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An Analysis of a 3-Factor Model proposed by the Danish Society of Actuaries for Forecasting and Risk Analysis

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Abstract

This paper provides the explicit solution to the three-factor diffusion model recently proposed by the Danish Society of Actuaries to the Danish industry of life insurance and pensions. The solution is obtained by use of the known general solution to multi-dimensional linear stochastic differential equation systems. With offset in the explicit solution we establish the conditional distribution of the future state variables which allows for exact simulation. Using exact simulation we illustrate how simulation of the system can be improved compared to a standard Euler scheme.

In order to analyze the effect of choosing the exact simulation scheme over the traditional Euler approximation scheme frequently applied by practitioners, we carry out a simulation study. We show that due to its recursive nature the Euler scheme becomes computationally expensive as it requires a small step size in order to minimize discretization errors. By using our exact simulation scheme one is able to cut these computational costs significantly and obtain even better forecasts. As probability density tail behavior is key to expected investment portfolio performance we further conduct a risk analysis in which we compare well known risk measures under both schemes. Finally, we conduct a sensitivity analysis and find that the relative performance of the two schemes depends on the chosen model parameter estimates.

1 Introduction

Financial asset value and return forecasts play a key role in the industry of life insurance and pensions (L&P) since the ex ante attractiveness of an L&P contract depends hereupon. With the mean-variance relationship of Markowitz (1952) in mind an L&P company is able to increase the expected return on an investment portfolio by increasing its underlying risk. This potential issue produces a need for transparency and consistency across L&P valuation schemes. Therefore the Danish Society of Actuaries (DSA) has recently issued a proposed guideline in DSA (2014) to ensure consistent estimation of the expected future payments across contract types and L&P companies. The main contribution of their report is to propose the recommendation of a dynamic model for modeling expected portfolio returns, henceforth referred to as the DSA-model.

The proposed model is highly relevant since L&P companies have been shifting policy holders with traditional participating/with-profits contracts towards market value based unit linked contracts, as the L&P companies tend to prefer more market based liabilities with fewer guarantees (Johannesson & Matthiesen, 2014). This shifting can be seen as a sound decision since the market value based unit linked contracts have an increased transparency, the investment climate has changed, and the markets' preferences and requirements are better suited for these products (Jørgensen, 2007).

The unit linked contracts are characterized by the assets' placement on individual accounts offering relatively variable and not necessarily smooth returns. As the return on the traditional participating/with-profits contracts consists primarily of a basic interest rate, which is often guaranteed, and a fairly stable annual bonus reflecting the success of the investment strategy undertaken, the shift causes a need for lucid communication due to a radical change in risk profile (DSA, 2014). It is not hard to imagine how the unit linked contracts can be presented as relatively more attractive when comparing expected future payments, since an increase in the expected return can be driven by an underlying risk increase which is not necessarily communicated. Since the forecasted value is highly dependent on the underlying financial model, regulators should ensure that a common valuation framework is in place. As matters stand, Danish regulators do in fact provide industry assumptions to be used in forecasting. Unfortunately they only concern expected returns and not risks which is probably the motivation behind proposing the DSA-model.

Since the overall purpose of the DSA-model is to allow for forecasting future values of stocks, fixed income securities, and derivatives the model must have certain generally accepted features. A standing consensus in the literature is that there exists an equity premium as first identified by Mehra & Prescott (1985), and that the expected equity rate of return is time-varying (Bollerslev *et al.*, 1988; Harvey, 1989; Lettau & Ludvigson, 2002). These empirics are taken into account by modeling the equity return with a stochastic equity premium. Specifically, this implies that the equity returns themselves are mean-reverting

which is empirically backed by the long-run predictability found by among others Fama & Schwert (1977) and Campbell & Shiller (1988). Further, a proper model for pricing fixed income securities should feature a stochastic interest rate and since the literature on dynamic interest rate models is vast (see e.g. Merton (1973); Vasicek (1977); Dothan (1978); Cox *et al.* (1985); Hull & White (1990) or for an empirical comparison Chan *et al.* (1992)) several possibilities exist.

Even though the DSA-model is empirically appropriate and theoretically tractable potential challenges exist regarding its implementation as performance and computation costs are highly dependent on the choice of simulation method. Therefore this paper focuses on the implementation of the DSA-model and more specifically on the importance of choosing the right simulation scheme. We show that a naive implementation of the model using the traditional Euler approximation scheme becomes computationally costly due to its recursive nature. As an alternative we derive the conditions needed for the so-called exact simulation which samples from the exact conditional distribution of future values of the state variables.

In order to evaluate on the choice of simulation scheme we conduct an analysis in which we compare the forecasting ability and computational expenses of both schemes. Since investors care about downside risk and upside potential we also analyze the tail behavior of the simulated probability density functions. In the risk analysis we mimic a real world pension contract scheme and consider well known risk measures such as Value-at-Risk and conditional Value-at-Risk and we investigate the upside potential through corresponding measures. As we expect our result to be dependent on the used model parameter estimates we accompany the risk analysis with a parameter sensitivity analysis.

Our main contribution is to provide the explicit solution and the exact simulation scheme of the DSA-model which should ease its implementation in the L&P industry. and thereby allow the L&P companies to cut their computational costs without losing performance.

The remainder of this paper is organized as follows. In section 2 we describe the financial model proposed by DSA and derive the distribution of future values of the state variables. The Euler and the exact simulation schemes are also provided in section 2. Section 3 contains the simulation study in which we analyze performance and risk through the forecasting ability, computational expenses, tail behavior, and model parameter sensitivity. Finally, in section 4 we provide some concluding remarks.

2 Methodology

When doing *exact simulation* the conditional distribution of the simulated state variables coincides with the conditional distribution of the continuous-time processes on the simulation time grid, and thus one obtains unbiased estimates of the future state variables. In the following we introduce the DSA-model and study the performance of the exact simulation scheme and compare it to the Euler approximation scheme often used in simulation of stochastic differential equations whose conditional distribution of the future state variables is unknown.

2.1 The DSA-model

The DSA-model is a three-factor diffusion model with uncertainty modeled by two Brownian Motions and has been studied in both Wachter (2002) and Munk *et al.* (2004) but with the aim of deciding an optimal strategy in a utility based consumption-portfolio choice framework. Our use of the model is somewhat different since we focus on characterizing future portfolio values and thus a dynamic modeling of asset prices. In that relation, a limitation of the model is that it only contains two primary assets: stocks and bonds and hence other assets must be approximately allocated to the stock/bond asset classes. As DSA (2014) states, the model is only constructed around two asset types such that the most risky asset of the model should contain the risky asset classes of the portfolio such as traditional stocks and likewise the least risky asset should contain different bond types of various durations.

Considering the DSA-model, the stock price dynamics take offset in a modified Black & Scholes (1973) setting in the sense that the relative drift consists of a risk-free part and a risky part. Specifically, it is the sum of a short rate following the Vasicek (1977) model and a mean-reverting equity risk premium governed by an Ornstein-Uhlenbeck process.

