Random search for a dark resonance

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At a glance

Quantum systems can act as sensitive probes and field sensors [1], and the paradigm question in such applications concerns the scaling of the sensing error with the number of physical resources .

We propose an adaptive protocol to search for a dark resonance in e.g. a three level quantum system (see Fig. 1), and we show how the sensitivity of our scheme follows from a Lévy statistical analysis [2]. Such have been applied to e.g. animal foraging behavior and financial systems. Our work closely follows [3], investigating subrecoil laser cooling.

The work presents a statistical analysis of parameter estimation in quantum systems with non-ergodic dynamics where conventional methods relying on the central limit theorem (CLT), e.g the Cramér-Rao bound, are invalid.

Dark resonances

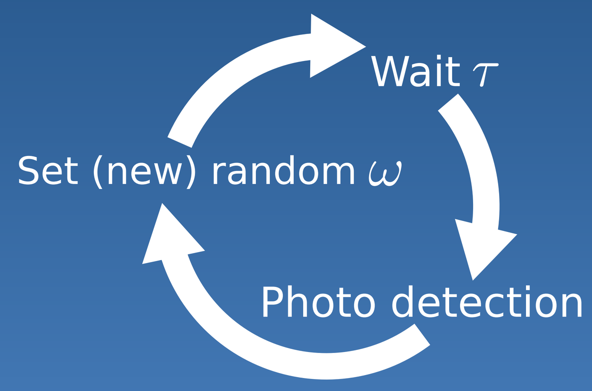
A vanishing fluorescence rate of an atomic system excited on exact resonance, e.g. in EIT (Fig. 2).

Due to narrow linewidths, they are sensitive probes of perturbations on the system; see, e.g., [3].

Our example : Λ-type three-level system Fig. 2 driven by two lasers ; one on resonance and one detuned by δ.
For δ = 0, the system is trapped in a dark state |ψ> = (|0> − |1>)/√2.
Close to resonance, the emission rate depends quadratically on δ.

Search protocol

As an alternative to a systematic scanning and accumulation of signal at different, discrete laser frequencies, we suggest a random search protocol (Fig. 1) in which the probe laser frequency may come arbitrarily close to the dark resonance. That event is witnessed by the complete absence of signal and suggests the following adaptive protocol for the duration T of the experiment:



Far from resonance, the probability for an early photon detection and a shift to a different frequency is high, while close to resonance, the photon emission rate is very small, and these frequencies are maintained for a long time.

The longer we probe the atom, the more likely are long intervals with frequencies close to the dark resonance.

The instantaneous, stochastically tuned laser frequency thus constitutes a good estimate of the atomic transition frequency.

Setup

$$\hat{H} = \delta\,|0\rangle\langle 0| + \frac{\Omega}{2}\left(|2\rangle\langle 0| + |0\rangle\langle 2|\right) + \frac{\Omega}{2}\left(|2\rangle\langle 1| + |1\rangle\langle 2|\right)$$

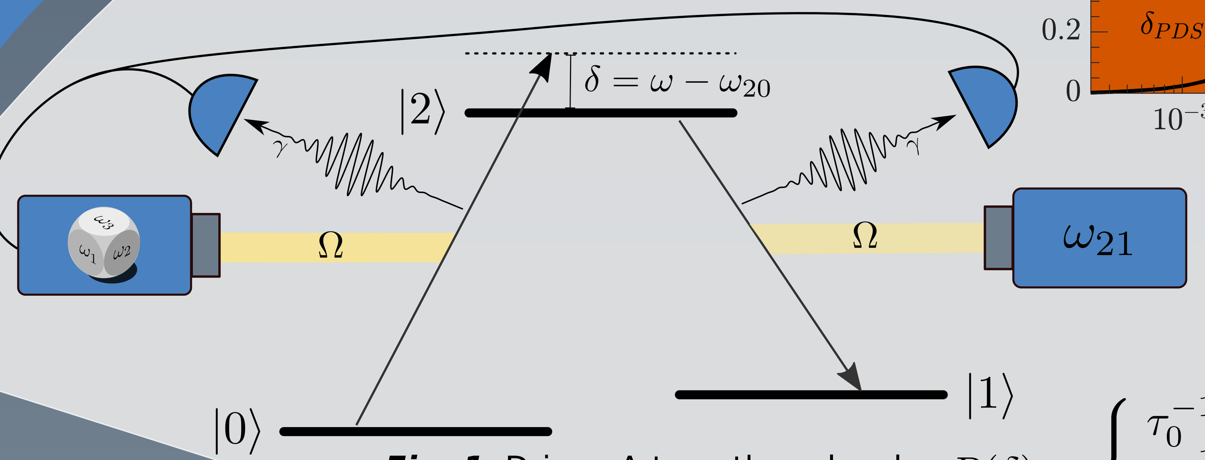


Fig. 1: Driven Λ-type three-level system subject to photo detection.

