Long-horizon Investing:
Pensions and Private Equity

PhD dissertation

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Søren Kærgaard Slipsager
Oslo, April 2018
To Mette
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SUMMARY

This dissertation comprises three chapters that cover two different economic topics. The first and second chapters consider long-horizon portfolio value forecasting in a pensions savings context, while the third chapter studies fee structures in private equity partnerships. At first glance, these topics seem rather dissimilar, but the common thread is an interest in quantitative investment portfolio management. In the former case through forecasting and risk analysis and in the latter case via asset pricing. Obviously, the topics differ in their target audiences as pension forecasting is relevant to the broad audience of pension savers, whereas mainly professional investors care about the valuation of private equity contracts. Nevertheless, all three chapters seek to answer questions relevant to the industry through academic approaches.

The first chapter, An analysis of a 3-factor model proposed by the Danish Society of Actuaries for forecasting and risk analysis (joint work with Peter Løchte Jørgensen),\(^1\) considers pension forecasting and risk analysis using the financial model proposed by the Danish Society of Actuaries in DSA (2014). The main driver behind the proposal and the thereof derived chapter, has been the life insurance and pension companies active shifting of policy holders with participating/with-profits contracts toward market value-based unit-linked contracts. The change in risk profile caused by replacing "guaranteed" participating/with-profits contracts with unit-linked contracts offering relatively variable returns has created a need for industry consistent risk modeling. The specific financial model has been considered previously in the life insurance and pensions literature by Jørgensen and Linnemann (2012), who use it for pension product evaluations. This chapter provides the explicit solution and an exact simulation scheme to the financial model that are usable for practitioners facing the model implementation. By use of a simulation study we illustrate that the exact simulation is generally superior to the traditional Euler simulation scheme in terms of forecasting accuracy and computation efficiency. This chapter conducts a risk analysis of a conventional real-world pension scheme using both simulation schemes and finds substantial relative differences in the considered risk measures. The risk analysis is accompanied by a sensitivity analysis that suggests that the differences in the risk measures are highly dependent on the chosen model parameter values.

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\(^1\)Published in *Scandinavian Actuarial Journal* 2016 (9), 837-857.
The second chapter, *The real risk in pension forecasting*,\(^2\) extends the financial model considered in chapter 1 by stochastic inflation and compares its pension forecasting ability to that of a corresponding model with deterministic inflation. The estimation of the extended model is performed with offset in Munk, Sørensen, and Nygaard (2004), who study the model in a utility-based consumption-portfolio choice framework. The results of this chapter suggest that the inclusion of the specific inflation dynamics leads to improved in-sample and out-of-sample forecasts and hence better pension forecasts. Additionally, the analysis indicates that the deterministic inflation approach, currently applied in the pension industry, causes misestimation in both distribution location and dispersion when forecasting real portfolio values. Through the examination of realistic expected real-value payout profiles of a pension saver, I find that the deterministic scheme underestimates downside risk and overestimates upside potential as the scheme does not account for the covariance between the nominal portfolio value and the price index.

The third chapter, *The valuation of catch-up provisions in fund managers' compensation contracts* (joint work Peter Løchte Jørgensen), studies the fee structures of private equity partnerships with an emphasis on the catch-up provision. The latter is a fund manager provision defined as a period in which the fund manager acquires profits at an increased rate until a predetermined profit-split threshold is reached. The chapter values the private equity partnerships' interests and computes catch-up induced value transfers. Due to an opaque secondary market for private equity securities causing the traditional contingent claims valuation approaches to break down, we perform the valuations under both effectively complete markets and incomplete markets. For the complete markets, we use the spanned fund risk model of Sørensen, Wang, and Yang (2014) and in the case of the incomplete markets, we use the good-deal bounds framework of Cochrane and Saa-Requejo (2000). In order to account for early fund dissolutions, both models are extended by stochastic premature fund termination. The results of this chapter suggests that unspanned fund risk gives rise to a substantial amount of valuation uncertainty in a typical private equity setting. Moreover, the results show that the costs of including a catch-up in a typical contract are of a non-negligible size. Finally, the chapter presents the catch-up implied value transfers as costs in terms of fund manager skill level (alpha), annual management fee, and carried interest rate.

\(^{2}\)Published in *Scandinavian Actuarial Journal* 2018 (3), 250-273.


References


Danish summary


afhængige af de valgte modelparame
terværdier.

Det andet kapitel, *The real risk in pension forecasting*² udvider den finansiel-
le model som vi betragtede i kapitel 1 med stokastisk inflation og sammenligner
modellens pensionsprognoseevne med den fra en tilsvarende model med determi-
nistisk inflation. Estimeringen af den udvidede model er udført med udgangspunkt
i Munk, Sørensen, og Nygaard (2004), som studerer modellen i et nyttebaseret forbrug-
porteføljevalgssystem. Kapitlets resultater viser, at inkluderingen af den specifikke
inflationsdynamik resulterer i forbedrede pensionsprognoser. Yderligere indikerer
analysen, at den deterministiske inflationstilgang, som i skrivende stund anvendes
i praksis, giver anledning til misestimering af både fordelingsplacering og –spred
ning ved prognosticering af reale porteføljeværdier. Ved at betragte en realistisk
pensionsopspørers forventede udbetalingsværdier i reale termer finder jeg, at den
deterministiske inflationstilgang underestimerer tabrisiko og overestimerer gevin-
spotentielle. Årsagen til dette er, at tilgangen ikke tager højde for kovariansen mellem
den nominelle porteføljeværdi og prisindekset.

Det tredje kapitel, *The valuation of catch-up provisions in fund managers’ com-
pensation contracts* (skrevet i samarbejde med Peter Løchte Jørgensen), studerer
afslønningsstrukturerne i kapitalfondspartnerskaber med et særligt fokus på catch-up
provisionen. Sidstnævnte er en forvalterprovision, der bestemmes som et led i prove-
nufordelen mellem investor og forvalter og er karakteriseret ved, at forvalteren i en
periode får en forøget andel af provenuet, indtil et forudbestemt provenudelingsmål
er nået. Kapitlet værdiansætter kapitalfondspartnerskabets fordringer og beregner
catch-up-forårsagede værdioverførsler. Grundet et uigennemsigtet sekundært mar-
ked for kapitalfondsaktier som medfører, at traditionelle modeller til værdiansættelse
af betingende fordringer bryder sammen, udfører vi værdiansættelserne under både
effektivt komplette markeder og ikke-komplette markeder. Til de komplette markeder
anvender vi ”spanned fund risk” modellen fra Sørensen, Wang, og Yang (2014), og i
tilfælde af ikke-komplette markeder anvender vi ”good-deal bounds” modellen fra
Cochrane og Saa-Requejo (2000). For at tage højde for utilsigted tidlig fondsluk-
ninger bliver begge modeller udvidet med stokastiske fortidige fondslukninger. Kapitlets
resultater antyder, at ikke-handlet fondsrisko giver anledning til stor værdianset-
sesusikkerhed i en typisk kapitalfondssammenhæng. Derudover viser resultaterne, at
inkludering af en catch-up i en typisk kontrakt fører til anselige omkostninger. Slutte-
ligt præsenterer kapitlet de catch-up-afledte værdioverførsler i form af forvalterens
evner (alfa), årlige administrationsafgifter og carried interest rater.

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Litteratur


AN ANALYSIS OF A THREE-FACTOR MODEL PROPOSED BY THE DANISH SOCIETY OF ACTUARIES FOR FORECASTING AND RISK ANALYSIS

PUBLISHED IN SCANDINAVIAN ACTUARIAL JOURNAL 2016 (9), 837-857

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Abstract

This paper provides the explicit solution to the three-factor diffusion model recently proposed by the Danish Society of Actuaries to the Danish industry of life insurance and pensions. The solution is obtained by use of the known general solution to multi-dimensional linear stochastic differential equation systems. With offset in the explicit solution, we establish the conditional distribution of the future state variables which allows for exact simulation. Using exact simulation, we illustrate how simulation of the system can be improved compared to a standard Euler scheme.

In order to analyze the effect of choosing the exact simulation scheme over the traditional Euler approximation scheme frequently applied by practitioners, we carry out a simulation study. We show that due to its recursive nature, the Euler scheme
becomes computationally expensive as it requires a small step size in order to minimize discretization errors. Using our exact simulation scheme, one is able to cut these computational costs significantly and obtain even better forecasts. As probability density tail behavior is key to expected investment portfolio performance, we further conduct a risk analysis in which we compare well-known risk measures under both schemes. Finally, we conduct a sensitivity analysis and find that the relative performance of the two schemes depends on the chosen model parameter estimates.

1.1 Introduction

Financial asset value and return forecasts play a key role in the industry of life insurance and pensions (L&P) since the ex ante attractiveness of an L&P contract depends hereupon. With the mean-variance relationship of Markowitz (1952) in mind an L&P company is able to increase the expected return on an investment portfolio by increasing its underlying risk. This potential issue produces a need for transparency and consistency across L&P valuation schemes. Therefore, the Danish Society of Actuaries (DSA) has recently issued a proposed guideline in DSA (2014) to ensure consistent estimation of the expected future payments across contract types and L&P companies. The main contribution of their report is to propose the recommendation of a dynamic model for modeling expected portfolio returns, henceforth referred to as the DSA model.

The proposed model is highly relevant since L&P companies have been shifting policy holders with traditional participating/with-profits contracts toward market value-based unit-linked contracts, as the L&P companies tend to prefer more market-based liabilities with fewer guarantees (Johannesson and Matthiesen, 2014). This shifting can be seen as a sound decision since the market value-based unit-linked contracts have an increased transparency, the investment climate has changed, and the markets’ preferences and requirements are better suited for these products (Jørgensen, 2007).

The unit-linked contracts are characterized by the assets’ placement on individual accounts offering relatively variable and not necessarily smooth returns. As the return on the traditional participating/with-profits contracts consists primarily of a basic interest rate, which is often guaranteed, and a fairly stable annual bonus reflecting the success of the investment strategy undertaken, the shift causes a need for lucid communication due to a radical change in risk profile (DSA, 2014). It is not hard to imagine how the unit-linked contracts can be presented as relatively more attractive when comparing expected future payments since an increase in the expected return can be driven by an underlying risk increase, which is not necessarily communicated. Since the forecasted value is highly dependent on the underlying financial model, regulators should ensure that a common valuation framework is in place. As matters stand, Danish regulators do in fact provide industry assumptions to be used in fore-
casting. Unfortunately, they only concern expected returns and not risks which is probably the motivation behind proposing the DSA model.

Since the overall purpose of the DSA model is to allow for forecasting future values of stocks, fixed income securities, and derivatives, the model must have certain generally accepted features. A standing consensus in the literature is that there exists an equity premium as first identified by Mehra and Prescott (1985), and that the expected equity rate of return is time-varying (Bollerslev, Engle, and Wooldridge, 1988; Harvey, 1989; Lettau and Ludvigson, 2002). These empirics are taken into account by modeling the equity return with a stochastic equity premium. Specifically, this implies that the equity returns themselves are mean-reverting which is empirically backed by the long-run predictability found by, among others, Fama and Schwert (1977) and Campbell and Shiller (1988). Further, a proper model for pricing fixed income securities should feature a stochastic interest rate and since the literature on dynamic interest rate models is vast (see e.g. Merton (1973); Vasicek (1977); Dothan (1978); Cox, Ingersoll Jr, and Ross (1985); Hull and White (1990) or for an empirical comparison Chan, Karolyi, Longstaff, and Sanders (1992)) several possibilities exist.

Even though the DSA model is empirically appropriate and theoretically tractable, potential challenges exist regarding its implementation as performance and computation costs are highly dependent on the choice of simulation method. Therefore, this paper focuses on the implementation of the DSA model and more specifically on the importance of choosing the right simulation scheme. We show that a naive implementation of the model using the traditional Euler approximation scheme becomes computationally costly due to its recursive nature. As an alternative, we derive the conditions needed for the so-called exact simulation which samples from the exact conditional distribution of future values of the state variables.

In order to evaluate on the choice of simulation scheme, we conduct an analysis in which we compare the forecasting ability and computational expenses of both schemes. Since investors care about downside risk and upside potential, we also analyze the tail behavior of the simulated probability density functions. In the risk analysis we mimic a real-world pension contract scheme and consider well known risk measures such as Value-at-Risk and conditional Value-at-Risk and we investigate the upside potential through corresponding measures. As we expect our result to be dependent on the used model parameter estimates, we accompany the risk analysis with a parameter sensitivity analysis.

Our main contribution is to provide the explicit solution and the exact simulation scheme of the DSA model which should ease its implementation in the L&P industry. and thereby allow the L&P companies to cut their computational costs without losing performance.

The remainder of this paper is organized as follows. In Section 1.2, we describe the financial model proposed by DSA and derive the distribution of future values of the state variables. The Euler and the exact simulation schemes are also provided in
Section 1.2. Section 1.3 contains the simulation study in which we analyze performance and risk through the forecasting ability, computational expenses, tail behavior, and model parameter sensitivity. Finally, in Section 1.4, we provide some concluding remarks.

1.2 Methodology

When doing \textit{exact simulation} the conditional distribution of the simulated state variables coincides with the conditional distribution of the continuous-time processes on the simulation time grid, and thus one obtains unbiased estimates of the future state variables. In the following, we introduce the DSA model and study the performance of the exact simulation scheme and compare it to the Euler approximation scheme, often used in simulation of stochastic differential equations whose conditional distribution of the future state variables is unknown.

1.2.1 The DSA model

The DSA model is a three-factor diffusion model with uncertainty modeled by two Brownian motions and has been studied in both Wachter (2002) and Munk, Sørensen, and Nygaard Vinther (2004) but with the aim of deciding an optimal strategy in a utility-based consumption-portfolio choice framework. Our use of the model is somewhat different since we focus on characterizing future portfolio values and thus a dynamic modeling of asset prices. In that relation, a limitation of the model is that it only contains two primary assets: stocks and bonds and hence other assets must be approximately allocated to the stock/bond asset classes. As DSA (2014) states, the model is only constructed around two asset types such that the most risky asset of the model should contain the risky asset classes of the portfolio such as traditional stocks and likewise the least risky asset should contain different bond types of various durations.

Considering the DSA model, the stock price dynamics take offset in a modified Black and Scholes (1973) setting in the sense that the relative drift consists of a risk-free part and a risky part. Specifically, it is the sum of a short rate following the Vasiceck (1977) model and a mean-reverting equity risk premium governed by an Ornstein-Uhlenbeck process.

Let the following stochastic processes represent the three state variables: the stock price $S = \{S_t\}_{t \geq 0}$, the equity risk premium $x = \{x_t\}_{t \geq 0}$, and the short-term interest rate $r = \{r_t\}_{t \geq 0}$. Further, let $(S, x, r)$ be defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and adapted to the filtration $\mathcal{F}_t$, which represents all available information at time $t \geq 0$. In the following, conditioning on time $t$ information will
be denoted by a subscript \( t \). The dynamics of the model is given by

\[
\frac{dS_t}{S_t} = (r_t + x_t) dt + \sigma_S dz_{1,t},
\]

(1.1)

\[
dx_t = \alpha (\xi - x_t) dt - \sigma_x dz_{1,t},
\]

(1.2)

\[
dr_t = \kappa (\theta - r_t) dt + \sigma_r dz_{2,t},
\]

(1.3)

where \( dz_{1,t} \) and \( dz_{2,t} \) are instantaneously correlated Brownian motions, i.e. \( dz_{1,t} \cdot dz_{2,t} = \rho dt \) with constant correlation coefficient \( \rho \in [-1;1] \).

As appears from equation (1.1) the stock price is described by a geometric Brownian motion (GBM) yet with the modification of a stochastic relative drift and a constant relative local volatility, \( \sigma_S > 0 \). The expected instantaneous rate of return of the stock is given by \( (r_t + x_t) \), which seems intuitively appealing since the instantaneous expected rate of return is decomposed into a risk-free short-rate part and an equity risk premium part. Including a stochastic equity premium in the relative drift of a stock was first introduced by Merton (1971) and gives the model the ability to capture a time-varying compensation for bearing equity risk. A large body of literature exists that supports a time-varying equity premium see e.g. Keim and Stambaugh (1986); Fama and French (1988); Bansal and Yaron (2004).

Considering the stochastic differential equation of \( x_t \) in (1.2) the first term implies a long-run regressive adjustment of the equity premium toward its long-run level, \( \xi \), with the \( \alpha > 0 \) parameter controlling the speed of adjustment. Further, the volatility of the Ornstein-Uhlenbeck process describing the equity risk premium, \( \sigma_x > 0 \), is assumed constant. Finally, the stock price and the equity risk premium are instantaneously perfectly negatively correlated, which gives rise to mean-reversion in the stock returns, i.e. high stock returns tend to be followed by relatively small expected returns.

Equation (1.3) describes the dynamics of the short rate following the traditional Vasicek model ensuring mean-reversion with long-term level \( \theta > 0 \), persistence parameter \( \kappa > 0 \), and a constant local volatility of \( \sigma_r > 0 \). Traditionally, it is seen as a weakness of the Vasicek model that it allows for negative interest rates, but taking the current interest rate levels into account this property should perhaps no longer be regarded as a weakness.

It is well known that the Vasicek model leads to zero-coupon bond prices being an exponential affine function of the current short rate, see e.g. Björk (2004). The time \( t \) price of a zero-coupon bond maturing at time \( T > t \) is given by

\[
P(r_t, t, T) = \exp\left\{ \theta - \frac{\sigma_r \Lambda}{\kappa} - \frac{\sigma_r^2}{2\kappa^2}\left( \Psi(\kappa, t, T) - (T - t) \right) \right. \\
- \frac{\sigma_r^2}{4\kappa} \Psi(\kappa, t, T)^2 - \Psi(\kappa, t, T)r_t \right\},
\]

(1.4)
where $\lambda$ denotes the market price of interest rate risk which we assume constant. Here, we have introduced the auxiliary function

$$\Psi(a, t, T) = \frac{1}{a} \left( 1 - e^{-a(T-t)} \right),$$

which will be heavily used in the remainder of this paper. Since $\Psi(\kappa, t, T) > 0$ the zero-coupon price becomes a convex decreasing function of the short rate. As derived in Vasicek (1977), the dynamics of the zero-coupon price $P_t^T = P(r_t, t, T)$ can be written as

$$dP_t^T = P_t^T \left( (r_t - \lambda \Psi(\kappa, t, T) \sigma_r) dt - \Psi(\kappa, t, T) \sigma_r dZ_{2,t} \right).$$

The stochastic shocks to the short rate are instantaneously negatively correlated with the zero-coupon price which is an appropriate feature of the model. After having stated the model dynamics, we turn to the derivation of the multivariate distribution of future values of the three state variables conditional on time $t$ information which we will need for the exact simulation.

### 1.2.2 The Conditional Distribution

The first step in deriving the conditional distribution of the state variables is to determine the explicit solution to the stochastic differential system. Subsequently, the obtained solution will reveal the distributions of the future values. All derivations regarding the explicit solution of the system and the conditional distribution are relegated to the appendix. The purpose of this subsection is solely to provide the main results related to the simulation task.

When dealing with a stock price modeled by a GBM, it is convenient to study the logarithmic version instead, and thus Itô’s Lemma is applied with the function $g(S, t) = \ln S$ and the process $y_t = g(S_t, t) = \ln S_t$ is defined. The dynamics of logarithmic stock price becomes

$$d \ln S_t = \left( r_t + x_t - \frac{1}{2} \sigma_S^2 \right) dt + \sigma_S dZ_{1,t},$$

and the explicit solution for $0 \leq t < T$ is

$$\ln S_T = \ln S_t + \theta (T-t) + (r_t - \theta) \Psi(\kappa, t, T) + \xi (T-t) \Psi(\alpha, t, T) - \frac{1}{2} \sigma_S^2 (T-t) - \sigma_x \int_t^T \Psi(\alpha, u, T) dZ_{1,u} + \sigma_S \int_t^T dZ_{1,u} + \sigma_r \int_t^T \Psi(\kappa, u, T) dZ_{2,u}.$$  

As appears, the future logarithmic stock returns are normally distributed with future values of $\ln S$ having conditional mean $m(S_t, x_t, r_t, t, T)$ and variance $V(t, T)$, i.e.

$$\ln S_T | \mathcal{F}_t \sim N\left( m(S_t, x_t, r_t, t, T), V(t, T) \right),$$  

(1.9)
where
\[
m(S_t, x_t, r_t, t, T) = \ln S_t + \theta (T - t) + (r_t - \theta) \Psi(\kappa, t, T) + \xi (T - t) + (x_t - \xi) \Psi(\alpha, t, T) - \frac{1}{2} \sigma_S^2 (T - t)
\] (1.10)

and
\[
V(t, T) = (T - t) \left( \frac{\sigma_x^2}{\alpha^2} + \sigma_S^2 + \frac{\sigma_r^2}{\kappa^2} + \frac{2 \rho \sigma_S \sigma_r}{\alpha \kappa} + \frac{2 \rho \sigma_x \sigma_r}{\alpha \kappa} - \frac{\sigma_x^2}{\alpha^2} \right)
\]
\[
- \frac{\sigma_x^2}{2 \alpha} \Psi(\alpha, t, T)^2 - \frac{\sigma_r^2}{2 \kappa} \Psi(\kappa, t, T)^2
\]
\[
+ \Psi(\alpha, t, T) \left( \frac{2 \sigma_x \sigma_S}{\alpha} + \frac{2 \rho \sigma_x \sigma_r}{\alpha \kappa} - \frac{\sigma_x^2}{\alpha^2} \right)
\]
\[
+ \Psi(\kappa, t, T) \left( \frac{2 \rho \sigma_r \sigma_S}{\kappa} - \frac{2 \rho \sigma_r \sigma_S}{\kappa} - \frac{\sigma_r^2}{\kappa^2} \right)
\]
\[
- \Psi(\alpha + \kappa, t, T) \frac{2 \rho \sigma_x \sigma_r}{\alpha \kappa}
\] (1.11)

It is seen that only the mean depends explicitly on the time \( t \) state variables and that neither the mean nor the variance depend directly on the calendar date \( t \) but only on the time to maturity. These seem to be reasonable properties, as changes in the distribution parameters should reflect changes in the economy, i.e. the state variables, and not the passage of time.

Turning to the equity risk premium and the short-rate processes both are modeled by Ornstein-Uhlenbeck processes with constant diffusion terms, and thus their explicit solutions are well known. The explicit solutions to equations (1.2) and (1.3) for \( 0 \leq t < T \) are given by
\[
x_T = x_t e^{-\alpha(T-t)} + \xi \left( 1 - e^{-\alpha(T-t)} \right) - \sigma_x \int_t^T e^{-\alpha(T-u)} \, dz_{1,u}, \tag{1.12}
\]
\[
r_T = r_t e^{-\kappa(T-t)} + \theta \left( 1 - e^{-\kappa(T-t)} \right) + \sigma_r \int_t^T e^{-\kappa(T-u)} \, dz_{2,u}. \tag{1.13}
\]
Both equations give rise to normally distributed future values and allow for easy determination of the conditional means
\[
E_t[ x_T ] = x_t e^{-\alpha(T-t)} + \xi \left( 1 - e^{-\alpha(T-t)} \right), \tag{1.14}
\]
\[
E_t[ r_T ] = r_t e^{-\kappa(T-t)} + \theta \left( 1 - e^{-\kappa(T-t)} \right). \tag{1.15}
\]
The conditional variances follow using that the integrands are deterministic functions of time
\[
\text{Var}_t[ x_T ] = \frac{\sigma_x^2}{2 \alpha} \left( 1 - e^{-2\alpha(T-t)} \right), \tag{1.16}
\]
In order to establish the complete conditional distribution of the state variables needed for exact simulation, we must derive the conditional covariances of the system \((\ln S_T, x_T, r_T)\). The covariances are as follows:

\[
\text{Cov}_t \left[ \ln S_T, x_T \right] = -\sigma_x \sigma_S \Psi \left( \alpha + \kappa, t, T \right) - \rho \sigma_x \sigma_r \Psi \left( \kappa, t, T \right),
\]

\[
\text{Cov}_t \left[ \ln S_T, r_T \right] = \rho \sigma_S \sigma_r \Psi \left( \kappa, t, T \right) - \rho \sigma_x \sigma_r \Psi \left( \alpha, t, T \right)
\]

and

\[
\text{Cov}_t \left[ x_T, r_T \right] = -\rho \sigma_x \sigma_r \Psi \left( \alpha + \kappa, t, T \right).
\]

As linear stochastic differential systems on this form are well known to the academic literature see e.g. Björk (2004), we are able to state a complete and tractable distribution of the future state variables. It has been shown that using the logarithmic stock price, the model is a three-factor Gaussian model and as the future values of the three state variables are normally distributed the application is straightforward. With reference to earlier equation numbers, the conditional distribution is given as:

\[
\begin{bmatrix}
\ln S_T \\
x_T \\
r_T
\end{bmatrix} \mid \mathcal{F}_t \sim \mathcal{N}
\begin{bmatrix}
(1.10) \\
(1.14) \\
(1.15)
\end{bmatrix},
\begin{bmatrix}
(1.11) & (1.18) & (1.19) \\
(1.18) & (1.16) & (1.20) \\
(1.19) & (1.20) & (1.17)
\end{bmatrix}.
\]

The following section contains the Euler discretization often used for quick-and-dirty simulation of stochastic processes as well as our recommended exact simulation scheme.

### 1.2.3 Simulation Schemes

When applying a simulation scheme with offset in dynamics modeled by continuous-time stochastic processes, as is the case here, the need for a discretization scheme arises. This is due to the practical implementation of simulation studies being carried out in discrete time steps, and thus one needs to "discretize" the continuous-time processes into a discrete-time process. Several discretization schemes exist for this purpose, see e.g. Asmussen and Glynn (2007). As stated in the outline, we consider the Euler approximation scheme, which is known for its simplicity and easy implementation, and compare it to the exact simulation scheme which in general requires some model tractability.
Besides being simple and easy to implement, the Euler approximation scheme offers a lot of flexibility since it is almost universally applicable (Glasserman, 2004). The scheme has two severe drawbacks: i) it is an approximation scheme which gives rise to a discretization error, and ii) it can be computationally expensive. The latter in the sense that in order to simulate a given value at a given point in time, one has to simulate intermediate values even though these are of no interest. The following subsection states the Euler approximation scheme for the three-factor model recommended by the DSA.

1.2.3.1 The Euler Approximation

Let the time-discretized approximation be carried out on time grid $0 = t_0 < t_1 < \cdots < t_m = T$ with fixed spacing, $h > 0$, such that $t_j = j \cdot h$ and let the simulated value of a given stochastic process, $Y$, be denoted by $\hat{Y}$.

The Euler approximation of the stochastic differential equations of the stock price, the equity risk premium, and the interest rate stated in equations (1.1), (1.2), and (1.3), will be given by:

$$\hat{S}(t_j + 1) = \hat{S}(t_j) + \left(\hat{r}(t_j) + \hat{x}(t_j)\right)\hat{S}(t_j) h + \sigma_S \hat{S}(t_j) \sqrt{\rho} \epsilon_{j+1}, \quad (1.22)$$

$$\hat{x}(t_j + 1) = \hat{x}(t_j) + \alpha \left(\xi - \hat{x}(t_j)\right) h - \sigma_x \sqrt{\rho} \epsilon_{j+1}, \quad (1.23)$$

$$\hat{r}(t_j + 1) = \hat{r}(t_j) + \kappa \left(\theta - \hat{r}(t_j)\right) h + \sigma_r \eta_{j+1}, \quad (1.24)$$

where $\{\epsilon_j\}_{j=1}^m$ and $\{\eta_j\}_{j=1}^m$ are sequences of i.i.d. standard normal variables with contemporaneous correlation $\rho$, implemented in practice by use of Cholesky factorization. From the equations it is clear that in order to approximate the value of, say, $S$ at $t_i$ where $0 < t_i \leq t_m$, one needs to approximate $S$ at $t_0, \ldots, t_{i-1}$ due to the recursive nature of the equation system.

1.2.3.2 Exact Simulation

When the dynamics of the model is sufficiently tractable and the distribution of the future state variables is available, one can apply the so-called exact simulation. As mentioned earlier, the term exact stems from the fact that future state variables will be distributed exactly as the dynamics of the model imply. Therefore, there will be no discretization error implying potentially biased estimates and further we are able to simulate unbiased values at any point in time with no need for intermediate values. When doing exact simulation from an arbitrary point $t_j$ and $\Delta t$ forward, where $0 \leq t_j < t_j + \Delta t \leq t_m$, one should use the distribution of the future state variables in equation (1.21) and the scheme becomes
\[
\hat{S}(t_i + \Delta t) = \exp \left\{ m \left( \hat{S}(t_i), \hat{x}(t_i), \hat{r}(t_i), t_i, t_i + \Delta t \right) + \sqrt{V(t_i, t_i + \Delta t)} \varepsilon \right\},
\]
(1.25)

\[
\hat{x}(t_i + \Delta t) = \hat{x}(t_i) e^{-\alpha \Delta t} + \xi \left( 1 - e^{-\alpha \Delta t} \right) + \sigma_x \sqrt{\frac{1}{2 \alpha} \left( 1 - e^{-2 \alpha \Delta t} \right)} \zeta,
\]
(1.26)

\[
\hat{r}(t_i + \Delta t) = \hat{r}(t_i) e^{-\kappa \Delta t} + \theta \left( 1 - e^{-\kappa \Delta t} \right) + \sigma_r \sqrt{\frac{1}{2 \kappa} \left( 1 - e^{-2 \kappa \Delta t} \right)} \eta,
\]
(1.27)

where \( \eta, \zeta \), and \( \xi \) standard normal variables satisfying the correlation structure of (1.21).

The linkage between the Euler scheme and exact simulation is the first-order Taylor approximation \( e^x \approx 1 + x \) at work in the Euler scheme.\(^1\) It is well known that this approximation is best for small values of \( x \). Ignoring our logarithmic transformation of the stock price, the Euler scheme, in general, closely resembles the exact simulation in the case of a small \( h \). The obvious downside of using a small \( h \) is that \( m \) has to be large which is computationally expensive.

### 1.3 Numerical Results

Since the aim of the DSA model is to ensure consistent valuation across offered contract types and L&P companies, a need for identical model parameter estimates arises. Given that the proposed model becomes industry standard, one of the trade associations or regulators must handle this calibration centrally. Furthermore, the calibration must correspond with the long-term industry assumptions of 2015 imposed by the Danish Insurance Association and the Danish Bankers Association. In this simulation study, we will use the parameter estimates provided by DSA (2014) as our reference. They are presented in table 1.1.

In our analysis, we follow DSA (2014) and use the long-term levels of the equity risk premium and the short rate as initial values \( x_0 = \xi = 0.0391 \), and \( r_0 = \theta = 0.0286 \). Finally, without loss of generality, we set the initial stock price \( S_0 = 100 \).

The first part of the analysis considers the probability densities of the time \( T = 1 \) horizon state variables using the different simulation schemes.

#### 1.3.1 Probability Densities

By simulating using the exact and the Euler scheme probability densities of time \( T \)-values of the three state variables are obtained. As it is hard to observe the differences between the simulation schemes in traditional density plots, we instead study deviations from the theoretical density. Using 50,000,000 simulations figure 1.1 depicts three deviation series computed as the theoretical density minus the

---

\(^1\)When applying the approximation to equation (1.25) one should compare the result to the Euler discretization of the logarithmic stock price.
1.3. NUMERICAL RESULTS

Table 1.1: Reference parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Numerical Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_S )</td>
<td>0.14</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.1</td>
</tr>
<tr>
<td>( \xi )</td>
<td>0.0391</td>
</tr>
<tr>
<td>( \sigma_x )</td>
<td>0.005</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>0.25</td>
</tr>
<tr>
<td>( \theta )</td>
<td>0.0286</td>
</tr>
<tr>
<td>( \sigma_r )</td>
<td>0.015</td>
</tr>
<tr>
<td>( \rho )</td>
<td>0</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>-0.25</td>
</tr>
</tbody>
</table>

respective sample density. We consider the sample densities using the exact scheme and the Euler scheme using both \( h = 1/12 \) and \( h = 1 \).

As appears from figure 1.1a depicting the deviations of the simulated distributions of the horizon stock price, the exact scheme yields a density that closely resembles the theoretical one. The Euler scheme with a step size of \( h = 1 \) produces a shifted and hence biased density, but as it appears using a smaller step size of \( h = 1/12 \) reduces the deviation. In both Euler cases, the Kolmogorov-Smirnov test rejects the null hypothesis that there is no significant difference between the sample density and the theoretical using a 1% significance level. Naturally, the shift emerges as a result of comparing the stochastic processes in equations (1.1) and (1.7). Nevertheless, this emphasizes the importance of replacing the quick-and-dirty Euler approach.

When considering the deviations of the simulated densities of the horizon short rate and equity risk premium, we are without this inconsistency. The figures 1.1b and 1.1c also indicate that a small step size is required for the Euler scheme to resemble the two theoretical densities. Applying the Kolmogorov-Smirnov test allows us to reject the null hypothesis for both Euler schemes for the short rate, but considering the risk premium, we cannot reject the null of the \( h = 1/12 \) case.

