}H_s^{T_{i+1}}\right), \tag{3.3}$$

where $H^{T_{i+1}}$ is a special *semimartingale* with canonical decomposition

$$H_t^{T_{i+1}} = \int_0^t \sqrt{c_s}\mathrm{d}W_s^{T_{i+1}} + \int_0^t\int_{\mathbb{R}} x(\mu^H - \nu^{T_{i+1}})(\mathrm{d}s, \mathrm{d}x). \tag{3.4}$$

Here $W^{T_{i+1}}$ is a $\mathbb{P}_{T_{i+1}}$-standard Brownian motion and $\nu^{T_{i+1}}$ is the $\mathbb{P}_{T_{i+1}}$-compensator of μ^H. The dynamics of an arbitrary LIBOR rate again evolves as a martingale under its corresponding forward measure; therefore, we specify the drift term of the forward LIBOR process $L(\cdot, T_i)$ as

$$b^L(s, T_i) = -\frac{1}{2}\lambda^2(s, T_i)c_s - \int_{\mathbb{R}} \left(\mathrm{e}^{\lambda(s,T_i)x} - 1 - \lambda(s, T_i)x\right)F_s^{T_{i+1}}(\mathrm{d}x). \tag{3.5}$$

The forward measure $\mathbb{P}_{T_{i+1}}$, which is defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T_{i+1}})$, is related to the terminal forward measure $\mathbb{P}_{T_*}$ via

$$\frac{\mathrm{d}\mathbb{P}_{T_{i+1}}}{\mathrm{d}\mathbb{P}_{T_*}} = \prod_{l=i+1}^{N} \frac{1 + \delta_l L(T_{i+1}, T_l)}{1 + \delta_l L(0, T_l)} = \frac{B(0, T_*)}{B(0, T_{i+1})} \prod_{l=i+1}^{N} \left(1 + \delta_l L(T_{i+1}, T_l)\right). \tag{3.6}$$

The $\mathbb{P}_{T_{i+1}}$-Brownian motion $W^{T_{i+1}}$ is related to the $\mathbb{P}_{T_*}$-Brownian motion via

$$\begin{aligned} W_t^{T_{i+1}} &= W_t^{T_{i+2}} - \int_0^t \alpha(s, T_{i+1})\sqrt{c_s}\mathrm{d}s = \dots \\ &= W_t^{T_*} - \int_0^t \left(\sum_{l=i+1}^{N} \alpha(s, T_l) \right) \sqrt{c_s}\mathrm{d}s, \end{aligned} \tag{3.7}$$

where

$$\alpha(t, T_l) = \frac{\delta_l L(t-, T_l)}{1 + \delta_l L(t-, T_l)} \lambda(t, T_l). \tag{3.8}$$

The $\mathbb{P}_{T_{i+1}}$-compensator of μ^H, $\nu^{T_{i+1}}$, is related to the $\mathbb{P}_{T_*}$-compensator of μ^H via

$$\begin{aligned} \nu^{T_{i+1}}(\mathrm{d}s, \mathrm{d}x) &= \beta(s, x, T_{i+1})\nu^{T_{i+2}}(\mathrm{d}s, \mathrm{d}x) = \dots \\ &= \left(\prod_{l=i+1}^{N} \beta(s, x, T_l) \right) \nu^{T_*}(\mathrm{d}s, \mathrm{d}x), \end{aligned} \tag{3.9}$$

where

$$\beta(t, x, T_l,) = \frac{\delta_l L(t-, T_l)}{1 + \delta_l L(t-, T_l)} \left(\mathrm{e}^{\lambda(t,T_l)x} - 1 \right) + 1. \tag{3.10}$$

Remark 3.1. Notice that the process $H^{T_{i+1}}$, driving the forward LIBOR rate $L(\cdot, T_i)$, and $H = H^{T_*}$ have the same *martingale* part and differ only in the *finite variation* part (drift). An application of Girsanov's theorem for semimartingales yields that the $\mathbb{P}_{T_{i+1}}$-finite variation part of H is

$$\int_0^{\cdot} c_s \sum_{l=i+1}^{N} \alpha(s, T_l)\mathrm{d}s + \int_0^{\cdot} \int_{\mathbb{R}} x \left(\prod_{l=i+1}^{N} \beta(s, x, T_l) - 1 \right) \nu^{T_*}(\mathrm{d}s, \mathrm{d}x).$$

3.2. **Option pricing and computational problems.** The main scope of a market model for interest rate derivatives is to adequately describe the dynamics of interest rates as they are reflected in prices of derivatives. Hence, a good market model should be easily calibrated to option prices of liquid derivatives, i.e. caps and at-the-money swaptions. Calibration requires the fast computation of option prices; either in closed-form or using semi-analytical methods (e.g. Fourier transform methods).

However, herein lies a major pitfall of the Lévy LIBOR model: the process $H^{T_{i+1}}$ driving the dynamics of $L(\cdot, T_i)$ is *not* a Lévy process under $\mathbb{P}_{T_{i+1}}$, or any other forward measure. One just has to observe that the compensator $\nu^{T_{i+1}}$ is random and not deterministic. Therefore, the characteristic function of the random variable $H_t^{T_{i+1}}$ is not available and Fourier methods cannot be used for option pricing. In other words, on top of the well-known problems of LMMs in pricing swaptions and other multi-LIBOR products, when the driving process has jumps even caplets *cannot* be priced in closed or semi-analytic form.

A remedy has been proposed in Eberlein and Özkan (2005) and further refined by Kluge (2005). They propose to "freeze" the random terms in the compensator, i.e. to replace them by their deterministic initial values. The approximate process is then a Lévy process and Fourier methods for option pricing can be applied. This method is equivalent to the "frozen drift" approximation, which does not however yield acceptable results.

3.3. **Terminal measure dynamics.** Once closed-form or semi-analytical methods are not available for option pricing, a Monte Carlo simulation is the next alternative. In this section we derive the dynamics of LIBOR rates under the terminal measure. This is the appropriate measure for simulations in LMMs.