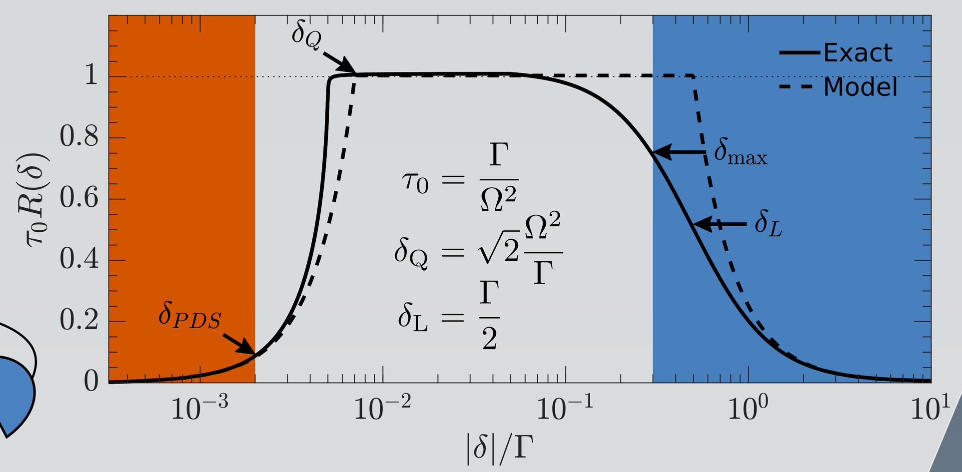
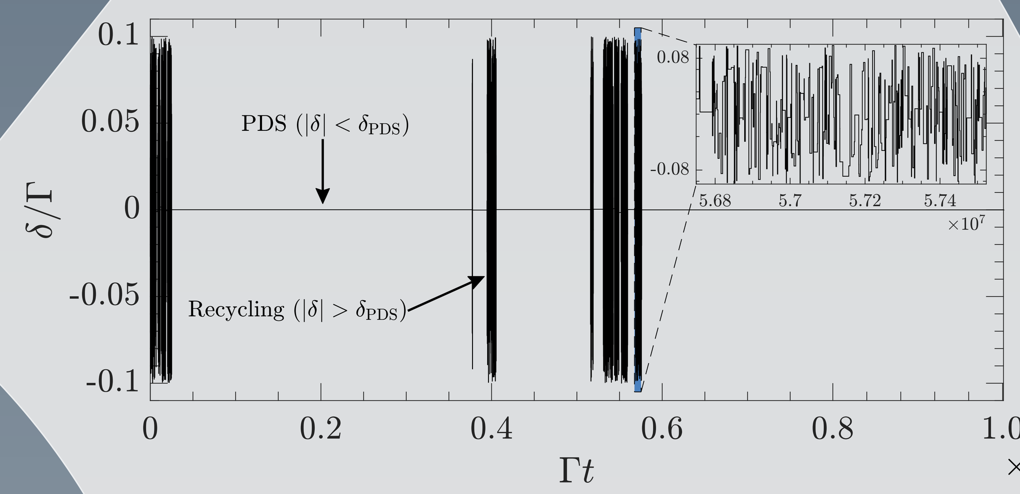


Fig. 2: Florescence rate as a function of detuning from dark resonance.

$$R(\delta) = \begin{cases} \tau_0^{-1}(\delta/\delta_Q)^2, & |\delta| < \delta_Q \\ \tau_0^{-1}, & \delta_Q < |\delta| < \delta_L \\ \tau_0^{-1}(\delta_L/\delta)^2, & \delta_L < |\delta| \end{cases}$$

Trajectories

Fig. 3: *Simulation of continuous monitoring and random frequency jumps.*



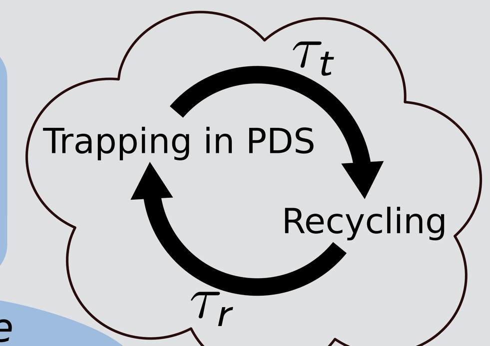
Few long intervals with small detuning interrupted by brief periods with larger, fluctuating values of δ.