The consequence of these density diversities is that investment portfolio and pension scheme valuation and risk profiling become dependent on the simulation scheme which is not in the spirit of DSA (2014) since it leaves room for inconsistency across the L&P industry. The following subsection compares the schemes with respect to computational costs.

1.3.2 Computation Costs

It is quite clear that the Euler scheme is potentially more costly to apply than the exact scheme due to dependence on intermediate values and need for simulation across the time dimension. To compare the performance of the two schemes, we use
the logarithmic stock price process in equation (1.7) and the reference parameters in table 1.1 and compute Monte Carlo estimates of the expected stock price at time $T = 1$, i.e. $\bar{S}_T = \frac{1}{N} \sum_{i=1}^{N} \hat{S}_i(t_m)$ for both schemes. Thereby, we are able to compare solely the two sampling schemes.

In the Euler scheme, we follow the optimal Duffie-Glynn rule that one should double the number of time steps and quadruple the number of simulations at the same time (Duffie and Glynn, 1995). As a measure of accuracy, we follow Broadie and Kaya (2006) and use the root mean square error (RMSE) as it takes both variance and bias into account. Given in table 1.2 and 1.3 are results for both schemes using 1000 trials in the determination of the RMSE.

As stated by Duffie and Glynn (1995) the RMSE is halved using the Duffie-Glynn rule and it is seen that there is a significant difference between the two schemes. First, a bias is present in the Euler scheme though it is of small magnitude.
### Table 1.2: Estimation of $\hat{S}_T$ using the Euler scheme.

<table>
<thead>
<tr>
<th>Simulations</th>
<th>Steps</th>
<th>$\hat{S}_T$</th>
<th>RMSE</th>
<th>Bias</th>
<th>RVs</th>
<th>Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20,000</td>
<td>12</td>
<td>106.950</td>
<td>0.106</td>
<td>-0.022</td>
<td>240,000</td>
<td>13</td>
</tr>
<tr>
<td>80,000</td>
<td>24</td>
<td>106.971</td>
<td>0.052</td>
<td>-0.001</td>
<td>1,920,000</td>
<td>123</td>
</tr>
<tr>
<td>320,000</td>
<td>48</td>
<td>106.969</td>
<td>0.026</td>
<td>-0.003</td>
<td>15,360,000</td>
<td>1,100</td>
</tr>
<tr>
<td>1,280,000</td>
<td>96</td>
<td>106.972</td>
<td>0.013</td>
<td>-0.002</td>
<td>122,880,000</td>
<td>8,444</td>
</tr>
</tbody>
</table>

### Table 1.3: Estimation of $\hat{S}_T$ using the exact scheme.

<table>
<thead>
<tr>
<th>Simulations</th>
<th>$\hat{S}_T$</th>
<th>RMSE</th>
<th>RVs</th>
<th>Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20,000</td>
<td>106.967</td>
<td>0.105</td>
<td>20,000</td>
<td>2</td>
</tr>
<tr>
<td>80,000</td>
<td>106.972</td>
<td>0.052</td>
<td>80,000</td>
<td>11</td>
</tr>
<tr>
<td>320,000</td>
<td>106.973</td>
<td>0.026</td>
<td>320,000</td>
<td>45</td>
</tr>
<tr>
<td>1,280,000</td>
<td>106.972</td>
<td>0.013</td>
<td>1,280,000</td>
<td>168</td>
</tr>
</tbody>
</table>

The most prominent result lies in the computation time as the exact scheme is highly superior to the Euler scheme. The weakness of the Euler scheme is its need for simulating intermediate values such that the total number of random variables (RVs) increases heavily e.g. when comparing the case of 320,000 simulations the exact scheme is about 24 times faster. Therefore, by use of the exact scheme, one can cut the computational costs without loosing performance, or even better, increase performance for a fixed computational budget.

### 1.3.3 An Example of a Pension Scheme

As the purpose of the DSA model is to estimate the value and risk profile of investment portfolios and pension contracts, we conduct a risk analysis in which we take a closer look at this estimation task under both schemes.

We try to mimic a real-world scenario in which an agent plans to make a monthly payment of 5,000 into an investment portfolio related to his pension scheme. As prescribed by the DSA model, we assume that the investment is allocated between stocks and bonds. Inspired by actual asset allocation behind many traditional pension contracts, we allocate 35% to stocks and 65% to bonds. Further, we assume that a monthly rebalancing is made such that the bonds of the portfolio after each rebalancing have a duration of 5 years. Finally, we use 50,000,000 simulations and consider a 40-year horizon as this corresponds to a traditional pension scheme from beginning to end.

In our analysis, we consider the simulated time $T$ portfolio value distribution and
the following measures: i) the Value-at-Risk (VaR), ii) the conditional Value-at-Risk\(^2\) (cVaR), iii) what we call the *upside* defined as the 95\% quantile, and iv) the *expected upside* computed as the average value in the best 5\% of the outcomes. Both downside risk measures will be computed as relative measures i.e. as deviations from their time \(T\) horizon mean values (Jorion, 2007). Table 1.4 contains the estimated values using the reference parameters.

**Table 1.4:** Estimated tail measures using reference parameters.

<table>
<thead>
<tr>
<th>Measure</th>
<th>Exact</th>
<th>Euler ((h = 1/12))</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>VaR</td>
<td>2,354,375</td>
<td>2,344,780</td>
<td>9,595</td>
</tr>
<tr>
<td>cVaR</td>
<td>2,766,315</td>
<td>2,756,375</td>
<td>9,940</td>
</tr>
<tr>
<td>Upside</td>
<td>10,473,906</td>
<td>10,441,373</td>
<td>32,533</td>
</tr>
<tr>
<td>Expected upside</td>
<td>11,566,198</td>
<td>11,520,938</td>
<td>45,261</td>
</tr>
</tbody>
</table>

Considering first the left tail of the simulated portfolio value distributions, the exact simulation yields almost the same VaR and cVaR as the Euler scheme. Thereby, there does not seem to be a large monetary difference in their estimation of downside risk when using the reference parameters. As we study the right tail density represented by the upside and the expected upside, we find that the Euler scheme produces a slight underestimation of the probability mass in the right tail.

In order to evaluate the distribution similarities, we use the Kolmogorov-Smirnov test which allows us to reject the null of similarity on a very low significance level. Thereby, we can conclude that when using the reference parameters provided by DSA (2014), the \(T\) value distributions become statistically different, but in monetary terms, the outcome is of relatively small magnitude.\(^3\)

Following this conclusion it becomes relevant to investigate whether the result is robust to changing model parameter estimates. In order to do so, we conduct a local sensitivity analysis in which we consider the relative differences of the risk measures, e.g. \(\left(\frac{\text{VaR}_{\text{Exact}}^k - \text{VaR}_{\text{Euler}}^k}{\text{VaR}_{\text{Exact}}^k}\right)\) for the \(k\)'th parameter evaluation. The sensitivity analysis is local in the sense that only one parameter is varied at the time and the effect measured is therefore partial and only direct effects are taken into account while interaction effects are ignored.

The following plots show the relative differences for the risk measures for various parameter values using 1,000,000 simulations and it should be noted that the legends in figures 1.2a and 1.3a apply to all eight plots. In figure 1.2, we first consider the parameters of the stock price and equity premium dynamics.

As appears from figure 1.2a, we see that for increased values of \(\sigma_S\) the underestimation of upside potential becomes more severe. Further, it is concerning that the

\(^2\)Also known as *expected shortfall* or *expected tail loss*.

\(^3\)A total payment of 2.4m is made and average terminal value is above 7m under both schemes.
1.3. **Numerical Results**

VaR and cVaR deviations remain for all our analyzed parameter values. Thereby, the Euler scheme seems to consistently underestimate downside risk when varying of $\sigma_S$. Having in mind that a representative stock price volatility could easily be around 20% this could be troublesome.

For the mean-reversion parameter of the equity premium, we have considered parameter values covering an interval from a half-life\(^4\) of approximately 14 years to 1 year. From figure 1.2b we see that the relative differences are not very sensitive toward partial changes in $\alpha$. However, the equity premium parameter values do matter as can be seen from figure 1.2c and 1.2d. First, the long-term equity premium heavily affects the relative differences as they all increase in $\xi$. Since a strong disagreement \(^4\)The half-life is the expected time it takes the short rate to move half the distance toward its long-term level.

---

**Figure 1.2:** Sensitivity plots for the parameters of the stock price and equity premium.

(a) Relative difference vs $\sigma_S$

(b) Relative difference vs $\alpha$

(c) Relative difference vs $\xi$

(d) Relative difference vs $\sigma_x$
on the magnitude of the long-run equity premium exists (Van Ewijk, De Groot, et al., 2012), this specific finding is key to have in mind when specifying the $\xi$ value. Finally, when considering the equity premium volatility, the relative differences of both the downside and the upside measures are highest for low values of $\sigma_x$. Hence, the Euler scheme underestimates downside risk and upside potential for low values of $\sigma_x$ and vice versa.

The last part of the sensitivity analysis concerns the parameters of the short-rate dynamics and the associated plots are depicted in figure 1.3.

![Sensitivity plots for the parameters of the short rate.](image)

**Figure 1.3:** Sensitivity plots for the parameters of the short rate.

When considering the speed of mean-reversion parameter in the short-rate dynamics we see from figure 1.3a that the downside risk measures slowly increase in $\kappa$. Using a high speed of mean-reversion therefore implies that the Euler scheme underestimates downside risk more heavily. It further appears that the misestimation of the upside potential of the investment portfolio seems insensitive toward changes
in $\kappa$. Overall, the Euler scheme seems to perform best for $\kappa \approx 0.1$ which corresponds to a half-life of approximately 7 years. Compared to the findings of Chan et al. (1992) this seems high as they estimate a half-life of around 4 years.

The second parameter we consider is the long-term short-rate parameter. As appears from figure 1.3b we obtain a similar result to that of the long-term equity premium parameter. The tendency is clear as all relative measures increase heavily in $\theta$.

In figure 1.3c we consider the sensitivity of the short-rate volatility. We find that the underestimation of downside risk is most pronounced for low values of $\sigma_r$ and the underestimation of the upside potential slowly increases in $\sigma_r$. In this case, we somehow face a trade-off between underestimation of downside risk or upside potential.

Finally, we consider the $\rho$-parameter controlling the correlation between the stock price and the short rate. Empirical studies have found that the stock-bond correlation is time-varying and can in fact be negative see e.g. Ilmanen (2003). In the DSA model this correlation is modeled through $\rho$ and thus both negative and positive values are considered. Figure 1.3d depicts the result of the sensitivity analysis in which we find that the relative differences of the risk measures depend strongly on $\rho$. Interestingly, we find that $\rho$ affects the downside risk measures such that the underestimation is increasing in $\rho$. Considering the upside measures, the degree of underestimation is present for the studied values of $\rho$ but most pronounced for a positive $\rho$.

Generally, we find that the Euler scheme underestimates both the downside risk and the upside potential of the investment portfolio. Furthermore, the magnitude is heavily dependent on several of the model parameters.

To sum up, the determination of the parameter values related to the stock price, equity premium, and short-rate dynamics has an impact on the relative performance of the exact simulation method and the Euler scheme. Therefore, the model calibration must be handled with care or regulators must ensure model consistency in the L&P industry.

### 1.4 Concluding Remarks

This paper studies the use of simulation schemes with respect to implementing the financial model recently proposed by the Danish Society of Actuaries. Specifically, we investigate the performance of the simple Euler scheme and due to its flaws, we provide the exact simulation scheme as an alternative.

Our first finding is that choice of simulation scheme matters with respect to obtaining consistent portfolio forecasts across the L&P industry. Typically, implementors would apply the straightforward Euler scheme due to its simplicity. We find that when doing this, the implementor faces the trade-off between accuracy and
computational costs. The Euler scheme is in fact very slow which is very inconvenient to the L&P industry.

Motivated by this, we derive the conditional distribution of the future state variables which allows for exact simulation. This is feasible due to the tractability of the DSA model and we are able to obtain closed-form solutions for the conditional means and the conditional variances of the multivariate distribution. The implementation of exact simulation of the DSA model is thus straightforward and we provide the concrete scheme.

In our analysis, we compare the Euler scheme with the exact scheme and find that for a fixed computational budget a higher accuracy is obtained when using the exact scheme. This result is due to the recursive nature of the Euler scheme which makes the number of simulations increase heavily. As one must simulate intermediate values even though they are of no interest, its application becomes computationally expensive. We therefore recommend that implementors seek to avoid the recursive structure, and thus choose exact simulation over the Euler scheme.

Finally, we conduct a risk analysis with offset in a real-world scenario in order to study the potential biases of the Euler scheme. In the analysis, we examine the well-known risk measures Value-at-Risk and conditional Value-at-Risk. Our analysis indicates that using the Euler scheme over the exact simulation does not give rise to a severe misestimation of downside risk when using the reference parameters. Specifically, we observe an underestimation of downside risk when using the Euler scheme, though it is of negligible magnitude in monetary terms. Since the upside potential of an investment portfolio is of high relevance to the investor, we study the behavior of the right tail under both schemes. It emerges from our analysis that the Euler scheme underestimates the upside potential, i.e. the probability mass of the right tail.

In order to study the effect of changing model parameter values, we conduct a local sensitivity analysis. By varying the model parameters, we show that the differences between the two schemes are quite sensitive toward certain model parameters. When considering realistic parameter values, our findings are qualitatively robust as the Euler scheme consistently underestimates downside risk and upside potential. Furthermore, for specific parameter settings, the underestimation is magnified which is not the intention of the DSA model.

Since the purpose of the DSA model is to allow for proper and objective valuation and risk analysis of investment portfolios in L&P companies we stress that a potential problem lies in the forecasts’ dependence on simulation scheme.

Acknowledgements

The authors are grateful to Elisa Nicolato and an anonymous referee for useful comments and suggestions and to Per Linnemann for encouraging this research.
1.5 References


Appendix

A.1 Derivations

Here, we derive the complete conditional covariance matrix of the future values of the state variables. We start out by stating the model on matrix form

\[ \text{d} X_t = (AX_t + c) \text{d} t + B \text{d} z_t; \]

\[
\begin{bmatrix}
\ln S_t \\
x_t \\
r_t
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 1 \\
0 & -\alpha & 0 \\
0 & 0 & -\kappa
\end{bmatrix}
\begin{bmatrix}
\ln S_t \\
x_t \\
r_t
\end{bmatrix} +
\begin{bmatrix}
-\frac{1}{2} \sigma_S^2 \\
\alpha \xi \\
\kappa \theta
\end{bmatrix} \text{d} t +
\begin{bmatrix}
\sigma_S & 0 \\
-\sigma_x & 0 \\
\rho \sigma_r & (1 - \rho^2)^{\frac{1}{2}} \sigma_r
\end{bmatrix}
\begin{bmatrix}
d \hat{z}_{1,t} \\
d \hat{z}_{2,t}
\end{bmatrix}.
\]

The Cholesky factorization has been applied such that the instantaneously correlated Brownian motions \( d z_{1,t} \) and \( d z_{2,t} \) have been replaced by \( d \hat{z}_{1,t} = d z_{1,t} \) and \( d \hat{z}_{2,t} = \rho d \hat{z}_{1,t} + (1 - \rho^2)^{\frac{1}{2}} d \hat{z}_{2,t} \) where \( d \hat{z}_{1,t} \) and \( d \hat{z}_{2,t} \) are uncorrelated Brownian motions.

Linear stochastic differential systems on this form are well known to the academic literature and can be found in e.g. Björk (2004). It can be shown that

\[ X_T = e^{A(T-t)} X_t + \int_t^T e^{A(T-u)} c \text{d} u + \int_t^T e^{A(T-u)} B \text{d} \hat{z}_u \]

holds for all \( T > t \geq 0 \). As appears, the future values of the three state variables are conditionally normally distributed

\[ X_T \mid \mathcal{F}_t \sim \mathcal{N} \left( E_t [ X_T ], \text{Var}_t [ X_T ] \right) \]

and to provide the complete conditional distribution, the following subsections contain the derivations of the conditional mean vector and covariance matrix of the future values of the state variables.

A.1.1 The Conditional Mean Vector

By use of the property that the increments of Brownian motions are normally distributed with mean zero, we can express the conditional mean vector as

\[ E_t [ X_T ] = E_t \left[ e^{A(T-t)} X_t + \int_t^T e^{A(T-u)} c \text{d} u \right]. \]

As appears, the entire content of the expectation operator is \( t \)-measurable and hence derivation is straightforward. Though we need to determine the exponential of the matrix \( A(T-t) \) which yields

\[
e^{A(T-t)} = \begin{bmatrix}
1 & \Psi(\alpha, t, T) & \Psi(\kappa, t, T) \\
0 & e^{-\alpha(T-t)} & 0 \\
0 & 0 & e^{-\kappa(T-t)}
\end{bmatrix}
\]
where we have introduced the auxiliary function $\Psi(a, t, T) = \frac{1}{a} \left(1 - e^{-a(T-t)}\right)$. By simple calculations the conditional mean vector becomes

$$E_t \begin{bmatrix} \ln S_T \\ x_T \\ r_T \end{bmatrix} =$$

$$\begin{bmatrix} \ln S_t + (T-t) \left(\theta + \xi - \frac{1}{2} \sigma_S^2\right) + (r_t - \theta) \Psi(\kappa, t, T) + (x_t - \xi) \Psi(\alpha, t, T) \\ x_t e^{-\alpha(T-t)} + \xi \left(1 - e^{-\alpha(T-t)}\right) \\ r_t e^{-\kappa(T-t)} + \theta \left(1 - e^{-\kappa(T-t)}\right) \end{bmatrix}. $$

### A.1.2 The Conditional Covariance Matrix

Following Björk (2004) the variance of a stochastic integral where the integrand satisfies some regularity conditions can be computed as

$$\text{Var}_t \left[ \int_t^T e^{A(T-u)} B d\tilde{z}_u \right] = \int_t^T E_t \left[ \left(e^{A(T-u)} B\right) \left(e^{A(T-u)} B\right)^T \right] du, \quad (A.1)$$

and hence it is clear that the instantaneous covariance matrix is given by

$$\Sigma_u = E_t \left[ \left(e^{A(T-u)} B\right) \left(e^{A(T-u)} B\right)^T \right].$$

Tedious calculations using matrix multiplication yield the entries $(\Sigma_{i,j,u})_{1 \leq i,j \leq 3}$ of the instantaneous covariance matrix

$$\Sigma_{11,u} = \left(\sigma_S - \sigma_x \Psi(\alpha, u, T) + \rho \sigma_r \Psi(\kappa, u, T)\right)^2 + \sigma_r^2 \left(1 - \rho^2\right) \Psi(\kappa, u, T)^2$$

$$= \sigma_S^2 + \sigma_x^2 \Psi(\alpha, u, T)^2 + \sigma_r^2 \Psi(\kappa, u, T)^2$$

$$- 2\sigma_S \sigma_x \Psi(\alpha, u, T) + 2\rho \sigma_S \sigma_r \Psi(\kappa, u, T)$$

$$- 2\rho \sigma_x \sigma_r \Psi(\alpha, u, T) \Psi(\kappa, u, T),$$

$$\Sigma_{12,u} = \Sigma_{21,u} = -\sigma_S \sigma_x e^{-\alpha(T-u)} + \sigma_r^2 e^{-\alpha(T-u)} \Psi(\alpha, u, T)$$

$$- \rho \sigma_x \sigma_r e^{-\alpha(T-u)} \Psi(\kappa, u, T),$$

$$\Sigma_{13,u} = \Sigma_{31,u} = \rho \sigma_S \sigma_r e^{-\kappa(T-u)} + \sigma_r^2 e^{-\kappa(T-u)} \Psi(\kappa, u, T)$$

$$- \rho \sigma_x \sigma_r e^{-\kappa(T-u)} \Psi(\alpha, u, T),$$

$$\Sigma_{23,u} = \Sigma_{32,u} = -\rho \sigma_x \sigma_r e^{-\alpha(\kappa)(T-u)}$$
\[ \Sigma_{22,u} = \sigma_x^2 e^{-2\alpha(T-u)}, \]
\[ \Sigma_{33,u} = \sigma_r^2 e^{-2\kappa(T-u)}. \]

### A.1.2.1 Covariances

The covariances conditional on time \( t \) information are derived by applying equation (A.1) element wise. Starting with the covariance between future values of the logarithmic stock price and the equity risk premium

\[
\text{Cov}_t(\ln S_T, x_T) = \int_t^T E_t[\Sigma_{12,u}] \, du
\]
\[
= -\sigma_x \sigma_S \int_t^T e^{-\alpha(T-u)} \, du + \sigma_x^2 \int_t^T e^{-\alpha(T-u)} \Psi(\alpha, u, T) \, du
\]
\[
- \sigma_x \sigma_r \int_t^T e^{-\alpha(T-u)} \Psi(\kappa, u, T) \, du
\]
\[
= -\sigma_x \sigma_S \Psi(\alpha, t, T) + \frac{\sigma_x^2}{\alpha} \left( \Psi(\alpha, t, T) - \Psi(2\alpha, t, T) \right)
\]
\[
- \frac{\sigma_x^2 \sigma_r}{\kappa} \left( \Psi(\alpha, t, T) - \Psi(\alpha + \kappa, t, T) \right).
\]

Secondly, the conditional covariance between future values of the logarithmic stock price and the interest rate is

\[
\text{Cov}_t(\ln S_T, r_T) = \int_t^T E_t[\Sigma_{13,u}] \, du
\]
\[
= \rho \sigma_S \sigma_r \int_t^T e^{-\kappa(T-u)} \, du - \rho \sigma_x \sigma_r \int_t^T e^{-\kappa(T-u)} \Psi(\alpha, u, T) \, du
\]
\[
+ \sigma_r^2 \int_t^T e^{-\kappa(T-u)} \Psi(\kappa, u, T) \, du
\]
\[
= \rho \sigma_S \sigma_r \Psi(\kappa, t, T) - \frac{\rho \sigma_x \sigma_r}{\alpha} \left( \Psi(\kappa, t, T) - \Psi(\alpha + \kappa, t, T) \right)
\]
\[
+ \frac{\sigma_r^2}{\kappa} \left( \Psi(\kappa, t, T) - \Psi(2\kappa, t, T) \right).
\]

Finally, the conditional covariance between the equity risk premium and the short rate is given by

\[
\text{Cov}_t(x_T, r_T) = \int_t^T E_t[\Sigma_{23,u}] \, du
\]
\[
= -\rho \sigma_x \sigma_r \int_t^T e^{-(\alpha + \kappa)(T-u)} \, du
\]
\[
= -\rho \sigma_x \sigma_r \Psi(\alpha + \kappa, t, T).
\]

After stating the covariances we turn to the variances.
A.1.2.2 Variances

The variances of future values of the state variables follows by considering the diagonal elements of $\Sigma_u$ and the conditional variance of future values of the logarithmic stock price follows here

\[
V(t, T) \equiv \text{Var}_t \left( \ln S_T \right) = \int_t^T E_t \left[ \Sigma_{13,u} \right] \, du
\]

\[
= \sigma_S^2(T-t) + \sigma_X^2 \int_t^T \Psi(\alpha, u, T)^2 \, du + \sigma_r^2 \int_t^T \Psi(\kappa, u, T)^2 \, du
\]

\[
- 2\sigma_S \sigma_X \int_t^T \Psi(\alpha, u, T) \, du + 2\rho \sigma_S \sigma_r \int_t^T \Psi(\kappa, u, T) \, du
\]

\[
- 2\rho \sigma_X \sigma_r \int_t^T \Psi(\alpha, u, T) \Psi(\kappa, u, T) \, du.
\]

By computing the integrals, the expression can be written as

\[
V(t, T) = \sigma_S^2(T-t) + \sigma_X^2 \left( \frac{T-t}{\alpha^2} - \frac{1}{2\alpha} \Psi(\alpha, t, T)^2 - \frac{1}{\alpha^2} \Psi(\alpha, t, T) \right)
\]

\[
+ \sigma_r^2 \left( \frac{T-t}{\kappa^2} - \frac{1}{2\kappa} \Psi(\kappa, t, T)^2 - \frac{1}{\kappa^2} \Psi(\kappa, t, T) \right)
\]

\[
- 2\sigma_S \sigma_X \left( \frac{T-t}{\alpha} - \frac{1}{\alpha} \Psi(\alpha, t, T) \right) + 2\rho \sigma_S \sigma_r \left( \frac{T-t}{\kappa} - \frac{1}{\kappa} \Psi(\kappa, t, T) \right)
\]

\[
- \frac{2\rho \sigma_X \sigma_r}{\alpha \kappa} \left( (T-t) - \Psi(\kappa, t, T) - \Psi(\alpha, t, T) + \Psi(\alpha + \kappa, t, T) \right).
\]

Finally, this can be rearranged nicely such that

\[
V(t, T) = (T-t) \left( \frac{\sigma_X^2}{\alpha^2} + \frac{\sigma_S^2}{\kappa^2} - \frac{2\sigma_X \sigma_S}{\alpha} + \frac{2\rho \sigma_X \sigma_r}{\alpha \kappa} - \frac{2\rho \sigma_X \sigma_r}{\alpha \kappa} \right)
\]

\[
- \frac{\sigma_X^2}{2\alpha} \Psi(\alpha, t, T)^2 - \frac{\sigma_r^2}{2\kappa} \Psi(\kappa, t, T)^2
\]

\[
+ \Psi(\alpha, t, T) \left( \frac{2\sigma_X \sigma_S}{\alpha} + \frac{2\rho \sigma_X \sigma_r}{\alpha \kappa} - \frac{\sigma_X^2}{\alpha^2} \right)
\]

\[
+ \Psi(\kappa, t, T) \left( \frac{2\rho \sigma_X \sigma_r}{\alpha \kappa} - \frac{2\rho \sigma_X \sigma_r}{\alpha \kappa} - \frac{\sigma_r^2}{\kappa^2} \right)
\]

\[
- \Psi(\alpha + \kappa, t, T) \frac{2\rho \sigma_X \sigma_r}{\alpha \kappa}.
\]

Since both the equity risk premium and the short rate follows Ornstein-Uhlenbeck processes their variances are quite familiar
\[ \text{Var}_t[x_T] = \int_t^T \text{E}_t[\Sigma_{22,u}] \, du \]
\[ = \frac{\sigma_x^2}{2\alpha} \left( 1 - e^{-2\alpha(T-t)} \right) \]

\[ \text{Var}_t[r_T] = \int_t^T \text{E}_t[\Sigma_{33,u}] \, du \]
\[ = \frac{\sigma_r^2}{2\kappa} \left( 1 - e^{-2\kappa(T-t)} \right) . \]

Expressions for the full covariance matrix of the future values of the state variables have now been provided.
THE REAL RISK IN PENSION FORECASTING

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Abstract

The purpose of this paper is to shed light on some of the flaws in the forecasting approach undertaken by the pension industry. Specifically, it considers the treatment of inflation and shows that the current modeling framework is too simplistic. I identify the flaws of the existing regulatory framework and provide an alternative full model framework constructed around the three-factor diffusion model recently proposed by the Danish Society of Actuaries.

By use of a simulation study I compare the deterministic inflation scheme applied in the industry to a stochastic scheme and show that the real value of the pension saver’s investment portfolio at retirement is highly dependent on the inflation scheme. As the deterministic scheme does not take state variable correlations into account it overestimates the expected portfolio value in real terms compared to the stochastic scheme. Moreover, the deterministic scheme gives rise to a more heavy-tailed distribution implying a misestimation of downside risk and upside potential. Finally, it is shown in a realistic case study that the pension saver’s expected retirement payout profile is heavily affected.
2.1 Introduction

In this paper, I present results illustrating that the deterministic view on inflation undertaken by the pension industry gives rise to misestimation of both distribution location and dispersion when forecasting real portfolio values. Due to its ignorance of the covariance between the nominal portfolio performance and the price level, excessively high probabilities are assigned to tail events affecting distribution location and dispersion.

The planning of saving and investing over the life cycle with an aim towards retirement is to many people a daunting task. Further complexities are added to the problem when considering the role of inflation whose importance to the actuary was emphasized in both Wilkie (1981) and Wilkie (1984). Even though pension savers tend to think in nominal terms they should be thinking in real terms as it is their purchasing power over the retirement period that matters. A common way to incorporate changes in the purchasing power is by using a consumer price index (CPI). In retirement planning, an easy and thus popular approach is to assume that the CPI grows deterministically at a fixed rate.

As an example, life insurance and pension (L&P) companies in Denmark apply a rate of inflation of 1% p.a. until 2019 and 2% p.a. afterward. These rates are specified by the Danish Insurance Association and the Danish Bankers Association in their so-called industry assumptions provided yearly. As the industry assumptions are mandatory to apply in forecasting they should be sufficiently realistic though implementable to the entire industry. Since the current industry assumptions do not include any take on risks, the Danish Society of Actuaries has recently proposed an improved financial model that is able to quantify equity and interest rate exposure (DSA, 2014).

In this paper, I analyze the impact of having a stochastic view on inflation relative to a deterministic view. In particular, I extend the financial model proposed by the Danish Society of Actuaries by a stochastic CPI and compare it to a deterministically evolving CPI. The extension is motivated by the realized rate of inflation, i.e. slope of the logarithmic CPI, being clearly non-constant as seen in figure 2.1, but also that several studies find significant correlations between inflation and important state variables (see e.g. Fama, 1975; Pennacchi, 1991; Brennan and Xia, 2002; Munk et al., 2004; Ang, Bekaert, and Wei, 2008; Haubrich, Pennacchi, and Ritchken, 2012). Naturally, a deterministic inflation scheme does not take such correlations into account.

The financial model has been constructed with the trade-off between tractability and the ability to capture empirically observed phenomena in mind. The following stylized facts are incorporated in the model: time-varying stock returns (e.g. Bollerslev et al., 1988), a mean-reverting equity risk premium (e.g. Mehra and Prescott, 1985 and Fama and Schwert, 1977), and time-varying bond returns modeled using a stochastic short rate.
Figure 2.1: Panel (a) depicts the logarithmic CPI over the period from June 1961 to September 2015 using data from the Bureau of Labor Statistics obtained through Robert Shiller’s homepage. The monthly realized rate of inflation and a 12-month moving average are found in panel (b). The logarithmic CPI has been normalized to start at zero and the values of panel (b) are all stated in per annum terms.

The continuous-time model of the term structure of interest rates was introduced by Merton (1970) and later improved by Vasicek (1977) to exhibit mean-reversion of the short rate and by Cox et al. (1985) to exhibit non-negativity. Both models are arbitrage-free and theoretically well-founded but have drawbacks as their assumptions on state variable dependency and market prices of risk are too limiting (see e.g. Dai and Singleton, 2000 and Duffee, 2002). Despite their limitations both models are frequently applied in practice as they are affine term structure models and, as demonstrated by Duffie and Kan (1996) they have yields being linear combinations of the state variables. Due to their empirical fit practitioners also embrace the class of term structure models derived from the famous yield curve representation by Nelson and Siegel (1987) (see e.g. Svensson, 1994; Diebold and Li, 2006; Gürkaynak, Sack, and Wright, 2007; Christensen, Diebold, and Rudebusch, 2011).

The development of the interest rates over the recent years has caused an increase in research on dealing with the zero lower bound. One popular approach introduced by Black (1995) is to specify the interest rate as the maximum of zero and a shadow rate which is allowed to be negative (see e.g. Kim and Singleton, 2012; Krippner, 2013; Christensen and Rudebusch, 2015; Lemke and Vladu, 2016). The main drawback of the shadow-rate models is that this option-like condition implies no closed-form pricing formulas. By contrast, well-specified quadratic term structure models (see e.g.

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1 As a number of key interest rates have in fact breached the zero lower bound, there is an ongoing debate over the location of the effective lower bound (see e.g. Agarwal and Kimball, 2015; Brunnermeier and Koby, 2016; Hills, Nakata, and Schmidt, 2016; Rogoff, 2016).
Ahn, Dittmar, and Gallant, 2002 or Leippold and Wu, 2002) and affine term structure models (e.g. Monfort, Pegoraro, Renne, and Roussellet, 2015) are able to handle the zero lower bound and at the same time deliver closed-form pricing formulas but at the cost of model complexity. As stated above the financial model of DSA (2014) has been specified with the model complexity and ease of communication in mind. For a thorough analysis of the financial model, the reader is referred to Jørgensen and Slipsager (2016).

The full model considered in this paper has previously been studied in Munk et al. (2004) but with the intention of determining an optimal strategy in a utility-based consumption-portfolio choice framework. In line with Sørensen (1999) and Brennan and Xia (2000) they consider a representative investor with power utility who maximizes his expected utility of wealth at a given investment horizon (e.g. at retirement) and they find that the optimal portfolio policy can be decomposed into a speculative term and a hedging term as originally prescribed by Merton (1971). Munk et al. (2004) show that the full model is able to simultaneously resolve why investment advisors recommend long-horizon investors to hold a higher fraction of stocks (the Samuelson (1963) puzzle), and risk averse investors to hold a higher bonds to stocks ratio (the Canner, Mankiw, and Weil (1997) puzzle). The mean-reverting stock returns in the model resolves the former (see e.g. Kim and Omberg, 1996; Wachter, 2002) whereas the stochastic interest rate resolves the latter (Brennan and Xia, 2000).