Starting with the dynamics of the LIBOR rate $L(\cdot, T_i)$ under the forward martingale measure $\mathbb{P}_{T_{i+1}}$, and using the connection between the forward and terminal martingale measures (cf. eqs. (3.7)–(3.10) and Remark 3.1), we have that the dynamics of the LIBOR rate $L(\cdot, T_i)$ under the terminal measure is given by

$$L(t, T_i) = L(0, T_i) \exp\left(\int_0^t b(s, T_i) \mathrm{d}s + \int_0^t \lambda(s, T_i) \mathrm{d}H_s \right), \tag{3.11}$$

where $H = (H_t)_{0 \le t \le T_*}$ is the $\mathbb{P}_{T_*}$-time-inhomogeneous Lévy process driving the LIBOR rates, cf. (3.1). The drift term $b(\cdot, T_i)$ has the form

$$\begin{aligned} b(s, T_i) = &-\frac{1}{2}\lambda^2(s, T_i) c_s - c_s \lambda(s, T_i) \sum_{l=i+1}^{N} \frac{\delta_l L(s-, T_l)}{1 + \delta_l L(s-, T_l)} \lambda(s, T_l) \\ &- \int_{\mathbb{R}} \left(\left(\mathrm{e}^{\lambda(s, T_i)x} - 1 \right) \prod_{l=i+1}^{N} \beta(s, x, T_l) - \lambda(s, T_i) x \right) F_s^{T_*}(\mathrm{d}x), \end{aligned} \tag{3.12}$$

where $\beta(s, x, T_l)$ is given by (3.10). Note that the drift term of (3.11) is random, therefore the log-LIBOR is a general semimartingale, and not a Lévy process. Of course, $L(\cdot, T_i)$ is not a $\mathbb{P}_{T_*}$-martingale, unless $i = N$ (where we use the conventions $\sum_{l=1}^{0} = 0$ and $\prod_{l=1}^{0} = 1$).

The equation for the dynamics under the terminal measure contains the next numerical problem, well-known for LMMs. The drift $b(\cdot, T_i)$ depends on all *subsequent* LIBOR rates in the tenor, yielding a dependence structure that has the form of a triangular matrix; cf. Table 3.1. In other words, in order to simulate any rate one has to start by simulating the last rate, save the path and proceed iteratively. This means that simulations are very slow, while the burden on the random access memory (RAM) is also significant.

The standard remedy to this problem is the so-called "'frozen drift" approximation, where one replaces the random terms in the drift by their deterministic initial values. This simplifies the simulations considerably, since rates are no longer state-dependent and can be simulated in parallel. However, this approximation is very crude and does not yield acceptable results; cf. section 6.

$L(t,T_1)$	$L(t,T_2)$	$\dots$	$L(t,T_k)$	$\dots$	$L(t,T_{N-2})$	$L(t,T_{N-1})$	$L(t,T_N)$
$L(t,T_N)$	$L(t,T_N)$	$\dots$	$L(t,T_N)$	$\dots$	$L(t,T_N)$	$L(t,T_N)$	
$L(t,T_{N-1})$	$L(t,T_{N-1})$	$\dots$	$L(t,T_{N-1})$	$\dots$	$L(t,T_{N-1})$		
$L(t,T_{N-2})$	$L(t,T_{N-2})$	$\dots$	$\dots$	$\dots$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$\dots$	$\dots$	$\dots$	$L(t,T_{k+1})$				
$L(t,T_3)$	$L(t,T_3)$	$\dots$					
$L(t,T_2)$							

TABLE 3.1. Matrix of dependencies for LIBOR rates

Moreover, an additional numerical problem arises in LMMs with jumps from the product term $\prod_l \beta(\cdot,\cdot,T_l)$. This term grows exponentially as a function of tenor dates N and makes the simulations even more time-consuming.

4. Picard approximation and drift expansion for the Lévy LIBOR model

The aim of this section is to derive approximation schemes for LIBOR models that can overcome the pitfalls of the the model – namely, the slow Monte Carlo simulations and the exponential growth of the product term. Firstly, using the idea of Picard iterations for the solution of SDEs, we derive approximate equations for the dynamics of LIBOR rates which are suitable for parallel computing. Secondly, by expanding and truncating the product term we can reduce the exponential to quadratic growth. Numerical examples show that these methods yield significant gain in computational time, while the loss in accuracy is very small.

4.1. **Picard iterations.** In order to derive approximation schemes for the dynamics of LIBOR rates it is more convenient to work with the logarithm of rates. Let us denote by Z the log-LIBOR rates, that is

$$\begin{aligned} Z(t,T_i) &:= \log L(t,T_i) \\ &= Z(0,T_i) + \int_0^t b(s,T_i)\mathrm{d}s + \int_0^t \lambda(s,T_i)\mathrm{d}H_s, \end{aligned} \tag{4.1}$$

where $Z(0,T_i) = \log L(0,T_i)$ for all $i \in \{1,\dots,N\}$. We can immediately deduce that $Z(\cdot,T_i)$ is a semimartingale and its triplet of predictable characteristics under $\mathbb{P}_{T_*}$, $\mathbb{T}(Z(\cdot,T_i)|\mathbb{P}_{T_*}) = (B^i, C^i, \nu^i)$, is described by

$$\begin{aligned} B^i &= \int_0^\cdot b(s,T_i)\mathrm{d}s \\ C^i &= \int_0^\cdot \lambda^2(s,T_i)c_s\mathrm{d}s \\ 1_A(x) * \nu^i &= 1_A\big(\lambda(s,T_i)x\big) * \nu^{T_*}, \qquad A \in \mathcal{B}(\mathbb{R}\backslash\{0\}). \end{aligned} \tag{4.2}$$

The assertion follows from the canonical decomposition of a semimartingale and the triplet of characteristics of the stochastic integral process; see, for example, Proposition 1.3 in Papapantoleon (2007).

Remark 4.1. Note that the martingale part of $Z(\cdot, T_i)$, i.e. the stochastic integral $\int_0^{\cdot} \lambda(s, T_i) \mathrm{d}H_s$, is a time-inhomogeneous Lévy process. However, the random drift term destroys the Lévy property of $Z(\cdot, T_i)$, as the increments are no longer independent.