Let the following stochastic processes represent the three state variables: the stock price $S = (S_t)_{t \geq 0}$, the equity risk premium $x = (x_t)_{t \geq 0}$, and the short-term interest rate $r = (r_t)_{t \geq 0}$. Further, let (S, x, r) be defined on the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P} \right)$ and adapted to the filtration $\mathcal{F}_t$, which represents all available information at time $t \geq 0$. In the following, conditioning on time t information will be denoted by a subscript t. The dynamics of the model is given by:

$$\frac{\mathrm{d}S_t}{S_t} = (r_t + x_t)\mathrm{d}t + \sigma_S \mathrm{d}z_{1,t} \tag{1}$$

$$\mathrm{d}x_t = \alpha(\xi - x_t)\mathrm{d}t - \sigma_x \mathrm{d}z_{1,t} \tag{2}$$

$$\mathrm{d}r_t = \kappa(\theta - r_t)\mathrm{d}t + \sigma_r \mathrm{d}z_{2,t} \tag{3}$$

where $\mathrm{d}z_{1,t}$ and $\mathrm{d}z_{2,t}$ are instantaneously correlated Brownian Motions, i.e. $\mathrm{d}z_{1,t} \cdot \mathrm{d}z_{2,t} = \rho \mathrm{d}t$ with constant correlation coefficient $\rho \in [-1; 1]$.

As appears from equation (1) the stock price is described by a Geometric Brownian Motion yet with the modification of a stochastic relative drift and a constant relative local volatility, $\sigma_S > 0$. The expected instantaneous rate of return of the stock is given by $(r_t + x_t)$, which seems intuitively appealing since the instantaneous expected rate of return is decomposed into a risk-free short rate part and an equity risk premium part. Including a stochastic equity premium in the relative drift of a stock was first introduced by Merton (1971) and gives the model the ability to capture a time-varying compensation for bearing equity risk. A large body of literature exists that supports a time-varying equity premium see e.g. Keim & Stambaugh (1986); Fama & French (1988); Bansal & Yaron (2004).

Considering the stochastic differential equation of x_t in (2) the first term implies a

long-run regressive adjustment of the equity premium towards its long-run level, ξ, with the $\alpha > 0$ parameter controlling the speed of adjustment. Further, the volatility of the Ornstein-Uhlenbeck process describing the equity risk premium, $\sigma_x > 0$, is assumed constant. Finally, the stock price and the equity risk premium are instantaneously perfectly negatively correlated, which gives rise to mean-reversion in the stock returns, i.e. high stock returns tend to be followed by relatively small expected returns.

Equation (3) describes the dynamics of the short rate following the traditional Vasicek model ensuring mean-reversion with long-term level $\theta > 0$, persistence parameter $\kappa > 0$, and a constant local volatility of $\sigma_r > 0$. Traditionally it is seen as a weakness of the Vasicek model that it allows for negative interest rates but taking the current interest rate levels into account this property should perhaps no longer be regarded as a weakness.

It is well known that the Vasicek model leads to zero-coupon bond prices being an exponential-affine function of the current short rate, see e.g. Björk (2004). The time t price of a zero-coupon bond maturing at time $T > t$ is given by:

$$P(r_t, t, T) = \exp\left\{ \left(\theta - \frac{\sigma_r \lambda}{\kappa} - \frac{\sigma_r^2}{2\kappa^2} \right) \left(\Psi(\kappa, t, T) - (T - t) \right) \right.$$
$$\left. - \frac{\sigma_r^2}{4\kappa} \Psi^2(\kappa, t, T) - \Psi(\kappa, t, T) r_t \right\} \tag{4}$$

where λ denotes the market price of interest rate risk which we assume constant. Here we have introduced the auxiliary function:

$$\Psi(a, t, T) = \frac{1}{a} \left(1 - e^{-a(T-t)} \right) \tag{5}$$

which will be heavily used in the remainder of this paper. Since $\Psi(\kappa, t, T) > 0$ the zero-coupon price becomes a convex decreasing function of the short rate. As derived in Vasicek (1977) the dynamics of the zero-coupon price $P_t^T = P(r_t, t, T)$ can be written as:

$$dP_t^T = P_t^T \left[(r_t - \lambda \Psi(\kappa, t, T)\sigma_r)\, dt - \Psi(\kappa, t, T)\sigma_r dz_{2t} \right]. \tag{6}$$

The stochastic shocks to the short rate are instantaneously negatively correlated with the zero-coupon price which is an appropriate feature of the model. After having stated the model dynamics we turn to the derivation of the multivariate distribution of future values of the three state variables conditional on time t information which we will need for the exact simulation.

2.2 The Conditional Distribution

The first step in deriving the conditional distribution of the state variables is to determine the explicit solution to the stochastic differential system. Subsequently the obtained solution will reveal the distributions of the future values. All derivations regarding the explicit solution

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of the system and the conditional distribution are relegated to the appendix. The purpose of this subsection is solely to provide the main results related to the simulation task.

When dealing with a stock price modeled by a Geometric Brownian Motion it is convenient to study the logarithmic version instead and thus Itô's Lemma is applied with the function $g(S,t) = \ln S$ and the process $y_t = g(S_t, t) = \ln S_t$ is defined. The dynamics of logarithmic stock price becomes

$$\mathrm{d} \ln S_t = \left(r_t + x_t - \frac{1}{2}\sigma_S^2 \right) \mathrm{d}t + \sigma_S \mathrm{d}z_{1,t}, \tag{7}$$

and the explicit solution for $0 \le t < T$ is:

$$
\begin{aligned}
\ln S_T = {}& \ln S_t + \theta (T - t) + (r_t - \theta)\, \Psi(\kappa, t, T) + \xi (T - t) + (x_t - \xi)\, \Psi(\alpha, t, T) \\
& - \frac{1}{2}\sigma_S^2 (T - t) - \sigma_x \int_t^T \Psi(\alpha, u, T)\, \mathrm{d}z_{1,u} \\
& + \sigma_S \int_t^T \mathrm{d}z_{1,u} + \sigma_r \int_t^T \Psi(\kappa, u, T)\, \mathrm{d}z_{2,u}.
\end{aligned}
\tag{8}
$$

As appears the future logarithmic stock returns are normally distributed with future values of $\ln S$ having conditional mean $m(S_t, x_t, r_t, t, T)$ and variance $V(t, T)$, i.e.

$$\ln S_T | \mathcal{F}_t \sim \mathcal{N}\left(m(S_t, x_t, r_t, t, T), V(t, T) \right) \tag{9}$$

where

$$
\begin{aligned}
m\left(S_t, x_t, r_t, t, T\right) = {}& \ln S_t + \theta (T - t) + (r_t - \theta)\, \Psi(\kappa, t, T) + \xi (T - t) \\
& + (x_t - \xi)\, \Psi(\alpha, t, T) - \frac{1}{2}\sigma_S^2 (T - t)
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
V(t, T) = {}& (T - t)\left[\frac{\sigma_x^2}{\alpha^2} + \sigma_S^2 + \frac{\sigma_r^2}{\kappa^2} - \frac{2\sigma_x \sigma_S}{\alpha} + \frac{2\rho\sigma_r\sigma_S}{\kappa} - \frac{2\rho\sigma_x\sigma_r}{\alpha\kappa} \right] \\
& - \frac{\sigma_x^2}{2\alpha}\Psi^2(\alpha, t, T) - \frac{\sigma_r^2}{2\kappa}\Psi^2(\kappa, t, T) \\
& + \Psi(\alpha, t, T)\left[\frac{2\sigma_x\sigma_S}{\alpha} + \frac{2\rho\sigma_x\sigma_r}{\alpha\kappa} - \frac{\sigma_x^2}{\alpha^2} \right] \\
& + \Psi(\kappa, t, T)\left[\frac{2\rho\sigma_x\sigma_r}{\alpha\kappa} - \frac{2\rho\sigma_r\sigma_S}{\kappa} - \frac{\sigma_r^2}{\kappa^2} \right] \\
& - \Psi\left(\alpha + \kappa, t, T\right) \frac{2\rho\sigma_x\sigma_r}{\alpha\kappa}.
\end{aligned}
\tag{11}
$$

It is seen that only the mean depends explicitly on the time t state variables and that neither

the mean nor the variance depend directly on the calendar date t but only on the time to maturity. This seems to be reasonable properties as changes in the distribution parameters should reflect changes in the economy, i.e. the state variables, and not the passage of time.