To illustrate the flaws of applying a deterministic inflation scheme this paper considers a simple stylized example which isolates the sole impact of the inflation scheme. Secondly, in order to increase the relevance of the analysis, the task of the pension industry is replicated by considering a real-world mimicking scenario. In both cases I consider the impact on the real value of the pension saver’s investment portfolio at the date of retirement, and in the latter case how it affects the real value payout profile during retirement.

This paper contributes to the L&P research by suggesting a specific dynamic model for the CPI and moreover by stressing the importance of modeling inflation stochastically when forecasting long-horizon real portfolio values. Finally, to make the complete model available to the L&P industry, this paper provides the parameter estimates needed for implementation.

### 2.2 Model Framework

#### 2.2.1 The Financial Model

In order to study the effect of choosing a stochastic inflation scheme over a deterministic scheme, a common underlying financial model is needed. The common financial model applied in this paper is the three-factor diffusion model, proposed for investment portfolio modeling by the Danish Society of Actuaries in DSA (2014). The model has also been studied in Munk et al. (2004), but in a different context. In
an L&P perspective, the model has been applied in Jørgensen and Slipsager (2016) to investment portfolio risk analysis and in Jørgensen and Linnemann (2009) and Linnemann, Bruhn, and Steffensen (2011) to L&P product analysis.

The investment opportunities according to the model comprise: a stock index, bonds with various durations, and a bank account (cash). With the objective of formulating the investment asset dynamics, I let the following stochastic processes denote three state variables: the stock index (with dividends reinvested) \( S_t = (S_t)_{t \geq 0} \), the equity risk premium \( x = (x_t)_{t \geq 0} \), and the nominal short-term interest rate \( r = (r_t)_{t \geq 0} \). Moreover, let the state variables be defined on the filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \) and adapted to the filtration \( \mathcal{F}_t \) representing all available information at time \( t \geq 0 \). The stochastic processes are characterized by the stochastic differential equations

\[
\begin{align*}
\frac{dS_t}{S_t} &= (r_t + x_t)dt + \sigma_S dz_{1,t}, \quad (2.1) \\
dx_t &= \alpha(\xi - x_t)dt - \sigma_x dz_{1,t}, \quad (2.2) \\
dr_t &= \kappa(\theta - r_t)dt + \sigma_r dz_{2,t}, \quad (2.3)
\end{align*}
\]

in which the uncertainty of the financial model is driven by two instantaneously correlated standard Brownian motions \( z_{1,t} \) and \( z_{2,t} \) with a constant correlation coefficient \( \rho_{S,r} \in [-1, 1] \).

As prescribed by equation, (2.1) the expected instantaneous rate of return on the nominal stock index consists of a risk-free short-rate part and as introduced by Merton (1971), a time-varying compensation for bearing equity risk (see e.g. Harvey, 1989b, or Lettau and Ludvigson, 2002). The relative local volatility of the stock index \( \sigma_S > 0 \) is to ease model implementation in the L&P industry constant.

The equity risk premium \( x_t \) is said to follow an Ornstein-Uhlenbeck process, ensuring mean-reversion with a long-term level \( \xi \), persistence parameter \( \alpha > 0 \), and a constant local volatility \( \sigma_x > 0 \). Moreover, the nominal stock index return and the equity risk premium are instantaneously negatively correlated implying mean-reversion in the index return, i.e. high index returns tend to be followed by relatively small expected returns and vice versa.

The nominal short rate is likewise characterized by an Ornstein-Uhlenbeck process causing a long-run regressive adjustment toward the long-run level \( \theta > 0 \) with speed of adjustment governed by \( \kappa > 0 \) and constant local volatility \( \sigma_r > 0 \).

Important to the investment portfolio modeling is that the term structure of interest rates has the same form as in Vasicek (1977) as this implies straightforward bond pricing. The nominal time \( t \) price of a zero-coupon bond maturing at time \( T \) is

\[
P(r_t, t, T) = \exp\left\{-a(T - t) - b(\kappa, T - t)r_t\right\}, \quad (2.4)
\]
where
\[ b(a, \tau) = \frac{1}{a} \left(1 - e^{-a\tau}\right), \quad (2.5) \]
\[ a(\tau) = \left(\theta - \frac{\sigma_r \lambda_r}{\kappa} - \frac{\sigma_r^2}{2\kappa^2}\right)(\tau - b(\kappa, \tau)) + \frac{\sigma_r^2}{4\kappa} b(\kappa, T - t)^2, \quad (2.6) \]
and where the constant parameter \( \lambda_r \) denotes the market price of interest rate risk.

An application of Itô’s lemma yields the following general bond dynamics
\[ \frac{dB_t}{B_t} = \left((r_t + \lambda_B) \, dt - \sigma_B \, dz_t \right). \quad (2.7) \]

The relationship between \( \lambda_B \) and the market price of interest rate risk is \( \lambda_B = -\lambda_r \sigma_r D(r_t, t) \) with \( D(r_t, t) \) being the bond duration, i.e. the elasticity of the nominal bond price with respect to the nominal short rate as \( D(r_t, t) = -\left(\partial B_t / \partial r_t\right) \left(1 / B_t\right) \). As the modeling of a bank account (cash) corresponds to a zero duration bond, the proposed model can also handle cash investments. An appropriate feature of the model is the instantaneous perfect negative correlation between the short-rate and the bond prices.

In the financial model both the nominal stock index value and nominal bond prices are log-normally distributed. Unfortunately, this implies that a portfolio defined as a linear combination of these does not have a log-normally distributed nominal value (see Dhaene, Vanduffel, Goovaerts, Kaas, and Vyncke, 2005 or Marín-Solano, Roch, Dhaene, Ribas, Bosch-Princep, and Vanduffel, 2010). Therefore, the analysis in this paper is based on Monte Carlo simulations opposed to using approximations as in e.g. Brigo, Mercurio, Rapisarda, and Scotti (2004) or Dufresne (2004).

### 2.2.2 Inflation Modeling

The policy holder should be concerned about his purchasing power during retirement and hence the real value of the investment portfolio. To compute the real value of the investment portfolio a deflator is needed. In this paper, the deflator is specified as a CPI which tracks the value of a certain basket of goods. In the following, \( \Psi_t \) will denote the time \( t \) value of the CPI. The real value of an asset will be determined as its nominal value deflated by \( \Psi_t \) such that the real value of the time \( t \) stock index, for instance, is \( S_t / \Psi_t \).

As the main purpose of the paper is to examine the effect of using a deterministic inflation scheme relative to a stochastic scheme, the CPI of the latter is said to follow
\[ \frac{d\Psi_t}{\Psi_t} = \pi_t \, dt + \sigma_\Psi \, dz_{3,t}, \quad (2.8) \]
\[ d\pi_t = \beta (\chi - \pi_t) \, dt + \sigma_\pi \, dz_{4,t}, \quad (2.9) \]
which are both adapted from Munk et al. (2004). As seen from equation (2.8), the CPI has a relative drift equal to \( \pi_t \) and a constant relative volatility \( \sigma_{\Psi} > 0 \). The former can be interpreted as the expected rate of inflation over the next instant.

In order to make the expected rate of inflation time-varying while being stationary, it is modeled as an Ornstein-Uhlenbeck process. Thereby, the expected instantaneous rate of inflation has a long-run mean of \( \chi \), a speed of mean-reversion controlled by \( \beta \), and a constant instantaneous volatility denoted by \( \sigma_{\pi} > 0 \).

As in the financial model, the uncertainty is generated by instantaneously correlated standard Brownian motions. The instantaneous shocks to the CPI and the expected rate of inflation are denoted by \( dz_{3,t} \) and \( dz_{4,t} \), respectively. Changes in the CPI and the expected rate of inflation are correlated with the return on the nominal stock index and the nominal interest rate through the instantaneous correlations between the standard Brownian motions. As an example, the instantaneous correlation between the nominal stock index shock and the CPI shock is denoted by \( \rho_{S,\Psi} \).

The nominal interest rate and inflation dynamics imply that the term structure of real interest rates is described by a two-factor model in case the market price of inflation risk is constant. Only in the specific case where the nominal interest rate and the expected inflation have identical speed of mean-reversion coefficients does the real interest rate follow a univariate Ornstein-Uhlenbeck process.

Throughout the paper the deterministic inflation scheme is characterized by a CPI following

\[
\Psi^D_T = \Psi^D_t \exp \left\{ i^T_t (T - t) \right\},
\]

(2.10)

where \( i^T_t \) is an inflation yield

\[
i^T_t = \left( \chi + \frac{\sigma_{\pi}^2}{2\beta^2} + \frac{\rho_{\psi,\pi} \sigma_{\Psi} \sigma_{\pi}}{\beta} \right) \frac{T - t - b (\beta, T - t)}{T - t} - \frac{\sigma_{\pi}^2 \ b (\beta, T - t)^2}{4\beta} \frac{T - t}{T - t}
\]

\[
+ \frac{\pi_t \ b (\beta, T - t)}{T - t},
\]

(2.11)

ensuring the same expected growth in the stochastic and deterministic deflators to ensure comparability of the two schemes. The derivation of \( i^T_t \) is described in the appendix.

### 2.3 Model Estimation

#### 2.3.1 The Empirical Model

The full model is estimated using a state space model and the Kalman filtering technique on U.S. data. The estimation is challenged by the equity risk premium, the short rate, and the expected rate of inflation being latent state variables. As all state variables in the model are Markov processes, the state space model approach is applied due to its ability to handle unobservable state variables following Markov
processes. From the explicit solution to the full model found in equation (A.1) in the appendix, it follows that future values of the state variables, denoted by the vector 

\[ X_T = \begin{bmatrix} \ln S_T, & x_T, & r_T, & \ln \Psi_T, & \pi_T \end{bmatrix}^\top \]

are conditionally normally distributed

\[ X_T | F_t \sim \mathcal{N} \left( \mu \left(X_t, T - t \right), \Sigma \left(T - t \right) \right), \quad (2.12) \]

with conditional mean vector \( \mu \left(X_t, T - t \right) \) and conditional variance matrix \( \Sigma \left(T - t \right) \).

Due to the normality, the system can be expressed as a linear Gaussian state space model allowing for the Kalman filter to be applied.

The underlying idea of a state space model is to model the observables through a series of latent states. Inspired by Munk et al. (2004) the observation equation that links the empirical observations to the state variables consists of: observed yields to maturity on 5 zero-coupon bonds with times to maturity \( \tau_1, \tau_2, \ldots, \tau_5 \), the logarithmic index value, and the logarithmic CPI. Using the methodology of Duan and Simonato (1999), the zero-coupon yields are related to the state variables through

\[ \tilde{y} \left(t, \tau_j \right) = \tau_j^{-1} a \left(\tau_j \right) + \tau_j^{-1} b \left(\kappa, \tau_j \right) r_t + \tilde{\varepsilon}_{t,j}, \quad j = 1, \ldots, 5, \quad (2.13) \]

which is recognizable as it corresponds to the yield in the Vasicek (1977) plus an uncorrelated noise term with zero mean and variance \( \sigma^2_{\varepsilon,j} \).

The observation equation is

\[ y_t = d + Z\delta_t + \varepsilon_t, \quad t = 1, \ldots, n, \quad (2.14) \]

with the empirical observations stored according to

\[ y_t = \begin{bmatrix} \tilde{y} \left(t, \tau_1 \right), & \tilde{y} \left(t, \tau_2 \right), & \tilde{y} \left(t, \tau_3 \right), & \tilde{y} \left(t, \tau_4 \right), & \tilde{y} \left(t, \tau_5 \right), & \ln S_t, & \ln \Psi_t \end{bmatrix}^\top. \quad (2.15) \]

Thereby, the observations are modeled as the sum of i) a time-invariant system vector \( d \) given by

\[ d = \begin{bmatrix} \tau_1^{-1} a \left(\tau_1 \right), & \tau_2^{-1} a \left(\tau_2 \right), & \tau_3^{-1} a \left(\tau_3 \right), & \tau_4^{-1} a \left(\tau_4 \right), & \tau_5^{-1} a \left(\tau_5 \right), & 0, & 0 \end{bmatrix}^\top, \quad (2.16) \]

ii) the state vector \( \delta_t \) pre-multiplied by the \( (7 \times 5) \) loading matrix \( Z \) with entry values \( \{ z_{j3} \}_{j=1}^5 = \tau_j^{-1} b \left(\kappa, \tau_j \right), \quad z_{61} = z_{74} = 1, \quad \text{and} \quad 0 \quad \text{in the remaining elements, and} \) iii) a measurement noise term \( \varepsilon_t \). As suggested in Harvey (1989b), \( \varepsilon_t \) is a vector of serially and cross-sectionally uncorrelated disturbances with zero mean and variance matrix \( H \), where \( H \) is a diagonal matrix with diagonal elements \( \{ h_{jj} \}_{j=1}^7 = \sigma^2_{\varepsilon,j} \). In the estimation I follow Munk et al. (2004) and model the shortest zero-coupon yield with a limited measurement error, in order to obtain a close fit to the short end of the term structure of interest rates. Munk et al. (2004) further assume that the logarithmic stock index and the logarithmic CPI are observed completely without measurement error, which I acknowledge, but in my implementation modify to a
very limited measurement error. In the estimation, the three corresponding standard deviations are therefore fixed at a value close to zero (0.0001).

The state equation governing the underlying state of nature is

\[ \delta_{t+1} = c + K\delta_t + \eta_t, \]  

where the state vector is \( \delta_t = [\ln S_t, x_t, r_t, \ln \Psi_t, \pi_t]^\top \). The vector \( c \), the coefficient matrix \( K \) and the shock \( \eta_t \) are determined in accordance to the explicit solution and its conditional moments and covariances (cf. the appendix). Explicitly, the time-invariant system vector \( c \) and matrix \( K \) states

\[
c = \begin{bmatrix}
\theta [\Delta t - b(\kappa, \Delta t)] + \xi [\Delta t - b(a, \Delta t)] - \frac{1}{2} \sigma_S^2 \Delta t \\
\xi \left(1 - e^{-a\Delta t}\right)
\end{bmatrix}
\]

and

\[
K = \begin{bmatrix}
1 & b(a, \Delta t) & b(k, \Delta t) & 0 & 0 \\
0 & e^{-a\Delta t} & 0 & 0 & 0 \\
0 & 0 & e^{-k\Delta t} & 0 & 0 \\
0 & 0 & 0 & 1 & b(\beta, \Delta t) \\
0 & 0 & 0 & 0 & e^{-\beta\Delta t}
\end{bmatrix}.
\]

Finally, the shocks \( \eta_t \) are serially uncorrelated and normally distributed with zero mean and variance matrix \( \Sigma(\Delta t) \).

The parameter estimation is carried out by maximum likelihood estimation as the log-likelihood function is easily obtained when using the recursive Kalman filter. As initial state vector I condition on the first set of observables and assume that the unobservables are jointly distributed according to their steady-state distribution. For a comprehensive discussion on the Kalman filter see Harvey (1989a) or Durbin and Koopman (2012).

### 2.3.2 Data

The full model is estimated using monthly data on U.S. nominal interest rates, equities, and inflation spanning the period from June 1961 to September 2015.

The nominal zero-coupon rates are obtained using the methodology and data set of Gürkaynak et al. (2007). The data set consists of parameter estimates for the model by Svensson (1994), which is an extension of the famous model by Nelson and Siegel (1987). By use of their parameter estimates, I construct time-series of
synthetic zero-coupon yields for five different times to maturity; 0.25, 1, 2, 5, and 10 years. All five series are recorded as end of month continuously compounded yields to maturity.

For the nominal stock index, I use the S&P 500 with dividends reinvested. In particular, cum dividend returns are constructed using end of month values of the index and its associated monthly dividends. Both series are along with the CPI series used in Shiller (2015) and acquired through Robert Shiller’s homepage. The basis of the Kalman filter application is thus 7 series of observables over 651 months.

### 2.3.3 Estimation Results

Table 2.1 presents the parameter estimates of the model with values stated in per annum terms.

The estimated stock index volatility $\sigma_S$ is 12.53%, which is considerably lower than the historical average of 17.65% in Ibbotson and Kim (2014), but it is comparable to that of Munk et al. (2004). For the equity premium dynamics I estimate the long-term level $\xi$ to be 5.83%, which lies between the 4.12% and 6.48% estimates of Wachter (2002) and Munk et al. (2004), respectively. The estimates of the associated degree of mean reversion coefficient $\alpha$ and the instantaneous volatility $\sigma_x$ are 0.0642 and 0.84%. Both estimates are also in line with the findings of Munk et al. (2004).

Figure 2.2 depicts the filtered time-series, resulting from the Kalman filtering accompanied by gray-shaded recessions obtained from the National Bureau of Economic Research. Panel (a) presents the filtered equity risk premium, which varies between 3.31% and 9.19% over the sample. The lowest compensation for taking on equity risk is observed during the Dot-com bubble in the early 2000s and the maximum is seen during the Great Recession from December 2007 until June 2009. Furthermore, the filtering captures the countercyclicality of the equity risk premium found in, among others, Lettau and Ludvigson (2001).

As regards the parameter estimates of the nominal interest rate dynamics, a long-term level $\theta$ of 4.64% is estimated, which is close to the steady-state level of 4.28% found by Haubrich et al. (2012). The estimates of the speed of adjustment coefficient $\kappa$, the constant instantaneous volatility $\sigma_r$, and the market price of interest rate risk $\lambda_r$ are 0.0717, 2.15%, and $-23.63\%$, respectively. The estimates imply a positive bond risk premium of 0.51% per unit of duration. The obtained bond risk premium as function of maturity, is for maturities below 10 years in line with that of Haubrich et al. (2012) but higher for longer maturities. Their higher degree of concavity corresponds to a higher degree of mean-reversion in the nominal interest rate. Thereby the $\kappa$ estimated in this paper lies between the estimate of Munk et al. (2004) and that backed out of Haubrich et al. (2012). Finally, the standard deviations of the measurement errors $\{\sigma_{\epsilon,j}\}_{j=1}^5$ are increasing in bond maturity and of the same magnitude as those found by Duan and Simonato (1999) in a Kalman filter based estimation of the Vasicek term.
### Table 2.1: Estimates of model parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock return process $dS_t/S_t = (r_t + x_t)dt + \sigma_S dz_{1,t}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_S$</td>
<td>0.1253</td>
<td>0.0011</td>
</tr>
<tr>
<td>Equity risk premium process $dx_t = \alpha(\xi - x_t)dt - \sigma_x dz_{1,t}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.0642</td>
<td>0.0241</td>
</tr>
<tr>
<td>$\xi$</td>
<td>0.0583</td>
<td>0.0087</td>
</tr>
<tr>
<td>$\sigma_x$</td>
<td>0.0084</td>
<td>0.0309</td>
</tr>
<tr>
<td>Nominal interest rate process $dr_t = \kappa(\theta - r_t)dt + \sigma_r dz_{2,t}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.0717</td>
<td>0.0018</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.0464</td>
<td>0.0011</td>
</tr>
<tr>
<td>$\sigma_r$</td>
<td>0.0215</td>
<td>0.0000</td>
</tr>
<tr>
<td>$\lambda_r$</td>
<td>-0.2363</td>
<td>0.0002</td>
</tr>
<tr>
<td>$\sigma_{r,1}$</td>
<td>0.0001</td>
<td>-</td>
</tr>
<tr>
<td>$\sigma_{r,2}$</td>
<td>0.0054</td>
<td>0.0011</td>
</tr>
<tr>
<td>$\sigma_{r,3}$</td>
<td>0.0075</td>
<td>0.0011</td>
</tr>
<tr>
<td>$\sigma_{r,4}$</td>
<td>0.0103</td>
<td>0.0011</td>
</tr>
<tr>
<td>$\sigma_{r,5}$</td>
<td>0.0119</td>
<td>0.0011</td>
</tr>
<tr>
<td>CPI process $d\Psi_t/\Psi_t = \pi_t dt + \sigma_\psi dz_{3,t}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_\psi$</td>
<td>0.0096</td>
<td>0.0012</td>
</tr>
<tr>
<td>Inflation process $d\pi_t = \beta(\chi - \pi_t)dt + \sigma_\pi dz_{4,t}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.4123</td>
<td>0.0100</td>
</tr>
<tr>
<td>$\chi$</td>
<td>0.0385</td>
<td>0.0067</td>
</tr>
<tr>
<td>$\sigma_\pi$</td>
<td>0.0253</td>
<td>0.0049</td>
</tr>
</tbody>
</table>

**Correlations**

| $\rho_{S,r}$ | 0.0195 | 0.0015 |
| $\rho_{S,\psi}$ | -0.0099 | 0.0016 |
| $\rho_{S,\pi}$ | 0.0358 | 0.0033 |
| $\rho_{r,\psi}$ | 0.0303 | 0.0016 |
| $\rho_{r,\pi}$ | 0.6654 | 0.0010 |
| $\rho_{\psi,\pi}$ | 0.0859 | 0.0277 |
structure model on U.S. data.

The dynamics of the CPI features a constant instantaneous volatility $\sigma_{\psi}$ whose estimate is 0.96% compared to 0.88% found in Haubrich et al. (2012) and 0.81% found in Munk et al. (2004). Regarding the dynamics of the expected rate of inflation, the adjustment coefficient $\beta$ is estimated to be 0.4123. The estimate implies a half-life for the expected rate of inflation shocks of 1.7 years, which is compatible with the 2.4 years found in Munk et al. (2004). The estimated long-run mean $\chi$ is 3.85%, which is consistent with the findings by Brennan and Xia (2002), Munk et al. (2004), and Ang et al. (2008), where the latter estimates the expected rate of inflation to be 3.94% using U.S. data over the period 1952-2004.

Panel (b) in figure 2.2 depicts the filtered values of the short rate and the expected rate of inflation. The depicted series indicates a positive correlation as expressed in the parameter estimates by the instantaneous correlation coefficient $\rho_{r,\pi} = 0.6654$. Munk et al. (2004) also find a high positive correlation between the nominal interest rate and the expected rate of inflation as their parameter estimate is 0.7975.

The filtered values of the nominal interest rate behave procyclically as found empirically in Fama (1990). By examination of the filtered values of expected rate of inflation, I find a large drop during the Great Recession, which was also identified by Haubrich et al. (2012). The specific month of the drop is December 2008 which encompasses the Federal Reserve’s key interest rate cut to the zero lower bound.

The instantaneous correlation between the logarithmic stock index and the expected rate of inflation is estimated to be 0.0358 which is different from that of
Munk et al. (2004) as their estimate is $-0.2465$. Even though the difference seems big, a global sensitivity analysis using the methodology of Saltelli, Annoni, Azzini, Campolongo, Ratto, and Tarantola (2010) shows that the parameter does not play an important role in long-horizon forecasting. The remaining four instantaneous correlation parameter estimates are all in line with those of Munk et al. (2004).

2.3.4 Model Fit and Forecasting

This section assesses the in-sample fit and out-of-sample forecasting ability of the estimated model. The estimated model featuring stochastic inflation is compared to the financial model in (2.1)-(2.3) accompanied by the deterministically evolving CPI in (2.10).

Using the estimated parameters of table 2.1 and the Kalman filter, one-step-ahead prediction errors are computed for all observables under both inflation schemes. As the prediction errors of the observables are correlated, standardized prediction errors are computed using the error variance matrix. According to Durbin and Koopman (2012) the use of standardized prediction errors facilitates diagnostic checking on individual series.

As the stock index dynamics are quite flexible and the stock index measurement error is limited, the filtered stock index closely resembles the observed stock index for both models in the empirical evaluation. The in-sample fit of the zero-coupon yields is generally satisfying but worst prior to the high interest rate environment of the early 1980s. Moreover, the fit of the zero-coupon yields is worsening in maturity, which is consistent with the measurement errors being increasing in maturity.

The main difference between the two models in the empirical evaluation is the fit of the logarithmic CPI series. Figure 2.3 presents the standardized logarithmic CPI prediction errors under both inflation schemes.

Given an infinitely long past history of observables, the standardized prediction errors of a well-specified model are standard normally and independently distributed (Harvey, 1989a). It is quite clear from panel (a) that the prediction errors under the deterministic inflation scheme are not independently standard normally distributed, whereas panel (b) indicates a better fit under stochastic inflation. When considering the largest errors (in terms of absolute values) under stochastic inflation, there is a clear indication of heteroscedasticity. In fact, the largest deviations occur around the Federal Reserve’s key interest rate cut in December 2008. Pennacchi (1991) also finds indications of heteroscedasticity in the prediction errors when facing a change in the monetary-policy regime.

The relative forecasting power of the models is evaluated based on the traditional root mean square error (RMSE) performance measure. In the RMSE computation deviations are assigned equal weights across the forecast horizon making it suitable for evaluating the long-horizon forecast ability.

As the CPI modeling constitutes the main difference between the two models, I
Figure 2.3: In-sample one-month-ahead prediction errors of the logarithmic CPI under the deterministic inflation scheme in panel (a) and under the stochastic inflation scheme in panel (b).

solely consider the RMSE of the forecasted CPIs and ignore the remaining observables. The in-sample fit considered above yields an RMSE of 0.165 and 0.000 under the deterministic inflation scheme and the stochastic inflation scheme, respectively. Thus the in-sample fit is way better under the stochastic inflation scheme. As a good in-sample fit does not necessarily imply good out-of-sample forecasting, I provide out-of-sample forecasts for both models.

Let $N$ denote the complete sample size ($N = 651$), $N^*$ the length of the estimation window, and $J$ the number of monthly out-of-sample forecasts, i.e. the length of the forecast window. The out-of-sample forecasting is performed by estimating the model over the estimation window ($N^* = N - J$) and comparing the out-of-sample forecasts to the $J$ observations left out of the estimation window.

In addition to the traditional out-of-sample forecasting in which all observables in the forecasting window are unknown, I also consider conditional forecasting. Conditional forecasting is a hybrid between in-sample and out-of-sample forecasting as it forecasts the CPI out-of-sample conditional on the realized zero-coupon yields and the stock index values. Thereby, it illustrates how well the out-of-sample forecasts of the models adjust to reliable exogenously given information about the observables. Table 2.2 presents RMSE for four different forecast horizons: 5 years, 10 years, 15 years, and 20 years.

As expected is the RMSE in general increasing with the length of the forecast window $J$. A comparison of the unconditional deterministic inflation scheme and the stochastic inflation scheme reveals that the stochastic inflation scheme model is also superior in its out-of-sample forecasting ability. There is a substantial difference between their RMSEs for the four considered windows. For the 5 year forecast window
2.4 The Role of Stochastic Inflation in Pension Planning

Table 2.2: RMSE based on in-sample fit and out-of-sample forecasts.

<table>
<thead>
<tr>
<th></th>
<th>In-sample</th>
<th>Out-of-sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>J</td>
<td>-</td>
<td>60 120 180 240</td>
</tr>
</tbody>
</table>

Panel A: Unconditional:
- Deterministic: 0.165 0.178 0.228 0.239 0.272
- Stochastic: 0.000 0.027 0.132 0.171 0.239

Panel B: Conditional:
- Deterministic: - 0.178 0.228 0.239 0.272
- Stochastic: - 0.018 0.069 0.043 0.094

The RMSE of the deterministic scheme is 650 percent that of the stochastic scheme whereas it is 14 percent for the 20 year forecast window.

As regards the conditional models, the availability of future yields and stock index values does not improve the deterministic model. This is of course due to the deterministic nature of CPI which ignores correlations. In contrast, the stochastic inflation scheme model improves tremendously. The RMSE is only 0.018 and 0.094 for the 5 and 20 year forecast windows, respectively.

The in-sample model fit and out-of-sample forecasting study indicates that it is beneficial to apply the financial model accompanied by the stochastic inflation scheme.

2.4 The Role of Stochastic Inflation in Pension Planning

In this section, the estimated model is applied in order to illustrate the impact of a stochastic inflation view on the pension saver’s investment portfolio forecast. By use of a simulation study it is demonstrated that a deterministic view yields flawed real portfolio value forecasts as it neglects important information in terms of the covariance between the nominal portfolio performance and the CPI. In order to isolate the effect of change in inflation scheme, a simple and stylized example is considered. In section 2.4.2, the analysis is extended to encompass a real-world mimicking scenario illustrating the consequences faced by the pension saver.

2.4.1 Stylized Example - a Glance at Retirement

The stylized example takes offset in a single individual who contributes by 50,000 at time $t = 0$ to an investment portfolio with a horizon of $T = 35$ years. The length of the investment period corresponds to the length of the accumulation phase of a typical pension saver. The retirement date forecast is considered below as this
is a typical point of reference for the pension saver. The investment portfolio is rebalanced monthly and uses an asset allocation of 60% in the stock index and 40% in bonds with an average time-denominated duration of 5 years (henceforth: the 60-40 portfolio). The analysis is based on 1,000,000 simulations across time of the five state variables with initial values specified such that $x_t, r_t, \text{ and } \pi_t$ start at their respective long-term levels and $S_t$ and $\Psi_t$ are without loss of generality initiated at one, i.e. $S_0 = 1, x_0 = \xi = 0.0583, r_0 = \theta = 0.0464, \Psi_0 = 1, \text{ and } \pi_0 = \chi = 0.0385$. 

To examine the effect of using a deterministic rate of inflation relative to a stochastic, series of real values are constructed using the same simulated nominal portfolio values $\{W_{i,T}\}_{i=1}^N$ but with either the CPI modeled using a deterministic rate of inflation $\Psi_{DT}$ by (2.10) or stochastically $\{\Psi_{Si,T}\}_{i=1}^N$ by (2.8). For each $i = 1, \ldots, N$ the real portfolio values under the deterministic scheme

$$W_{i,T}/\Psi_{DT}, \quad (2.20)$$

are compared to the corresponding real portfolio values under the stochastic scheme

$$W_{i,T}/\Psi_{i,T}. \quad (2.21)$$

Prior to analyzing the simulated nominal portfolio values consider the inflation yield in (2.11) evaluated in the estimated model parameters. Figure 2.4 plots $i_T^T$ as a function of $T$ which for $\pi_0 = \chi$ is increasing in $T$. This indicates that the use of the same constant rate of inflation for all maturities is probably too simplistic.

2.4.1.1 Density Discrepancies

In the following, it is shown that the distribution of real portfolio values at the retirement date is dependent on the inflation scheme applied. In particular, differences are found in the estimated distribution location and dispersion both relevant to the pension saver.

Figure 2.5 contains simulated real portfolio value distributions using both inflation schemes in panel (a) along with their implied density difference in panel (b). The density difference is computed as the density under the deterministic inflation scheme subtracted the density under the stochastic inflation scheme.

As can be seen the simulated distributions are heavy-tailed and exhibit positive skewness. Both properties that can also be seen in the nominal portfolio value distribution. A comparison of the distributions using the density difference plot shows that the simple deterministic scheme assigns higher probabilities to extreme outcomes.

---

2The behavior of $i_T^T$ is comparable to that of the zero-coupon yield curve in the Vasicek (1977) model, i.e. it can take on increasing, decreasing, and humped shapes.

3Using a simple constant rate of inflation instead of the deterministic inflation scheme does not change the conclusions of the paper as the deviations are of the same magnitude. Results are available upon request.
2.4. The Role of Stochastic Inflation in Pension Planning

Figure 2.4: The figure shows the annualized expected rate of inflation that must be applied in a deterministic setting in order to ensure consistency between the stochastic and deterministic CPIs.

Figure 2.5: Panel (a) plots the densities of simulated real portfolio values at retirement date using either the deterministic or stochastic inflation scheme. By subtracting the latter from the former, the difference plot in panel (b) is obtained.
and especially to lower outcomes. The sample kurtosis confirms that the deterministic scheme is more prone to producing tail-events than the stochastic scheme as the sample kurtosis is 98.26 under the deterministic scheme and 45.96 under the stochastic scheme. Following a decomposition of the sample kurtosis, figure 2.6 depicts the relative contribution from simulated values outside a given standard deviation.

Figure 2.6: The plot depicts a decomposition of the sample kurtosis based on the dispersion of the simulated real portfolio values. The relative contribution is measured as the kurtosis derived from simulated values outside a given standard deviation from its mean.

In both cases it appears that almost the entire kurtosis contribution can be attributed to simulated values that are more than 3 standard deviations away from the mean. Secondly, the extremity of the tails is more severe under the deterministic scheme as 50% of the relative contribution stems from simulated values more than 22.4 standard deviations away from the mean compared to 15.3 under the stochastic scheme.

As a measure of dispersion, the interquartile range (IQR), calculated as the difference between the upper and the lower quartiles, provides an outlier robust measure for expressing density diversities. The IQRs are 377,859 and 328,950 under the deterministic scheme and stochastic scheme, respectively. This indicates that the density discrepancy between the two deflating approaches remains even when accounting for extreme values.