The dynamics of log-LIBOR rates can be alternatively described as the solution to the following *linear* SDE

$$\begin{aligned} \mathrm{d}Z(t, T_i) &= b(t, T_i; Z(t))\mathrm{d}t + \lambda(t, T_i)\mathrm{d}H_t \\ Z(0, T_i) &= \log L(0, T_i) \end{aligned} \tag{4.3}$$

for all $i \in \{1, \dots, N\}$ and all $t \in [0, T_i]$. We have introduced the term $Z(\cdot)$ in the drift term $b(\cdot, T_i; Z(\cdot))$ to make explicit that the log-LIBOR rates depend on all subsequent rates in the tenor.

The idea behind the Picard approximation scheme is to approximate the dynamics of LIBOR rates by the Picard iterations for the SDE (4.3). The first Picard iteration for (4.3) is simply the initial value, i.e.

$$Z^{(0)}(t, T_i) = Z(0, T_i), \tag{4.4}$$

while the second Picard iteration is

$$\begin{aligned} Z^{(1)}(t, T_i) &= Z(0, T_i) + \int_0^t b(s, T_i; Z^{(0)}(s))\mathrm{d}s + \int_0^t \lambda(s, T_i)\mathrm{d}H_s \\ &= Z(0, T_i) + \int_0^t b(s, T_i; Z(0))\mathrm{d}s + \int_0^t \lambda(s, T_i)\mathrm{d}H_s. \end{aligned} \tag{4.5}$$

Since the drift term $b(\cdot, T_i; Z(0))$ is deterministic, as the random terms have been replaced with their initial values, we can easily deduce that the second Picard iterate $Z^{(1)}(\cdot, T_i)$ is a Lévy process.

Remark 4.2. Comparing (4.5) with (4.1) it becomes evident that we are approximating the semimartingale $Z(\cdot, T_i)$ with the time-inhomogeneous *Lévy process* $Z^{(1)}(\cdot, T_i)$.

4.2. **Application to LIBOR models.** We will now use the Picard iterations in order to deduce strong – i.e. pathwise – approximation schemes for the dynamics of LIBOR rates. More specifically, we will use the Picard iterates as proxies for the log-LIBOR rates in the drift term of the dynamics, cf. (3.12). Obviously, using the first Picard iterate $Z^{(0)}$ in the drift term we have just recovered the "frozen drift" approximation.

Let us denote by $\widehat{Z}(\cdot, T_i)$ the approximate log-LIBOR rate stemming from using the second Picard iterate $Z^{(1)}$. The dynamics of the *approximate* log-LIBOR rate is

$$\widehat{Z}(t, T_i) = Z(0, T_i) + \int_0^t b(s, T_i; Z^{(1)}(s))\mathrm{d}s + \int_0^t \lambda(s, T_i)\mathrm{d}H_s, \tag{4.6}$$

where the drift term is provided by

$$b(s,T_i;Z^{(1)}(s)) = -\frac{1}{2}\lambda^2(s,T_i)c_s - c_s\lambda(s,T_i)\sum_{l=i+1}^{N}\frac{\delta_l \mathrm{e}^{Z^{(1)}(s-,T_l)}}{1+\delta_l \mathrm{e}^{Z^{(1)}(s-,T_l)}}\lambda(s,T_l)$$
$$- \int_{\mathbb{R}} \left(\left(\mathrm{e}^{\lambda(s,T_i)x} - 1 \right) \prod_{l=i+1}^{N} \widehat{\beta}(s,x,T_l) - \lambda(s,T_i)x \right) F_s^{T_*}(\mathrm{d}x), \tag{4.7}$$

with

$$\widehat{\beta}(t,x,T_l,) = \frac{\delta_l \exp\left(Z^{(1)}(t-,T_l)\right)}{1+\delta_l \exp\left(Z^{(1)}(t-,T_l)\right)}\left(\mathrm{e}^{\lambda(t,T_l)x} - 1\right) + 1. \tag{4.8}$$

The main advantage of this Picard approximation is that the resulting SDE for $\widehat{Z}(\cdot,T_i)$ can be simulated more easily than the equation for $Z(\cdot,T_i)$. Indeed, contrary to the dynamics of $Z(\cdot,T_i)$, the dynamics of $\widehat{Z}(\cdot,T_i)$ depend only on the Lévy processes $Z^{(1)}(\cdot,T_l)$, $i+1 \le l \le N$, which are *independent* of each other. Compare also the "dependence matrix" for the approximate rates (Table 4.1) with Table 3.1. Hence, we can use *parallel computing* to simulate all approximate LIBOR rates simultaneously. This significantly increases the speed of the Monte Carlo simulations while, as the numerical examples reveal, the empirical performance is very satisfactory.

$\widehat{L}(t,T_1)$	$\widehat{L}(t,T_2)$	$\dots$	$\widehat{L}(t,T_k)$	$\dots$	$\widehat{L}(t,T_{N-1})$	$L(t,T_N)$
$Z^{(1)}(t,T_N)$	$Z^{(1)}(t,T_N)$	$\dots$	$Z^{(1)}(t,T_N)$	$\dots$	$Z^{(1)}(t,T_N)$	
$Z^{(1)}(t,T_{N-1})$	$Z^{(1)}(t,T_{N-1})$	$\dots$	$Z^{(1)}(t,T_{N-1})$	$\dots$		
$Z^{(1)}(t,T_{N-2})$	$Z^{(1)}(t,T_{N-2})$	$\dots$	$\dots$	$\dots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
$\dots$	$\dots$	$\dots$	$Z^{(1)}(t,T_{k+1})$			
$Z^{(1)}(t,T_3)$	$Z^{(1)}(t,T_3)$	$\dots$				
$Z^{(1)}(t,T_2)$						

TABLE 4.1. Matrix of dependencies for approximate LIBOR rates

Remark 4.3. Note that the Picard approximation can be also used in case one wants to apply P(I)DE methods for the valuation of derivatives in LIBOR models, and yields an analogous simplification of the problem.

Remark 4.4. Let us point out that the Picard approximation scheme (4.6)–(4.8) we have developed is *universal* and can be applied to any LMM. We can replace the Lévy process H driving the dynamics of LIBOR rates by a general semimartingale X with random predictable characteristics (thus also incorporating stochastic volatility). Subject to certain assumptions X has the canonical decomposition

$$X_t = \int_0^t \sqrt{c_s}\mathrm{d}W_s + \int_0^t \int_{\mathbb{R}} x(\mu^H - \nu)(\mathrm{d}s,\mathrm{d}x); \tag{4.9}$$

compare with (3.4). Then we can construct an LMM driven by the semimartingale X following the steps for the Lévy LMM in Eberlein and Özkan (2005). We can also follow analogously all the steps for the Picard approximation; in this case, the Picard iterate $Z^{(1)}$ will have the same dynamics as in (4.5), with H replaced by X.