Turning to the equity risk premium and the short rate processes both are modeled by Ornstein-Uhlenbeck processes with constant diffusion terms and thus their explicit solutions are well known. The explicit solutions to equation (2) and (3) for $0 \leq t < T$ are given by:

$$x_T = x_t e^{-\alpha(T-t)} + \xi \left(1 - e^{-\alpha(T-t)} \right) - \sigma_x \int_t^T e^{-\alpha(T-u)} \, \mathrm{d}z_{1,u} \tag{12}$$

$$r_T = r_t e^{-\kappa(T-t)} + \theta \left(1 - e^{-\kappa(T-t)} \right) + \sigma_r \int_t^T e^{-\kappa(T-u)} \, \mathrm{d}z_{2,u}. \tag{13}$$

Both equations give rise to normally distributed future values and allow for easy determination of the conditional means:

$$\mathrm{E}_t \left[x_T \right] = x_t e^{-\alpha(T-t)} + \xi \left(1 - e^{-\alpha(T-t)} \right) \tag{14}$$

$$\mathrm{E}_t \left[r_T \right] = r_t e^{-\kappa(T-t)} + \theta \left(1 - e^{-\kappa(T-t)} \right). \tag{15}$$

The conditional variances follow using that the integrands are deterministic functions of time:

$$\mathrm{Var}_t \left[x_T \right] = \frac{\sigma_x^2}{2\alpha} \left(1 - e^{-2\alpha(T-t)} \right) \tag{16}$$

$$\mathrm{Var}_t \left[r_T \right] = \frac{\sigma_r^2}{2\kappa} \left(1 - e^{-2\kappa(T-t)} \right). \tag{17}$$

In order to establish the complete conditional distribution of the state variables needed for exact simulation we must derive the conditional covariances of the system $(\ln S_T, x_T, r_T)$. The covariances are as follows:

$$\mathrm{Cov}_t \left(\ln S_T, x_T \right) = -\sigma_x \sigma_S \Psi \left(\alpha, t, T \right) + \frac{\sigma_x^2}{\alpha} \left[\Psi \left(\alpha, t, T \right) - \Psi \left(2\alpha, t, T \right) \right] \tag{18}$$
$$- \frac{\sigma_x \sigma_r \rho}{\kappa} \left[\Psi \left(\alpha, t, T \right) - \Psi \left(\alpha + \kappa, t, T \right) \right],$$

$$\mathrm{Cov}_t \left(\ln S_T, r_T \right) = \rho \sigma_S \sigma_r \Psi \left(\kappa, t, T \right) - \frac{\rho \sigma_x \sigma_r}{\alpha} \left[\Psi \left(\kappa, t, T \right) - \Psi \left(\alpha + \kappa, t, T \right) \right] \tag{19}$$
$$+ \frac{\sigma_r^2}{\kappa} \left[\Psi \left(\kappa, t, T \right) - \Psi \left(2\kappa, t, T \right) \right],$$

and

$$\mathrm{Cov}_t \left(x_T, r_T \right) = -\rho \sigma_x \sigma_r \Psi \left(\alpha + \kappa, t, T \right). \tag{20}$$

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As linear stochastic differential systems on this form are well known to the academic literature see e.g. Björk (2004), we are able to state a complete and tractable distribution of the future state variables. It has been shown that by using the logarithmic stock price the model is a three-factor Gaussian model and as the future values of the three state variables are normally distributed the application is straightforward. With reference to earlier equation numbers the conditional distribution is given as:

$$
\left. \begin{pmatrix} \ln S_T \\ x_T \\ r_T \end{pmatrix} \right| \mathcal{F}_t \sim \mathcal{N} \left(\begin{pmatrix} (10) \\ (14) \\ (15) \end{pmatrix}, \begin{pmatrix} (11) & (18) & (19) \\ (18) & (16) & (20) \\ (19) & (20) & (17) \end{pmatrix} \right). \tag{21}
$$

The following section contains the Euler discretization often used for quick-and-dirty simulation of stochastic processes as well as our recommended exact simulation scheme.

2.3 Simulation Schemes

When applying a simulation scheme with offset in dynamics modeled by continuous-time stochastic processes, as is the case here, the need for a discretization scheme arises. This is due to the practical implementation of simulation studies being carried out in discrete time steps and thus one needs to "discretize" the continuous-time processes into a discrete-time process. Several discretization schemes exists for this purpose, see e.g. Asmussen & Glynn (2007). As stated in the outline we consider the Euler approximation scheme, which is known for its simplicity and easy implementation, and compare it to the exact simulation scheme which in general requires some model tractability.

Besides being simple and easy to implement, the Euler approximation scheme offers a lot of flexibility since it is almost universally applicable (Glasserman, 2004). The scheme has two severe drawbacks: i) it is an approximation scheme which gives rise to a discretization error, and ii) it can be computationally expensive. The latter in the sense that in order to simulate a given value at a given point in time one has to simulate intermediate values even though these are of no interest. The following subsection states the Euler approximation scheme for the three-factor model recommended by the DSA.

2.3.1 The Euler Approximation

Let the time-discretized approximation be carried out on time grid $0 = t_0 < t_1 < \cdots < t_m = T$ with fixed spacing, $h > 0$, such that $t_j = j \cdot h$ and let the simulated value of a given stochastic process, Y, be denoted by $\hat{Y}$.

The Euler approximation of the stochastic differential equations of the stock price, the equity risk premium, and the interest rate stated in equation (1), (2), and (3), will be given by: $\hat{S}(0) = S_0$, $\hat{x}(0) = x_0$, $\hat{r}(0) = r_0$, and for $j = 0, \ldots, m-1$,

$$\hat{S}\left(t_{j+1}\right) = \hat{S}\left(t_j\right) + \left(\hat{r}\left(t_j\right) + \hat{x}\left(t_j\right)\right)\hat{S}\left(t_j\right)h + \sigma_S\hat{S}\left(t_j\right)\sqrt{h}\varepsilon_{j+1} \tag{22}$$

$$\hat{x}\left(t_{j+1}\right) = \hat{x}\left(t_j\right) + \alpha\left(\xi - \hat{x}\left(t_j\right)\right)h - \sigma_x\sqrt{h}\varepsilon_{j+1} \tag{23}$$

$$\hat{r}\left(t_{j+1}\right) = \hat{r}\left(t_j\right)\kappa\left(\theta - \hat{r}\left(t_j\right)\right)h + \sigma_r\sqrt{h}\eta_{j+1}, \tag{24}$$

where $\{\varepsilon_j\}_{j=1}^m$ and $\{\eta_j\}_{j=1}^m$ are sequences of i.i.d. standard normal variables with contemporaneous correlation ρ, implemented in practice by use of Cholesky factorization. From the equations it is clear that in order to approximate the value of, say, S at t_i where $0 < t_i \leq t_m$, one needs to approximate S at $t_0, \ldots, t_{i-1}$ due to the recursive nature of the equation system.