These extreme outcomes naturally affect the distribution mean. The theoretical mean of the real portfolio value can be decomposed into

\[
\tilde{W}_T \equiv E_t \left[ W_T / \Psi_T^D \right] = E_t \left[ W_T \exp \left\{ -t(T-t) \right\} \right] = E_t \left[ W_T \right] \left\{ E_t \left[ \Psi_T^D \right]\right\}^{-1},
\]

under the deterministic scheme and into

\[
\hat{W}_T \equiv E_t \left[ W_T / \Psi_T^S \right] = E_t \left[ W_T \right] E_t \left[ (\Psi_T^S)^{-1} \right] + \text{Cov}_t \left[ W_T, (\Psi_T^S)^{-1} \right],
\]

under the stochastic scheme.
under the stochastic scheme. This obviously leads to potential differences as

\[
\tilde{W}_T - \hat{W}_T = E_t \left[ W_T \left( \left\{ E_t \left[ \Psi^{S}_T \right] \right\}^{-1} - E_t \left[ \left( \Psi^{S}_T \right)^{-1} \right] \right) - \text{Cov}_t \left[ W_T, \left( \Psi^{S}_T \right)^{-1} \right] \right],
\]

where the parenthesis is non-positive due to Jensen's inequality.

The simulation study shows that the mean of the real portfolio values is highest under the deterministic scheme, i.e. \( \tilde{W}_T - \hat{W}_T > 0 \), indicating the presence of non-negligible negative covariance between the nominal portfolio value and the inverse stochastic CPI. The average real portfolio value under the deterministic scheme is 427,240 compared to 387,421 under the stochastic scheme.

The positive correlation between the retirement date nominal portfolio value and the CPI is ignored by the deterministic scheme. The implication is that under the stochastic scheme bad states (low nominal portfolio values) are accompanied by low price levels (low CPI) resulting in real portfolio values higher than under the corresponding deterministic view, and vice versa. Thereby, the deterministic CPI fails to account for the fact that the CPI reduces the upside but also provides an insurance against bad states. For the real portfolio values the implication of the covariance ignorance of the deterministic scheme is twofold as both location and dispersion discrepancies arise.

### 2.4.1.2 Examining the Asset Allocation

In this section, I illustrate the robustness of the misestimation of the retirement sample mean and IQR for various asset allocations. As alternatives to the traditional 60-40 portfolio I consider two asset allocations that are characterized by constant portfolio weights: i) stocks only and ii) bonds only. Due to their increasing popularity I also include: iii) target date funds and iv) the risk parity approach, both having time-varying portfolio weights.

As regards the target date funds the savings are invested in an equity allocation that follows the industry average of Yang, Acheson, Holt, Rupp, and Spica (2015) as presented in table 2.3.

<table>
<thead>
<tr>
<th>Years to target</th>
<th>+25</th>
<th>20</th>
<th>15</th>
<th>10</th>
<th>5</th>
<th>0</th>
<th>-5</th>
<th>-10</th>
<th>-15</th>
<th>-20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity allocation (%)</td>
<td>88</td>
<td>79</td>
<td>72</td>
<td>62</td>
<td>53</td>
<td>44</td>
<td>36</td>
<td>31</td>
<td>28</td>
<td>27</td>
</tr>
</tbody>
</table>

As was the case for the stylized example the remaining funds are invested in a bond portfolio with an average time-denominated duration of 5 years.

The risk parity approach is known for its asset allocation ensuring an equal risk contribution from the portfolio components. In the construction of the portfolio I
follow Maillard, Roncalli, and Teiletche (2010) and let the asset \( i \) weight\(^4\) be given by the ratio of the inverse of asset \( i \)’s return volatility to the sum of the inverse portfolio asset return volatilities. The asset return volatilities are in the implementation estimated using a rolling window of length 1 year.

Presented in table 2.4 are the sample means and IQRs obtained from simulated distributions using each of the five asset allocations.

**Table 2.4:** Sample mean and IQR under the deterministic and stochastic inflation schemes for five different asset allocations.

<table>
<thead>
<tr>
<th></th>
<th>Deterministic</th>
<th>Stochastic</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: 60-40 portfolio</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>427,240</td>
<td>387,421</td>
<td>39,819</td>
</tr>
<tr>
<td>IQR</td>
<td>377,859</td>
<td>328,950</td>
<td>48,908</td>
</tr>
<tr>
<td><strong>Panel B: Stocks only</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>791,019</td>
<td>683,001</td>
<td>108,018</td>
</tr>
<tr>
<td>IQR</td>
<td>684,916</td>
<td>605,266</td>
<td>79,649</td>
</tr>
<tr>
<td><strong>Panel C: Bonds only</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>148,963</td>
<td>145,426</td>
<td>3,537</td>
</tr>
<tr>
<td>IQR</td>
<td>118,020</td>
<td>102,912</td>
<td>15,108</td>
</tr>
<tr>
<td><strong>Panel D: Target date funds</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>511,015</td>
<td>456,791</td>
<td>54,224</td>
</tr>
<tr>
<td>IQR</td>
<td>452,213</td>
<td>398,816</td>
<td>53,397</td>
</tr>
<tr>
<td><strong>Panel E: Risk parity</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>316,631</td>
<td>293,525</td>
<td>23,107</td>
</tr>
<tr>
<td>IQR</td>
<td>275,727</td>
<td>238,926</td>
<td>36,801</td>
</tr>
</tbody>
</table>

As it appears, the differences in sample mean and IQR under the two inflation schemes are preserved but change character across asset allocations. At first glance it looks like the problems are heavily mitigated for bond-like portfolios as the absolute difference is largest for the portfolio fully invested in the stock index. However, another picture emerges when taking the change in magnitude into account. In order to do so relative measures are constructed by dividing the differences by the corresponding estimate under the deterministic scheme. The results are shown in figure 2.7.

Firstly, it appears that each asset allocation has positive relative differences in the estimated sample mean, but this is most pronounced in case of portfolios tilted towards stocks. By decomposing the covariance term in (2.24) it is seen that the

---

\(^4\)The weights of the average risk parity portfolio are 0.42 and 0.58 for stocks and bonds, respectively.
standard deviation of the nominal portfolio return amplifies the problem, i.e. volatile portfolios yield larger relative differences and vice versa.

Secondly, the dispersion discrepancy between the two inflation schemes illustrated by the relative difference in IQR is seen quite distinctly. The relative difference in the outlier-robust IQR lies in the range between 11.6% and 13.3%, which indicates that the distance between the upper and lower quartiles of the simulated distribution is 11.6% larger under the deterministic scheme as a minimum. The dispersion discrepancy is therefore of a considerable size and it is a concern due to its presence across asset allocations.

![Relative differences](image)

**Figure 2.7:** Relative differences of mean and IQR when comparing inflation schemes for the five asset allocations.

### 2.4.1.3 Tail Analysis

When investing over a long horizon, the pension saver naturally faces the possibilities that the real value of his portfolio declines (downside risk) and increases (upside potential). Below I illustrate how the use of the deterministic inflation scheme leads to overestimating both downside risk and upside potential.

To analyze the density discrepancies’ impact on the density tails I consider two real value downside risk measures: i) the well-known Value-at-Risk, and ii) the conditional Value-at-Risk (cVaR), both on a 95% significance level and as deviations from their time $T$ mean, i.e. as relative measures. For the portfolio potential I consider iii) the upside (US) corresponding to the 95% quantile, and iv) the conditional upside (cUS) computed as the average of the 5% best outcomes. Table 2.5 presents the four estimated tail measures for the five asset allocations.

From panel A it appears that a 60-40 portfolio leads to a VaR of 381,479 and 323,180 under the deterministic and stochastic schemes, respectively. The absolute difference is 58,299 in real terms, which is a considerable amount of time $T$ wealth and corresponds to a 15% discrepancy between the schemes. Across the five asset
Table 2.5: Value-at-Risk (VaR), conditional Value-at-Risk (cVaR), upside (US), and conditional upside (cUS) under the deterministic and stochastic inflation schemes for the five asset allocations.

<table>
<thead>
<tr>
<th>Panel A: 60-40 portfolio</th>
<th>Deterministic</th>
<th>Stochastic</th>
<th>Difference</th>
<th>Relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>VaR</td>
<td>381,479</td>
<td>323,180</td>
<td>58,299</td>
<td>0.15</td>
</tr>
<tr>
<td>cVaR</td>
<td>395,599</td>
<td>340,555</td>
<td>55,044</td>
<td>0.14</td>
</tr>
<tr>
<td>US</td>
<td>1,374,371</td>
<td>1,104,374</td>
<td>269,998</td>
<td>0.20</td>
</tr>
<tr>
<td>cUS</td>
<td>2,309,856</td>
<td>1,683,878</td>
<td>625,978</td>
<td>0.27</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Stocks only</th>
<th>Deterministic</th>
<th>Stochastic</th>
<th>Difference</th>
<th>Relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>VaR</td>
<td>744,734</td>
<td>616,958</td>
<td>127,776</td>
<td>0.17</td>
</tr>
<tr>
<td>cVaR</td>
<td>761,120</td>
<td>637,864</td>
<td>123,256</td>
<td>0.16</td>
</tr>
<tr>
<td>US</td>
<td>2,840,198</td>
<td>2,244,110</td>
<td>596,088</td>
<td>0.21</td>
</tr>
<tr>
<td>cUS</td>
<td>5,482,477</td>
<td>3,859,465</td>
<td>1,623,012</td>
<td>0.30</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C: Bonds only</th>
<th>Deterministic</th>
<th>Stochastic</th>
<th>Difference</th>
<th>Relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>VaR</td>
<td>115,664</td>
<td>101,721</td>
<td>13,943</td>
<td>0.12</td>
</tr>
<tr>
<td>cVaR</td>
<td>123,689</td>
<td>110,831</td>
<td>12,858</td>
<td>0.10</td>
</tr>
<tr>
<td>US</td>
<td>384,376</td>
<td>331,111</td>
<td>53,265</td>
<td>0.14</td>
</tr>
<tr>
<td>cUS</td>
<td>547,070</td>
<td>440,860</td>
<td>106,209</td>
<td>0.19</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel D: Target date funds</th>
<th>Deterministic</th>
<th>Stochastic</th>
<th>Difference</th>
<th>Relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>VaR</td>
<td>467,425</td>
<td>395,615</td>
<td>71,810</td>
<td>0.15</td>
</tr>
<tr>
<td>cVaR</td>
<td>481,623</td>
<td>413,286</td>
<td>68,337</td>
<td>0.14</td>
</tr>
<tr>
<td>US</td>
<td>1,723,755</td>
<td>1,386,476</td>
<td>337,279</td>
<td>0.20</td>
</tr>
<tr>
<td>cUS</td>
<td>3,051,918</td>
<td>2,216,695</td>
<td>835,224</td>
<td>0.27</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel E: Risk parity</th>
<th>Deterministic</th>
<th>Stochastic</th>
<th>Difference</th>
<th>Relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>VaR</td>
<td>273,587</td>
<td>233,898</td>
<td>39,689</td>
<td>0.15</td>
</tr>
<tr>
<td>cVaR</td>
<td>286,020</td>
<td>248,867</td>
<td>37,154</td>
<td>0.13</td>
</tr>
<tr>
<td>US</td>
<td>960,116</td>
<td>782,221</td>
<td>177,895</td>
<td>0.19</td>
</tr>
<tr>
<td>cUS</td>
<td>1,528,547</td>
<td>1,139,666</td>
<td>388,881</td>
<td>0.25</td>
</tr>
</tbody>
</table>
allocations significant VaR and cVaR differences appear in both monetary and relative terms. The highest absolute differences are observed for a portfolio fully tilted towards stocks contrary to the bond portfolio which yields the lowest difference. Dependent on the risk measure the deterministic inflation scheme’s overestimation of downside risk is between 10% and 17%.

As regards the upside measures US and cUS, the absolute differences are 269,998 and 625,978, respectively. Hence, the deterministic scheme also heavily overestimates the upside potential in both absolute and relative terms. The US overestimation lies between 14% and 21% and the cUS overestimation is between 19% and 30%.

The deviations are attributed to the deterministic scheme ignoring the covariance between the nominal portfolio value and the CPI. All in all, a downside misestimation of at least 10% and an upside misestimation of at least 14% seem unsatisfying.

2.4.2 A Simulated Pension Scheme

I now consider a specific case mimicking a realistic real world scenario in order to translate the previous findings into relevant quantities. In particular, I conduct a simulation study on a single individual who makes monthly contributions to an investment portfolio associated with his pension contract. The monthly contribution is determined as 6% of the yearly gross salary starting at 50,000 and is assumed to grow deterministically by 2% p.a. (real growth) and stochastically by the inflation modeled by equation (2.8) (nominal growth). The contribution rate of 6% decomposes into a default personal rate of 3% adapted from Choi, Laibson, Madrian, and Metrick (2004) and an equivalent amount in employer contribution. Like in the above example, the contribution period is chosen to be 35 years, corresponding to a 30 year old policy holder who works until the age of 65. The accumulated funds are invested in the highly popular monthly rebalanced target date funds. Specifically, the equity allocation follows that of table 2.3 and the remaining funds are invested in bonds with an average time-denominated duration of 5 years. The withdrawal period is set to 20 years after retirement and he withdraws funds according to a yearly determined fractionated annuity-due with a conservative interest rate of 2%. Finally, the simulation study is based on 500,000 simulations across time of the five state variables with the same initial values as in the stylized example.

Figure 2.8 depicts the pension saver’s outlook in terms of the distribution of real portfolio values at the retirement date along with his expected retirement payout profile.

From panel (a) it is observed that the choice of inflation scheme affects the real portfolio value distribution as in the stylized example. Upon examination of the tails, it also appears that the deterministic scheme is prone to producing excessively extreme outcomes.

Table 2.6 presents the retirement date distribution averages, IQRs, and tail measures under both inflation schemes.
Figure 2.8: This figure shows the pension saver’s outlook in the real-world example. Panel (a) depicts the densities of simulated real portfolio values at retirement date deflated by either stochastic or deterministic inflation; and panel (b) illustrates the average payout profile under both schemes.

Table 2.6: Time $T = 35$ retirement date forecast for the real-world scenario.

<table>
<thead>
<tr>
<th></th>
<th>Deterministic</th>
<th>Stochastic</th>
<th>Difference</th>
<th>Relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>527,390</td>
<td>480,882</td>
<td>46,508</td>
<td>0.09</td>
</tr>
<tr>
<td>IQR</td>
<td>430,080</td>
<td>345,233</td>
<td>84,846</td>
<td>0.20</td>
</tr>
<tr>
<td>VaR</td>
<td>435,452</td>
<td>343,960</td>
<td>91,492</td>
<td>0.21</td>
</tr>
<tr>
<td>cVaR</td>
<td>457,341</td>
<td>367,967</td>
<td>89,373</td>
<td>0.20</td>
</tr>
<tr>
<td>US</td>
<td>1,542,602</td>
<td>1,200,483</td>
<td>342,120</td>
<td>0.22</td>
</tr>
<tr>
<td>cUS</td>
<td>2,495,573</td>
<td>1,753,671</td>
<td>741,902</td>
<td>0.30</td>
</tr>
</tbody>
</table>

It is seen that the mean is overestimated by 9% and the distance between the upper and lower quartiles is overestimated by 20%. The tail measures indicate that substantial discrepancies exist as the downside risk measures are misestimated by approximately 20% and the upside is around 22% too optimistic.

Panel (b) of figure 2.8 presents the expected payout profile faced by the pension saver during retirement. It is seen that the retirement date misestimation translates directly into the payout profile as the deterministic scheme overestimates the average funds available for withdrawal during the entire retirement period. Moreover, the overestimation increases over time due to the conservative choice of annuity interest rate that backloads the payouts. If a more aggressive interest rate is applied, the reverse is observed and hence the overestimation is not driven by the choice of annuity interest rate.

Both phenomena, i) the time $T = 35$ distributional discrepancy and ii) the overoptimistic payout profile, are caused by the deterministic scheme’s failure to account for
the positive correlation between nominal portfolio value and the CPI, meaning that good nominal portfolio states tend to be accompanied by high CPI levels, yielding a relatively lower real portfolio value.

2.5 Conclusion

This paper considers pension forecasting with a focus on the way the choice of inflation model affects the real value pension forecasts. The analysis is conducted by comparing the real value forecasts of two models that have an underlying financial model in common but differ by their inflation model. The applied financial model is adapted from a recent proposal by the Danish Society of Actuaries on pension forecasting (DSA, 2014) and is therefore relevant to the pension industry. Regarding the inflation models, I analyze the effect of having a stochastic view on inflation relative to the deterministic view that is often favored by regulators.

The full model featuring stochastic inflation is estimated using the Kalman filtering technique on U.S. data and the in-sample fit and out-of-sample forecasting ability are evaluated. Firstly, I find that the model produces filtered values of the equity risk premium, the short rate, and the expected rate of inflation in line with certain historical events, and secondly that the in-sample fit of the full model is quite good. Moreover, when considering the out-of-sample forecasting ability, it is found that the full model is superior to the model with a deterministic inflation scheme.

By use of a simulation study I find that when comparing the real value forecasts of the two models their distributions differ in both location and dispersion. Firstly, as the deterministic inflation scheme is prone to producing heavily overoptimistic forecasts it overestimates the expected real portfolio value. Secondly, the deterministic inflation scheme’s ignorance of the correlation between nominal portfolio performance and the consumer price level leads to an excessively heavy-tailed real value distribution.

As a robustness check I consider five popular asset allocations and find that the distributional differences in location and dispersion are present across asset allocations. When considering the downside risk and the upside potential of the five asset allocations I find that relative to the stochastic scheme, the deterministic inflation scheme overestimates the downside risk measures by 10-17% and the upside measures by 14-30%.

Finally, I construct a setup that mimics a real-world pension saving scenario and conduct a simulation study in which I also consider the payout profile faced by the pension saver during retirement. As in the previous analysis I find a significant density discrepancy between the two inflation schemes as the estimated sample means and interquartile ranges under the deterministic scheme are 9% and 20% higher, respectively. Moreover, the downside risk measures differ by 20-21% and the upside measures by 22-30%. The differences translate directly into the payout profile making it excessively optimistic under the deterministic inflation scheme.
The disconcerting outcome of the analysis is that portfolio forecasts conducted by use of the deterministic inflation scheme are overoptimistic with respect to expected retirement value and upside potential compared to the stochastic inflation scheme considered in this paper. This is disappointing since pension saving is all about planning and an overoptimistic forecast would lead to pension savers to save too little for retirement.

In order to overcome the challenge of replacing the deterministic inflation scheme, this paper provides a fully estimated model framework for real value portfolio forecasting, easily applicable to the pension industry.

Acknowledgements

The author is grateful to Per Linnemann for encouraging this research.
2.6 References


Appendix

A.1 Solving the System

The purpose of this appendix is to provide the explicit solution to the linear stochastic differential system as the Kalman filter application and simulation study both take offset in this. The explicit solution constitutes the first step in the derivation of the complete conditional distribution of the future state variables.

In order to derive the explicit solution the model is stated on matrix form

\[ dX_t = \begin{bmatrix} \ln S_t \\ x_t \\ r_t \\ \ln \Psi_t \\ \pi_t \end{bmatrix} dt + \begin{bmatrix} \alpha \xi \\ -\kappa \theta \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} dt + C \\ \tilde{d}z_1, t \\ \tilde{d}z_2, t \\ \tilde{d}z_3, t \\ \tilde{d}z_4, t \end{bmatrix}, \]

Here, the Cholesky decomposition has been applied such that the instantaneously correlated Brownian motions of the model \( \{dz_j, t\}_{j=1}^4 \) have been replaced by independent standard Brownian motions \( \{\tilde{d}z_j, t\}_{j=1}^4 \) and the matrix \( C \) accounts for the correlations.

The explicit solution to the linear stochastic differential system is adapted from Björk (2004) and states

\[ X_T = e^{A(T-t)}X_t + \int_t^T e^{A(T-u)}c \, du + \int_t^T e^{A(T-u)}B \, \tilde{d}z_u, \quad T > t \geq 0. \]

which gives rise to conditional normally distributed state variables

\[ X_T | \mathcal{F}_t \sim N\left( \mu(X_t, T-t), \Sigma(T-t) \right), \quad (A.1) \]

with the conditional moments and covariances presented below. Note that both the conditional mean and variance depend on how far into the future you are looking, the length \( T-t \), and not directly on \( T \) or \( t \), which is a reasonable property as variation should enter through the state variables and not through calendar time.

A.1.1 The conditional mean vector

As the uncertainty of the model is driven solely by standard Brownian motions, the conditional mean vector, \( \mu(X_t, T-t) \), follows easily
A.1.2 The conditional variances

The diagonal elements of $\Sigma(T-t)$ are derived using the conventional treatment of the stochastic integrals, see e.g. Munk (2013).

$$V_t \left[ \ln S_T \right] = (T-t) \left( \frac{\sigma_x^2}{\alpha^2} + \frac{\sigma_S^2}{\kappa^2} - \frac{2 \sigma_x \sigma_S}{\alpha} + \frac{2 \rho_{S,r} \sigma_r \sigma_S}{\kappa} - \frac{2 \rho_{S,r} \sigma_x \sigma_r}{\alpha \kappa} \right)$$

$$- b(\alpha, T-t)^2 \frac{\sigma_x^2}{2 \alpha} - b(\kappa, T-t)^2 \frac{\sigma_r^2}{2 \kappa}$$

$$+ b(\alpha, T-t) \left( \frac{2 \sigma_x \sigma_S}{\alpha} + \frac{2 \rho_{S,r} \sigma_x \sigma_S}{\alpha \kappa} - \frac{\sigma_x^2}{\alpha^2} \right)$$

$$+ b(\kappa, T-t) \left( \frac{2 \rho_{S,r} \sigma_r \sigma_S}{\alpha \kappa} - \frac{2 \rho_{S,r} \sigma_r \sigma_S}{\kappa} - \frac{\sigma_r^2}{\kappa^2} \right)$$

$$- b(\alpha + \kappa, T-t) \frac{2 \rho_{S,r} \sigma_x \sigma_r}{\alpha \kappa}.$$
A.1.3 The conditional covariances

The ten conditional covariances in the off-diagonal of \( \Sigma (T - t) \) are as follows

\[
\begin{align*}
\text{Cov}_t [\ln S_T, x_T] &= b(\alpha, T - t) \left( -\sigma_x \sigma_S + \frac{\sigma^2_x}{\alpha} - \frac{\sigma_x \sigma_r \rho_{S,r}}{\kappa} \right) - b(2\alpha, T - t) \frac{\sigma^2_x}{\alpha} \\
&+ b(\alpha + \kappa, T - t) \frac{\sigma_x \sigma_r \rho_{S,r}}{\kappa}, \\
\text{Cov}_t [\ln S_T, r_T] &= b(\kappa, T - t) \left( \rho_{S,r} \sigma_S \sigma_r - \frac{\rho_{S,r} \sigma_x \sigma_y}{\alpha} + \frac{\sigma^2_r}{\alpha} \right) + b(\alpha + \kappa, T - t) \frac{\rho_{S,r} \sigma_x \sigma_r}{\alpha} \\
&- b(2\kappa, T - t) \frac{\sigma^2_r}{\kappa}, \\
\text{Cov}_t [\ln S_T, \ln \Psi_T] &= (T - t) \left( \rho_{S,\Psi} \sigma_S \sigma_\Psi - \frac{\rho_{S,\Psi} \sigma_x \sigma_\Psi}{\alpha} + \frac{\rho_{S,\pi} \sigma_S \sigma_\pi}{\beta} - \frac{\rho_{S,\pi} \sigma_x \sigma_\pi}{\alpha \beta} \right) \\
&+ \frac{\rho_{r,\Psi} \sigma_r \sigma_\Psi}{\kappa} + \frac{\rho_{r,\pi} \sigma_r \sigma_\pi}{\beta \kappa} \\
&+ b(\alpha, T - t) \left( \frac{\rho_{S,\Psi} \sigma_x \sigma_\Psi}{\alpha} + \frac{\rho_{S,\pi} \sigma_x \sigma_\pi}{\alpha \beta} \right) \\
&- b(\beta, T - t) \left( \frac{\rho_{S,\pi} \sigma_S \sigma_\pi}{\beta} - \frac{\rho_{S,\pi} \sigma_x \sigma_\pi}{\alpha \beta} + \frac{\rho_{r,\pi} \sigma_r \sigma_\pi}{\beta \kappa} \right) \\
&- b(\kappa, T - t) \left( \frac{\rho_{r,\pi} \sigma_r \sigma_\pi}{\beta \kappa} + \frac{\rho_{r,\Psi} \sigma_r \sigma_\Psi}{\kappa} \right) \\
&- b(\alpha + \beta, T - t) \frac{\rho_{S,\pi} \sigma_x \sigma_\pi}{\alpha \beta} + b(\beta + \kappa, T - t) \frac{\rho_{r,\pi} \sigma_r \sigma_\pi}{\beta \kappa}, \\
\text{Cov}_t [\ln S_T, \pi_T] &= b(\beta, T - t) \left( \rho_{S,\pi} \sigma_S \sigma_\pi + \frac{\rho_{r,\pi} \sigma_r \sigma_\pi}{\kappa} - \frac{\rho_{S,\pi} \sigma_x \sigma_\pi}{\alpha} \right) \\
&+ b(\alpha + \beta, T - t) \frac{\rho_{S,\pi} \sigma_x \sigma_\pi}{\alpha} - b(\beta + \kappa, T - t) \frac{\rho_{r,\pi} \sigma_r \sigma_\pi}{\kappa}, \\
\text{Cov}_t [x_T, r_T] &= -\rho_{S,r} \sigma_x \sigma_r b(\alpha + \kappa, T - t), \\
\text{Cov}_t [x_T, \ln \Psi_T] &= -b(\alpha, T - t) \left( \rho_{S,\Psi} \sigma_x \sigma_\Psi + \frac{\rho_{S,\pi} \sigma_x \sigma_\pi}{\beta} \right) + b(\alpha + \beta, T - t) \frac{\rho_{S,\pi} \sigma_x \sigma_\pi}{\beta}, \\
\text{Cov}_t [x_T, \pi_T] &= -\rho_{S,\pi} \sigma_x \sigma_\pi b(\alpha + \beta, T - t),
\end{align*}
\]
\[ \text{Cov}_t [r_T, \ln \Psi_T] = b(\kappa, T-t) \left( \rho_{r,\Psi} \sigma_r \sigma_{\Psi} + \frac{\rho_{r,\pi} \sigma_r \sigma_{\pi}}{\beta} \right) - b(\beta + \kappa, T-t) \frac{\rho_{r,\pi} \sigma_r \sigma_{\pi}}{\beta}, \]

\[ \text{Cov}_t [r_T, \pi_T] = \rho_{r,\pi} \sigma_r \sigma_{\pi} b(\beta + \kappa, T-t), \]

\[ \text{Cov}_t [\ln \Psi_T, \pi_T] = b(\beta, T-t) \left( \rho_{\psi,\pi} \sigma_{\psi} \sigma_{\pi} + \frac{\sigma_{\pi}^2}{\beta} \right) - b(2\beta, T-t) \frac{\sigma_{\pi}^2}{\beta}. \]

By the conditional mean vector and the elements of the conditional variance matrix the conditional distribution in equation (A.1) has now been fully stated.

### A.2 Inflation Yield

The derivation of the long-run expected rate of inflation, \( \iota \), is comparable to that described on pp. 450-452 in Munk (2013), and takes its offset in the dynamics

\[ \frac{d\Psi_t}{\Psi_t} = \pi_t dt + \sigma_{\Psi} dz_{3,t}, \]

\[ d\pi_t = \beta (\chi - \pi_t) dt + \sigma_{\pi} dz_{4,t}, \]

and the relation

\[ E_t \left[ \frac{\Psi_T}{\Psi_t} \right] = e^{\iota_t (T-t)}, \tag{A.2} \]

in which I have introduced \( \iota_T \) as the continuously compounded rate implied by the annualized expected gross rate of inflation.

\[ \pi_T = \pi_t e^{-\beta(T-t)} + \chi \left( 1 - e^{-\beta(T-t)} \right) + \int_t^T \sigma_{\pi} e^{-\beta(T-v)} dv_{4,v}, \]

by integrating I get

\[ \int_t^s \pi_u du = \int_t^s \pi_t e^{-\beta(u-t)} du + \int_t^s \chi \left( 1 - e^{-\beta(u-t)} \right) du 
+ \int_t^s \int_t^u \sigma_{\pi} e^{-\beta(u-v)} dv_{4,v} du. \]

By use of the Fubini Theorem

\[ \int_t^s \pi_u du = \int_t^s \pi_t e^{-\beta(u-t)} du + \int_t^s \chi \left( 1 - e^{-\beta(u-t)} \right) du 
+ \int_t^s \int_t^u \sigma_{\pi} e^{-\beta(u-v)} dv_{4,v} du + \int_t^s \int_t^u \sigma_{\pi} e^{-\beta(u-v)} dv_{4,v} du 
= \pi_t b(\beta, s-t) + \chi (s-t - b(\beta, s-t)) + \int_t^s \sigma_{\pi} b(\beta, s-u) dv_{4,u}. \tag{A.3} \]

From the CPI dynamics I have

\[ \frac{\Psi_T}{\Psi_t} = \exp \left\{ \int_t^T \left( \pi_u - \frac{1}{2} \sigma_{\Psi}^2 \right) du + \int_t^T \sigma_{\Psi} dv_{3,u} \right\}, \]
and by inserting (A.3) I get
\[
\frac{\Psi_T}{\Psi_t} = \exp\left\{ \pi_t b(\beta, T-t) + \chi \left( T - t - b(\beta, T-t) \right) \right. \\
+ \int_t^T \sigma_b(\beta, T-u) d\zeta_{4,u} - \int_t^T \frac{1}{2} \sigma^2_\Psi du + \int_t^T \sigma_\Psi d\zeta_{5,u} \left. \right\}.
\]

The expected gross rate of inflation is then
\[
E_t \left[ \frac{\Psi_T}{\Psi_t} \right] = \exp\left\{ \pi_t b(\beta, T-t) + \chi \left( T - t - b(\beta, T-t) \right) - \frac{1}{2} \sigma^2_\Psi (T-t) \right. \\
+ \frac{1}{2} \left( \sigma^2_\pi \left( \frac{T-t}{\beta^2} - \frac{1}{2\beta} b(\beta, T-t)^2 - \frac{1}{\beta^2} b(\beta, T-t) \right) + \sigma^2_\Psi (T-t) \right) \\
\left. + 2 \rho_{34} \sigma_\Psi \sigma_\pi \left( \frac{T-t}{\beta} - \frac{1}{\beta} b(\beta, T-t) \right) \right\}.
\]

By comparing the relation in (A.2) with (A.4) I get
\[
i_t^T = \left( \chi + \frac{\sigma^2_\pi}{2\beta^2} + \frac{\rho_{34} \sigma_\Psi \sigma_\pi}{\beta} \right) \frac{T-t - b(\beta, T-t)}{T-t} - \frac{\sigma^2_\Psi b(\beta, T-t)^2}{4\beta} \frac{T-t}{T-t} \\
+ \frac{\pi_t b(\beta, T-t)}{T-t},
\]
which is the expression used in the analysis in section 2.4.
CHAPTER 3

THE VALUATION OF CATCH-UP PROVISIONS IN FUND MANAGERS’ COMPENSATION CONTRACTS

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Abstract

This paper shows how the catch-up provision that is often embedded in private equity compensation contracts shifts a substantial amount of value from the investor to the fund manager. Due to the presence of an illiquid and opaque secondary market for private equity securities we value the contracts under both spanned and unspanned risks in the underlying fund. Moreover, we introduce random premature fund closings which lead to liquidation risk in both the spanned and unspanned fund risk models.

Examples with realistic parameter values indicate that for a zero-alpha fund under spanned fund risk the inclusion of a 100% catch-up in a typical contract shifts 2.25% and 3.37% of the value of the investor’s investment proceeds claim to the fund manager when the fund is unlevered and financed by 2/3 debt, respectively. When we account for market incompleteness through 0.05 unspanned fund risk, the catch-up implied value transfer lies in the range of 1.62% – 2.89% for the unlevered fund and in the range of 3.19% – 3.45% for the levered fund.

The introduction of liquidation risk through an exponentially distributed termination date reduces the 100% catch-up value transfer for an unlevered fund to 1.92% under spanned fund risk and to the range 1.39% – 2.47% under 0.05 unspanned fund risk.
risk when the termination intensity equals 0.05. For the 2/3 debt financed fund the
spanned fund risk value transfer equals 2.82% whereas the unspanned fund risk value
transfer lies in the interval 2.70% – 2.81%.

Additionally, we find that the marginal value transfer is decreasing in the catch-up
rate and that the decision on whether to include a catch-up or not is very important in
value-transfer terms compared to the decision on the specific catch-up rate. Finally,
we compute breakeven values of alpha, the fixed annual management fee, and the
carried interest rate for various catch-up specifications.

3.1 Introduction

The catch-up provision is a fund manager compensation element derived from an
increased rate of profit acquisition subject to a predetermined profit-split target be-
tween the investor and the fund manager. In this paper we present results indicating
that the catch-up provision often included in private equity (PE) fee structures shifts
substantial value from the investor to the fund manager. Our main result suggests
that for an unlevered fund the inclusion of a 100% catch-up provision to the typical PE
fee structure transfers between 1.62% – 2.89% of the investor's investments proceeds
derived value to the fund manager, when accounting for market incompleteness, un-
expected prematurely fund dissolutions, and fund manager skill levels. In contractual
terms, we find that the inclusion of a 100% catch-up in a typical contract corresponds
to an increase in the annual management fee by 18 basis points and to an increase in
the carried interest rate by 13.81 percentage points.