4.3. **Drift expansion.** This part is devoted to the integral term in the drift of the LIBOR dynamics; see (3.12) again. Obviously this is a problem solely related to LMMs driven by jump processes.

Let us introduce the following shorthand notation for convenience:

$$\lambda_l := \lambda(s, T_l) \quad \text{and} \quad L_l := L(s, T_l). \tag{4.10}$$

We denote by $\mathbb{A}$ the part of the drift term that stems from the jumps, i.e.

$$\mathbb{A} = \int_{\mathbb{R}} \left(\left(\mathrm{e}^{\lambda_i x} - 1 \right) \prod_{l=i+1}^{N} \left(1 + \frac{\delta_l L_l}{1 + \delta_l L_l} \left(\mathrm{e}^{\lambda_l x} - 1 \right) \right) - \lambda_i x \right) F_s^{T_*}(\mathrm{d}x). \tag{4.11}$$

In theory, one could simply employ a straightforward numerical integration to compute $\mathbb{A}$. However this is not feasible in practice since a numerical integration should be performed at each step of the Monte Carlo simulation. An alternative solution is to express $\mathbb{A}$ in terms of the cumulant generating function of (the jump part) of H.

Observe that a product of the form $\prod_{l=1}^{N}(1 + \alpha_l)$ appears, where $\alpha_l := \frac{\delta_l L_l}{1+\delta_l L_l}(\mathrm{e}^{\lambda_l x} - 1)$. This product can be expressed in terms of so-called *elementary symmetric polynomials*. Let $k \leq N$, then the elementary symmetric polynomial of degree k in N variables is given by

$$\varepsilon_k(\alpha_1, \dots, \alpha_N) = \sum_{1 \leq i_1 < \dots < i_k \leq N} \alpha_{i_1} \times \dots \times \alpha_{i_k}. \tag{4.12}$$

Hence we have that

$$\prod_{l=1}^{N}(1 + \alpha_l) = 1 + \varepsilon_1(\alpha_1, \dots, \alpha_N) + \dots + \varepsilon_N(\alpha_1, \dots, \alpha_N). \tag{4.13}$$

One can immediately deduce that, while the drift term stemming from the diffusion part is a first order polynomial in $\frac{\delta_l L_l}{1+\delta_l L_l}$, $\mathbb{A}$ is an N-th order polynomial. More importantly, the number of terms on the RHS of (4.13) is 2^N. Hence, we need to perform 2^N computations in order to calculate the drift of the LIBOR rates. Since N is the length of the tenor, it becomes apparent that this calculation is feasible only for short tenors. If, for example, $N = 40$ this amounts to more than 1 trillion computations.

In order to make this computation more feasible we will truncate the RHS of (4.13) at the first or second order. The first order approximation of the product term is

$$\begin{aligned}
\mathbb{A}' &= \int_{\mathbb{R}} \Big(\big(e^{\lambda_i x} - 1 \big) \Big(1 + \varepsilon_1(\alpha_{i+1}, \ldots, \alpha_N) - \lambda_i x \Big) \Big) F_s^{T_*}(dx) \\
&= \int_{\mathbb{R}} \left(\big(e^{\lambda_i x} - 1 \big) \left(1 + \sum_{l=i+1}^{N} \frac{\delta_l L_l}{1+\delta_l L_l} \big(e^{\lambda_l x} - 1 \big) \right) - \lambda_i x \right) F_s^{T_*}(dx) \\
&= \int_{\mathbb{R}} \big(e^{\lambda_i x} - 1 - \lambda_i x \big) F_s^{T_*}(dx) \\
&\quad + \sum_{l=i+1}^{N} \frac{\delta_l L_l}{1+\delta_l L_l} \int_{\mathbb{R}} \big(e^{\lambda_i x} - 1 \big) \big(e^{\lambda_l x} - 1 \big) F_s^{T_*}(dx) \\
&= \kappa(\lambda_i) + \sum_{l=i+1}^{N} \frac{\delta_l L_l}{1+\delta_l L_l} \Big(\kappa(\lambda_i + \lambda_l) - \kappa(\lambda_i) - \kappa(\lambda_l) \Big),
\end{aligned} \tag{4.14}$$

and the order of the error is

$$\mathbb{A} = \mathbb{A}' + O\big(N \|L\|^2\big). \tag{4.15}$$

Similarly the second order approximation is provided by

$$\begin{aligned}
\mathbb{A}'' &= \int_{\mathbb{R}} \Big(\big(e^{\lambda_i x} - 1 \big) \Big(1 + (\varepsilon_1 + \varepsilon_2)(\alpha_{i+1}, \ldots, \alpha_N) - \lambda_i x \Big) \Big) F_s^{T_*}(dx) \\
&= \kappa(\lambda_i) + \sum_{l=i+1}^{N} \frac{\delta_l L_l}{1+\delta_l L_l} \Big(\kappa(\lambda_i + \lambda_l) + \kappa(\lambda_i) + \kappa(\lambda_l) \Big) \\
&\quad + \sum_{i+1 \le k < l \le N} \frac{\delta_l L_l}{1+\delta_l L_l} \frac{\delta_k L_k}{1+\delta_k L_k} \\
&\qquad \times \Big(\kappa(\lambda_i + \lambda_l + \lambda_k) - \kappa(\lambda_i + \lambda_l) - \kappa(\lambda_i + \lambda_k) \\
&\qquad\quad - \kappa(\lambda_k + \lambda_l) + \kappa(\lambda_i) + \kappa(\lambda_l) + \kappa(\lambda_k) \Big),
\end{aligned} \tag{4.16}$$

and the order of the error is

$$\mathbb{A} = \mathbb{A}'' + O\big(N^2 \|L\|^3\big). \tag{4.17}$$

Since the LIBOR rate is an order of magnitude smaller than the number of tenor dates, these approximations are justified. Indeed, numerical results show that truncation at the second order yields very satisfying results, while the gain in computational time is very significant.