2.3.2 Exact Simulation

When the dynamics of the model is sufficiently tractable and the distribution of the future state variables is available one can apply the so-called exact simulation. As mentioned earlier the term *exact* stems from the fact that future state variables will be distributed exactly as the dynamics of the model imply. Therefore there will be no discretization error implying potentially biased estimates and further we are able to simulate unbiased values at any point in time with no need for intermediate values. When doing exact simulation from an arbitrary point t_i and Δt forward, where $0 \leq t_i < t_i + \Delta t \leq t_m$, one should use the distribution of the future state variables in equation (21) and the scheme becomes

$$\hat{S}\left(t_i + \Delta t\right) = \exp\left\{m\left(\hat{S}\left(t_i\right), \hat{x}\left(t_i\right), \hat{r}\left(t_i\right), t_i, t_i + \Delta t\right) + \sqrt{V\left(t_i, t_i + \Delta t\right)}\varepsilon\right\} \tag{25}$$

$$\hat{x}\left(t_i + \Delta t\right) = \hat{x}\left(t_i\right)e^{-\alpha\Delta t} + \xi\left(1 - e^{-\alpha\Delta t}\right) + \sigma_x\sqrt{\frac{1}{2\alpha}\left(1 - e^{-2\alpha\Delta t}\right)}\zeta \tag{26}$$

$$\hat{r}\left(t_i + \Delta t\right) = \hat{r}\left(t_i\right)e^{-\kappa\Delta t} + \theta\left(1 - e^{-\kappa\Delta t}\right) + \sigma_r\sqrt{\frac{1}{2\kappa}\left(1 - e^{-2\kappa\Delta t}\right)}\eta, \tag{27}$$

where η, ζ, and η standard normal variables satisfying the correlation structure of (21).

The linkage between the Euler scheme and exact simulation is the first-order Taylor approximation $e^x \approx 1 + x$ at work in the Euler scheme[1]. It is well known that this approximation is best for small values of x. Ignoring our logarithmic transformation of the stock price the Euler scheme in general closely resembles the exact simulation in the case of a small h. The obvious downside of using a small h is that m has to be large which is computationally expensive.

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[1]When applying the approximation to equation (25) one should compare the result to the Euler discretization of the logarithmic stock price.

3 Numerical Results

Since the aim of the DSA-model is to ensure consistent valuation across offered contract types and L&P companies a need for identical model parameter estimates arises. Given that the proposed model becomes industry standard one of the trade associations or regulators must handle this calibration centrally. Furthermore, the calibration must correspond with the long-term industry assumptions of 2015 imposed by the Danish Insurance Association and the Danish Bankers Association. In this simulation study we will use the parameter estimates provided by DSA (2014) as our reference. They are presented in table 1.

Table 1: Reference parameters.

Parameter	Numerical Value
σ_S	0.14
α	0.1
ξ	0.0391
σ_x	0.005
κ	0.25
θ	0.0286
σ_r	0.015
ρ	0
λ	-0.25

In our analysis we follow DSA (2014) and use the long-term levels of the equity risk premium and the short rate as initial values $x_0 = \xi = 0.0391$, and $r_0 = \theta = 0.0286$. Finally, without loss of generality we set the initial stock price $S_0 = 100$.

The first part of the analysis considers the probability densities of the time $T = 1$ horizon state variables using the different simulation schemes.

3.1 Probability Densities

By simulating using the exact and the Euler scheme probability densities of time T-values of the three state variables are obtained. As it is hard to observe the differences between the simulation schemes in traditional density plots we instead study deviations from the theoretical density. Using 50,000,000 simulations figure 1 depicts three deviation series computed as the theoretical density minus the respective sample density. We consider the sample densities using the exact scheme and the Euler scheme using both $h = 1/12$ and $h = 1$.

As appears from figure 1a depicting the deviations of the simulated distributions of the horizon stock price, the exact scheme yields a density that closely resembles the theoretical. The Euler scheme with a step size of $h = 1$ produces a shifted and hence biased density but as it appears using a smaller step size of $h = 1/12$ reduces the deviation. In both Euler

cases the Kolmogorov-Smirnov test rejects the null hypothesis that there is no significant difference between the sample density and the theoretical using a 1% significance level. Naturally, the shift emerges as a result of comparing the stochastic processes in equation (1) and (7). Nevertheless this emphasizes the importance of replacing the quick-and-dirty Euler approach.

When considering the deviations of the simulated densities of the horizon short rate and equity risk premium we are without this inconsistency. The figures 1b and 1c also indicate that a small step size is required for the Euler scheme to resemble the two theoretical densities. Applying the Kolmogorov-Smirnov test allows us to reject the null hypothesis for both Euler schemes for the short rate but considering the risk premium we cannot reject the null of the $h = 1/12$ case.

The consequence of these density diversities is that investment portfolio and pension scheme valuation and risk profiling becomes dependent on the simulation scheme which is not in the spirit of DSA (2014) since it leaves room for inconsistency across the L&P industry. The following subsection compares the schemes with respect to computational costs.
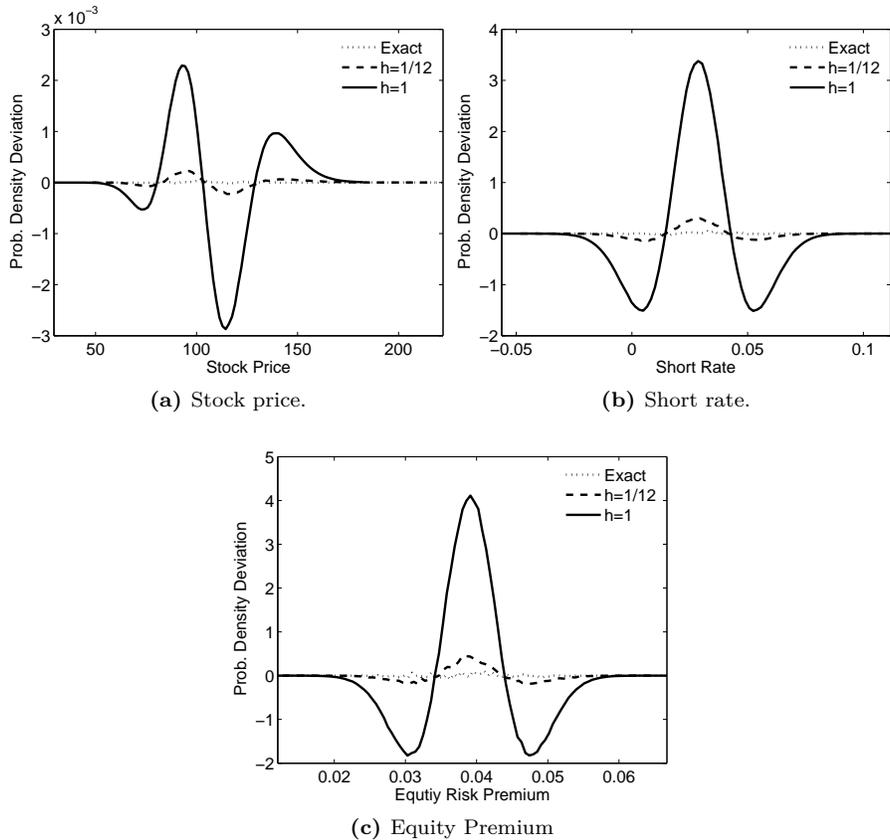

(a) Stock price.

(b) Short rate.

(c) Equity Premium

Figure 1: Simulated and theoretical densities.

3.2 Computation Costs

It is quite clear that the Euler scheme is potentially more costly to apply than the exact scheme due to dependence on intermediate values and need for simulation across the time dimension. To compare the performance of the two schemes we use the logarithmic stock price process in equation (7) and the reference parameters in table 1 and compute Monte Carlo estimates of the expected stock price at time $T = 1$, i.e. $\bar{S}_T = \frac{1}{N} \sum_{i=1}^{N} \hat{S}_i (t_m)$ for both schemes. Thereby we are able to compare solely the two sampling schemes.