When investment committees consider participation in PE partnerships, they
face highly complex fee structures that make the ex-ante investment evaluation
troublesome. PE partnerships are characterized by the investor typically referred to as
the limited partner (LP) committing capital and the fund manager referred to as the
general partner (GP) carrying out fund investments in line with the fund agreement.
The purpose of the fee structure is twofold as it governs the sharing of profits between
the partners and, due to the obvious principle-agent problem, it must incentivize
the GP to deliver the desired effort and take an appropriate amount of risk (Gilson,
2003). Even though the basis of the PE fee structures is generally fairly standard
certain contractual terms are up for negotiation resulting in a substantial degree of
fee structure heterogeneity (Bartlett and Swan, 2000; Fleischer, 2005). Typically, the
fee structures consist of a fixed scheme and a variable performance based scheme.
The latter serves as an incentive-aligning element and can include the catch-up
which can be described as an increased rate of profit-acquiring kicking in once the
profits exceed a prespecified threshold. Obviously, the objective of the catch-up is
therefore to make the GP more eager to breach the threshold, thereby aligning the
incentives of the GP and LP.

The purpose of this paper is to make PE fund investing more transparent by valu-
3.1. Introduction

Valuing the partnerships’ interests with a particular focus on the catch-up. The valuation is further complicated by the market incompleteness that arises from an illiquid and opaque secondary market for PE fund shares and which causes traditional contingent claims valuation approaches to break down. In particular, the investor’s inability to perfectly hedge his fund exposure violates the replicating portfolio argument underlying the option pricing methodology of Black and Scholes (1973) and Merton (1973). Therefore, we adopt the spanned fund risk valuation framework of Sørensen, Wang, and Yang (2014), which relies on the assumption that there exists an asset useful for hedging the otherwise unspanned fund specific risk. In case no such asset exists and the investor faces truly unspanned fund risk, we are unable to put an exact value on the contingent claim without specifying the investor’s utility function. Instead we make use of the preference-free good-deal bound framework of Cochrane and Saa-Requejo (2000), which is based on an assumption of investors wanting to buy abnormally high Sharpe ratio deals (good deals). Throughout the paper, we refer to the two models as the spanned and unspanned fund risk model, respectively. In both frameworks we allow the GP to deliver alpha. This alpha is not arbitraged away since the LP cannot invest in the PE assets directly but only through the PE fund. The LP is therefore willing to take on fund specific risk when the GP is sufficiently skilled.

As PE funds are subject to early dissolutions through, among other things, GP bankruptcy, misconduct, or the departure of key personnel (Breslow and Schwartz, 2015), we introduce stochastic premature fund termination into both the spanned and unspanned fund risk models. Stochastic fund termination is often applied in hedge fund studies (see e.g. Goetzmann, Ingersoll, and Ross, 2003 and Drechsler, 2014) and has previously been applied in the PE literature by Metrick and Yasuda (2010) and Choi, Metrick, and Yasuda (2012) as a modeling compromise of the GP’s investment exit behavior. Regarding the latter, the underlying assumption that the investment realization time is independent of the investment value is highly dubious, and hence our results are biased in favor of the LP or mainly attributed to early fund dissolutions. On that note, our modeling approach is based on the specific dynamics governing the evolvement of the asset values of our economy, and consequently we will not be focusing on the fund manager’s effort and risk-taking behavior.

The principal-agent literature on effort, risk-taking incentives, and compensation is vast and in relation to this paper we refer the reader to e.g. Carpenter (2000) for the dynamic portfolio choice problem faced by a risk-averse manager who is paid an unhedgeable call option, Axelson, Strömberg, and Weisbach (2009) for a study on financing and optimal GP compensation, Kandel, Leshchinskii, and Yuklea (2011) for an investigation of the GP’s investment realization timing, Chung, Sensoy, Stern, and Weisbach (2012) for an empirical assessment of the direct and indirect pay-for-performance for PE funds, and Drechsler (2014) for the risk choice of a hedge fund manager facing a high-water mark.

A significant challenge in PE valuation is how to cope with the fact that the PE
fund shares are nontradable assets in a model perspective. The pricing of options written on a non-traded underlying poses a recurring problem in the academic literature. In relation to our work, Björk and Slinko (2006) provide an in-depth study and formalization of the good-deal bounds framework of Cochrane and Saa-Requejo (2000) in which they also introduce jumps to the dynamics of the underlying. Bayraktar and Young (2008) consider a writer of an option on a non-traded underlying and show that the bid and ask prices correspond to the good-deal bounds of Cochrane and Saa-Requejo (2000) and Björk and Slinko (2006). Moreover, Bayraktar and Young (2008) extend the framework to capture stochastic volatility in the underlying. Černý (2003) takes a utility-based approach and recommends replacing the Sharpe ratio in the good-deal bound constraint by a generalized Sharpe ratio based on a suitable utility function, but he also shows that for Itô price processes the bounds are invariant to the choice of reward-for-risk measure. Likewise, Bernardo and Ledoit (2000) derive a set of pricing bounds with offset in the no-arbitrage condition and a maximum gain-loss ratio of the economy.

This paper contributes to the PE literature in three ways. First, it is the first paper that applies the preference-free good-deal bounds framework to valuation of PE compensation contracts when unspanned fund risk is present. Second, we extend the spanned fund risk model of Sørensen et al. (2014) and the good-deal bounds framework by liquidation risk through premature fund termination. Third, this paper is to the best of our knowledge the first that focuses on the catch-up provision and presents its costs in terms of the implied value transfer and breakeven parameters.

The remainder of the paper is organized as follows. Section 3.2 introduces the PE contract and its fee structures in terms of the partners’ payoff functions. Moreover, the section carefully describes the catch-up and its properties through a stylized example. Subsequently, Section 3.3 introduces our valuation framework that consists of the spanned and unspanned fund risk models. Section 3.4 presents the numerical results, and Section 3.5 concludes.

### 3.2 Contract Structures

In the following, we introduce a contract setup that allows us to value and analyze PE fund contracts and the catch-up feature. The contract framework is closely related to that of Sørensen et al. (2014) but with alterations that fit our data set of 25 PE fund contracts and that allow for premature fund termination. The data set has primarily been used in the early stages of the research project and has given us an indication of a typical contract. Section 3.2.1 presents the fee structures through the payoff functions of the LP, the GP, and the debtholders, and Section 3.2.2 illustrates the role and features of the catch-up provision via a stylized example.

As mentioned in the introduction PE fund contracts typically contain a fixed management fee scheme and a variable performance based scheme. The treatment
and valuation of the management fees are relatively straightforward as the GP is normally paid a fixed amount periodically on some prespecified basis (e.g. committed, contributed, or invested capital). The valuation of the variable performance-based scheme is, however, potentially more complicated as it naturally depends on the contract specifications and the fund performance.

We consider a partnership to which the LP commits an amount of capital $X_0$ at fund initiation, referred to as time $t = 0$. The main part of $X_0$ is earmarked investments and a relatively small amount is dedicated to paying management fees. In particular, the GP who is responsible for investing has $I_0$ at his disposal at fund initiation, and $\varphi m T X_0$ is dedicated to paying management fees and thus left uncalled. The amount $\varphi m T X_0$ stems from the annual management fee $\varphi m \in [0, 1]$ being paid on the basis of $X_0$ and the fund maturity being $T$ years. The division of committed capital is summarized in the relation

$$X_0 = I_0 + \varphi m T X_0. \quad (3.1)$$

The introduction of a stochastic fund termination date gives rise to premature termination and thus not all management fees being paid to the GP. In case the fund is dissolved early, the remainder of the uncalled capital is assumed to go to the LP. Naturally, the lifetime of the fund, denoted $u$, is bounded from above by the maturity, i.e. $u \leq T$.

It is well known that some PE funds rely heavily on their ability to apply leverage which we incorporate by use of a leverage ratio $l \equiv D_0/I_0$ in which $D_0$ represents the debt obtained at time $t = 0$. The PE fund thus acquires $V_0 = D_0 + I_0$ assets at fund initiation. For simplicity, we assume that the obtained loan is of the balloon type implying that the creditors should be repaid

$$D(u) = (1 + y_0)^u D_0, \quad (3.2)$$

at time $u$ where $y_0$ represents the periodically compounded yield.

A key component of PE fund agreements and PE fee structures is the preferred return which constitutes an important threshold for the GP as profits are not shared before all debt is repaid and the preferred return has been delivered (Fleischer, 2005). The preferred return is highly fund specific but often characterized by a hurdle rate of return $r_H > 0$ paid on e.g. committed capital, invested capital, or total contributed capital (Bartlett and Swan, 2000). In line with Mathonet and Meyer (2008) and consistent with the internal rate of return approach applied in practice, the periodically compounded preferred return value of a $u$-period investment is defined as

$$K(u) = I_0 \left(1 + r_H\right)^u + \frac{\varphi m X_0}{r_H} \left((1 + r_H)^u - 1\right). \quad (3.3)$$

The first term captures the repayment of the initially invested capital $I_0$ periodically compounded by the hurdle rate. Likewise, the second term corresponds to periodic hurdle rate compounded value of the management fees paid up until time $u$. 
Throughout this paper, we consider a traditional PE performance based scheme characterized by a carried interest component (carry) and a potential catch-up provision. Carry is a commonly used PE term and corresponds to the GP’s share of the distributed profits. Thus the aggressiveness of the performance scheme depends on the carried interest rate parameter $\phi_p \in [0, 1]$ and the profit base on which carry is paid. As stated in the introduction, the catch-up serves as an incentivizing component as it allows the GP to acquire profits in excess of the preferred return at a rate $\phi_c$ which is higher than the carried interest rate. When the catch-up has been fully utilized, meaning that the GP has received $\phi_c$ of the total amount of the profits, the GP again acquires profits at the rate $\phi_p$.

### 3.2.1 The Payoffs

When the fund is dissolved either through premature liquidation or regular termination at maturity, the value of the proceeds from the sale of the PE assets is distributed according to the ranking of the claims. In case the fund relies on leverage, the debtholders have the highest seniority of the partners. The LP’s claim is junior to that of the debtholders but senior to that of the GP. We present the partners’ payoff functions according to their seniority.

At time $u$ the debtholders receive the payoff

$$\Phi^D_u = \min \left[ D(u), V_u \right].$$  \hspace{1cm} (3.4)

The first case represents the situation where the proceeds from the sale of the PE assets are sufficient to cover the promised balloon payment $D(u)$, and the second the bad state where the proceeds are insufficient and hence fully assigned to the debtholders.

The LP’s time $u$ payoff in the presence of a catch-up is

$$\Phi^P_{Lu} = \begin{cases} 0 & V_u \leq D(u) \\ V_u - D(u) & D(u) \leq V_u \leq D(u) + K(u) \\ V_u - D(u) - \phi_c \left( V_u - \left( D(u) + K(u) \right) \right) & D(u) + K(u) \leq V_u \leq \Phi(u) \\ V_u - D(u) - \phi_p \left( V_u - \left( D(u) + I_0 + \phi_m u X_0 \right) \right) & V_u \geq \Phi(u), \end{cases}$$

where

$$\Phi(u) \equiv D(u) + \frac{\phi_c K(u) - \phi_p \left( I_0 + \phi_m u X_0 \right)}{\phi_c - \phi_p}.$$  \hspace{1cm} (3.5)

The first case of equation (3.5) represents the bad state where the proceeds from the sale of the PE assets are less than the balloon payment owed to the creditors. In the second case, the proceeds are sufficiently large to cover the debt payment, but too small to fully pay the LP’s preferred return. The third case represents the situation in which the GP utilizes the catch-up. The final case represents the state where the catch-up is fully utilized and the LP and GP share the remaining proceeds in line with
3.2. Contract Structures

$\varphi_p$. The linkage between the third and fourth case becomes clear when considering the preferred return limit, $\lim_{u \to 0} K(u) = I_0 + \varphi_m u X_0$.

The structure of the claim is derived from our data set of PE fund contracts and nests that of Sørensen et al. (2014) in the absence of early fund dissolution, i.e. when $u = T$. In particular, the "upper attachment point" $\bar{\varphi}(u)$ is obtained from the catch-up "continuity" condition

$$\varphi_c \left( \bar{\varphi}(u) - (D(u) + K(u)) \right) = \varphi_p \left( \bar{\varphi}(u) - (D(u) + I_0 + \varphi_m u X_0) \right),$$

(3.7)

which states that the GP receives $\varphi_c$ of distributed profits until he has received $\varphi_p$ of distributed investment proceeds net of debt and contributed capital $I_0 + \varphi_m u X_0$.

Note that the contract payoff only makes sense for $\varphi_c > \varphi_p$.

The LP's payoff in equation (3.5) can be expressed in "option-form" as

$$\Phi_{LP}^u = \left( V_u - D(u) \right)^+ - \varphi_c \left( V_u - P(u) \right)^+ + \left( \varphi_c - \varphi_p \right) \left( V_u - \bar{\varphi}(u) \right)^+, \quad (3.8)$$

where we have introduced $P(u) \equiv D(u) + K(u)$. Throughout the article, we let $\max \{ A, 0 \}$.

The LP thereby holds a portfolio of European call options on $V_u$ with different strike prices and participation rates. In particular, the LP is long a European call with strike $D(u)$, short a $\varphi_c$ share in a European call with strike price $P(u)$, and long a $(\varphi_c - \varphi_p)$ share in a European call with strike price $\bar{\varphi}(u)$. It is clear that the choice of preferred return bases $(I_0$ and $X_0)$ in equation (3.3) affects the valuation complexity as path-dependency can be introduced through the strike prices. Time-varying strike prices are quite common in the hedge fund industry through high-water mark contracts but not in the PE industry.

A contract without a catch-up provision is, in this framework, characterized by the catch-up speed $\varphi_c$ going towards the carried interest $\varphi_p$. In the absence of a catch-up, the LP's payoff reads

$$\Phi_{LP,0}^u \equiv \lim_{\varphi_c \downarrow \varphi_p} \Phi_{LP}^u = \left( V_u - D(u) \right)^+ - \varphi_p \left( V_u - P(u) \right)^+, \quad (3.9)$$

as $\lim_{\varphi_c \downarrow \varphi_p} \bar{\varphi}(u) = \infty$. Additionally, it can easily be shown that $\Phi_{LP,0}^u - \Phi_{LP}^u \geq 0$.

If a catch-up prevails, the GP's performance based time $u$ claim referred to as the incentive fee is given as

$$\Phi_{GP}^u = \begin{cases} 0 & V_u \leq P(u) \\ \varphi_c \left( V_u - P(u) \right) & P(u) \leq V_u \leq \bar{\varphi}(u) \\ \varphi_p \left( V_u - (D(u) + I_0 + \varphi_m u X_0) \right) & V_u \geq \bar{\varphi}(u) \end{cases} \quad (3.10)$$

The first case represents the bad state in which the proceeds from the sale of the PE assets are less than the sum of the creditors' balloon payment and the LP's preferred
return. The second case captures the intermediate state in which the proceeds are large enough to activate the catch-up but too low to fully utilize it. The final case represents the case where the catch-up is fully utilized and the remainder of the proceeds is distributed between the GP and the LP according to $\varphi_p$.

Stated in option-form, the payoff reads

$$
\Phi^\mathrm{GP}_u = \varphi_c \left(V_u - P(u)\right)^+ - \left(\varphi_c - \varphi_p\right) \left(V_u - \bar{\varphi}(u)\right)^+. 
$$

(3.11)

The payoff function for the GP’s catch-up based performance fee is similar to that of a portfolio of European call options on the underlying fund value $V_u$. In particular, the payoff resembles that of a long $\varphi_c$ share in a European call option with strike price $P(u)$ and a short $\left(\varphi_c - \varphi_p\right)$ share of European call option with strike price $\bar{\varphi}(u)$.

The corresponding payoff for a contract with carried interest rate $\varphi_p$ and no catch-up feature is in option-form

$$
\Phi^\mathrm{GP,0}_u \equiv \lim_{\varphi_c \downarrow \varphi_p} \Phi^\mathrm{GP}_u = \varphi_p \left(V_u - P(u)\right)^+. 
$$

(3.12)

It can easily be shown that $\Phi^\mathrm{GP}_u - \Phi^\mathrm{GP,0}_u \geq 0$ holds.

### 3.2.2 The Catch-up Provision

Through the following stylized example we show that the inclusion of a catch-up increases the GP’s profit potential by construction, and that for a well-performing fund delivering high fund returns the specific catch-up rate is less important.

The stylized example is based on the fee structure of a typical contract from our data set and includes no debtholders as the fund does not rely on leverage ($l = 0$). The fee structure contains a fixed management fee of $\varphi_m = 0.02$ of committed capital paid annually over the lifetime of the fund and a variable performance-based scheme characterized by the preferred return in equation (3.3), a hurdle rate of $r_H = 0.08$, and a carried interest rate of $\varphi_p = 0.2$. Moreover, we assume that committed capital is divided according to equation (3.1) and without loss of generality, we let $I_0 = 100$.

Finally, we consider a fund that is not exposed to early dissolution and whose assets grow at 13% per annum for $T = 10$ years.

By use of the payoff functions in equations (3.5) and (3.10) figure 3.1 depicts the decompositions of the distributed proceeds in the presence of a 100% catch-up in panel (a) and a 50% catch-up in panel (b). For both contracts the LP’s preferred return has the highest priority and as the terminal fund value is sufficiently high the preferred return is paid in full. Then the 100% catch-up component in panel (a) is determined by the GP solely receiving proceeds until the catch-up has been fully utilized, i.e. until the GP has received 20% of the distributed proceeds net of contributed capital. As the fund is terminated at time $T$ called capital, $I_0 + \varphi_m T X_0$, equals committed capital $X_0$. In panel (b), the catch-up speed is only 50% and thus the GP and LP share the proceeds in excess of the preferred return $K(T)$ equally until
3.2. Contract Structures

the GP has received 20% of the distributed proceeds net of contributed capital. In both cases, the terminal fund value is high enough to ensure full catch-up utilization, and the carried interest components are thus characterized by \( \varphi_p \).

A comparison of panels (a) and (b) shows that the GP’s payoff obtained through the catch-up provision is lowest in panel (a) that has the highest catch-up parameter. The explanation behind this counterintuitive result is simply that a smaller portion of the proceeds is distributed in panel (a) before the GP has received 20% of the distributed proceeds net of committed capital as he is the sole collector. As the yearly fund return is sufficiently high, the GP receives the same total incentive fee in both cases and the catch-up speed parameter \( \varphi_c \) thus solely affects distribution priority. In short, a high \( \varphi_c \) implies a better positioning for the GP and thus affects the ex ante valuation positively.

Figure 3.2 depicts the required catch-up speed \( \varphi_c \) and net fund return (p.a.) that ensure a full utilization of the catch-up provision. The plot illustrates that the contractual parameter \( \varphi_c \) is most important for low return funds and less important for high return funds. From the GP’s point of view, the distribution of proceeds relative to the catch-up speed is too aggressive below and to the left of the curve.

Figure 3.3 depicts the GP’s and LP’s payoffs under four different contracts: one without a catch-up and three with catch-ups and rates \( \varphi_c \in \{0.25, 0.50, 1.00\} \). Panel (a) contains the payoffs of the GP, whereas panel (b) contains those of the LP.

First, consider the GP’s payoff in panel (a). The four contracts face the same preferred return, implying that the incentive fee yields a positive payoff once the terminal fund value exceeds the threshold at \( K(T) \). The catch-up speed parameter \( \varphi_c \) determines the steepness of the payoff function in the range between the preferred return threshold and the common path reached by contracts with catch-up provisions, i.e.
Figure 3.2: Stylized example: The required $\varphi_c$ that fully utilizes the GP’s 100% catch-up.

Figure 3.3: Stylized example: The GP’s and LP’s payoffs when the contract has no catch-up provision or catch-ups with rates $\varphi_c \in \{0.25, 0.50, 1.00\}$.

$V_T \in \left( K(T), \bar{\varphi}(T) \right)$. The common path of catch-up contracts is characterized by the fund return being high enough to fully utilize the catch-up. Thus a low $\varphi_c$ contract requires a higher terminal fund value to fully utilize the catch-up as low $\varphi_c$-contracts suffer from the catch-up rate being low relative to the profit sharing governed by $\varphi_p$.

Second, examine panel (b) that presents the opposite story of panel (a), namely the LP’s payoff. In order to ease the reading of the plot and since the LP collects all proceeds below $K_T$, terminal fund values below 220 are not considered in the plot. The catch-up affects the payoff negatively as the payoff steepness is decreasing in $\varphi_c$ for $V_T \in \left( K(T), \bar{\varphi}(T) \right)$. The payoffs of the catch-up contracts reaches a common path that is lower than that of the contract without catch-up.

Finally, in both panels (a) and (b), we observe fundamental differences in terms
3.3. Valuation Framework

of the common paths between contracts with and without catch-ups. The inability of
the no-catch-up contracts to reach the common payoff path is due to the GP being
paid \( \varphi \) of the proceeds in excess of the preferred return, \( V_T - K(T) \), whereas catch-
up contracts target the total profit base \( V_T - X_0 > V_T - K(T) \). Hence, irrespective of
the catch-up rate, the inclusion of a catch-up provision allows the GP to increase the
profit potential by construction.

3.3 Valuation Framework

In this section, we present the spanned and unspanned fund risk models that make
up our valuation framework. We first outline the setting that is common for both
models whereas section 3.3.1 relates to the spanned fund risk model and section 3.3.2
to the unspanned fund risk model.

Let prices admit no arbitrage, let the stochastic process \( S = (S_t)_{t \geq 0} \) denote the
price of the traded public equity, and let the \( V = (V_t)_{t \geq 0} \) denote value of the non-
traded PE fund \( V \). Let both the state variables \( (S, V) \) be defined on the filtered proba-
bility space \( (\Omega, F, (F_t)_{t \geq 0}, P) \) and adapted to the filtration \( F_t \), representing all available
information at time \( t \geq 0 \). Conditioning on information is denoted by a subscript \( t \).

We assume that \( S \) and \( V \) evolve according to the dynamics

\[
\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dz_t, \quad (3.13)
\]

\[
\frac{dV_t}{V_t} = \mu_V dt + \sigma_{Vz} dz_t + \sigma_{Vw} dw_t, \quad (3.14)
\]

where \( \mu_S \) and \( \mu_V \) characterize the relative drift of \( S \) and \( V \), respectively. The uncertainty of the model is driven by the two independent standard Brownian motions
\( z \) and \( w \). The latter represents the fund specific risk as a \( z \)-shock affects both \( S \) and
\( V \) through \( \sigma_z > 0 \) and \( \sigma_{Vz} > 0 \), respectively, whereas a \( w \)-shock solely affects \( V \) via
\( \sigma_{Vw} > 0 \). In addition to the marketed asset \( S \) and the fund \( V \), the economy also
contains an instantaneously riskless asset \( B = (B_t)_{t \geq 0} \) following

\[
\frac{dB_t}{B_t} = rd t, \quad B_0 = 1. \quad (3.15)
\]

Sørensen et al. (2014) suggest modeling the GP’s value-adding skill (alpha) explicitly
and as the unspanned fund risk model is also suited for this, \( \mu_V \) is decomposed into

\[
\mu_V = r + \alpha + \beta (\mu_S - r). \quad (3.16)
\]

Here, \( \alpha \) represents the GP’s skill level and \( \beta \) the instantaneous CAPM beta measuring
the sensitivity of the fund return to the excess return of the marketed asset. The beta
of the model is given by

\[
\beta = \frac{\sigma_{Vz}}{\sigma_S}, \quad (3.17)
\]
where

\[ \sigma_V^2 \equiv \sigma_{Vz}^2 + \sigma_{Vw}^2. \tag{3.18} \]

Consider a contingent claim subject to premature termination introduced through an independent exponentially distributed termination date. In the absence of a premature termination, the claim pays \( \Phi(V_T) \) at time \( T \), whereas in case of an intermediate termination at time \( u \) it immediately pays \( H(V_u, u) \). As implicitly assumed by Metrick and Yasuda (2010) and Choi et al. (2012), we let liquidation risk be diversifiable through holding a large portfolio of random maturity assets. A generic claim \( f_t = f(V_t, t) \) subject to a stochastic termination date evolves according to

\[
df_t = \left( \frac{\partial f}{\partial t}(V_t, t) + \frac{\partial f}{\partial V}(V_t, t) \mu_V V_t + \frac{1}{2} \frac{\partial^2 f}{\partial V^2}(V_t, t) \sigma_V^2 V_t^2 \right) dt
\]

\[
+ \frac{\partial f}{\partial V}(V_t, t) \sigma_{Vz} V_t \, dz_t + \frac{\partial f}{\partial V}(V_t, t) \sigma_{Vw} V_t \, dw_t
\]

\[
+ \left( H(V_t, t) - f_{t-} \right) dN_t, \tag{3.19}
\]

where \( dN_t \) is the increment to a point process with values \( N = 0 \) and \( N = 1 \) before and after termination, respectively. The point process is assumed to be independent of other random variables and governs premature termination by the intensity parameter \( \lambda \) satisfying \( \mathbb{E}_t[dN_t] = \lambda dt \). The premature termination is characterized by the stopping time \( \tau = \inf \{ t : N_t > 0 \} \) implying that the fund dissolution time equals \( u = \min \{ \tau, T \} \).

In line with the payoffs in section 3.2, we are interested in valuing a European call option written on a non-traded underlying, but subject to stochastic premature termination. In particular, we consider the payoff functions

\[
H(V_u, u) = \max \left[ V_u - F(u), 0 \right], \quad \Phi(V_T) = \max \left[ V_T - F(T), 0 \right]. \tag{3.20}
\]

As stated in the introduction we will see that when the fund specific \( w \)-risk can be hedged we are able to put an exact price on such an option contrary to a set of bounds as will be the case under unspanned fund risk.

Prior to valuing the European call option in the spanned and unspanned fund risk models, we present the valuation formulas for the GP’s management fees and the risky debt as these formulas are common for both models.

**Management Fee Valuation**  Assuming that the GP receives \( \varphi_m X_0 dt \) each instant, the time \( t = 0 \) value of the management fees is

\[
\mathcal{M}_0 = \begin{cases} 
X_0 \varphi_m \left( 1 - e^{-(r + \lambda)T} \right) / (r + \lambda) & \text{with liquidation risk} \\
X_0 \varphi_m \left( 1 - e^{-rT} \right) / r & \text{without liquidation risk},
\end{cases} \tag{3.21}
\]

where it is required that both \( r \) and \( \lambda \) are positive.
3.3. Valuation Framework

Debt Valuation  As the debt is risky, the creditors are entitled to receive either \( \min \left[V_T, D(T)\right] \) at maturity or \( \min \left[V_\tau, D(\tau)\right] \) at time \( \tau \) in case of early dissolution. Since \( \min \left[x, y\right] = x - \max \left[x - y, 0\right] \), the debtholder’s claim can be expressed as a claim on the fund value subtracted a European call option on the fund value with a strike price equal to the outstanding debt, both subject to the same stochastic termination.

Throughout the paper we assume that the debtholders are well-diversified and do not require any compensation for taking on the fund specific risk. This implies that debt pricing becomes identical under both the spanned and unspanned fund risk models. Following the spanned fund risk model approach in appendix A.1, the time \( t = 0 \) value of the debt subject to liquidation risk is

\[
D_0 = \begin{cases} 
V_0 \left( \lambda - \alpha e^{-\left(\lambda - \alpha\right)T} \right) / (\lambda - \alpha) - C_0 \left( \alpha, \lambda, D(\cdot) \right) & \text{with liquidation risk, } \lambda \neq \alpha \\
V_0 \left( 1 + \lambda T \right) & \text{with liquidation risk, } \lambda = \alpha \\
V_0 e^{\alpha T} - C_0 \left( \alpha, \lambda, D(\cdot) \right) & \text{without liquidation risk.}
\end{cases}
\]  
(3.22)

\( C_0 \left( \alpha, \lambda, D(\cdot) \right) \) is the value of a European call option written on \( V \) with strike \( D(\cdot) \) but subject to stochastic liquidation. It will be defined more precisely in equation (3.24) below.

Finally, we assume competitive debt pricing, i.e. the applied yield \( y_0 \) and the associated credit spread \( \zeta_0 = y_0 - r \) are determined by the value of the debt at issuance \( D(y_0) \) equaling the amount borrowed \( D_0 \).

3.3.1 The Spanned Fund Risk Model

In the spanned fund risk model of Sørensen et al. (2014) a hedge asset with stochastic process \( S' = (S'_t)_{t \geq 0} \) is introduced. The asset does not carry any risk premium and has the dynamics

\[
\frac{dS'_t}{S'_t} = rd t + \sigma S'_t dw_t.
\]  
(3.23)

The presence of the hedge asset implies that the fund specific risk \( w \) is traded, thus making the market effectively complete.

We consider the pricing of a European call option written on the non-traded underlying \( V \) with strike \( F(\cdot) \) subject to an exponentially distributed termination date with intensity \( \lambda \). In particular, the option pays off according to the equations in (3.20). The time \( t = 0 \) price of such a European call option is

\[
C_0 \left( \alpha, \lambda, F(\cdot) \right) = \begin{cases} 
\int_0^T \lambda e^{-\lambda v} BS(v, F(v), -\alpha) \, dv + e^{-\lambda T} BS(T, F(T), -\alpha) & \text{with liquidation risk} \\
BS(T, F(T), -\alpha) & \text{without liquidation risk},
\end{cases}
\]  
(3.24)
where

\[
BS\{v, F(v), q\} \equiv V_0 e^{-qv} N\left(d\{v, F(v), q\}\right)
- F(v) e^{-rv} N\left(d\{v, F(v), q\} - \sigma_V \sqrt{v}\right),
\]

(3.25)

and

\[
d\{v, F(v), q\} \equiv \ln\left(\frac{V_0}{F(v)}\right) + \left(r - q + \frac{1}{2} \sigma_V^2 v\right) \sigma_V \sqrt{v}.
\]

(3.26)

\(N(\cdot)\) denotes the cumulative distribution function of a standard normal variable. We note that \(BS\{v, F(v), q\}\) corresponds to the Black-Scholes price of a European call option on \(V\) with maturity \(v\) and strike price \(F(v)\) when \(V\) pays a continuous dividend yield \(q\). In this paper, we denote the Black-Scholes price of a zero-strike call option by \(BS\{v, 0, q\}\). The derivation of the pricing formula in (3.24) can be found in appendix A.1. The model nests the Sørensen et al. (2014) model when \(\lambda = 0\) and the Black-Scholes model when \(\lambda = \alpha = 0\).

Using the payoffs in equations (3.8) and (3.9), the time \(t = 0\) value of the LP’s partnership interests is

\[
\gamma^\text{LP}\_0 = \mathcal{R}_0 - \mathcal{M}_0,
\]

(3.27)

where the value of the LP’s investment proceeds claim is given by

\[
\mathcal{R}_0 = \begin{cases}
C_0(\alpha, \lambda, D(\cdot)) - \varphi_c C_0(\alpha, \lambda, P(\cdot)) \\
+ \left(\varphi_c - \varphi_p\right) C_0(\alpha, \lambda, \bar{\varphi}(\cdot)) & \text{with catch-up} \\
C_0(\alpha, \lambda, D(\cdot)) - \varphi_p C_0(\alpha, \lambda, P(\cdot)) & \text{without catch-up}.
\end{cases}
\]

(3.28)

In case the contract does not include a catch-up we add a superscript 0 to the value measures. \(\gamma^\text{LP,0}_0\) and \(\mathcal{R}^0_0\) thus denote the no-catch-up values of LP’s total claim and investment proceeds claim, respectively.

The time \(t = 0\) value of the GP’s total claim is the sum of the value of the incentive fee and the value of the management fees

\[
\gamma^\text{GP}\_0 = J_0 + \mathcal{M}_0.
\]

(3.29)

The value of the GP’s incentive fee is derived using the payoffs in equations (3.11) and (3.12) and reads

\[
J_0 = \begin{cases}
\varphi_c C_0(\alpha, \lambda, P(\cdot)) - \left(\varphi_c - \varphi_p\right) C_0(\alpha, \lambda, \bar{\varphi}(\cdot)) & \text{with catch-up} \\
\varphi_p C_0(\alpha, \lambda, P(\cdot)) & \text{without catch-up}.
\end{cases}
\]

(3.30)
3.3. Valuation Framework

Despite including stochastic fund termination the concept of value additivity applied in Sørensen et al. (2014) prevails in the model since

\[
D_0 + V_0^{LP} + V_0^{GP} = V_0 = \begin{cases} 
V_0 \left( \lambda - \alpha e^{- (\lambda - \alpha) T} \right) / (\lambda - \alpha) & \text{with liquidation risk, } \lambda \neq \alpha \\
V_0 \left( 1 + \lambda T \right) & \text{with liquidation risk, } \lambda = \alpha \\
V_0 e^{\alpha T} & \text{without liquidation risk,}
\end{cases}
\]

(3.31)

where \( V_0 \) is interpreted as the economic value of the fund. In addition, the capital structure irrelevance principle of Modigliani and Miller (1958) holds in the model as the economic value of the fund does not depend on the financing but only on \( V_0, \lambda, \) and \( \alpha \).