5. Derivatives on LIBOR rates

In this section we briefly describe the derivatives we will use for the numerical illustration. Namely we use caplets and swaptions, which are the most liquid derivatives in the interest rate markets. We will also use forward rate agreements (FRAs) as a benchmark for the different approximations because they have model independent values. Of course, since we have developed a pathwise approximation, we could also consider many other derivatives –

especially path-dependent ones – for the illustration. We avoid this for the sake of brevity.

5.1. **Caplets.** The price of a caplet with strike K maturing at time T_i, using the relationship between the terminal and the forward measures (3.6), can be expressed as

$$\begin{aligned}\mathbb{C}_0(K,T_i) &= \delta_i B(0,T_{i+1})\,\mathbb{E}_{\mathbb{P}_{T_{i+1}}}[(L(T_i,T_i)-K)^+]\\ &= \delta B(0,T_{i+1})\,\mathbb{E}_{\mathbb{P}_{T_*}}\Big[\frac{\mathrm{d}\mathbb{P}_{T_{i+1}}}{\mathrm{d}\mathbb{P}_{T_*}}\big|_{\mathcal{F}_{T_i}}(L(T_i,T_i)-K)^+\Big]\\ &= \delta B(0,T_*)\,\mathbb{E}_{\mathbb{P}_{T_*}}\Big[\prod_{l=i+1}^{N}\big(1+\delta L(T_i,T_l)\big)(L(T_i,T_i)-K)^+\Big]. \quad (5.1)\end{aligned}$$

This equation will provide the actual prices of caplets corresponding to simulating the full SDE (Euler discretization) for the LIBOR rates. In order to calculate the Picard approximation prices for a caplet we have to replace $L(\cdot,T_\cdot)$ in (5.1) with $\widehat{L}(\cdot,T_\cdot)$.

5.2. **Swaptions.** A payer (resp. receiver) swaption can be viewed as a put (resp. call) option on a coupon bond with exercise price 1; cf. section 16.2.3 and 16.3.2 in Musiela and Rutkowski (1997). Consider a payer swaption with strike rate K, where the underlying swap starts at time T_i and matures at T_m ($i < m \le N$). The time-T_i value is

$$\begin{aligned}\mathbb{S}_{T_i}(K,T_i,T_m) &= \left(1-\sum_{k=i+1}^{m} c_k B(T_i,T_k)\right)^+\\ &= \left(1-\sum_{k=i+1}^{m}\left(c_k\prod_{l=i}^{k-1}\frac{1}{1+\delta L(T_i,T_l)}\right)\right)^+, \quad (5.2)\end{aligned}$$

where

$$c_k = \begin{cases}\delta_k K, & i+1\le k\le m-1,\\ 1+\delta_k K, & k=m.\end{cases} \quad (5.3)$$

Then, the time-0 value of the swaption is obtained by taking the $\mathbb{P}_{T_i}$-expectation of its time-T_i value, that is

$$\begin{aligned}\mathbb{S}_0 &= \mathbb{S}_0(K,T_i,T_m)\\ &= B(0,T_i)\,\mathbb{E}_{\mathbb{P}_{T_i}}\left[\left(1-\sum_{k=i+1}^{m}\left(c_k\prod_{l=i}^{k-1}\frac{1}{1+\delta L(T_i,T_l)}\right)\right)^+\right]\\ &= B(0,T_*)\\ &\quad\times\mathbb{E}_{\mathbb{P}_{T_*}}\left[\prod_{l=i}^{N}\big(1+\delta L(T_i,T_l)\big)\left(1-\sum_{k=i+1}^{m}\left(c_k\prod_{l=i}^{k-1}\frac{1}{1+\delta L(T_i,T_l)}\right)\right)^+\right],\end{aligned}$$

hence

$$\mathbb{S}_0 = B(0,T_*)\,\mathbb{E}_{\mathbb{P}_{T_*}}\left[\left(-\sum_{k=i}^{m}\left(c_k\prod_{l=k}^{N}\big(1+\delta L(T_i,T_l)\big)\right)\right)^+\right], \quad (5.4)$$

where $c_i := -1$.

5.3. **Forward Rate Agreements.** A forward rate agreement with strike K and notional value of 1 with expiry at time T_i, has the value of $\delta_i(K - L(T_i,T_i))$ at expiry. The time zero value of the contract has the model independent value of

$$\mathbb{F}_0(K,T_i) = \delta_i B(0,T_{i+1})[K - L(0,T_i)], \tag{5.5}$$

or zero if the contract is struck at-the-money ($K = L(0,T_i)$). In the following section we will compare the known true values with simulated prices generated from the terminal measure expectation of the payoff, i.e.

$$\mathbb{F}_0(K,T_i) = \delta B(0,T_*)\,\mathbb{E}_{\mathbb{P}_{T_*}}\Big[\prod_{l=i+1}^{N}\big(1+\delta L(T_i,T_l)\big)(K - L(T_i,T_i))\Big]. \tag{5.6}$$

6. Numerical illustration

The aim of this section is to demonstrate the accuracy and efficiency of the Picard approximation scheme and the drift expansions in the valuation of options in the Lévy LIBOR model. In the first section we demonstrate the accuracy of our methods compared to a standard Euler discretization of the LIBOR SDE in pricing caplets and swaptions. In the second section we estimate the speed of our method; finally we compare with alternative approximations.

We will consider a simple example with a flat volatility structure of $\lambda(\cdot,T_i) = 18\%$ and zero coupon rates generated from a flat term structure of interest rates: $B(0,T_i) = \exp(-0.04\cdot T_i)$. We consider a tenor structure with 6 month increments ($\delta_i = \frac{1}{2}$).

The driving Lévy process H is a normal inverse Gaussian (NIG) process with parameters $\alpha = \bar{\delta} = 12$ and $\mu = \beta = 0$, resulting in a process with mean zero and variance 1. We denote by μ^H the random measure of jumps of H and by $\nu(\mathrm{d}t,\mathrm{d}x) = F(\mathrm{d}x)\mathrm{d}t$ the $\mathbb{P}_{T_*}$-compensator of μ^H, where F is the Lévy measure of the NIG process. The necessary conditions are then satisfied for term structures up to 30 years of length because $M = \alpha$, hence $\sum_{i=1}^{60}|\lambda(\cdot,T_i)| = 10.8 < \alpha$.