In the Euler scheme we follow the optimal Duffie-Glynn rule that one should double the number of time steps and quadruple the number of simulations at the same time (Duffie & Glynn, 1995). As a measure of accuracy we follow Broadie & Kaya (2006) and use the root mean square error (RMSE) as it takes both variance and bias into account. Given in table 2 and 3 are results for both schemes using 1000 trials in the determination of the RMSE.

Table 2: Estimation of $\bar{S}_T$ using the Euler scheme.

Simulations	Steps	$\bar{S}_T$	RMSE	Bias	RVs	Time (seconds)
20,000	12	106.950	0.106	-0.022	240,000	13
80,000	24	106.971	0.052	-0.001	1,920,000	123
320,000	48	106.969	0.026	-0.003	15,360,000	1,100
1,280,000	96	106.972	0.013	-0.002	122,880,000	8,444

Table 3: Estimation of $\bar{S}_T$ using the exact scheme.

Simulations	$\bar{S}_T$	RMSE	RVs	Time (seconds)
20,000	106.967	0.105	20,000	2
80,000	106.972	0.052	80,000	11
320,000	106.973	0.026	320,000	45
1,280,000	106.972	0.013	1,280,000	168

As stated by Duffie & Glynn (1995) the RMSE is halved using the Duffie-Glynn rule and it is seen that there is a significant difference between the two schemes. First, a bias is present in the Euler scheme though it is of small magnitude.

The most prominent result lies in the computation time as the exact scheme is highly superior to the Euler scheme. The weakness of the Euler scheme is its need for simulating intermediate values such that the total number of random variables (RVs) increases heavily e.g. when comparing the case of 320,000 simulations the exact scheme is about 24 times faster. Therefore by use of the exact scheme one can cut the computational costs without loosing performance, or even better, increase performance for a fixed computational budget.

3.3   An Example of a Pension Scheme

As the purpose of the DSA-model is to estimate the value and risk profile of investment portfolios and pension contracts we conduct a risk analysis in which we take a closer look at this estimation task under both schemes.

We try to mimic a real world scenario in which an agent plans to make a monthly payment of 5,000 into an investment portfolio related to his pension scheme. As prescribed by the DSA-model we assume that the investment is allocated between stocks and bonds. Inspired by actual asset allocation behind many traditional pension contracts we allocate 35% to stocks and 65% to bonds. Further, we assume that a monthly rebalancing is made such that the bonds of the portfolio after each rebalancing has a duration of 5 years. Finally, we use 50,000,000 simulations and consider a 40 year horizon as this corresponds to a traditional pension scheme from beginning to end.

In our analysis we consider the simulated time T portfolio value distribution and the following measures: i) the Value-at-Risk (VaR), ii) the conditional Value-at-Risk[2] (cVaR), iii) what we call the *upside* defined as the 95% quantile, and iv) the *expected upside* computed as the average value in the best 5% of the outcomes. Both downside risk measures will be computed as relative measures i.e. as deviations from their time T horizon mean values (Jorion, 2007). Table 4 contains the estimated values using the reference parameters.

Table 4:   Estimated tail measures using reference parameters.

Measure	Exact	Euler ($h = 1/12$)	Difference
VaR	2,354,375	2,344,780	9,595
cVaR	2,766,315	2,756,375	9,940
Upside	10,473,906	10,441,373	32,533
Expected upside	11,566,198	11,520,938	45,261

Considering first the left tail of the simulated portfolio value distributions the exact simulation yields almost the same VaR and cVaR as the Euler scheme. Thereby there does not seem to be a large monetary difference in their estimation of downside risk when using the reference parameters. As we study the right tail density represented by the upside and the expected upside we find that the Euler scheme produces a slight underestimation of the probability mass in the right tail.

In order to evaluate the full sample distribution similarities we use the Kolmogorov-Smirnov test which allows us to reject the null of similarity on a very low significance level. Thereby we can conclude that when using the reference parameters provided by DSA (2014), the T value distributions become statistically different but in monetary terms the outcome is of relatively small magnitude[3].

---

[2]Also known as *expected shortfall* or *expected tail loss*.

[3]A total payment of 2.4m is made and average terminal value is above 7m under both schemes.

Following this conclusion it becomes relevant to investigate whether the result is robust to changing model parameter estimates. In order to do so we conduct a local sensitivity analysis in which we consider the relative differences of the risk measures, e.g. $\left(\text{VaR}^k_{\text{Exact}} - \text{VaR}^k_{\text{Euler}}\right)/\text{VaR}^k_{\text{Exact}}$ for the k'th parameter evaluation. The sensitivity analysis is local in the sense that only one parameter is varied at the time and the effect measured is therefore partial and only direct effects are taken into account while interaction effects are ignored.

The following plots show the relative differences for the risk measures for various parameter values using 1,000,000 simulations and it should be noted that the legends in figure 2a and 3a apply to all eight plots. In figure 2 we first consider the parameters of the stock price and equity premium dynamics.

(a) Stock price volatility.

(b) Mean-reversion in the equity premium.

(c) Long-term equity premium.
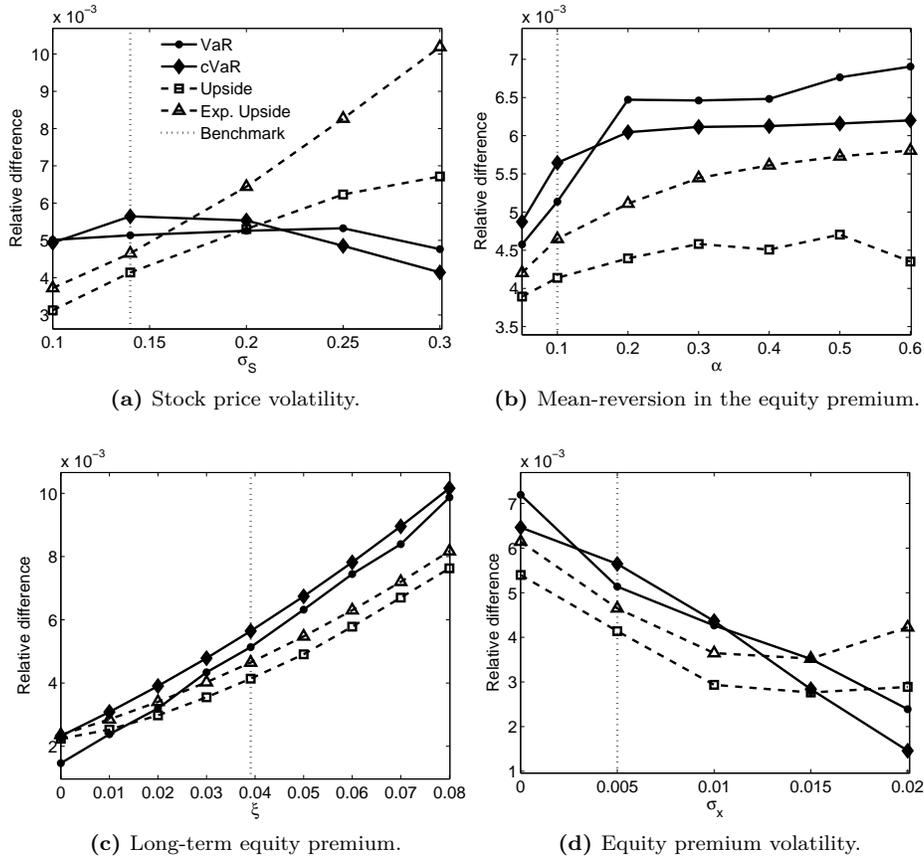
(d) Equity premium volatility.