We also consider what we name the relative catch-up value defined as

\[
C_0 \equiv - \left( R_0 - \bar{R}_0^0 \right) / R_0^0 = \frac{\left( \phi_c - \phi_p \right) C_0(\alpha, \lambda, P(\cdot)) - C_0(\alpha, \lambda, \bar{\phi}(\cdot))}{C_0(\alpha, \lambda, D(\cdot)) - \varphi_p C_0(\alpha, \lambda, P(\cdot))},
\]

(3.32)

as a measure of the relative value transfer from the LP to the GP in case a catch-up is added to the contract. The relative catch-up value thus defines the LP’s costs of including a catch-up.

In order to ease the interpretation of the relative catch-up value consider the case of no liquidation risk. The relative catch-up value can then be interpreted as a \( \left( \phi_c - \phi_p \right) \) scaling of the ratio between the values of two European option strategies whose options have different strike prices and are written on the same underlying which happens to pay a continuous dividend yield \( -\alpha \).

The numerator in (3.32) corresponds to a "bull call spread" option strategy consisting of a long position in an out-of-the-money call option with strike price \( P(\cdot) \) and a short position in an out-of-the-money call option with strike price \( \bar{\phi}(\cdot) \). The strategy has a positive payoff when the value of the underlying is greater than \( P(\cdot) \) though with a capped payoff at values greater than \( \bar{\phi}(\cdot) \). Obviously, this payoff resembles that of the catch-up and thus a bet on the value of the PE fund assets being above the sum of the debt repayment and the preferred return at maturity.

The denominator in (3.32) corresponds to long position in a call option with strike price \( D(\cdot) \) accompanied by a \( \varphi_p \)-participation in a short call option with strike price \( P(\cdot) \). Thus the strategy pays off when the value of the underlying is greater than \( D(\cdot) \) but with a \( \varphi_p \) payoff reduction when the value of the underlying is above \( P(\cdot) \).

The relative catch-up value is highest when the numerator strategy is highly valued compared to the denominator strategy and thus depends on the location of the thresholds \( \{ D(\cdot), P(\cdot), \bar{\phi}(\cdot) \} \) and the GP’s ability to grow the fund \( \alpha \).

When interested in how \( \alpha \) affects the relative catch-up value one must consider the derivative \( \partial C_0 / \partial \alpha \). In case of leverage, the credit spread depends on \( \alpha \) causing
3. The Valuation of Catch-up Provisions

The thresholds \( \{D(\cdot), P(\cdot), \phi(\cdot)\} \) to depend on \( \alpha \) as well. This implies that we must rely on numerical approximations of the derivative when the fund relies on leverage. Likewise, in the presence of liquidation risk the derivative contains integrals that must be computed numerically. Nevertheless, in the absence of leverage and liquidation risk the sensitivity is given by

\[
\frac{\partial C_0}{\partial \alpha} = \left( \phi_c - \phi_p \right) TV_0 e^{aT} \left( \frac{N\left(d\left(T, K(T), -\alpha\right)\right) - N\left(d\left(T, \phi(T), -\alpha\right)\right)}{V_0 e^{aT} - \phi_p BS\left(T, K(T), -\alpha\right)} + \frac{1 - \phi_p N\left(d\left(T, K(T), -\alpha\right)\right)}{V_0 e^{aT} - \phi_p BS\left(T, K(T), -\alpha\right)} \right) \right) \] 

(3.33)

Appendix A.4 contains the derivation and the expressions used for computing the derivative in the presence of leverage and under liquidation risk.

Finally, we are interested in how changes in the liquidation intensity \( \lambda \) affect the relative catch-up value and hence we consider the derivative \( \frac{\partial C_0}{\partial \lambda} \). As in the previous case the derivative is determined numerically when the fund relies on leverage, but for the unlevered fund we apply

\[
\frac{\partial C_0}{\partial \lambda} = \left( \phi_c - \phi_p \right) \left( \frac{\frac{\partial}{\partial \lambda} C_0(\alpha, \lambda, K(\cdot)) - \frac{\partial}{\partial \lambda} C_0(\alpha, \lambda, \phi(\cdot))}{V_0 - \phi_p C_0(\alpha, \lambda, K(\cdot))} \right) \] 

(3.34)

The derivation is found in appendix A.4 along with the expressions for \( \frac{\partial}{\partial \lambda} C_0(\alpha, \lambda, F(\cdot)) \) and \( \frac{\partial}{\partial \lambda} V_0 \).

3.3.2 The Unspanned Fund Risk Model

As the LP and GP can neither trade in the option nor the underlying asset they are facing unspanned fund risk. We follow the standard approach for pricing derivatives and value the claims relative to the assets traded in the financial market. Unfortunately, the prevailing market incompleteness implies that we cannot put an exact price on the option. In order to cope with the unspanned fund risk, we adopt the good-deal bounds methodology introduced by Cochrane and Saa-Requejo (2000) which allows us to derive a set of bounds encircling model consistent prices. The good-deal bounds are founded in the assumption that besides pure arbitrage opportunities, investors will always engage in high Sharpe ratio-deals (so-called good deals).

The good-deal bounds approach originally proposed by Cochrane and Saa-Requejo (2000) uses the state price deflator pricing mechanism (see e.g. Munk, 2013), but in
order to ensure consistency between the two valuation frameworks applied in this paper, we base our analyses on the refined good-deal bounds framework of Björk and Slinko (2006) which relies on a martingale measure approach. It is well established that in the presence of a riskless asset, a risk-neutral measure exists if and only if prices admit no arbitrage. Moreover, the uniqueness of the measure is conditional on the market being complete which is why valuation in incomplete markets turns troublesome.

Like we did in the previous section, we consider the pricing of a European call option written on the non-traded underlying \( V \) with strike \( F(\cdot) \) subject to an exponentially distributed termination date with intensity \( \lambda \). The option payoffs follow from the equations in (3.20).

Given a Sharpe ratio bound \( A \) and a positive instantaneous market Sharpe ratio \( SR_S \), the time \( t = 0 \) good-deal bounds are obtained by setting \( a = 1 \) and \( a = -1 \) in

\[
C^\pm_0 (\eta(a), \lambda, F(\cdot)) = \begin{cases}
\int_0^T \lambda e^{-\lambda v} BS(v, F(v), -\eta(a)) \, dv + e^{-\lambda T} BS(T, F(T), -\eta(a)) & \text{with liquidation risk} \\
BS(T, F(T), -\eta(a)) & \text{without liquidation risk}
\end{cases}
\]

where

\[
\eta(a) \equiv \alpha + a\sigma_V w \sqrt{A^2 - SR_S^2}, \quad (3.36)
\]

\[
SR_S \equiv \frac{\mu_S - r}{\sigma_S}. \quad (3.37)
\]

As in the spanned fund risk model, \( BS(v, F(v), q) \) represents the Black-Scholes price and follows from (3.25). The superscript \( \pm \) is our shorthand notation for capturing both bounds in one line. The derivation of the valuation formula is found in appendix A.2.

Compared to the spanned fund risk model, the introduction of unspanned fund risk gives rise to a risk adjustment in the "dividend yield" of the Black-Scholes prices. Thus the unspanned fund risk valuation can be seen as a natural extension of the spanned fund risk model accounting for both the GP's skill level and market incompleteness through the level of unspanned fund risk. Due to the unspanned \( w \)-risk and the difference in expected growth rates of \( S \) and \( V \), the drift of the underlying asset plays a role in the option valuation which is not the case in the Black-Scholes model.

The time \( t = 0 \) unspanned fund risk bounds of the value of the LP's total claim are given as the good-deal bounds of the LP's proceeds claim subtracted the value of the management fees,

\[
V_0^{LP, \pm} = \mathcal{R}_0^{\pm} - \mathcal{M}_0. \quad (3.38)
\]
The first term is given by
\[
\mathcal{R}_0^+ = \begin{cases} 
C_0^+ (\eta(a), \lambda, D(\cdot)) - \varphi_c C_0^+ (\eta(a), \lambda, P(\cdot)) \\
+ (\varphi_c - \varphi_p) C_0^+ (\eta(a), \lambda, \bar{\varphi}(\cdot))
\end{cases}
\]
with catch-up (3.39)
\[
C_0^+ (\eta(a), \lambda, D(\cdot)) - \varphi_p C_0^+ (\eta(a), \lambda, P(\cdot))
\]
without catch-up.
whereas the second term follows from equation (3.21).

The time \( t = 0 \) value of the GP's total claim is the sum of the unspanned fund risk bounds of the incentive fee and the value of the management fees,

\[
\mathcal{J}^{GP, \pm}_0 = \mathcal{J}^{\pm}_0 + \mathcal{M}_0.
\] (3.40)

The unspanned fund risk bounds of the GP's incentive fee are

\[
\mathcal{J}_0^\pm = \begin{cases} 
\varphi_c C_0^+ (\eta(a), \lambda, P(\cdot)) - (\varphi_c - \varphi_p) C_0^+ (\eta(a), \lambda, \bar{\varphi}(\cdot)) \\
\varphi_p C_0^+ (\eta(a), \lambda, P(\cdot))
\end{cases}
\]
with catch-up
\[
\mathcal{J}_0^\pm = \begin{cases} 
C_0^+ (\eta(a), \lambda, D(\cdot)) - \varphi_p C_0^+ (\eta(a), \lambda, P(\cdot))
\end{cases}
\]
without catch-up.
In the unspanned fund risk model, value additivity and the Modigliani and Miller (1958) theorem do not prevail due to the creditors' spanned fund risk debt pricing. In particular we have

\[
\mathcal{D}_0 + \mathcal{V}_{0,LP}^{GP, \pm} + \mathcal{V}_{0,GP}^{GP, \pm} \neq \mathcal{V}_0^+, \quad (3.42)
\]
where

\[
\mathcal{V}_{0}^{\pm} = \begin{cases} 
\frac{V_0 (\lambda - \eta(a) e^{-(\lambda - \eta(a))T})}{\lambda - \eta(a)} \\
V_0 e^{\eta(a)T}
\end{cases}
\]
with liquidation risk
\[
\mathcal{V}_{0}^{\pm} = \begin{cases} 
\frac{V_0 (\lambda - \eta(a) e^{-(\lambda - \eta(a))T})}{\lambda - \eta(a)} \\
V_0 e^{\eta(a)T}
\end{cases}
\]
without liquidation risk.
Finally, like in the spanned fund risk model we consider the counterpart of the relative catch-up value, namely the relative catch-up bounds defined as

\[
\mathcal{C}_0^{\pm} \equiv - \left( \mathcal{R}_0^{\pm} - \mathcal{R}_0^{\pm,0} \right) / \mathcal{J}_0^{\pm,0}
\]
and

\[
\mathcal{C}_0^{\pm} = \frac{\varphi_c - \varphi_p}{\varphi_c} C_0^+ (\eta(a), \lambda, \bar{\varphi}(\cdot)) - \varphi_p C_0^+ (\eta(a), \lambda, P(\cdot)).
\] (3.44)
These bounds serve as the limits for the value transfer implied by including a catch-up in the contract.

With the valuation formulas at hand for both the spanned and unspanned fund risk model, we turn to analyzing the PE fund contracts. In our application the valuations in (3.24) and (3.35) are computed by use of numerical integration procedures.

3.4 Numerical Results

In this section we first illustrate the potential shortcomings in PE contract valuation and fee structure evaluation related to assuming that the financial market spans all
fund risks. Subsequently, we investigate how different levels of unspanned fund risk, liquidation risk, and a combination hereof, affect the valuation of the PE partnerships’ interests. Additionally, we determine the value transfer implied by the inclusion of a catch-up in various settings. In order to ease the interpretation of such value transfers, we compute corresponding breakeven values of the fund manager skill level, the annual management fee rate, and the carried interest rate.

3.4.1 The Typical Contract and Model Parameter Values

The basis for our analysis is a GP compensation contract characterized by an annually paid fixed management fee of $\varphi_m = 0.02$ of committed capital, and an incentive fee described by a carry rate of $\varphi_p = 0.20$ and a catch-up with rate $\varphi_c = 1.00$. For the GP to receive the incentive fee, the investment proceeds must exceed the preferred return specified in equation (3.3). Regarding the fund specifications, the maturity is set to $T = 10$ and committed capital is assumed to be divided according to equation (3.1) with an initial investment of $I_0 = 100$. As we are interested in the impact of leverage, we consider both an unlevered fund, i.e. $l = 0$, and a levered fund. Specifically, we consider a levered fund that has a capital structure characterized by 2/3 debt and 1/3 equity, which corresponds to using a leverage factor of two, i.e. $l = 2$.

Prior to the analysis we establish the conditions of the economy by means of the model parameter values. As the marketed asset $S$ we consider the public stock market which we assume grows according to an expected nominal return of $\mu_S = 0.10$ and fluctuates with a volatility of $\sigma_S = 0.20$. These public equity parameter values are obtained from Ibbotson (2011) and reflects the historical 1926-2010 averages of US data. As we assume an interest rate of $r = 0.04$, the implied public equity risk premium is 0.06, which is in line with the average of the meta-analysis of Van Ewijk et al. (2012) and the time-series average of Dimson, Marsh, and Staunton (2002). The resulting Sharpe ratio of the public equity is $h_S = 0.3$.

We consider the PE fund as the non-traded asset of the model and thus the $V$ parameters must reflect PE investments. For the fund volatility we follow Sørensen et al. (2014) and use the return volatility of $\sigma_V = 0.25$, which they derive from Metrick and Yasuda (2010). According to equation (3.16) the expected fund return $\mu_V$ depends on the asset beta $\beta$. As PE fund $\beta$s are hard to estimate and closely tied to the fund characteristics, we abstain from specifying $\beta$ directly. Instead, we specify the unspanned fund risk $\sigma_{Vw}$, i.e. the volatility that stems from the unhedgeable risk source. Throughout the paper we consider both $\sigma_{Vw} = 0.05$ and $\sigma_{Vw} = 0.10$ representing a relatively hedgeable fund and a fund that is harder to hedge, respectively.

Cochrane and Saa-Requejo (2000) recommend using highly correlated assets in the good-deal framework as it tightens the bounds. The implied instantaneous correlation coefficients between $dV_t / V_t$ and $dS_t / S_t$ in our applications are 0.98 and 0.92, which indicates satisfying matches between the public stock market and the non-traded PE fund. Finally, we follow Cochrane and Saa-Requejo (2000) and consider
3.4.2 Valuation under Spanned Fund Risk

Table 3.1 presents the spanned fund risk contract valuations and credit spreads when \( \alpha \) is varied and no liquidation risk prevails. Panel A contains the case of the unlevered fund and panel B that of the levered fund. With the exception of the relative catch-up value \( C_0 \), the results are close to those presented in Sørensen et al. (2014) but serve the purpose of being a benchmark for the unspanned fund risk model here.

### Table 3.1: Spanned fund risk valuations in the absence of liquidation risk.

The table contains spanned fund risk valuations for the unlevered and levered fund in panels A and B, respectively. The columns refer to the valuations of the GP's incentive fee \( I_0 \), the GP's management fees \( M_0 \), the GP's total claim \( V_{GP0} \), the LP's proceeds claim \( R_0 \), the LP's total claim \( V_{LP0} \), and the total fund \( V_0 \). The table also shows the relative catch-up values \( C_0 \) and the credit spreads \( \zeta_0 \).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( I_0 )</th>
<th>( M_0 )</th>
<th>( V_{GP0} )</th>
<th>( R_0 )</th>
<th>( V_{LP0} )</th>
<th>( C_0 )</th>
<th>( \zeta_0 )</th>
<th>( V_0 )</th>
</tr>
</thead>
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<tr>
<td>Panel A: Unlevered (( l = 0 ))</td>
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<td></td>
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<td>4.05</td>
<td>20.60</td>
<td>24.66</td>
<td>86.43</td>
<td>65.83</td>
<td>2.00%</td>
<td>-</td>
<td>90.48</td>
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<td>25.78</td>
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<td>2.25%</td>
<td>-</td>
<td>100.00</td>
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<td>20.60</td>
<td>27.15</td>
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<td>-</td>
<td>110.52</td>
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<td>20.60</td>
<td>28.79</td>
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<td>20.60</td>
<td>30.74</td>
<td>124.85</td>
<td>104.25</td>
<td>2.98%</td>
<td>-</td>
<td>134.99</td>
</tr>
<tr>
<td>Panel B: Levered (( l = 2 ))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.01</td>
<td>9.74</td>
<td>20.60</td>
<td>30.34</td>
<td>61.71</td>
<td>41.11</td>
<td>3.39%</td>
<td>4.72%</td>
<td>271.45</td>
</tr>
<tr>
<td>0.00</td>
<td>13.91</td>
<td>20.60</td>
<td>34.51</td>
<td>86.09</td>
<td>65.49</td>
<td>3.37%</td>
<td>3.59%</td>
<td>300.00</td>
</tr>
<tr>
<td>0.01</td>
<td>18.76</td>
<td>20.60</td>
<td>39.36</td>
<td>112.80</td>
<td>92.19</td>
<td>3.32%</td>
<td>2.77%</td>
<td>331.55</td>
</tr>
<tr>
<td>0.02</td>
<td>24.36</td>
<td>20.60</td>
<td>44.97</td>
<td>142.06</td>
<td>121.45</td>
<td>3.27%</td>
<td>2.17%</td>
<td>366.42</td>
</tr>
<tr>
<td>0.03</td>
<td>30.82</td>
<td>20.60</td>
<td>51.42</td>
<td>174.14</td>
<td>153.53</td>
<td>3.19%</td>
<td>1.70%</td>
<td>404.96</td>
</tr>
</tbody>
</table>

Contract parameters: \( I_0 = 100 \), \( T = 10 \), \( r_H = 0.06 \), \( \phi_m = 0.02 \), \( \phi_p = 0.20 \).
Model parameters: \( r = 0.04 \), \( \sigma_V = 0.25 \).

Irrespective of capital structure, the value of the fund \( V_0 \) increases in \( \alpha \). The straightforward interpretation of this result is that the more the GP is able to grow the PE assets in excess of the market growth, the larger the investment proceeds at maturity.

As the LP benefits from the large investment proceeds, the LP's total claim \( V_{LP0} \) is also increasing in \( \alpha \). In particular, the value of the LP's investment proceeds claim \( R_0 \) is increasing in \( \alpha \) whereas the value of the management fees \( M_0 \) is unresponsive. As emphasized by Sørensen et al. (2014), the LP does not cover his initial investment of \( I_0 = 100 \) in case the GP does not generate value \( (\alpha = 0) \) as the LP's total claim
valuations are 74.22 and 65.49 for an unlevered and levered fund, respectively. Thus, for a PE investment to break even from the LP’s point of view, the fund manager must be able to deliver a sufficiently high $\alpha$.

Similar to the LP, the GP experiences a positive relationship between the value of his total claim $V_{GP}^0$ and $\alpha$. Naturally, the GP benefits from large investment proceeds through the incentive fee and hence its value $J_0$ increases in $\alpha$. In case the GP generates no $\alpha$ the value of the management fees constitutes 80% and 60% of the value of the GP’s total claim for an unlevered and levered fund, respectively.

In terms of $R_0$ and $V_{LP}^0$, the introduction of leverage makes the LP worse off for low-alpha funds and better off for high-alpha funds. The reasons are that i) the creditors break-even which causes a negative relationship between $\alpha$ and the credit spread $\zeta_0$ and thus an inverse relationship between $\alpha$ and the thresholds $\{D(T), P(T), \bar{\phi}(T)\}$, and that ii) the value of the PE assets are positively related to $\alpha$. Consequently, when the fund manager is unable to grow the PE assets in a convincing manner, expensive debt will be incurred and the thresholds will be hard to breach and vice versa. This implies that $D_0/V_0 > D_0/V_0$, $D_0/V_0 = D_0/V_0$, and $D_0/V_0 < D_0/V_0$ when $\alpha < 0$, $\alpha = 0$, and $\alpha > 0$, respectively.

Contrary to the LP, the GP benefits from leverage as the value of the incentive fee increases relative to the unlevered case. The reason for this is that the GP benefits from the higher PE asset volatility caused by leverage. Finally, the value of the management fees is insensitive with respect to leverage in this case as their capital base $X_0$ remains unchanged.

As stated above the results of this section mainly constitutes a benchmark for the later valuations. But to the best of our knowledge, the column referring to the relative catch-up value $C_0$ in table 3.1 has not been considered before. For the considered alpha levels, embedding a 100% catch-up in the PE contract leads to a significant transfer of value from the LP’s proceeds claim to the GP’s incentive fee. In fact, the relative catch-up values $C_0$ lie in the intervals $[2.00\%, 2.98\%]$ for the unlevered fund and in the range $[3.19\%, 3.39\%]$ for the levered fund. Section 3.4.5 examines the catch-up implied value transfer under unspanned fund risk and liquidation risk.

### 3.4.3 Valuation under Unspanned Fund Risk

In this section, we introduce unspanned fund risk and examine the valuation uncertainty on the LP’s total claim. Specifically, we consider the unspanned fund risk bounds $V_{LP,0}^{\pm}$ relative to the spanned fund risk benchmark $V_{LP}^0$ and the implied interval that spans model consistent relative valuations expressed in terms of $V_{LP}^0$. Figure 3.4 depicts the relative bounds of the LP’s total claim $V_{LP,0}^{\pm}/V_{LP}^0$ for varying alpha levels. Panel (a) contains the case of 0.10 unspanned fund risk whereas panel (b) presents the case of 0.05 unspanned fund risk.

For both the high and low degree of unspanned fund risk and irrespective of financing, the relative bounds of the LP’s total claim exhibit a trumpet-like shape.
Due to the plot scaling, the bounds of the unlevered fund appear to be constant, but close examination shows that they do in fact exhibit a similar trumpet-like shape. The shape implies that the LP’s valuation uncertainty is highest for funds with low-skilled GPs (low $\alpha$) and converges towards some limit interval. For both the levered and unlevered fund, the bounds of the LP’s claim obey the limit

$$\lim_{\alpha \to \infty} \left[ \frac{\gamma_{LP}^{0,\pm}}{\gamma_{LP}^{0}} \right] = e^{a \sigma_{V, w} \sqrt{A^2 - SR^2 T}}, \tag{3.45}$$

where $a \in \{-1, 1\}$ is the auxiliary bound parameter. For 0.10 and 0.05 unspanned fund risk, the limiting $\gamma_{LP}^{0,\pm}/\gamma_{LP}^{0}$ bounds are [59%, 168%] and [77%, 130%], respectively. Equation (3.45) shows that the limiting bounds depend exponentially on the unspanned fund risk parameter $\sigma_{V, w}$, which is consequently highly influential.

When considering an unlevered fund with $\alpha = 0$, the valuation uncertainty interval is [51%, 178%] in the presence of 0.10 unspanned fund risk. Thus the LP’s claim could be worth as much as 178% of the spanned fund risk valuation $\gamma_{LP}^{0}$ but as little as 51% of the very same. In case the fund exhibits only 0.05 unspanned fund risk, the spanned fund risk valuation $\gamma_{LP}^{0}$ is still flawed as the unspanned fund risk model cannot reject valuations in the interval [73%, 134%] of $\gamma_{LP}^{0}$. The alarming consequence of the latter is that even when there is only a little unhedgeable fund risk, the spanned fund risk model may get the ex-ante valuation wrong. For example, for an unlevered fund with 0.05 unspanned fund risk, the good-deal bounds are [53.84, 99.75] in value terms making the right endpoint quite close to a breakeven valuation ensuring $\gamma_{LP}^{0,\pm} \approx I_0$. Thereby, the earlier statement on the LP’s dilution at fund initiation and need for alpha might not hold when taking unspanned fund risk into account.

In case the fund relies on leverage, the uncertainty results are exacerbated.
particular, when the fund is levered and $\alpha = 0$, the unspanned fund risk model cannot reject valuations inside the interval $[9\%, 306\%]$ of $\gamma_{LP}^0$ in case of 0.10 unspanned fund risk or in the range $[44\%, 184\%]$ of $\gamma_{LP}^0$ in case of 0.05 unspanned fund risk. In value terms the intervals correspond to $[6.14, 200.39]$ and $[28.93, 120.76]$, respectively. According to the unspanned fund risk model, the LP’s total claim could consequently have a value close to zero in case the fund is levered and its risk hard to hedge, but it might also have a value of up to 306% of the spanned fund risk benchmark. A significant amount of valuation uncertainty prevails for the fund with an unspanned fund risk of 0.05 as the LP’s total claim could be worth as little as $44\%$ of the spanned fund risk valuation and as much as $184\%$ of the very same.

When the fund relies on leverage and the GP is skilled, the valuation uncertainty is significantly reduced. In particular, when $\alpha = 0.03$ the valuation intervals shrink to $[30\%, 235\%]$ and $[59\%, 157\%]$ of $\gamma_{LP}^0$ for 0.10 and 0.05 unspanned fund risk, respectively. Thereby, it seems very important to take unspanned fund risk into account when the GP is unable to generate value (low $\alpha$).

Finally, it is noteworthy that for a levered fund with 0.10 unspanned fund risk, the lower bound of the LP’s total claim $\gamma_{LP}^0$ is negative at very low $\alpha$ levels. The negativity is driven by a passive valuation of the LP’s proceeds claim making the claim worth less than the GP’s management fees.

### 3.4.4 Liquidation Risk

Table 3.2 presents spanned fund risk valuations for a fund exposed to liquidation risk through an exponentially distributed premature termination date. Panel A contains results concerning an unlevered fund whereas panel B contains those of a levered fund. The panels present valuations for three different GP skill levels $\alpha \in \{-0.01, 0.00, 0.01\}$ representing a value destroying GP, a zero value-adding GP, and a value-adding GP. For each of the considered $\alpha$, levels we conduct valuations for the cases of no liquidation risk and liquidation risk with intensity parameter $\lambda \in \{0.05, 0.10, 0.15\}$. The latter corresponds to expected fund lifetimes of approximately 8, 6, and 5 years, respectively.\(^1\)

For both types of financing and across alpha levels we find that the values of the GP’s partnership interests are decreasing in the termination intensity $\lambda$. The inverse behavior stems from the values of both the incentive fee $J_0$ and the management fees $M_0$ having a negative relationship with $\lambda$. Due to its dual role in $\gamma_{GP}^0$ and $\gamma_{LP}^0$ the latter shifts value directly from the GP to the LP as $\lambda$ increases, whereas the former can be thought of as missing out on proceeds from potential value creation.

Besides the termination intensity’s influence on the value of the management fees, the value of the LP’s total claim is affected by $\lambda$ through its relationship with value of the proceeds claim $R_0$. In the $\alpha$ cases considered, we find that $R_0$ depends positively

\(^1\)See appendix A.3 for the derivation of the expected fund lifetime.
Table 3.2: The effect of liquidation risk when all fund risks are spanned.
The table contains spanned fund risk valuations for the unlevered and levered funds in panels A and B, respectively. Each panel presents valuations of the partnerships’ interests for three different GP skill levels $\alpha \in \{-0.01, 0.00, 0.01\}$ and for the cases of no liquidation risk and liquidation risk with intensities $\lambda \in \{0.05, 0.10, 0.15\}$. The columns refer to the valuations of the GP’s incentive fee $I_0$, the GP’s management fees $M_0$, the GP’s total claim $V_{\text{GP}}^0$, the LP’s proceeds claim $R_0$, the GP’s total claim $V_{\text{LP}}^0$, and the total fund $V_0$. The table also shows the relative catch-up values $C_0$ and the credit spreads $\zeta_0$.

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>$\lambda$</th>
<th>$I_0$</th>
<th>$M_0$</th>
<th>$N_0$</th>
<th>$V_{\text{GP}}^0$</th>
<th>$R_0$</th>
<th>$V_{\text{LP}}^0$</th>
<th>$C_0$</th>
<th>$\zeta_0$</th>
<th>$V_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: Unlevered ($l = 0$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.01</td>
<td>-</td>
<td>4.05</td>
<td>20.60</td>
<td>24.66</td>
<td>86.43</td>
<td>65.83</td>
<td>2.00%</td>
<td>-</td>
<td>90.48</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.01</td>
<td>0.05</td>
<td>3.68</td>
<td>16.48</td>
<td>20.16</td>
<td>88.80</td>
<td>72.32</td>
<td>1.70%</td>
<td>-</td>
<td>92.48</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.01</td>
<td>0.10</td>
<td>3.37</td>
<td>13.45</td>
<td>16.83</td>
<td>90.56</td>
<td>77.11</td>
<td>1.48%</td>
<td>-</td>
<td>93.94</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.01</td>
<td>0.15</td>
<td>3.13</td>
<td>11.19</td>
<td>14.32</td>
<td>91.88</td>
<td>80.69</td>
<td>1.31%</td>
<td>-</td>
<td>95.01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>-</td>
<td>5.18</td>
<td>20.60</td>
<td>25.78</td>
<td>94.82</td>
<td>74.22</td>
<td>2.25%</td>
<td>-</td>
<td>100.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>0.05</td>
<td>4.56</td>
<td>16.48</td>
<td>21.05</td>
<td>95.44</td>
<td>78.95</td>
<td>1.92%</td>
<td>-</td>
<td>100.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>0.10</td>
<td>4.09</td>
<td>13.45</td>
<td>17.54</td>
<td>95.91</td>
<td>82.46</td>
<td>1.66%</td>
<td>-</td>
<td>100.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>0.15</td>
<td>3.72</td>
<td>11.19</td>
<td>14.91</td>
<td>96.28</td>
<td>85.09</td>
<td>1.46%</td>
<td>-</td>
<td>100.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.01</td>
<td>-</td>
<td>6.54</td>
<td>20.60</td>
<td>27.15</td>
<td>103.98</td>
<td>83.37</td>
<td>2.50%</td>
<td>-</td>
<td>110.52</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.01</td>
<td>0.05</td>
<td>5.63</td>
<td>16.48</td>
<td>22.11</td>
<td>102.61</td>
<td>86.13</td>
<td>2.13%</td>
<td>-</td>
<td>108.24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.01</td>
<td>0.10</td>
<td>4.94</td>
<td>13.45</td>
<td>18.39</td>
<td>101.66</td>
<td>88.20</td>
<td>1.84%</td>
<td>-</td>
<td>106.59</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.01</td>
<td>0.15</td>
<td>4.40</td>
<td>11.19</td>
<td>15.59</td>
<td>100.98</td>
<td>89.79</td>
<td>1.61%</td>
<td>-</td>
<td>105.38</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Panel B: Levered ($l = 2$) |
|-------|----------|-----------|------|-------|------|-----------------|------|-----------------|------|-----------|------|
| -0.01 | -        | 9.74      | 20.60| 30.34 | 61.71| 41.11           | 3.39%| 4.72%           | 271.45|
| -0.01 | 0.05     | 9.17      | 16.48| 25.66 | 68.27| 51.78           | 2.75%| 4.41%           | 277.44|
| -0.01 | 0.10     | 8.67      | 13.45| 22.12 | 73.13| 59.68           | 2.31%| 4.12%           | 281.81|
| -0.01 | 0.15     | 8.22      | 11.19| 19.41 | 76.81| 65.62           | 1.99%| 3.85%           | 285.04|
| 0.00  | -        | 13.91     | 20.60| 34.51 | 86.09| 65.49           | 3.37%| 3.59%           | 300.00|
| 0.00  | 0.05     | 12.41     | 16.48| 28.89 | 87.59| 71.11           | 2.82%| 3.42%           | 300.00|
| 0.00  | 0.10     | 11.23     | 13.45| 24.69 | 88.77| 75.31           | 2.41%| 3.25%           | 300.00|
| 0.00  | 0.15     | 10.29     | 11.19| 21.48 | 89.71| 78.52           | 2.09%| 3.09%           | 300.00|
| 0.01  | -        | 18.76     | 20.60| 39.36 | 112.80| 92.19           | 3.32%| 2.77%           | 331.55|
| 0.01  | 0.05     | 16.14     | 16.48| 32.63 | 108.58| 92.10           | 2.86%| 2.69%           | 324.73|
| 0.01  | 0.10     | 14.17     | 13.45| 27.62 | 105.61| 92.16           | 2.48%| 2.59%           | 319.78|
| 0.01  | 0.15     | 12.65     | 11.19| 23.84 | 103.50| 92.31           | 2.17%| 2.49%           | 316.14|

Contract parameters: $I_0 = 100$, $T = 10$, $r_H = 0.08$, $\varphi_m = 0.02$, $\varphi_c = 1.00$, $\varphi_p = 0.20$.
Model parameters: $r = 0.04$, $\sigma_V = 0.25$. 
3.4. Numerical Results

on \( \lambda \) when \( \alpha = 0.00 \) and \( \alpha = -0.01 \) whereas a negative relationship prevails when \( \alpha = 0.01 \). The former is due to the LP benefiting from early fund termination when the GP is either value destroying or zero value-adding, while the latter is due to the LP missing out on potential proceeds from value generating activities when the fund is closed prematurely. In other words, the low GP skill level implies that delivering a decent return becomes less probable when expected fund lifetime is short. Finally, we find that irrespective of financing the relative catch-up value \( C_0 \) decreases in \( \lambda \).