The NIG Lévy process is a pure-jump process with canonical decomposition

$$H = \int_0^{\cdot}\int_{\mathbb{R}} x(\mu^H - \nu)(\mathrm{d}s,\mathrm{d}x). \tag{6.1}$$

The cumulant generating function of the NIG distribution, for all $u \in \mathbb{C}$ with $|\Re u| \le \alpha$, is

$$\kappa(u) = \bar{\delta}\alpha - \bar{\delta}\sqrt{\alpha^2 - u^2}. \tag{6.2}$$

6.1. **Accuracy of the methods.** The Picard approximation should be considered primarily as a parallelizable alternative to the standard Euler discretization of the model. This will therefore be the benchmark to which we compare. In order to avoid Monte Carlo error we use the same discretization grid (5 steps per tenor increment) and the same pseudo random numbers

(50000 paths) for each method. The pseudo random numbers are generated from the NIG distribution using the standard methodology described in Glasserman (2003).

Starting with caplets we can see in Figure 6.1 the difference between the Euler discretization and the Picard approximation expressed in price (left) and implied volatility (right). The difference in price is small with max. errors around half of a percentage of a basis point. On the right we see somewhat larger errors with a maximum slightly below 1 basis point of implied volatility for low strike mid-maturity caplets. Implied volatility is normally quoted in units of 1 basis point while bid-ask spreads are usually around at least 5 bp of implied volatility. The errors are therefore at acceptable levels. Note also that in experiments not shown we found that the levels and patterns of the errors are insensitive to the number of discretization points as well the number of paths.

The errors display a non-monotonic behavior as a function of maturity with peaks around mid-maturity. The non-monotonicity can be explained by the fact that volatility dominates the price of options in the short end, making the drift, and any error in it, less relevant. As maturity increases the importance of the drift grows relative to volatility but the state dependence becomes less critical as the number of "live" rates decreases. These two opposing effects result in the mid-maturity peak that we observe. This pattern is also noted in the study by Joshi and Stacey (2008).

As we established in (4.13) the number of terms needed to calculate the drift grows at a rate 2^N. In market applications N is often as high as 60 reflecting a 30 year term structure with a 6 month tenor increment. At this level even the calculation of one drift term becomes infeasible and this necessitates the use of the approximations introduced in (4.14) and (4.16). We investigate the errors introduced by the drift expansion by comparing them with the full numerical solution obtained as before using the true drift from (3.12). The results are plotted in Figure 6.2. Here we can see that the first order drift expansion adds errors of fairly low magnitude, whereas the second order drift expansion performs significantly better with errors so small that they can be disregarded. We also plot the Picard approximation alone as well as combined with the second order drift expansion and these are similarly indistinguishable.

Continuing on with swaptions, in Figure 6.3 we plot again differences in implied volatility between our methods and the Euler discretization. We observe even smaller errors than for caplets, most likely because swaptions span a broad range of maturities which smooths out the higher errors in the mid-maturity region.

6.2. **Computational speed.** In terms of computational time the largest gain by far is realized when using the drift expansions in (4.14) and (4.16). In the example above the CPU time for the full numerical solution is 1.5 hours; after applying the first order and second order drift expansion it drops to 1.3 and 27.2 seconds respectively.

The Picard approximation in itself does not contribute to the computational speed unless parallelization is employed. In fact, it is slightly but insignificantly slower as the auxiliary processes $Z^{(1)}$ have to be evolved along

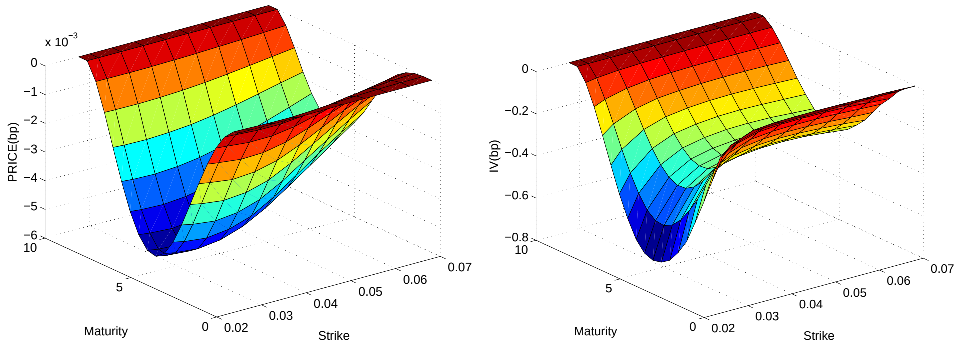

FIGURE 6.1. Difference in price (left) and implied caplet volatility (right) between the Euler discretization and the Picard approximation (in basis points).

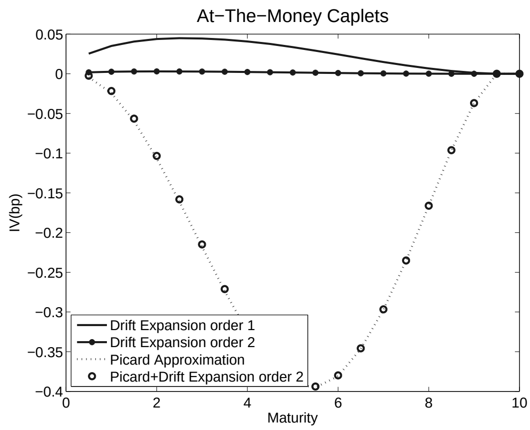

FIGURE 6.2. Difference in implied caplet volatility (in basis points) between Euler discretization and 4 other methods.

with the rates. On the left in Figure 6.4, we plot CPU time as a function of the number of paths for the Picard approximation and the full Euler discretization. In both cases the second order drift approximation scheme in (4.16) is employed. The computations are done in Matlab running on an Intel i7 processor with the capability of running 8 processes simultaneously. Here we see the typical linear behavior as the number of paths are increased; notice though that the Picard approximation has significantly lower slope. On the right in Figure 6.4 we plot CPU time as a function of the number of rates. One can see CPU time exponentially increasing, revealing that large gains in computational time are realizable when using the Picard approximation scheme and the drift expansion.