Figure 2: Sensitivity plots for the parameters of the stock price and equity premium.

As appears from figure 2a we see that for increased values of σ_S the underestimation of upside potential becomes more severe. Further it is concerning that the VaR and cVaR deviations remains for all our analyzed parameter values. Thereby the Euler scheme seems

to consistently underestimate downside risk when varying of σ_S. Having in mind that a representative stock price volatility could easily be around 20% this could be troublesome.

For the mean-reversion parameter of the equity premium we have considered parameter values covering an interval from a half-life[4] of approximately 14 years to 1 year. From figure 2b we see that the relative differences are not very sensitive towards partial changes in α. However, the equity premium parameter values do matter as can be seen from figure 2c and 2d. First the long-term equity premium heavily affects the relative differences as they all increase in ξ. Since a strong disagreement on the magnitude of the long-run equity premium exists (Van Ewijk *et al.*, 2012), this specific finding is key to have in mind when specifying the ξ value. Finally, when considering the equity premium volatility the relative differences of both the downside and the upside measures are highest for low values of σ_x. Hence the Euler scheme underestimates downside risk and upside potential for low values of σ_x and vice versa.

The last part of the sensitivity analysis concerns the parameters of the short rate dynamics and the associated plots are depicted in figure 3.

When considering the speed of mean-reversion parameter in the short rate dynamics we see from figure 3a that the downside risk measures slowly increase in κ. Using a high speed of mean reversion therefore implies that the Euler scheme underestimates downside risk more heavily. It further appears that the misestimation of the upside potential of the investment portfolio seems insensitive towards changes in κ. Overall the Euler scheme seems to perform best for $\kappa \approx 0.1$ which corresponds to a half-life of approximately 7 years. Compared to the findings of Chan *et al.* (1992) this seems high as they estimate a half-life of around 4 years.

The second parameter we consider is the long-term short rate parameter. As appears from figure 3b we obtain a similar result to that of the long-term equity premium parameter. The tendency is clear as all relative measures increase heavily in θ.

In figure 3c we consider the sensitivity of the short rate volatility. We find that the underestimation of downside risk is most pronounced for low values of σ_r and the underestimation of the upside potential slowly increases in σ_r. In this case we somehow face a trade-off between underestimation of downside risk or upside potential.

Finally, we consider the ρ-parameter controlling the correlation between the stock price and the short rate. Empirical studies have found that the stock-bond correlation is time-varying and can in fact be negative see e.g. Ilmanen (2003). In the DSA-model this correlation is modeled through ρ and thus both negative and positive values are considered. Figure 3d depicts the result of the sensitivity analysis in which we find that the relative differences of the risk measures depends strongly on ρ. Interestingly we find that the sign of ρ affects the downside risk measures such that $\rho > 0$ implies that the Euler scheme underestimates downside risk while the opposite is found for $\rho > 0$. Considering the upside measures the degree of underestimation is present for the studied values of ρ but most pronounced for a positive ρ.

---

[4]the half-life is the expected time it takes the short rate to move half the distance towards its long-term level.

(a) Mean-reversion in the short rate.

(b) Long-term short rate.

(c) Short rate volatility.

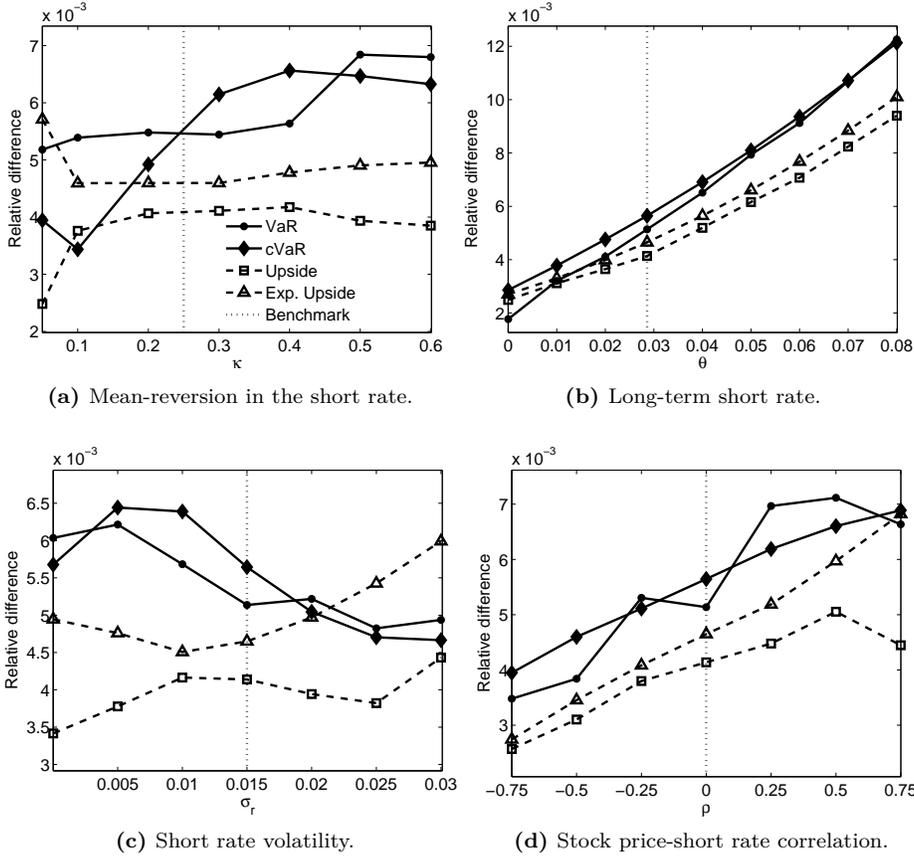**(d)** Stock price-short rate correlation.

Figure 3: Sensitivity plots for the parameters of the short rate.

Generally, we find that the Euler scheme underestimates both the downside risk and the upside potential of the investment portfolio. Furthermore, the magnitude is heavily dependent on several of the model parameters.

To sum up, the determination of the parameter values related to the stock price, equity premium, and short-rate dynamics has an impact on the relative performance of the exact simulation method and the Euler scheme. Therefore the model calibration must be handled with care or regulators must ensure model consistency in the L&P industry.

4   Concluding Remarks

This paper studies the use of simulation schemes with respect to implementing the financial model recently proposed by the Danish Society of Actuaries. Specifically, we investigate the performance of the simple Euler scheme and due to its flaws we provide the exact simulation scheme as an alternative.

Our first finding is that choice of simulation scheme matters with respect to obtaining consistent portfolio forecasts across the L&P industry. Typically, implementors would apply the straightforward Euler scheme due to its simplicity. We find that when doing this the implementor faces the trade-off between accuracy and computational costs. The Euler scheme is in fact very slow which is very inconvenient to the L&P industry.

Motivated by this we derive the conditional distribution of the future state variables which allows for exact simulation. This is feasible due to the tractability of the DSA-model and we are able to obtain closed-form solutions for the conditional means and the conditional variances of the multivariate distribution. The implementation of exact simulation of the DSA-model is thus straightforward and we provide the concrete scheme.

In our analysis we compare the Euler scheme with the exact scheme and find that for a fixed computational budget a higher accuracy is obtained when using the exact scheme. This result is due to the recursive nature of the Euler scheme which makes the number of simulations increase heavily. As one must simulate intermediate values even though they are of no interest its application becomes computationally expensive. We therefore recommend that implementors seek to avoid the recursive structure and thus choose exact simulation over the Euler scheme.