The choice of financing affects the way in which the introduction of liquidation risk and subsequent changes in liquidation intensity affects the value of the LP's total claim. However, qualitatively, the dependencies are identical for the value destroying and zero value-adding GP across financing. Specifically, the value of the LP's total claim is increasing in liquidation risk for \( \alpha = 0.00 \) and \( \alpha = -0.01 \). Conversely, the introduction of leverage changes the positive relationship between \( V_{LP}^0 \) and \( \lambda \) present for the unlevered fund to a "trough" (first decreasing, then increasing). The introduction of liquidation risk with \( \lambda = 0.05 \) implies that the LP's loss in \( R_0 \) is larger than the gain from a lower \( M_0 \), hence the total effect is negative. Subsequent increases in \( \lambda \) imply that the total value of the LP's claim increases since the LP benefits from leverage and the value of the management fees decreases. Naturally, we find that the debt pricing is affected positively as the higher the \( \lambda \), the earlier the debtholders receive their payment on average, which implies lower credit spreads.

We find \( \lambda = 0.05 \) to be the most realistic case considered and in this case for \( \alpha = 0 \), we find a value of the LP's total claim equaling 78.95 and 71.11 when the fund is unlevered and levered, respectively. In the absence of liquidation risk, the corresponding valuations are 74.22 and 65.49 and hence lower. Following-up on the \( \gamma_{LP}^0 = I_0 \) breakeven discussion in the previous section, we find that for \( \lambda \) equal to 0.05, 0.10, and 0.15 the breakeven alphas are 0.027, 0.029, and 0.030, respectively, for the unlevered fund. For the levered fund the breakeven alphas are 0.014, 0.015, and 0.016, respectively.

Figure 3.5 considers the valuation uncertainty of the LP's total claim when early fund termination is stochastic through the unspanned fund risk bounds relative to the corresponding spanned fund risk valuations. Panels (a) and (c) depict the 0.10 unspanned fund risk results, whereas panels (b) and (d) depict the lower 0.05 unspanned fund risk results. Moreover, the liquidation intensity \( \lambda = 0.05 \) is applied in the upper panels (a) and (b) whereas \( \lambda = 0.10 \) prevails in the lower panels (c) and (d).

When we compare panels (a) - (b) to the no-liquidation cases in figure 3.4, it appears that the introduction of a small liquidation risk (\( \lambda = 0.05 \)) does not cause substantial changes to the conclusions from the previous section regarding the valuation uncertainty in neither the 0.05 nor 0.10 unspanned fund risk case. For example, the relative 0.05 unspanned fund risk bounds of the LP's claim are [79%, 125%] when considering an unlevered fund with \( \lambda = 0.05 \) and a GP with \( \alpha = 0.00 \). In the absence of liquidation risk the unspanned fund risk model cannot reject the corresponding
Figure 3.5: Unspanned fund risk bounds of the LP’s claim divided by the corresponding spanned fund risk model valuations. In panels (a) and (b), we use an intensity of $\lambda = 0.05$, whereas we let $\lambda = 0.15$ in panels (c) and (d). Panels (a) and (c) differ from (b) and (d) in their unspanned fund risk. The former two use $\sigma_{V^w} = 0.10$ whereas the latter two use $\sigma_{V^w} = 0.05$. 
valuations in the range $[73\%, 134\%]$ of $V_0$. Thus, a tiny narrowing of the relative unspanned fund risk bounds appears across alpha levels, irrespective of financing.

Panels (c) and (d) show that a large liquidation risk ($\lambda = 0.15$) reduces the valuation uncertainty in both the 0.10 and 0.05 unspanned fund risk cases as the width of the bounds decreases significantly. When increasing $\lambda$ to 0.15 in the example above, the corresponding relative 0.05 unspanned fund risk bounds are characterized by the narrower $[87\%, 115\%]$ range. That said, the reduction is indeed small taking the extremity of a 10 years maturity fund having an expected lifetime of 5 years into account.

3.4.5 The Catch-up Implied Value Transfer

Panel (a) of figure 3.6 presents the relation between the GP’s $\alpha$ and the relative catch-up value $C_0$, while panel (b) shows the corresponding sensitivities $\partial C_0 / \partial \alpha$. Both panels display results for the unlevered and levered funds with and without liquidation risk. The curves without markers represent values for a fund that is not subject to liquidation risk, whereas those with circle markers describe the cases of liquidation risk with intensity parameter $\lambda = 0.05$.

The sensitivities for the unlevered fund that is not subject to liquidation risk follow from equation (3.33). In the presence of liquidation risk, we must rely on the expressions presented in appendix A.4.1 which rely on numerical procedures. Irrespective of liquidation risk, our competitive debt-pricing assumption implies that the sensitivities for the levered fund are obtained through numerical derivative approximations.

3.4.5.1 Valuation under Spanned Fund Risk

Consider the spanned fund risk valuations in figure 3.6 concerning a fund not exposed to liquidation risk. When we compare the relative catch-up value profiles of the unlevered and levered funds there are some striking differences. For $\alpha < 0.037$ the relative catch-up value for an unlevered fund is lower than the corresponding of a levered fund, whereas for $\alpha > 0.037$ the relative catch-up values of the unlevered fund are the highest. Thus the profile of costs implied by the inclusion of a catch-up depends both on the capital structure and the GP’s skill level. In case $\alpha = 0$, for example, the relative catch-up value equals 2.25% for an unlevered fund and 3.37% for a levered fund, whereas the corresponding costs for $\alpha = 0.04$ are 3.19% and 3.11%, respectively.

For the unlevered fund the relative catch-up value is increasing in $\alpha$ for reasonable $\alpha$ levels and has a maximum value of 3.82% at obscure $\alpha = 0.099$. According to panel (b), the derivative for the unlevered fund is maximized and the relative catch-up value curve is the steepest around $\alpha = 0$. The decreasing behavior of the derivative leads to a diminishing marginal effect of $\alpha$ on $C_0$ for $\alpha > 0$. 
The introduction of leverage complicates the relative catch-up valuations and the derivative evaluations as $\alpha$ affects the equilibrium credit spread $\zeta_0$ determining the thresholds $\{D(T), P(T), \phi(T)\}$. In particular, for $\alpha < -0.041$, the potential debtholders are unable to break even by demanding a default risk premium ensuring that the value of their debt equals the issued debt. Thus the model captures the debt market failure of credit rationing. For slightly higher alpha levels, the GP immediately benefits from the introduction of a catch-up as two effects benefiting both the LP and the GP kick in: i) thresholds are lowered, and ii) value generation is higher. The former affects how early the catch-up kicks in whereas the latter affects at what pace the catch-up is utilized. From the derivative in panel (b), we find that the $\alpha$ sensitivity is very high near the credit rationing asymptote and that the levered fund has its maximum relative catch-up value of 3.41% at $\alpha = -0.022$ where $\zeta_0 = 6.87\%$.

For $\alpha > 0$ and irrespective of financing, the redundancy of the catch-up increases in $\alpha$ as both the levered and the unlevered funds experience downward sloping derivatives and thus diminishing marginal effects of $\alpha$ on $C_0$ for $\alpha > 0$. The diminishing marginal effect can loosely be explained through the option strategies in the numerator and denominator of the relative catch-up value expression in equation (3.32). Both the numerator bull call spread and the denominator strategy benefit from increases in $\alpha$, i.e. the GP's ability to grow the PE assets. As the bull call spread payoff is capped and the denominator strategy payoff uncapped, the fraction between the two is decreasing in $\alpha$ at high $\alpha$ levels.

### 3.4.5.2 Valuation under Unspanned Fund Risk

In this section, we examine the relative catch-up valuation uncertainty in the absence of liquidation risk and for reasonable $\alpha$ levels. The degree of valuation uncertainty is characterized by the difference between the ratio pairs $C_0^+ / C_0$ and $C_0^- / C_0$. Figure 3.7 depicts the relative catch-up valuation uncertainties for an unlevered and a levered fund. In particular, panel (a) presents the results for a fund with 0.10 unspanned fund risk while panel (b) depicts those of a fund with 0.05 unspanned fund risk.

Irrespective of the degree of unspanned fund risk, the valuation uncertainty results for the unlevered and levered cases are qualitatively different. In particular, in the absence of leverage, the width between the ratio pairs $C_0^+ / C_0$ and $C_0^- / C_0$ decreases in $\alpha$, whereas it increases under leverage.

For the unlevered fund, the valuation uncertainty decreases in $\alpha$. When unspanned fund risk equals 0.10, the valuation uncertainty interval reads $[44\%, 161\%]$ at $\alpha = -0.01$ and when $\alpha$ increases to 0.03, the interval tightens to $[57\%, 127\%]$. Naturally, the uncertainty increases in the level of unspanned volatility and when the level is 0.05, the corresponding intervals shrink to $[69\%, 132\%]$ and $[79\%, 117\%]$. Finally, the valuation uncertainty for an unlevered fund with $\alpha = 0$ is characterized by the intervals $[47\%, 152\%]$ and $[72\%, 128\%]$ when unspanned volatility is 0.10 and 0.05, respectively. In relative catch-up bound terms, these intervals correspond to
3.4. Numerical Results

Figure 3.6: Spanned fund risk evaluations for unlevered and levered funds. Panel (a) depicts relative catch-up valuations $C_0$ for various $\alpha$ levels and panel (b) depicts the corresponding derivatives $\partial C_0 / \partial \alpha$.

Figure 3.7: Relative catch-up valuation ratios $C_{\alpha} / C_0$ in the absence of liquidation risk. Panel (a) shows the case $\sigma_{Vw} = 0.10$ and panel (b) that of $\sigma_{Vw} = 0.05$. 
1.06%, 3.42%] and [1.62%, 2.89%].

Compared to the unlevered setting the introduction of leverage reduces the misme-
stimation of the value transfer across the considered \( \alpha \) levels. At \( \alpha = 0 \), for example, the
valuation uncertainty intervals are [87%, 103%] when unspanned volatility amounts
to 0.10 and [95%, 103%] when there is only 0.05 unspanned volatility. In terms of
relative catch-up bounds, the two intervals are [2.94%, 3.45%] and [3.19%, 3.45%].
Thus in the presence of leverage the spanned fund risk model does a decent job in
determining the value transfer implied by embedding a catch-up.

### 3.4.5.3 Liquidation Risk

From panel (a) in figure 3.6, we note that irrespective of financing, the introduction
of liquidation risk decreases the relative catch-up values across \( \alpha \) levels. Moreover,
the difference between the relative catch-up values when the fund is and is not
exposed to liquidation risk, i.e. the distance between the relative catch-up curves,
is most pronounced at intermediate \( \alpha \) levels for the unlevered fund and at low \( \alpha \)
levels for the levered fund. Thus, ignoring liquidation risk at these \( \alpha \) levels leads to an
overestimation of the relative catch-up value.

Compared to the case without liquidation risk, the relative catch-up curve of the
unlevered fund with liquidation risk experiences a relative catch-up maximum value
of 3.47% at the slightly higher \( \alpha = 0.108 \). From the sensitivities in panel (b) in figure
3.6, we note that the two relative catch-up curves have equal slopes around \( \alpha = 0.05 \).
Moreover, the relative catch-up curve under liquidation risk is less steep for \( \alpha < 0.05 \)
and steeper for \( \alpha > 0.05 \). For the considered \( \alpha \) levels no credit rationing is present
under liquidation risk as debt is issued over the entire \( \alpha \) range. It must be emphasized
that credit rationing is, however, still present though at lower \( \alpha \) levels. The relative
catch-up curve of the levered fund subject to liquidation risk has a maximum value
of 2.87% at the realistic \( \alpha = 0.018 \) where \( \xi_0 = 2.23\% \). The sensitivities of the levered
fund is decreasing more slowly than its no-liquidation-risk counterpart.

In order to consider the role of the termination intensity, we consider the relation
between \( \lambda \) and the relative catch-up value \( C_0 \) and the sensitivities \( \partial C_0 / \partial \lambda \). Like
we did in section 3.4.4, we consider both an unlevered and a levered fund with
\( \alpha \in \{-0.01, 0.00, 0.01\} \). For the unlevered fund the derivative is obtained through
equation (3.34) that relies on numerical integration. In the presence of leverage, the
derivative is approximated numerically. Figure 3.8 presents the relative catch-up
valuations in panel (a) and the sensitivities in panel (b).

For the considered \( \alpha \) levels the relative catch-up value is higher for a fund that
relies on leverage than for an unlevered fund across \( \lambda \) levels. Moreover, we find that for
both types of financing, the relative catch-up values are decreasing in \( \lambda \). In addition
to the decreasing behavior, the differences in relative catch-up values across \( \alpha \) levels
are preserved for the unlevered fund. Obviously, this is reflected in the corresponding
derivatives being relatively similar in value terms. For the levered fund the relative
catch-up values are similar for low $\lambda$ levels and diverge as $\lambda$ increases.

According to the derivatives, levered funds with $\alpha = -0.01$ experience the largest drops in relative catch-up values from increases in $\lambda$. Thus a marginal increase in the likelihood of early dissolution is most costly to the value destroying GP in terms of the relative catch-up value.

In terms of costs we find that for an unlevered fund in the somewhat extreme $\lambda = 0.15$ case, the inclusion of a catch-up transfers 1.31%, 1.46%, and 1.61% of $R_0^0$ when $\alpha$ is $-0.01$, 0.00, and 0.01, respectively. In the more realistic case where $\lambda = 0.05$, the corresponding costs are 1.70%, 1.92%, and 2.13% of $R_0^0$.

When the fund is levered and subject to $\lambda = 0.15$, the costs are 2.75%, 2.82%, and 2.86% of $R_0^0$ for $\alpha$ is $-0.01$, 0.00, and 0.01, respectively. When $\lambda = 0.05$ prevails, the corresponding catch-up costs are magnified to 1.99%, 2.09%, and 2.17%.

Figure 3.9 presents the valuation uncertainty pairs $C^+ / C_0$ and $C^- / C_0$ when $\alpha$ is varied and liquidation risk is present. We consider liquidation risk characterized by intensity $\lambda = 0.05$ and an unspanned fund risk level of 0.10 and 0.05. The former case is depicted in panel (a) while the latter case is presented in panel (b).

The introduction of liquidation risk clearly affects the valuation uncertainty. That said, when comparing the bounds of the unlevered fund in figure 3.9 to those in figure 3.7, the bounds seem hardly unaffected. The levered fund, on the other hand, experiences a substantial change as the bounds cross over when $\alpha$ is around 0.011. In both the 0.10 and 0.05 unspanned fund risk cases, the crossing lies beneath the $C^+ / C_0 = 1$ line, indicating that the relative catch-up bounds of the unspanned fund risk model are below the relative catch-up value of the spanned fund risk model.

The explanation behind the equality of the unspanned fund risk bounds is found by rewriting $C^+ = C_0$ as

$$\begin{align*}
\frac{R_0^+ - R_0^0}{R_0^0} &= \frac{R_0^- - R_0^0}{R_0^-} \\
\text{aggressive catch-up} & \quad \text{passive no-catch-up contract} \\
\text{passive catch-up} & \quad \text{aggressive no-catch-up contract}
\end{align*}$$

As appears the aggressive valuation of the catch-up and the passive valuation of the no-catch-up contract balances the passive valuation of the catch-up and the aggressive valuation of the no-catch-up contract. It is important to note that a large catch-up implied value transfer can be caused by either a highly valued catch-up (numerator) or a low no-catch-up contract valuation (denominator). The underlying structure and its dependence on $\alpha$ are hard to interpret as a comparison of $C^+ / C_0$ and $C^- / C_0$ when $\alpha$ changes corresponds to relating multiple option strategies when the dividend yield of a common underlying changes.

### 3.4.6 The Catch-up Rate

Table 3.3 presents the valuations of the partnership interests when the contract includes no catch-up and when it contains catch-ups with $\varphi_c \in \{0.25, 0.50, 0.75, 1.00\}$. 
Chapter 3. The Valuation of Catch-up Provisions

Figure 3.8: Spanned fund risk evaluations for unlevered and levered funds with $\alpha \in \{-0.01, 0.00, 0.01\}$. Panel (a) depicts the relative catch-up valuations $C_0$ for various $\lambda$ levels and panel (b) the corresponding derivatives $\partial C_0 / \partial \lambda$.

Figure 3.9: Relative catch-up valuation ratios $C_0^\pm / C_0$ in the presence of liquidation risk. Panel (a) shows the case $\sigma_{Vw} = 0.10$ and panel (b) that of $\sigma_{Vw} = 0.05$. 
Table 3.3: The effect of varying the catch-up rate when all risks are spanned and liquidation risk is present.

The table reports the spanned fund risk valuations for five catch-up specifications when premature fund termination dates are exponentially distributed with intensity $\lambda = 0.05$. We consider a contract with no catch-up and four choices of catch-up contracts $\varphi_c \in \{0.25, 0.50, 0.75, 1.00\}$. The columns refer to the valuations of the GP’s incentive fee $I_0$, the GP’s management fees $M_0$, the GP’s total claim $V_{GP}^0$, the LP’s proceeds claim $R_0$, the LP’s total claim $V_{LP}^0$, and the total fund $V_0$. The table also shows the relative catch-up values $C_0$ and the credit spreads $\zeta_0$.

<table>
<thead>
<tr>
<th>$\varphi_c$</th>
<th>$I_0$</th>
<th>$M_0$</th>
<th>$V_{GP}^0$</th>
<th>$R_0$</th>
<th>$V_{LP}^0$</th>
<th>$C_0$</th>
<th>$\zeta_0$</th>
<th>$V_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: Unlevered ($l = 0$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>-</td>
<td>2.70</td>
<td>16.48</td>
<td>19.18</td>
<td>97.30</td>
<td>80.82</td>
<td>-</td>
<td>-</td>
<td>100.00</td>
</tr>
<tr>
<td>0.25</td>
<td>3.32</td>
<td>16.48</td>
<td>19.80</td>
<td>96.68</td>
<td>80.20</td>
<td>0.63%</td>
<td>-</td>
<td>100.00</td>
</tr>
<tr>
<td>0.50</td>
<td>4.27</td>
<td>16.48</td>
<td>20.75</td>
<td>95.73</td>
<td>79.25</td>
<td>1.61%</td>
<td>-</td>
<td>100.00</td>
</tr>
<tr>
<td>0.75</td>
<td>4.47</td>
<td>16.48</td>
<td>20.96</td>
<td>95.53</td>
<td>79.04</td>
<td>1.82%</td>
<td>-</td>
<td>100.00</td>
</tr>
<tr>
<td>1.00</td>
<td>4.56</td>
<td>16.48</td>
<td>21.05</td>
<td>95.44</td>
<td>78.95</td>
<td>1.92%</td>
<td>-</td>
<td>100.00</td>
</tr>
<tr>
<td>Panel B: Levered ($l = 2$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-</td>
<td>9.86</td>
<td>16.48</td>
<td>26.35</td>
<td>90.14</td>
<td>73.65</td>
<td>-</td>
<td>3.42%</td>
<td>300.00</td>
</tr>
<tr>
<td>0.25</td>
<td>11.40</td>
<td>16.48</td>
<td>27.88</td>
<td>88.60</td>
<td>72.12</td>
<td>1.70%</td>
<td>3.42%</td>
<td>300.00</td>
</tr>
<tr>
<td>0.50</td>
<td>12.25</td>
<td>16.48</td>
<td>28.73</td>
<td>87.75</td>
<td>71.27</td>
<td>2.65%</td>
<td>3.42%</td>
<td>300.00</td>
</tr>
<tr>
<td>0.75</td>
<td>12.36</td>
<td>16.48</td>
<td>28.85</td>
<td>87.64</td>
<td>71.15</td>
<td>2.77%</td>
<td>3.42%</td>
<td>300.00</td>
</tr>
<tr>
<td>1.00</td>
<td>12.41</td>
<td>16.48</td>
<td>28.89</td>
<td>87.59</td>
<td>71.11</td>
<td>2.82%</td>
<td>3.42%</td>
<td>300.00</td>
</tr>
</tbody>
</table>

Contract parameters: $I_0 = 100$, $T = 10$, $r_H = 0.08$, $\varphi_m = 0.02$, $\varphi_p = 0.20$. Model parameters: $r = 0.04$, $\alpha = 0$, $\sigma_V = 0.25$, $\lambda = 0.05$.

The valuations are carried out using the spanned fund risk model with a stochastic liquidation characterized by $\lambda = 0.05$. Like we did in the previous sections, we rely on the relative catch-up value $C_0$ as a cost measure. Panel A contains the results for an unlevered fund and panel B those of a levered fund.

Irrespective of financing, the value of the management fees remains constant across catch-up specifications and the catch-up implied value transfers happen via the incentive fee $I_0$ and the LP’s profit claim $R_0$. Naturally, the inclusion of a catch-up transfers value from the LP to the GP as $I_0$ increases in $\varphi_c$ and $R_0$ decreases in $\varphi_c$.

The costs of embedding a catch-up $C_0$ are higher if the fund is levered than if the fund is unlevered. For example, including a 25% catch-up transfers 0.63% of $R_0$ to the LP in the absence of leverage, whereas the value transfer is 1.70% in case of leverage. For both types of financing, the marginal change in $C_0$ diminishes with $\varphi_c$. Thus, the changes in the value transfer from altering the catch-up rate are greatest at low catch-up rates and vice versa. Moreover, the decision on whether to include a catch-up or not proves important. In fact, when the fund does not rely on leverage,
an increase in the catch-up rate from 50% to 100% implies a change in \( C_0 \) of 0.31%, which is small compared to the 0.63% implied by embedding a 25% catch-up. For the levered fund, the costs are magnified and the catch-up more costly as the former 50% catch-up rate increase transfers 0.17% of \( \mathcal{R}_0^0 \) whereas the inclusion of a 25% catch-up transfers 1.70% of \( \mathcal{R}_0^0 \).

### 3.4.7 Breakeven Parameters

In this section, we present breakeven parameters that are useful for relating the value of the catch-up to more tangible and well-known parameters. In particular, we compute breakeven parameters related to \( \alpha, \varphi_m, \) and \( \varphi_p \). All computations focus on the valuation of the LP’s total claim and are carried out under liquidation risk with intensity \( \lambda = 0.05 \).

#### 3.4.7.1 The Breakeven \( \alpha \)

We compute the \( \alpha \) that in spanned fund risk valuation terms ensures equivalence between a contract without a catch-up subject to \( \alpha \) and a \( \varphi_c \) catch-up contract subject to \( \alpha_0 \). Specifically, we calculate the change

\[
\Delta \alpha \equiv \alpha_0 - \alpha^* \tag{3.47}
\]

where \( \alpha^* \) is the breakeven alpha that solves

\[
\mathcal{V}_0^{LP,0}(\alpha^*) = \mathcal{V}_0^{LP}(\alpha_0, \varphi_c). \tag{3.48}
\]

In the numerical optimization, we assume that the debtholders do not revalue the debt taking the \( \alpha \) into account. This assumption implies that the breakeven alphas are conditional on the debt issuance and thus mimics the discrete fee structure decision made independently of the financing process. Thus \( \Delta \alpha \) represents the reduction in \( \alpha_0 \) that the LP implicitly suffers when including a \( \varphi_c \) catch-up and hence it can be interpreted as the LP’s cost in terms of \( \alpha \). Table 3.4 presents the values of \( \Delta \alpha \) expressed in basis points (bps) for various catch-up rates and base alpha levels \( \alpha_0 \). In particular, panel A holds the no-leverage case and panel B holds the leverage case.

In both the unlevered and levered cases \( \Delta \alpha \) increases with \( \alpha_0 \), i.e. the higher the base alpha, the greater the catch-up implied value transfer and the associated "alpha dilution" of the LP. For example, in the absence of leverage, the inclusion of a 100% catch-up dilutes 23.29 bps alpha when \( \alpha_0 = -0.01 \) and 34.15 bps alpha when \( \alpha_0 = 0.03 \). The introduction of leverage lowers the alpha dilution as a corresponding catch-up costs only 16.59 bps and 20.24 bps when \( \alpha_0 = -0.01 \) and \( \alpha_0 = 0.03 \), respectively. In fact, this is the case for the \( \varphi_c \in \{0.50, 0.75, 1.00\} \) catch-ups but for the low catch-up rate (\( \varphi_c = 0.25 \)) the opposite is found. Thus the inclusion of a low rate catch-up harms the LP relatively more if the fund is levered.
When we consider the changes across the catch-up rates, the alpha reductions are increasing in the catch-up rate for both unlevered and levered funds. In line with our previous findings we find diminishing marginal effects and the inclusion of a 25% catch-up is more expensive than increasing the catch-up rate from 50% to 100%. For example, the change in alpha dilution from increasing the rate from 50% to 100% when the fund is unlevered and $\alpha_0 = 0.00$ is 4.16 bps, whereas the inclusion of a 25% catch-up dilutes 8.53 bps.

In figure 3.10 we present alpha reduction bounds computed using the unspanned fund risk model with 0.05 unspanned fund risk instead of the spanned fund risk valuations in equation (3.48). In the unlevered case, we observe quite wide bounds, indicating a high degree of uncertainty compared to the levered case. For example, for $\alpha_0 = 0.00$ and $\varphi_c = 1.00$, we compute $\Delta \alpha$ to be in the range $[19.16, 33.10]$ bps in the absence of leverage and in the range $[16.40, 18.71]$ bps in the presence of leverage.

When we consider either the unlevered or levered fund some of the alpha reduction intervals are overlapping. For example, the intervals concerning the 50% and 100% catch-ups are overlapping across the considered base alpha levels regardless of financing. It is therefore evident that a high degree of valuation uncertainty prevails across catch-up rates.

### 3.4.7.2 The Breakeven $\varphi_m$

Using the spanned fund risk model, we calculate the management fee add-on that in value terms corresponds to embedding a catch-up for various base management fees $\varphi_{m,0}$. Specifically, we consider the management fee add-on

$$\Delta \varphi_m \equiv \varphi_m^* - \varphi_{m,0} \quad (3.49)$$

where $\varphi_m^*$ solves

$$\gamma_{0}^{\text{LP},0} (\varphi_m^*) = \gamma_{0}^{\text{LP}} (\varphi_c, \varphi_{m,0}). \quad (3.50)$$

Thus, $\Delta \varphi_m$ corresponds to an implicit increase in $\varphi_{m,0}$ caused by embedding a $\varphi_c$ catch-up. Table 3.5 contains the management fee add-ons expressed in bps for various catch-up rates and base management fees $\varphi_{m,0}$.

For both the unlevered and the levered funds the breakeven management fee add-on is decreasing in the base case management fee $\varphi_{m,0}$. When the base case contract contains a high management fee $\varphi_{m,0}$ the preferred return is high, which makes it harder for the GP to breach. Additionally, it becomes harder for the GP to fully utilize the catch-up lowering the value of the catch-up.

For a given $\varphi_{m,0}$ the management fee add-ons are increasing in the catch-up rate $\varphi_c$ and in line with our previous findings they are most pronounced when going from a no-catch-up contract to the 25% catch-up contract. When we compare the unlevered case to the levered case, we find that the breakeven $\Delta \varphi_m$ is increasing
Table 3.4: Breakeven costs in terms of alpha
Spanned fund risk model $\Delta \alpha$ reductions for various catch-up rates $\phi_c$ and base alpha levels $\alpha_0$. The results are expressed in basis points.

<table>
<thead>
<tr>
<th>$\alpha_0$ \ $\phi_c$</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: Unlevered ($l = 0$)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.01</td>
<td>7.38</td>
<td>19.41</td>
<td>22.12</td>
<td>23.29</td>
</tr>
<tr>
<td>0.00</td>
<td>8.53</td>
<td>21.85</td>
<td>24.76</td>
<td>26.01</td>
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<td>30.14</td>
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<td>0.03</td>
<td>12.58</td>
<td>29.43</td>
<td>32.75</td>
<td>34.15</td>
</tr>
<tr>
<td><strong>Panel B: Levered ($l = 2$)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.01</td>
<td>9.79</td>
<td>15.49</td>
<td>16.28</td>
<td>16.59</td>
</tr>
<tr>
<td>0.00</td>
<td>10.63</td>
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<td>17.43</td>
<td>17.75</td>
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<td>0.03</td>
<td>13.00</td>
<td>19.16</td>
<td>19.94</td>
<td>20.24</td>
</tr>
</tbody>
</table>

Parameters: $I_0 = 100$, $T = 10$, $r_H = 0.08$, $\phi_m = 0.20$, $\phi_p = 0.20$, $r = 0.04$, $\sigma_V = 0.25$.

Figure 3.10: Unspanned fund risk model $\Delta \alpha$ reductions for various catch-up rates across base alpha levels. Panel (a) presents the unlevered case and panel (b) the levered case.
3.4. Numerical Results

in leverage. For example, for an unlevered fund and a base annual management fee of $\phi_{m,0} = 0.02$, including a 100% catch-up corresponds to increasing the annual management fee by 17.81 bps but in case the fund is levered, the breakeven add-on increases to 24.14 bps.

Figure 3.11 depicts the breakeven management fee add-ons according to the unspanned fund risk model with 0.05 unspanned fund risk.

Irrespective of financing, the bounds and the width of the bounds decrease in $\phi_{m,0}$, i.e. the higher the base management fee, the more certainty on the breakeven $\Delta \phi_m$ at a low level. We find a high degree of breakeven management fee add-on uncertainty as the bounds overlap across catch-up rates and financing. For example, for the industry standard $\phi_{m,0} = 0.02$ and 100% catch-up, we compute $\Delta \phi_m$ to be in the range $[10.69, 27.65]$ bps in the absence of leverage and in the range $[15.18, 35.75]$ bps in the presence of leverage.

3.4.7.3 The Breakeven $\phi_p$

Finally, we consider the breakeven carried interest rate add-on defined as

$$\Delta \phi_p \equiv \phi_p^* - \phi_{p,0}$$  \hspace{1cm} (3.51)

where $\phi_p^*$ is the carry rate that solves

$$\mathcal{V}^{LP,0}_0 (\phi_p^*) = \mathcal{V}^{LP}_0 \left( \phi_c, \phi_{p,0} \right).$$  \hspace{1cm} (3.52)

Thus, $\Delta \phi_p$ represents the carried interest rate add-on that is implicitly added to the base contract in value terms in case a $\phi_c$ catch-up is included. Table 3.6 presents the breakeven carry rate add-ons for various catch-up rates and base case interest rates $\phi_{p,0}$ expressed in percentage points (pps).

The financing decision affects the breakeven carry rate add-ons significantly as the introduction of leverage deflates their values. Thus embedding a catch-up to the contract implicitly increases the base carry rate relatively more in the absence of leverage.

For both types of financing, the breakeven carry rate add-ons exhibit a concave shape, i.e. increasing at low base carry rates with diminishing marginal changes. In fact, the marginal effects are negative when the base carry rate is close to the catch-up rate and since it does not make sense to talk about a catch-up acquiring profits at a lower rate than the carry rate, $\phi_p^*$ is not defined for $\phi_{p,0} \geq \phi_c$.

Naturally, the breakeven carry rates are increasing in the catch-up rate since the inclusion of a high rate catch-up is costly and corresponds to a significant carry rate add-on. Again we find diminishing marginal effects such that it is cheaper to increase the catch-up rate at high levels than to include a low rate catch-up. For example, for the unlevered fund and $\phi_{p,0} = 0.20$, increasing a 50% catch-up to 100% costs 2.19 pps whereas including a catch-up corresponds to increasing $\phi_{p,0}$ by 4.56 pps. For the
Table 3.5: Breakeven costs in terms of management fee
Spanned fund risk model breakeven management fee add-ons $\Delta \phi_m$ for various catch-up rates $\phi_c$ and base case management fees $\phi_{m,0}$. The results are expressed in basis points.

<table>
<thead>
<tr>
<th>$\phi_{m,0}$(\backslash) $\phi_c$</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: Unlevered ($l = 0$)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>11.56</td>
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<td>10.08</td>
<td>11.43</td>
<td>11.95</td>
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<tr>
<td>0.04</td>
<td>2.34</td>
<td>6.17</td>
<td>7.19</td>
<td>7.42</td>
</tr>
<tr>
<td><strong>Panel B: Levered ($l = 2$)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>23.75</td>
<td>35.63</td>
<td>37.03</td>
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<tr>
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<td>7.73</td>
<td>12.34</td>
<td>13.05</td>
<td>13.36</td>
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</table>

Parameters: $l_0 = 100$, $T = 10$, $r_H = 0.08$, $\varphi_p = 0.20$, $r = 0.04$, $\alpha = 0$, $\sigma_V = 0.25$.

**Figure 3.11**: Unspanned fund risk model $\Delta \phi_m$ add-ons for various catch-up rates across base case levels. Panel (a) presents the results for the unlevered fund and panel (b) for the levered fund.
### Table 3.6: Breakeven costs in terms of carry rate

Spanned fund risk model breakeven carried interest rate add-ons $\Delta \varphi_p$ for various catch-up rates $\varphi_c$ and base case carried interest rates $\varphi_{p,0}$. The results are expressed in percentage points.