Competition: Trapping in pseudo dark state (PDS) vs. recycling

Broad τ_t distribution

$$P_t(\tau_t) \underset{\text{large }\tau_t}{\simeq} \frac{\mu\tau_b^{\mu}}{\tau_t^{1+\mu}},$$

where $\mu = 1/2$



Finite mean recycling time

$$\langle\tau_r\rangle = \langle n\rangle\,\tau_0, \quad \text{where}$$

$$\langle n\rangle = \frac{\delta_{max}}{\delta_{PDS}}$$

Statistical results

Due to the distribution of short and very long time intervals, the dynamics is not ergodic and the estimation performance follows Lévy statistics; see [2] for details.

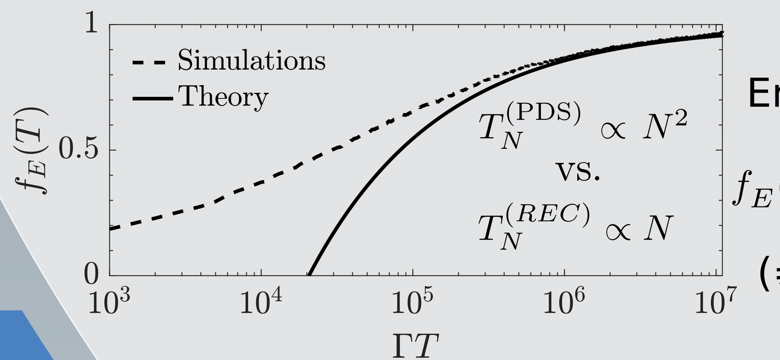


Fig. 4: *Proportion of trajectories with |δ|< δ$_{PDS}$ as a function of time.*

Ensemble average

$$f_E(T) \simeq 1 - \frac{\sin(\pi\mu)}{\pi}\frac{\langle\tau_r\rangle}{\tau_b^{\mu}T^{1-\mu}}$$

(≠ time average)

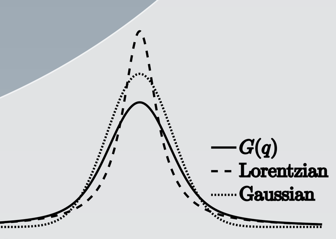
$f_E(T) \to 1 \Rightarrow$ Random search is successful.

Asymptotic estimation error:

$$\delta_T = \delta_Q\left(\frac{\tau_0}{T}\right)^{\mu},$$

with $R(\delta_T) = T^{-1}$

(broken ergodicity)



Distribution of trajectories in PDS

$$\mathcal{P}(\delta, T) = h(T)G(\delta/\delta_T)$$

Explicit T dependence (no stationary form) ↔ broken ergodicity

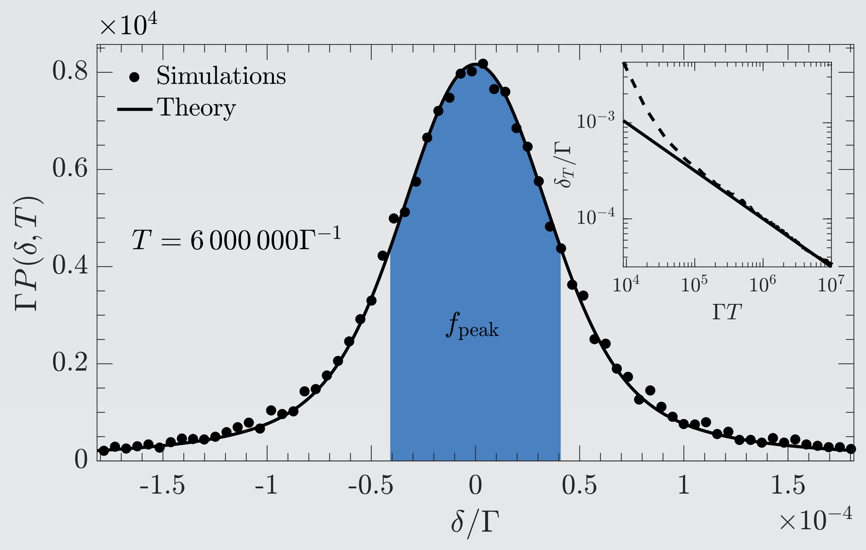


Fig. 5: *Asymptotic frequency distribution. Blue area (59%~0.82σ) have |δ|<δ$_T$.*