Finally, we conduct a risk analysis with offset in a real world scenario in order to study the potential biases of the Euler scheme. In the analysis we examine the well known risk measures Value-at-Risk and conditional Value-at-Risk. Our analysis indicates that using the Euler scheme over the exact simulation does not give rise to a severe misestimation of downside risk when using the reference parameters. Specifically, we observe an underestimation of downside risk when using the Euler scheme though it is of negligible magnitude in monetary terms. Since the upside potential of an investment portfolio is of high relevance to the investor we study the behavior of the right tail under both schemes. It emerges from our analysis that the Euler scheme underestimates the upside potential, i.e. the probability mass of the right tail.

In order to study the effect of changing model parameter values we conduct a local sensitivity analysis. By varying the model parameters we show that the differences between the two schemes are quite sensitive towards certain model parameters. When considering realistic parameter values our findings are qualitatively robust as the Euler scheme consistently underestimates downside risk and upside potential. Furthermore, for specific parameter settings the underestimation is magnified which is not the intention of the DSA-model.

Since the purpose of the DSA-model is to allow for proper and objective valuation and risk analysis of investment portfolios in L&P companies we stress that a potential problem lies in the forecasts' dependence on simulation scheme.

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Appendix. Derivations

Here we derive the complete conditional covariance matrix of the future values of the state variables. We start out by stating the model on matrix form $\mathrm{d}X_t = (AX_t + c)\,\mathrm{d}t + B\mathrm{d}z$:

$$
\mathrm{d}
\begin{bmatrix} \ln S_t \\ x_t \\ r_t \end{bmatrix}
=
\left(
\begin{bmatrix} 0 & 1 & 1 \\ 0 & -\alpha & 0 \\ 0 & 0 & -\kappa \end{bmatrix}
\begin{bmatrix} \ln S_t \\ x_t \\ r_t \end{bmatrix}
+
\begin{bmatrix} -\frac{1}{2}\sigma_S^2 \\ \alpha\xi \\ \kappa\theta \end{bmatrix}
\right)\mathrm{d}t
+
\begin{bmatrix} \sigma_S & 0 \\ -\sigma_x & 0 \\ \rho\sigma_r & \left(1-\rho^2\right)^{\frac{1}{2}}\sigma_r \end{bmatrix}
\begin{bmatrix} \mathrm{d}\bar{z}_{1,t} \\ \mathrm{d}\bar{z}_{2,t} \end{bmatrix}.
$$

The Cholesky factorization has been applied such that the instantaneously correlated Brownian Motions $\mathrm{d}z_{1,t}$ and $\mathrm{d}z_{2,t}$ have been replaced by $\mathrm{d}z_{1,t} = \mathrm{d}\bar{z}_{1,t}$ and $\mathrm{d}z_{2,t} = \rho\mathrm{d}\bar{z}_{1,t} + \left(1-\rho^2\right)^{\frac{1}{2}}\mathrm{d}\bar{z}_{2,t}$ where $\mathrm{d}\bar{z}_{1,t}$ and $\mathrm{d}\bar{z}_{2,t}$ are uncorrelated Brownian Motions.

Linear stochastic differential systems on this form are well known to the academic literature and can be found in e.g. Björk (2004). It can be shown that

$$
X_T = e^{A(T-t)}X_t + \int_t^T e^{A(T-u)}c\,\mathrm{d}u + \int_t^T e^{A(T-u)}B\,\mathrm{d}\bar{z}_u \tag{28}
$$

holds for all $T > t \geq 0$. As appears the future values of the three state variables are conditionally normally distributed

$$
X_T|\,\mathcal{F}_t \sim \mathcal{N}\left(\mathrm{E}_t\left[X_T\right], \mathrm{Var}_t\left[X_T\right]\right)
$$

and to provide the complete conditional distribution the following subsections contain the derivation of the conditional mean vector and covariance matrix of the future values of the state variables.

The Conditional Mean Vector

By use of the property that the increments of Brownian Motions are normally distributed with mean zero we can express the conditional mean vector as

$$
\mathrm{E}_t\left[X_T\right] = \mathrm{E}_t\left[e^{A(T-t)}X_t + \int_t^T e^{A(T-u)}c\,\mathrm{d}u\right].
$$

As appears the entire content of the expectation operator is t-measurable and hence derivation is straightforward. Though we need to determine the exponential of the matrix $A\left(T-t\right)$ which yields:

$$
e^{A(T-t)} =
\begin{bmatrix}
1 & \Psi\left(\alpha,t,T\right) & \Psi\left(\kappa,t,T\right) \\
0 & e^{-\alpha(T-t)} & 0 \\
0 & 0 & e^{-\kappa(T-t)}
\end{bmatrix}
$$

where we have introduced the auxiliary function $\Psi(a, t, T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right)$. By simple calculations the conditional mean vector becomes:

$$
E_t \begin{bmatrix} \ln S_T \\ x_T \\ r_T \end{bmatrix} = \begin{bmatrix} \ln S_t + \theta\,(T-t) + (r_t - \theta)\,\Psi(\kappa, t, T) + \xi\,(T-t) + (x_t - \xi)\,\Psi(\alpha, t, T) - \frac{1}{2}\sigma_S^2\,(T-t) \\ x_t e^{-\alpha(T-t)} + \xi\left(1 - e^{-\alpha(T-t)}\right) \\ r_t e^{-\kappa(T-t)} + \theta\left(1 - e^{-\kappa(T-t)}\right) \end{bmatrix}.
$$

The Conditional Covariance Matrix

Following Björk (2004) the variance of a stochastic integral where the integrand satisfies some regularity conditions can be computed as:

$$
\mathrm{Var}_t\left[\int_t^T e^{A(T-u)} B \, \mathrm{d}\bar{z}_u\right] = \int_t^T E_t\left[\left(e^{A(T-u)}B\right)\left(e^{A(T-u)}B\right)^\top\right]\mathrm{d}u, \qquad (29)
$$

and hence it is clear that the instantaneous covariance matrix is given by:

$$
\Sigma_u = E_t\left[\left(e^{A(T-u)}B\right)\left(e^{A(T-u)}B\right)^\top\right].
$$

Tedious calculations using matrix multiplication yields the individual entries $(\Sigma_{ij,u})_{1 \le i,j \le 3}$ of the instantaneous covariance matrix:

$$
\begin{aligned}
\Sigma_{11,u} &= \left(\sigma_S - \sigma_x \Psi(\alpha, u, T) + \rho\sigma_r \Psi(\kappa, u, T)\right)^2 + \sigma_r^2\left(1 - \rho^2\right)\Psi(\kappa, u, T)^2 \\
&= \sigma_S^2 + \sigma_x^2 \Psi(\alpha, u, T)^2 + \sigma_r^2 \Psi(\kappa, u, T)^2 \\
&\quad - 2\sigma_S\sigma_x \Psi(\alpha, u, T) + 2\rho\sigma_S\sigma_r \Psi(\kappa, u, T) \\
&\quad - 2\rho\sigma_x\sigma_r \Psi(\alpha, u, T)\Psi(\kappa, u, T),
\end{aligned}
$$

$$
\Sigma_{12,u} = \Sigma_{21,u} = -\sigma_S\sigma_x e^{-\alpha(T-u)} + \sigma_x^2 e^{-\alpha(T-u)}\Psi(\alpha, u, T) - \rho\sigma_r\sigma_x e^{-\alpha(T-u)}\Psi(\kappa, u, T),
$$