<table>
<thead>
<tr>
<th>$\varphi_{p,0}$</th>
<th>$\varphi_c$</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
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<td>0.05</td>
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<td>5.81</td>
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<tr>
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<tr>
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<td>-</td>
<td>12.95</td>
<td>17.43</td>
<td>19.20</td>
<td></td>
</tr>
<tr>
<td><strong>Panel B: Levered ($l = 2$)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>0.05</td>
<td>1.29</td>
<td>1.32</td>
<td>1.33</td>
<td>1.33</td>
<td></td>
</tr>
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<td>2.62</td>
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<tr>
<td>0.25</td>
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<td>5.75</td>
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<tr>
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<td>-</td>
<td>6.42</td>
<td>7.25</td>
<td>7.52</td>
<td></td>
</tr>
</tbody>
</table>

Parameters: $I_0 = 100$, $T = 10$, $r_H = 0.08$, $\varphi_m = 0.02$, $r = 0.04$, $\alpha = 0$, $\sigma_V = 0.25$.

levered fund, the corresponding increases are 0.32 pps and 3.11 pps. Finally, the costs in terms of breakeven carry rate add-on of adding a 100% catch-up when $\varphi_{p,0} = 0.2$ are 13.81 pps and 5.16 pps in the absence and presence of leverage, respectively.

Figure 3.12 presents the breakeven carry rate add-on bounds obtained using the unspanned fund risk model with 0.05 unspanned fund risk.

We find that the unspanned fund risk model bounds obey concavity similar to that of the spanned fund risk model presented in table 3.6. Similarly, the breakeven bounds are wider and located at substantially higher levels in the unlevered case than in the levered case. The tendency of overlapping observed in the previous two sections is not that pronounced for the carry rate add-on. In fact, there is only a very little overlapping for base carry rates above 0.20.

For the industry standard $\varphi_{p,0} = 0.20$, the carry rate add-on of a 100% catch-up lies in the interval $[12.11, 15.31]$ pps in the absence of leverage and in the range $[4.44, 5.83]$ pps under leverage.
Figure 3.12: Unspanned fund risk model $\Delta \varphi_{p}$ add-ons for various catch-up rates across base case levels (pps). Panel (a) presents the results for the unlevered fund and panel (b) for the levered fund.

3.5 Conclusion

In this paper, we study the role of the catch-up provision that is often included in fund managers’ compensation contracts in private equity partnerships.

In order to analyze the impact of the catch-up we present a contract and valuation framework that can be used for valuing private equity contracts and their associated fee structures. The contract setting and the payoff functions build on the framework of Sørensen et al. (2014) but with the extension of a timing dimension that enables us to handle early fund dissolutions.

As the secondary market of private equity securities is characterized by illiquidity and opaqueness, the replicating portfolio argument underlying traditional contingent claims valuation approaches seems dubious in this case. Hence, we distinguish between the hedgeable fund risk spanned by the financial market and unhedgeable unspanned fund risk. As our benchmark model we apply the spanned fund risk model of Sørensen et al. (2014) and in order to account for unspanned fund risk we adapt the good-deal bounds model of Cochrane and Saa-Requejo (2000). We introduce liquidation risk into both models by allowing for premature fund dissolutions through exponentially distributed fund termination dates. The liquidation risk is inspired by the private equity studies of Metrick and Yasuda (2010) and Choi et al. (2012) and the hedge fund studies of Goetzmann et al. (2003) and Drechsler (2014).

With the spanned and unspanned fund risk models at hand, we value the private equity partnerships’ interests with a particular focus on the catch-up provision. Our analysis is based on a typical contract in our data set that coincides with the so-called 2-and-20 contract as the fund manager receives a fixed management fee of 2% of committed capital and a performance fee of 20% of fund profits above a given
threshold. As the typical contract also contains a 100% catch-up provision, we base our analysis on such a contract.

In our benchmark spanned fund risk valuation with no liquidation risk, we demonstrate, in line with the findings of Sørensen et al. (2014), that the investor’s claim at fund initiation is worth substantially less than the invested amount of capital and thus the fund manager must be able to deliver alpha for the investment to break even. The value transfer implied by including a catch-up provision, which is not considered by Sørensen et al. (2014), amounts to $2\% - 2.98\%$ for an unlevered fund and to $3.19\% - 3.39\%$ for a levered fund financed with $2/3$ debt for base case parameter values. By use of the unspanned fund risk model we illustrate the importance of taking market incompleteness into account. When we consider the investor’s claim, we find a wide range of model consistent valuations and cannot reject that the investor breaks even at fund establishment.

The introduction of liquidation risk via an exponentially distributed premature fund termination date mainly results in higher spanned fund risk valuations of the investor’s claim. These increases are driven by the fact that the cost in terms of missed investment proceeds are lower than the reductions in management fees. Finally, the introduction of liquidation risk does not affect the valuation bounds of the investor’s claim in the unspanned fund risk model. In fact, a small liquidation risk barely changes the bounds and a large and probably unrealistic liquidation risk only reduces the width of the bounds slightly.

Regarding the catch-up provision, we find that in the spanned fund risk model and in the absence of liquidation risk, its value transfer is maximized at an abnormally high alpha level for unlevered funds and at a slightly negative alpha level for levered funds. As the introduction of a small liquidation risk changes the latter case to a slightly positive and reasonable alpha level, the value transfer can be at its maximum value for many funds. Importantly, we find that in most cases the changes in catch-up implied value transfer are diminishing in alpha, which underlines that the redundancy of the catch-up increases in the fund manager’s ability to grow the fund. Finally, in our base case, the catch-up implied value transfer for an unlevered fund equals $1.92\%$ according to the spanned fund risk model and lies in the interval $1.39\% - 2.47\%$ according to the unspanned fund risk model. For the $2/3$ debt financed fund, the corresponding spanned fund risk valuation equals $2.82\%$ and the unspanned fund risk bounds are $2.70\% - 2.81\%$. We generally find tighter implied value transfer bounds and thus lower valuation uncertainty when the fund is levered.

When considering the catch-up implied value transfer, we find it marginally decreasing in the catch-up rate. The decision on whether to include a catch-up at all becomes much more important than its size. Moreover, we find that leverage magnifies these effects.

Finally, our last contribution is to provide breakeven parameter values for the typical contract which will be useful for the practitioner when considering the inclu-
sion of a catch-up. The breakeven parameters are those that apply to an alternative contract which ensures that the LP is indifferent in value terms between including a catch-up and taking the alternative contract. For base case parameters and an unlevered (a levered) fund, the inclusion of a 100% catch-up corresponds to: i) reducing alpha by 26.21 (17.75) bps, ii) increasing the annual management fee by 17.81 (24.14) bps, or iii) increasing the carry rate by 13.81 (5.16) pps.

**Acknowledgements**

The authors are grateful to Thomas Kokholm, Agatha Murgoci, and Alice Bucciol for helpful comments and suggestions.
3.6 References


Appendix

A.1 Pricing in the Spanned Fund Risk Model

The purpose of this section is to value a European call option written on the non-traded PE fund that is subject to an independent exponentially distributed premature liquidation date. In order to do so, we apply the technique of Jennergren and Näslund (1996) to the model framework of Sørensen et al. (2014).

Let $V_t$ denote the value of the PE fund at time $t$ with dynamics

$$\frac{dV_t}{V_t} = \mu_V dt + \sigma_{Vz} dz_t + \sigma_{Vw} dw_t,$$

where $z$ and $w$ are independent standard Brownian motions representing market exposure and fund specific risk, respectively.

Assume that the economy consists of three traded assets whose prices admit no arbitrage: i) a publicly traded market portfolio denoted $S_t$, ii) a tradable asset useful for hedging fund-specific risk denoted $S'_t$, and iii) a money market account denoted $B_t$. The public equity is assumed to follow a geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dz_t.$$

By use of the CAPM argument investors are not compensated for taking on $w$-risk, and the hedge asset dynamics is

$$\frac{dS'_t}{S'_t} = rd t + \sigma_S dw_t.$$

Finally, let the money market account evolve according to

$$dB_t = r B_t dt, \quad B_0 = 1.$$

We consider the claim $G(V_t, t)$ and value it using a replicating portfolio argument. The first step is to construct a portfolio of $\Delta_t$ shares of public equity $S_t$, $\Delta'_t$ shares of the hedging asset $S'_t$, $\Delta^B_t$ units in the money market account $B_t$, and one unit of the claim $G(V_t, t)$, and derive its dynamics. The value and dynamics of such a portfolio are given by

$$Q_t = \Delta_t S_t + \Delta'_t S'_t + \Delta^B_t B_t + G(V_t, t),$$

and

$$dQ_t = \Delta_t dS_t + \Delta'_t dS'_t + \Delta^B_t dB_t + dG(V_t, t).$$
Since the claim is subject to a stochastic termination independent of the fund value, its dynamics reads

$$dG(V_t, t) = \left( \frac{\partial G}{\partial t} (V_t, t) + \frac{\partial G}{\partial V} (V_t, t) \mu_V V_t + \frac{\partial^2 G}{\partial V^2} (V_t, t) \frac{\sigma_V^2 V_t^2}{2} \right) dt$$

$$+ \frac{\partial G}{\partial V} (V_t, t) \sigma_{VZ} V_t dz_t + \frac{\partial G}{\partial V} (V_t, t) \sigma_{VW} V_t dw_t$$

$$+ \left( H(V_t, t) - G(V_{t-}, t-) \right) dN_t,$$

where $dN_t$ is the increment to a point process with values $N = 0$ and $N = 1$ before and after termination, respectively. The independent point process governs premature liquidation via the intensity parameter $\lambda$ satisfying $E_t [dN_t] = \lambda dt$. The early dissolution is characterized by the claim holder receiving an immediate payoff $H(V_t, t)$ contrary to $G(V_T, T) = \Phi(V_T)$ at maturity.

The dynamics of the portfolio $Q_t$ is

$$dQ_t = \left( \Delta_t S_t \mu_S + \left( \Delta_t^r S_t + \Delta_t^B B_t \right) r + \frac{\partial G}{\partial t} (V_t, t) + \frac{\partial G}{\partial V} (V_t, t) \mu_V V_t + \frac{\partial^2 G}{\partial V^2} (V_t, t) \frac{\sigma_V^2 V_t^2}{2} \right) dt$$

$$+ \left( \Delta_t S_t \sigma_S + \frac{\partial G}{\partial V} (V_t, t) \sigma_{VZ} V_t \right) dz_t + \left( \Delta_t^r S_t^r \sigma_{S^r} + \frac{\partial G}{\partial V} (V_t, t) \sigma_{VW} V_t \right) dw_t$$

$$+ \left( H(V_t, t) - G(V_{t-}, t-) \right) dN_t.$$

The second step of the replicating argument is to choose $\Delta_t$ and $\Delta_t^r$ such that the portfolio neither loads on $z$ nor $w$. The quantities that give rise to zero diffusion risk are

$$\Delta_t^* = -\frac{\sigma_{VZ}}{\sigma_S} S_t \frac{\partial G}{\partial V} (V_t, t),$$

and

$$\Delta_t^{r*} = -\frac{\sigma_{VW}}{\sigma_{S^r}} S_t^r \frac{\partial G}{\partial V} (V_t, t).$$

When evaluated at $\Delta_t^*$ and $\Delta_t^{r*}$, the expected rate of return on the portfolio must equal that of the money market account as we consider the liquidation risk diversifiable and hence not priced.

$$E [dQ_t] = \left( r Q_t - G(V_t, t) r + V_t \frac{\partial G}{\partial V} (V_t, t) \left( \mu_V - \frac{\sigma_{VZ}}{\sigma_S} (\mu_S - r) \right) + \frac{\partial G}{\partial t} (V_t, t) \right.$$

$$\left. + \frac{\partial^2 G}{\partial V^2} (V_t, t) \frac{\sigma_V^2 V_t^2}{2} + \lambda \left( H(V_t, t) - G(V_{t-}, t-) \right) \right) dt = rQ_t dt.$$
\[
\frac{\partial G}{\partial t}(V_t, t) + \frac{\partial G}{\partial V}(V_t, t) (r + \alpha) V_t + \frac{1}{2} \frac{\partial^2 G}{\partial V^2} \sigma^2 V_t^2 \lambda \left( H(V_t, t) - G(V_{t-}, t-) \right) - G(V_t, t) r = 0,
\]
with terminal condition
\[G(V_T, T) = \Phi(V_T).\]

As in Jennnergren and Näslund (1996), we obtain the solution through the Feynman-Kac representation. The price of the claim is
\[G(V_t, t) = \tilde{E}_t \left[ \int_T^T e^{-(\lambda + r)(v-t) \lambda H(V_v, v) d v + e^{-(\lambda + r)(T-t)} \Phi(V_T)} \right],\]
where \(V_t\) satisfies the stochastic differential equation
\[dV_t = (r + \alpha) V_t dt + \sigma V_t d\tilde{W}_t.\]

The risk-adjusted probability measure \(\tilde{P}\) is characterized by
\[d\tilde{W}_t = \rho d\tilde{w}_t + \sqrt{1-\rho^2} d\tilde{z}_t\]
where \(\rho\) is the instantaneous correlation coefficient between \(dz_t\) and \(dw_t\). The Brownian motion differentials are obtained via Girsanov’s theorem
\[d\tilde{z}_t = dz_t + \frac{\mu_S - r}{\sigma_S} dt, \quad d\tilde{w}_t = dw_t.\]

In our case, the payoffs are given by
\[H(V_v, v) = \max \left[ V_v - F(v), 0 \right],\]
\[\Phi(V_T) = \max \left[ V_T - F(T), 0 \right],\]
where \(F(v)\) is a deterministic function of time. The price of the European call option subject to stochastic termination is
\[C_t(\alpha, \lambda, F(\cdot)) = G(V_t, t) = \int_T^T \lambda e^{-\lambda(v-t)} BS(v, F(v), -\alpha) d v + e^{-\lambda(T-t)} BS(T, F(T), -\alpha),\]
where \(BS(v, F(v), q)\) is the Black-Scholes price given in (3.25).

### A.2 Pricing in the Unspanned Fund Risk Model

We assume that premature termination is governed by the independent point process \(N = (N_t)_{t \geq 0}\) with values \(N = 0\) and \(N = 1\) before and after liquidation, respectively, such that \(\tau = \inf \{ t : N_t > 0 \}\) denotes the time of stochastic termination. Furthermore,
we assume that the point process is specified by the intensity parameter $\lambda$ satisfying $E_t[\, dN_t] = \lambda \, dt$. If $\tau \leq T$ (early fund dissolution), the claim holder receives the immediate payoff $H(V_\tau, \tau)$, and if $\tau > T$ the claim holder receives $\Phi(V_T)$ at maturity. Finally, we assume that the liquidation risk is diversifiable and thus not compensated.

Contrary to the spanned fund risk model, the unspanned fund risk model is characterized by its inability to put an exact price on the considered claim. Instead of an exact price, the unspanned fund risk model provides a set of market consistent value bounds. In order to derive these bounds, we must assume that the model is free of arbitrage and hence there exists a risk-neutral martingale measure $Q$ which is not necessarily unique. Following Björk and Slinko (2006) and Murgoci (2013), the good-deal bounds are defined as the optimal value processes for the following maximization and minimization problems. The first bound follows from the problem

$$\max_{\{h_z, h_w\}} E_t^Q \left[ e^{-r(T-t)} H(V_\tau, \tau) 1_{[\tau \leq T]} + e^{-r(T-t)} \Phi(V_T) 1_{[\tau > T]} \right],$$

with $Q$ dynamics

$$\frac{dS_t}{S_t} = (\mu_S + h_z \sigma_S) \, dt + \sigma_S dz_t^Q,$$

$$\frac{dV_t}{V_t} = (\mu_V + h_z \sigma_V + h_w \sigma_V) \, dt + \sigma_V dz_t^Q + \sigma_V dw_t^Q,$$

and conditions

$$\mu_S + h_z \sigma_S = r,$$

$$h_w^2 \leq A^2 - (\mu_S - r)^2.$$

The second bound is found by replacing the maximum operator in (A.1) by the minimum operator. Above $1_E$ represents an indicator function associated with the event $E$, and $\{h_z, h_w\}$ are the Girsanov kernels from

$$dz_t = h_z \, dt + dz_t^Q, \quad dw_t = h_w \, dt + dw_t^Q.$$

Thus $h_z$ and $h_w$ may be interpreted as (the negative of) the market prices of $z$ and $w$-risks, respectively.

As in Björk and Slinko (2006) we solve the optimal control problem in (A.1)-(A.5) by use of dynamic programming. Thus, let the optimal value function be defined as

$$J_t = \sup_{\{h_z, h_w\}} \left[ E_t^Q \left[ e^{-r(T-t)} H(V_\tau, \tau) 1_{[\tau \leq T]} + e^{-r(T-t)} \Phi(V_T) 1_{[\tau > T]} \right] \right].$$

By use of $J_t = J(V_t, t)$ the Bellman equation corresponding to a discrete-time optimization with $\Delta t$ increments is

$$J(V_t, t) = \sup_{\{h_z, h_w\}} \left[ e^{-r \Delta t} E_t^Q \left[ J(V_{t+\Delta t}, t + \Delta t) \right] \right].$$
and can be rewritten as
\[
e^{r\Delta t} - 1 \frac{J(V_t, t)}{\Delta t} = \sup_{\{h_z, h_w\}} \left[ \frac{1}{\Delta t} E^Q \left[ J(V_{t+\Delta t}, t+\Delta t) - J(V_t, t) \right] \right].
\]
When \( \Delta t \to 0 \), the left-hand side approaches \( r J(V_t, t) \) and the right-hand side goes to the supremum of the drift rate of \( J \) under \( Q \).

\[
r J(V_t, t) = \sup_{\{h_z, h_w\}} \left[ \frac{\partial J}{\partial t}(V_t, t) + \frac{\partial J}{\partial V}(V_t, t) \left( \mu_V + \sigma_{Vz} h_z + \sigma_{Vw} h_w \right) V_t \right.
\]
\[
\left. + \frac{\partial^2 J}{\partial V^2}(V_t, t) \frac{\sigma^2_V V^2_t}{2} + \lambda \left( H(V_t, t) - J(V_{t-}, t-) \right) \right].
\]

Thereby the optimal value function will on \([t, T]\) satisfy the Hamilton-Jacobi-Bellmann equation
\[
\frac{\partial J}{\partial t}(V_t, t) + \sup_{\{h_z, h_w\}} \left[ AJ(V_t, t) \right] = r J(V_t, t), \tag{A.6}
\]
subject to (A.4) and (A.5), and terminal condition
\[
J(V_T, T) = \Phi(V_T).
\]
The infinitesimal operator (A.6) is given by
\[
AJ(V_t, t) \equiv \frac{\partial J}{\partial t}(V_t, t) \left( \mu_V + \sigma_{Vz} h_z + \sigma_{Vw} h_w \right) V_t + \frac{\partial^2 J}{\partial V^2}(V_t, t) \frac{V^2_t \sigma^2_V}{2}
\]
\[
+ \lambda \left( H(V_t, t) - J(V_{t-}, t-) \right).
\]
The good-deal bound is found by solving the embedded static maximization problem first and subsequently the PDE.

Consider first the static maximization problem. For a fixed \( V_t \) and \( t \), the problem reads
\[
\sup_{\{h_z, h_w\}} \left[ AJ(V_t, t) \right]
\]
subject to (A.4) and (A.5). The optimal (negative of) the market price of \( z \) risk is obtained through the martingale condition and is
\[
h^*_z = \frac{r - \mu_S}{\sigma_S}.
\]
Thereby, \( h^*_w \) is found by forming the Lagrangian
\[
\mathcal{L} = \frac{\partial J}{\partial V}(V_t, t) \sigma_{Vw} h_w - m \left( h_w - \sqrt{A^2 - \left( \frac{\mu_S - r}{\sigma_S} \right)^2} \right).
\]
The optimal $h_w$ satisfies
\[
\frac{\partial L}{\partial h_w} = \frac{\partial J}{\partial V}(V_t, t) \sigma_{Vw} - m^* = 0,
\]
and
\[
m^* \left( h_w^* - \sqrt{A^2 - \left( \frac{\mu_S - r}{\sigma_S} \right)^2} \right) = 0.
\]
As $m^* > 0$, the optimal $h_w$ reads
\[
h_w^* = \sqrt{A^2 - \left( \frac{\mu_S - r}{\sigma_S} \right)^2}.
\]
With the optimal kernels at hand we continue by solving the PDE given by
\[
r J(V_t, t) = \frac{\partial J}{\partial t}(V_t, t) + \frac{\partial J}{\partial V}(V_t, t) \left( r + \eta \right) V_t + \frac{\partial^2 J}{\partial V^2}(V_t, t) \frac{V_t^2 \sigma_V^2}{2}
\]
\[
+ \lambda \left( H(V_t, t) - J(V_{t-}, t-) \right),
\]
where
\[
\eta \equiv \alpha + \sigma_{Vw} \sqrt{A^2 - \text{SR}_S^2},
\]
\[
\sigma_V^2 \equiv \sigma_{Vz}^2 + \sigma_{Vw}^2,
\]
\[
\text{SR}_S \equiv \frac{\mu_S - r}{\sigma_S}.
\]
The second good-deal bound is characterized by a minus in front of $\sigma_{Vw} \sqrt{A^2 - \text{SR}_S^2}$ in $\eta$ and in line with Cochrane and Saa-Requejo (2000) we introduce an auxiliary parameter $a \in \{-1, 1\}$ such that
\[
\eta(a) \equiv \alpha + a \sigma_{Vw} \sqrt{A^2 - \text{SR}_S^2}.
\]
The solution $J(V_t, t)$ to the value problem is obtained through the Feynman-Kac representation and states
\[
J(V_t, t) = E_t^{Q^*} \left[ \int_t^T e^{-\lambda \left( r + \eta(a) \right) (v-t)} \lambda H(V_v, v) \, dv + e^{-\lambda \left( r + \eta(a) \right) (T-t)} \Phi(V_T) \right],
\]
where $V_t$ satisfies the stochastic differential equation
\[
dV_t = \left( r + \eta(a) \right) V_t \, dt + \sigma_V V_t \, dW_t^{Q^*}.
\]
The risk-adjusted probability measure $Q^*$ is characterized by
\[
dW_t^{Q^*} = \rho \, dw_t^{Q^*} + \sqrt{1 - \rho^2} \, dz_t^{Q^*},
\]
A.3. Expected Fund Lifetime

Let \( t = 0 \) and consider the fund lifetime

\[ T = \min \{ \tau, T \} = 1_{\{ \tau \geq T \}} T + 1_{\{ \tau < T \}} \tau. \]

From the cdf of the exponentially distributed \( \tau \), the expected fund lifetime is

\[
E[ T ] = E \left[ 1_{\{ \tau \geq T \}} T + 1_{\{ \tau < T \}} \tau \right] \\
= E \left[ 1_{\{ \tau \geq T \}} T \right] + E \left[ 1_{\{ \tau < T \}} \tau \right] \\
= \left( 1 - (1 - e^{-\lambda T}) \right) T + E \left[ 1_{\{ \tau < T \}} \tau \right].
\]

The last term is found by rewriting \( 1_{\{ \tau < T \}} \tau \) using the auxiliary function

\[
1_{\{ \tau < T \}} \tau = g(\tau) = \begin{cases} 
\tau & \text{if } \tau < T \\
0 & \text{otherwise}. 
\end{cases}
\]

The expected value equals

\[
E \left[ g(\tau) \right] = \int_{-\infty}^{\infty} g(\tau) P_{\tau}(\tau) \, d\tau \\
= \int_{-\infty}^{T} \tau P_{\tau}(\tau) \, d\tau + \int_{T}^{\infty} 0 \cdot P_{\tau}(\tau) \, d\tau \\
= \int_{-\infty}^{T} \tau P_{\tau}(\tau) \, d\tau.
\]
From $\tau$ being exponentially distributed with pdf

$$
P_{\tau}(v) = \begin{cases} 
\lambda \exp^{-\lambda v} & v \geq 0 \\
0 & v < 0,
\end{cases}
$$

we get

$$
E[g(\tau)] = \int_{-\infty}^{T} v \cdot P_{\tau}(v) \, dv \\
= \int_{-\infty}^{0} v \cdot 0 \, dv + \int_{0}^{T} v \cdot \lambda e^{-\lambda v} \, dv \\
= -Te^{-\lambda T} - \frac{1}{\lambda} e^{-\lambda T} + \frac{1}{\lambda}.
$$

Thereby the expected fund lifetime is

$$
E[T] = \frac{1}{\lambda} \left(1 - e^{-\lambda T}\right).
$$

### A.4 Relative Catch-up Value Derivatives

In this section we derive the sensitivity of the relative catch-up with respect to changes in $\alpha$ and $\lambda$ when the fund does not rely on leverage.

Before turning to the derivatives, we present the limit of the value of a European call option when the obtained debt goes towards zero as well as the relative catch-up value in the absence of leverage since these are needed in our derivations. The former is

$$
\lim_{D \to 0} C_0(\alpha, \lambda, D(\cdot)) = V_0 = \begin{cases} 
V_0 \left(\lambda - \alpha e^{-(\lambda - \alpha)T}\right) / (\lambda - \alpha) & \text{with liquidation risk, } \lambda \neq \alpha \\
V_0(1 + \lambda T) & \text{with liquidation risk, } \lambda = \alpha \\
V_0 e^{\alpha T} & \text{without liquidation risk},
\end{cases}
$$

and the latter

$$
C_0 = \left(\varphi_c - \varphi_p\right) \frac{C_0(\alpha, \lambda, K(\cdot)) - C_0(\alpha, \lambda, \bar{\varphi}(\cdot))}{V_0 - \varphi_p C_0(\alpha, \lambda, K(\cdot))}.
$$

#### A.4.1 $\partial C_0 / \partial \alpha$

Consider the derivative

$$
\frac{\partial C_0}{\partial \alpha} = \frac{\partial}{\partial \alpha} \left(\varphi_c - \varphi_p\right) \frac{C_0(\alpha, \lambda, K(\cdot)) - C_0(\alpha, \lambda, \bar{\varphi}(\cdot))}{V_0 - \varphi_p C_0(\alpha, \lambda, K(\cdot))}.
$$

In order to apply the quotient rule, let

$$
C_0 = \frac{f(\alpha)}{g(\alpha)},
$$
such that

\[ \frac{\partial C_0}{\partial \alpha} = \frac{f'g - g'f}{g^2}. \]

The \( f' \) and \( g' \) derivatives are

\[ f' = \left( \varphi_c - \varphi_p \right) \left( \frac{\partial}{\partial \alpha} C_0(\alpha, \lambda, K(\cdot)) - \frac{\partial}{\partial \alpha} C_0(\alpha, \lambda, \bar{\varphi}(\cdot)) \right), \]

\[ g' = \frac{\partial}{\partial \alpha} V_0 - \varphi_p \frac{\partial}{\partial \alpha} C_0(\alpha, \lambda, K(\cdot)), \]

and thus we need to determine the derivatives \( \frac{\partial}{\partial \alpha} C_0(\alpha, \lambda, F(\cdot)) \) and \( \frac{\partial}{\partial \alpha} V_0 \).

### A.4.1.1 With Liquidation Risk

In the presence of liquidation risk we have

\[
\frac{\partial}{\partial \alpha} C_0(\alpha, \lambda, F(\cdot)) = \int_0^T \lambda e^{-\lambda v} \frac{\partial}{\partial \alpha} BS(v, F(v), -\alpha) \, dv \\
+ e^{-\lambda T} \frac{\partial}{\partial \alpha} BS(T, F(T), -\alpha). \tag{A.10}
\]

The derivative of the Black-Scholes price with respect to \( \alpha \) follows from

\[ \frac{\partial}{\partial \alpha} BS(v, F(v), -\alpha) = \frac{\partial}{\partial q} BS(v, F(v), q) \frac{\partial q}{\partial \alpha}, \]

where \( q = -\alpha \). By use of the Black-Scholes "greek" \( \psi \) defined as

\[ \psi(v, F(v), q) = \frac{\partial}{\partial q} BS(v, F(v), q) = -V_0 v e^{-q v} N \left( d(v, F(v), q) \right), \]

we get

\[ \frac{\partial}{\partial \alpha} BS(v, F(v), -\alpha) = V_0 v e^{\alpha v} N \left( d(v, F(v), -\alpha) \right). \]

When we use this in equation (A.10), we get

\[
\frac{\partial}{\partial \alpha} C_0(\alpha, \lambda, F(\cdot)) = \int_0^T \lambda e^{-(\lambda - \alpha)v} V_0 v N \left( d(v, F(v), -\alpha) \right) \, dv \\
+ e^{-(\lambda - \alpha)T} V_0 TN \left( d(T, F(T), -\alpha) \right). \tag{A.11}
\]

In the presence of liquidation risk we compute the sensitivity in equation (A.9) using numerical approximations of (A.11) and the derivative

\[
\frac{\partial}{\partial \alpha} V_0 = \left\{ \begin{array}{ll}
V_0 \left( \lambda - \left( \lambda (1 + \alpha T) - \alpha^2 T \right) e^{-(\lambda - \alpha)T} \right) / (\lambda - \alpha)^2 & \lambda \neq \alpha \\
V_0 \left( T + T^2 \lambda / 2 \right) & \lambda = \alpha.
\end{array} \right.
\]
A.4.1.2 Without Liquidation Risk

In the absence of liquidation risk we have

\[ C_0(\alpha, \lambda, F(\cdot)) = BS(T, F(T), -\alpha), \]

and thus the derivative

\[ \frac{\partial}{\partial \alpha} C_0(\alpha, \lambda, F(\cdot)) = \frac{\partial}{\partial \alpha} BS(T, F(T), -\alpha) = -\psi(T, F(T), -\alpha). \]

Since

\[ \frac{\partial}{\partial \alpha} V_0 = TV_0e^{\alpha T}, \]

the sensitivity of the relative catch-up value with respect to \( \alpha \) is

\[ \frac{\partial C_0}{\partial \alpha} = (\varphi_c - \varphi_p)TV_0e^{\alpha T}\left( \frac{N\left( d(T, K(T), -\alpha) \right) - N\left( d(T, \varphi(T), -\alpha) \right)}{V_0e^{\alpha T} - \varphi_pBS(T, K(T), -\alpha)} + \frac{\left( 1 - \varphi_pN\left( d(T, K(T), -\alpha) \right) \right)\left( BS(T, \varphi(T), -\alpha) - BS(T, K(T), -\alpha) \right)}{\left( V_0e^{\alpha T} - \varphi_pBS(T, K(T), -\alpha) \right)^2} \right). \]

A.4.1.3 \( \frac{\partial C_0}{\partial \lambda} \)

As in the previous section, we make use of the quotient rule and by the equation (A.8) the derivative of interest is

\[
\frac{\partial C_0}{\partial \lambda} = \left( \varphi_c - \varphi_p \right) T V_0 e^{\alpha T} \left[ \frac{\frac{\partial}{\partial \lambda} C_0(\alpha, \lambda, K(\cdot)) - \frac{\partial}{\partial \lambda} C_0(\alpha, \lambda, \varphi(\cdot))}{V_0 - \varphi_p C_0(\alpha, \lambda, K(\cdot))} \right] + \left( 1 - \varphi_p N\left( d(T, K(T), -\alpha) \right) \right) \left( \frac{\frac{\partial}{\partial \lambda} V_0 - \varphi_p \frac{\partial}{\partial \lambda} C_0(\alpha, \lambda, K(\cdot))}{V_0 - \varphi_p C_0(\alpha, \lambda, K(\cdot))} \right) \left( C_0(\alpha, \lambda, K(\cdot)) - C_0(\alpha, \lambda, \varphi(\cdot)) \right). \tag{A.12}
\]

From equation (A.7) we obtain the derivative

\[
\frac{\partial V_0}{\partial \lambda} = \begin{cases} 
V_0 \left[ \left( 1 + Tae^{-(\lambda - \alpha)T} \right)(\lambda - \alpha) - \left( \lambda - \alpha e^{-(\lambda - \alpha)T} \right) \right] / (\lambda - \alpha)^2 & \lambda \neq \alpha \\
V_0 T & \lambda = \alpha.
\end{cases}
\]
The derivative of the call option price with respect to $\lambda$ follows from the use of the Leibniz integral rule

$$
\frac{\partial}{\partial \lambda} C_0(\alpha, \lambda, F(\cdot)) = \int_0^T \frac{\partial}{\partial \lambda} \left( Ae^{-\lambda v} \right) \text{BS}(v, F(v), -\alpha) \, dv \\
+ \frac{\partial}{\partial \lambda} \left( e^{-\lambda T} \right) \text{BS}(T, F(T), -\alpha) \\
= \int_0^T e^{-\lambda v} (1 - \lambda v) \text{BS}(v, F(v), -\alpha) \, dv \\
- Te^{-\lambda T} \text{BS}(T, F(T), -\alpha). \tag{A.13}
$$

We are now able to compute values the of $\partial C_0 / \partial \lambda$ in equation (A.12) by use of numerical procedures for the call prices and their derivatives in (A.13).
Declaration of co-authorship

Full name of the PhD student: Søren Kærgaard Slipsager

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Signature of the PhD student
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| Authors: | Slipsager, Søren Kærgaard and Jørgensen, Peter Løchte |

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