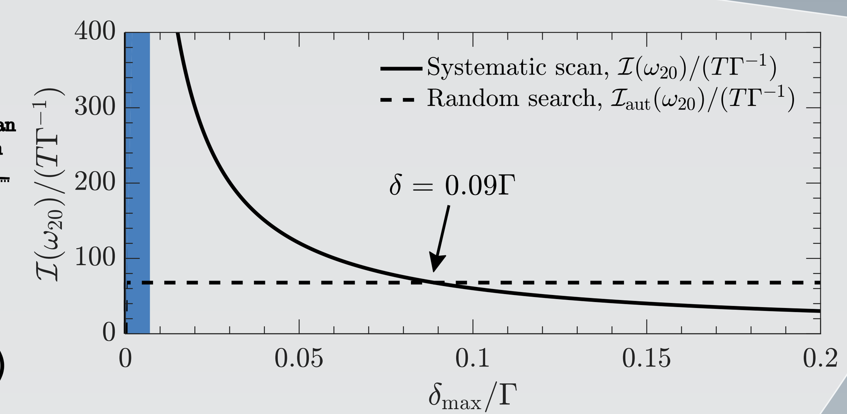
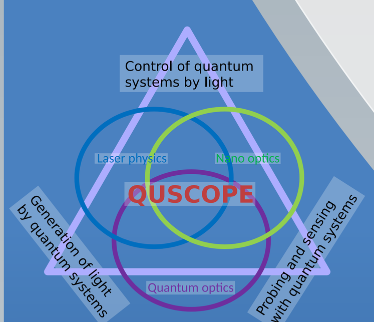


Fig. 6: *Comparison of information per. time to systematic frequency scan as a function of search width δ$_{max}$.*

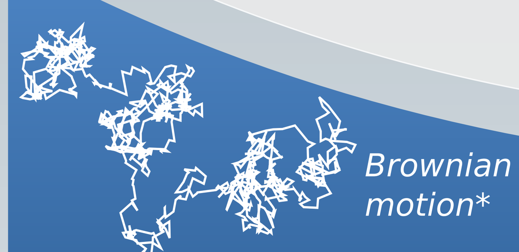
$$h(T) = \left(\frac{\tau_{PDS}}{\tau_b}\right)^{\mu}\frac{\sin(\pi\mu)}{\pi\mu\delta_T}$$



Outlook

While our example concerns a rate R(δ) with a quadratic dip around δ = 0, our analysis applies equally to other systems. This formalism allows better than 1/√T scaling of the error δ_T in estimating a parameter θ, if a process is found for which the rate depends on θ as R(θ) ∝ θ^{α} with 0 < α < 2.

Metrology protocols have been proposed with similar feedback and adaptive elements, and which show convergence faster than 1/√T (or 1/√N); see, e.g. , [1,6]. Since adaptive schemes may generally induce non-ergodic dynamics, we believe that our analysis will be relevant to a number of such protocols where standard statistical analyses are inadequate.



Broad distributions and Lévy statistics

Power law decay with tail:

$$P(\tau) \underset{\text{large }\tau}{\simeq} \frac{\mu\tau_b^{\mu}}{\tau^{1+\mu}}, \quad \text{where } \mu > 0.$$

Sum of independent realizations:

$$T_N = \sum_{i=1}^{N}\tau_i$$

For μ > 2 the CLT ensures that T_N is normally distributed and $T_N \to \langle\tau\rangle N$.

For μ < 2 the variance of τ is infinte and a generalized CLT by Lévy and Gnedenko [5] states that T_N follows a Lévy distribution. If 1 < μ < 2, $\langle\tau\rangle$ is finite and $T_N \to \langle\tau\rangle N$.
In this work μ < 1 and even the mean is undefined. Then

$$T_N = \xi\tau_b N^{1/\mu}$$

is dominated by a few single terms.

References

[1] V. Giovanetti, S. Lloyd and L. Maccone, Phys. Rev. Lett. 96, 010401 (2006).

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[4] A. Nagel et. al., Europhys. Lett. 66, 2593 (1991).

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[6] M. Mehboudi, L. A. Correa and A. Sanpera, Phys. Rev. A 94, 042121 (2016).