Needless to say, the speed advantages of the Picard approximation are only partially revealed in these graphs since we are using desktop computer; further speed increases can of course be realized as the access to more CPUs (or clusters of PCs) becomes available. This is already part of the infrastructure of many large financial institutions.

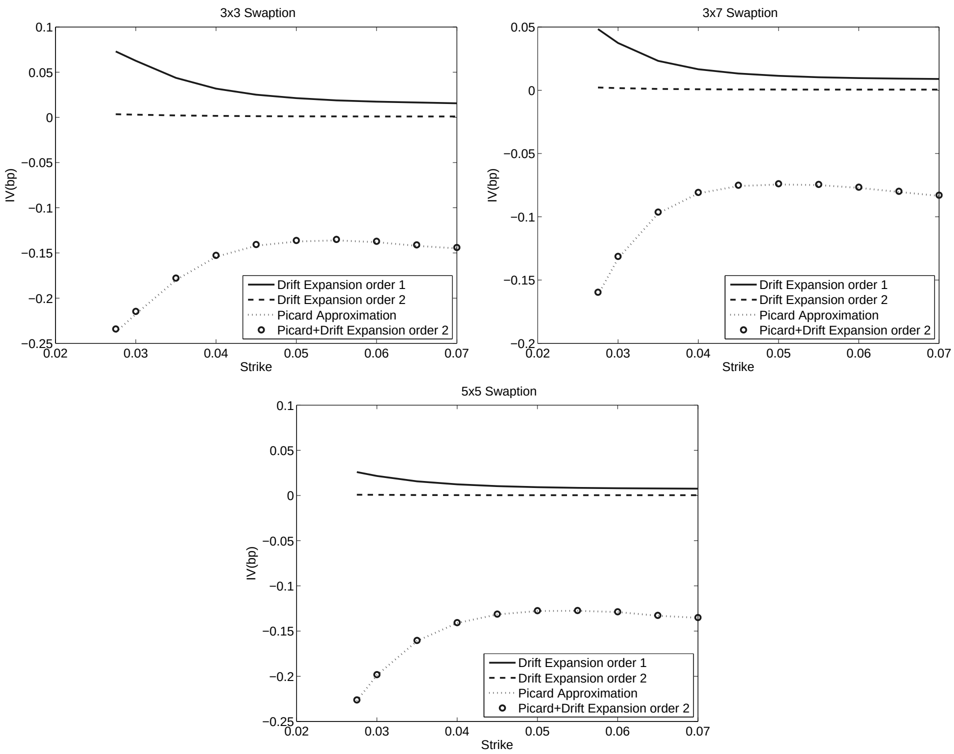

FIGURE 6.3. Difference in implied swaption volatility (basis points) between Euler discretization and 4 other methods.

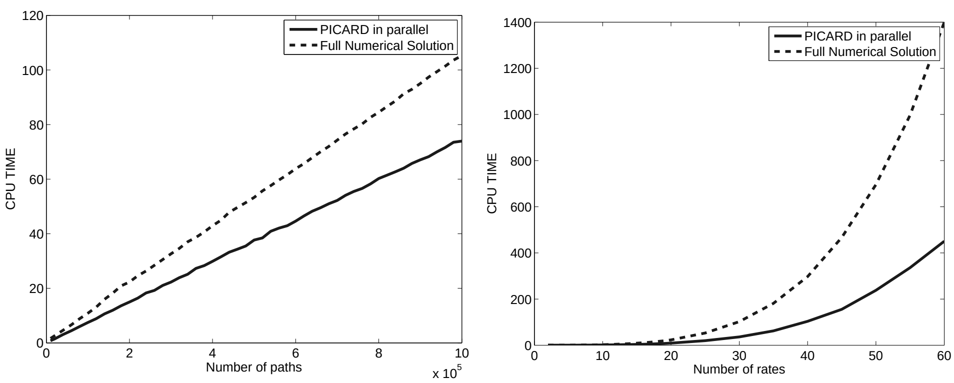

FIGURE 6.4. CPU time as a function of the number of paths (left) and the number of rates N (right).

6.3. **Comparison with other methods.** Unfortunately, very little work has been done in the area of approximations for LMMs driven by general semimartingales, leaving us without any standard method to compare with, other than the frozen drift approximation. As mentioned in the introduction the existing work has focused mainly on the log-normal case and the case of finite intensity jump-diffusion models. However, some of the techniques applied to the log-normal case can be adapted to our setup as well.

Assume we want to simulate the LIBOR rates from time t to time $t+h$, where $t+h \leq T_i$. We have

$$L(t+h, T_i) = L(t, T_i) \exp\left(\int_t^{t+h} b(s, T_i; Z(s))\mathrm{d}s + \int_t^{t+h} \lambda(s, T_i)\mathrm{d}H_s \right),$$

where $b(\cdot, T_i; Z(\cdot))$ is the state dependent drift function defined in (3.12); cf. also (4.3). The standard (log)-Euler scheme leads to

$$\int_t^{t+h} b(s, T_i; Z(s))\mathrm{d}s \approx b(t, T_i; Z(t))\, h. \tag{6.3}$$

This can be further refined as noted first in **?** (see also Hunter, Jäckel, and Joshi 2001, and Joshi and Stacey 2008 and the references therein) by using instead

$$\int_t^{t+h} b(s, T_i; Z(s))\mathrm{d}s \approx \left(\frac{1}{2} b(t, T_i; Z(t)) + \frac{1}{2} b(t+h, T_i; Z(t+h)) \right) h. \tag{6.4}$$

The second term in the parenthesis requires the knowledge of the LIBOR rates $L(t+h, T_{i+1}), \ldots, L(t+h, T_N)$; one therefore has to simulate these rates. This can be done in a separate simulation step and the procedure is known in the LMM literature as *predictor-corrector* (PC) method. One can also note that when rate i is evolved under the terminal measure it only depends on rates $k > i$. Furthermore, if we start with $i = N$ we have no state-dependence in the drift. We can then generate realizations of $L(t+h, T_N)$ without discretization error and these can be used in the drift of rate $N-1$ as described above. Realizations of rate $N-1$ can then be generated with the corrected drift from (3.12), which can be subsequently used in the drift of rate $N-2$, and so forth. This latter method is referred to as *iterative predictor-corrector* (IPC).

IPC has been found to often outperform PC in the log-normal case studied in Joshi and Stacey (2008). It is also slightly more efficient since it does not require a separate simulation step for the rates at time $t+h$.

Remark 6.1. One should point out that PC and IPC are alternative *discretization schemes* to the Euler discretization.

We can also combine PC and IPC with the Picard approximation by merely using $b(\cdot, T_i; Z^{(1)}(\cdot))$ instead of $b(\cdot, T_i; Z(\cdot))$ in (6.4). Furthermore, PC and IPC will actually be equivalent when applied to the Picard approximation since the drift term does not involve the rates themselves, but the auxiliary processes $Z^{(1)}$, which makes the order in which the rates are evolved irrelevant (see section 4.2).

We compare PC and IPC with our methods in 3 different cases. Since we do not have the true price of caplets or swaptions we instead compare prices of at-the-money forward rate agreements (FRA) since these all have a known model independent price of zero. Here we keep Monte Carlo error sufficiently low by simulating 5 million paths, employing antithetic sampling as the only variance redaction measure. Looking at the top left of Figure 6.5 we have kept the discretization grid dense at 5 steps per tenor period.

Here we see prices which for all methods are sufficiently small at max levels of 2.5% of a basis point. Furthermore the different methods are more or less indistinguishable – certainly in statistical terms with 95% confidence limit halfwidths ranging from about 0.004 bp in the short end to to 0.03 basis points in the long end. The frozen drift case is left out in these graphs since the errors/prices are so big that they dominate all other methods.

The second graph in the top right of Figure 6.5 shows prices for a discretization grid of 1 step per tenor period. Here the prices for Picard and Euler clearly reflect the higher discretization error and the PC and IPC methods are indistinguishable, and significantly lower than Picard and Euler. Note that prices are around the same level as in the 5 step per tenor period discretization.

Finally, we price each FRA using a single long discretization step from time zero to the maturity of each contract, also referred to as "long-stepping". In this case the Picard and Euler methods are analogous to the frozen drift approximation, while PC and Picard+PC are equivalent to each other; hence we draw only PC, IPC, and frozen drift. We see, somewhat surprisingly that the frozen drift slightly outperforms PC and IPC which contradicts the results in the log-normal case previously studied in the literature. Nevertheless the errors are quite high and beyond an acceptable level for all methods.

The general conclusion one can draw from these graphs is that Picard+PC with 1 discretization step per tenor period would be the preferable choice. The errors are indistinguishable from regular PC and IPC, but the method has the advantage that prices can be parallelized in the maturity dimension, and the gains in computational time shown in the previous section can be realized.

7. Conclusion

This paper derives new approximation methods for Monte Carlo simulation in LIBOR models. The methods address the problem of speed in Monte Carlo simulations by allowing for parallel computing through Picard approximations. In particular, our method decouples the interdependence of the rates when moving them forward in time in a simulation, meaning that computations can be parallelized in the maturity dimension. Furthermore, the largest computational load in the model stems from the exponential growth of the drift terms. We reduce this growth to quadratic through truncated expansions of the product terms that appear in the drift. The accuracy is very high if the second order expansion is employed and we showed that it reduces the computational load immensely.

Numerical methods for LIBOR market models driven by general semimartingales are still in their infancy. As we have demonstrated there is still work to be done in this area, in particular developing algorithms for pricing derivatives using long time-stepping. Predictor-corrector methods do not perform very well in this case and a finer discretization grid needs to be employed.

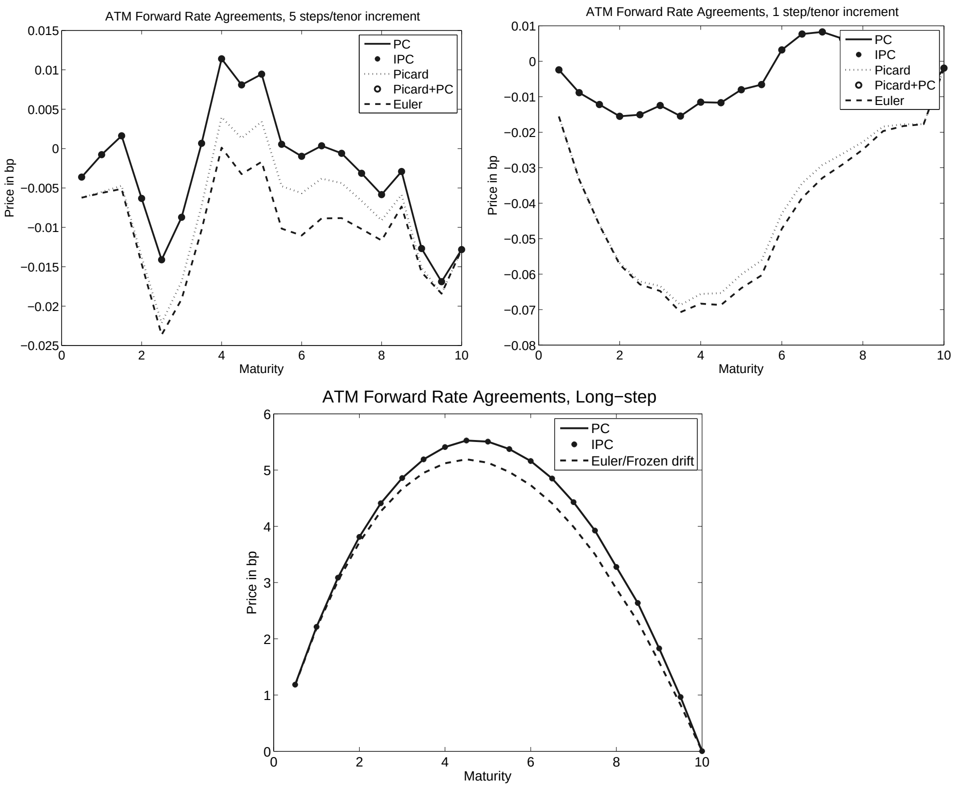

FIGURE 6.5. Prices for at-the-money Forward Rate Agreements (in basis points). The true price is zero.

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