$$
\Sigma_{13,u} = \Sigma_{31,u} = \rho\sigma_S\sigma_r e^{-\kappa(T-u)} - \rho\sigma_x\sigma_r e^{-\kappa(T-u)}\Psi(\alpha, u, T) + \sigma_r^2 e^{-\kappa(T-u)}\Psi(\kappa, u, T),
$$

$$
\Sigma_{23,u}\Sigma_{32,u} = -\rho\sigma_x\sigma_r e^{-(\alpha+\kappa)(T-u)}
$$

$$
\Sigma_{22,u} = \sigma_x^2 e^{-2\alpha(T-u)},
$$

and

$$
\Sigma_{33,u} = \sigma_r^2 e^{-2\kappa(T-u)}.
$$

Covariances

The covariances conditional on time t information are derived by applying equation (29) element wise. Starting with the covariance between future values of the logarithmic stock price and the equity risk premium

$$
\begin{aligned}
\mathrm{Cov}_t \left(\ln S_T, x_T \right) &= \int_t^T \mathrm{E}_t \left[\Sigma_{12,u} \right] \mathrm{d}u \\
&= -\sigma_x \sigma_S \int_t^T e^{-\alpha(T-u)} \, \mathrm{d}u + \sigma_x^2 \int_t^T e^{-\alpha(T-u)} \Psi(\alpha, u, T) \, \mathrm{d}u \\
&\quad - \sigma_x \sigma_r \int_t^T e^{-\alpha(T-u)} \Psi(\kappa, u, T) \rho \, \mathrm{d}u \\
&= -\sigma_x \sigma_S \Psi \left(\alpha, t, T \right) + \frac{\sigma_x^2}{\alpha} \left[\Psi \left(\alpha, t, T \right) - \Psi \left(2\alpha, t, T \right) \right] \\
&\quad - \frac{\sigma_x \sigma_r \rho}{\kappa} \left[\Psi \left(\alpha, t, T \right) - \Psi \left(\alpha + \kappa, t, T \right) \right].
\end{aligned}
$$

Secondly, the conditional covariance between future values of the logarithmic stock price and the interest rate is:

$$
\begin{aligned}
\mathrm{Cov}_t \left(\ln S_T, r_T \right) &= \int_t^T \mathrm{E}_t \left[\Sigma_{13,u} \right] \mathrm{d}u \\
&= \rho \sigma_S \sigma_r \int_t^T e^{-\kappa(T-u)} \, \mathrm{d}u - \rho \sigma_x \sigma_r \int_t^T e^{-\kappa(T-u)} \Psi(\alpha, u, T) \, \mathrm{d}u \\
&\quad + \sigma_r^2 \int_t^T e^{-\kappa(T-u)} \Psi(\kappa, u, T) \, \mathrm{d}u \\
&= \rho \sigma_S \sigma_r \Psi \left(\kappa, t, T \right) - \frac{\rho \sigma_x \sigma_r}{\alpha} \left[\Psi \left(\kappa, t, T \right) - \Psi \left(\alpha + \kappa, t, T \right) \right] \\
&\quad + \frac{\sigma_r^2}{\kappa} \left[\Psi \left(\kappa, t, T \right) - \Psi \left(2\kappa, t, T \right) \right].
\end{aligned}
$$

Finally, the conditional covariance between the equity risk premium and the short rate is given by:

$$
\begin{aligned}
\mathrm{Cov}_t \left(x_T, r_T \right) &= \int_t^T \mathrm{E}_t \left[\Sigma_{23,u} \right] \mathrm{d}u \\
&= \text{-}\rho \sigma_x \sigma_r \int_t^T e^{-(\alpha+\kappa)(T-u)} \, \mathrm{d}u \\
&= -\rho \sigma_x \sigma_r \Psi \left(\alpha + \kappa, t, T \right).
\end{aligned}
$$

After stating the covariances we turn to the variances.

Variances

The variances of future values of the state variables follows by considering the diagonal elements of Σ_u and the conditional variance of future values of the logarithmic stock price follows here:

$$V(t,T) \equiv \mathrm{Var}_t(\ln S_T) = \int_t^T \mathrm{E}_t[\Sigma_{13,u}] \, \mathrm{d}u$$

$$= \sigma_S^2(T-t) + \sigma_x^2 \int_t^T \Psi(\alpha, u, T)^2 \, \mathrm{d}u + \sigma_r^2 \int_t^T \Psi(\kappa, u, T)^2 \, \mathrm{d}u$$

$$- 2\sigma_S\sigma_x \int_t^T \Psi(\alpha, u, T) \, \mathrm{d}u + 2\rho\sigma_S\sigma_r \int_t^T \Psi(\kappa, u, T) \, \mathrm{d}u$$

$$- 2\rho\sigma_x\sigma_r \int_t^T \Psi(\alpha, u, T)\Psi(\kappa, u, T)\mathrm{d}u.$$

By computing the integrals the expression can be written as:

$$V(t,T) = \sigma_S^2(T-t) + \sigma_x^2\left[\frac{T-t}{\alpha^2} - \frac{1}{2\alpha}\Psi^2(\alpha,t,T) - \frac{1}{\alpha^2}\Psi(\alpha,t,T)\right]$$

$$+ \sigma_r^2\left[\frac{T-t}{\kappa^2} - \frac{1}{2\kappa}\Psi^2(\kappa,t,T) - \frac{1}{\kappa^2}\Psi(\kappa,t,T)\right]$$

$$- 2\sigma_S\sigma_x\left[\frac{T-t}{\alpha} - \frac{1}{\alpha}\Psi(\alpha,t,T)\right] + 2\rho\sigma_S\sigma_r\left[\frac{T-t}{\kappa} - \frac{1}{\kappa}\Psi(\kappa,t,T)\right]$$

$$- \frac{2\rho\sigma_x\sigma_r}{\alpha\kappa}\left[(T-t) - \Psi(\kappa,t,T) - \Psi(\alpha,t,T) + \Psi(\alpha+\kappa,t,T)\right].$$

Finally, this can be rearranged nicely such that:

$$V(t,T) = (T-t)\left[\frac{\sigma_x^2}{\alpha^2} + \sigma_S^2 + \frac{\sigma_r^2}{\kappa^2} - \frac{2\sigma_x\sigma_S}{\alpha} + \frac{2\rho\sigma_r\sigma_S}{\kappa} - \frac{2\rho\sigma_x\sigma_r}{\alpha\kappa}\right]$$

$$- \frac{\sigma_x^2}{2\alpha}\Psi^2(\alpha,t,T) - \frac{\sigma_r^2}{2\kappa}\Psi^2(\kappa,t,T)$$

$$+ \Psi(\alpha,t,T)\left[\frac{2\sigma_x\sigma_S}{\alpha} + \frac{2\rho\sigma_x\sigma_r}{\alpha\kappa} - \frac{\sigma_x^2}{\alpha^2}\right]$$

$$+ \Psi(\kappa,t,T)\left[\frac{2\rho\sigma_x\sigma_r}{\alpha\kappa} - \frac{2\rho\sigma_r\sigma_S}{\kappa} - \frac{\sigma_r^2}{\kappa^2}\right]$$

$$- \Psi(\alpha+\kappa,t,T)\frac{2\rho\sigma_x\sigma_r}{\alpha\kappa}.$$

Since both the equity risk premium and the short rate follows Ornstein-Uhlenbeck processes their variances are quite familiar:

$$\mathrm{Var}_t\left[x_T\right] = \int_t^T \mathrm{E}_t\left[\Sigma_{22,u}\right]\mathrm{d}u$$

$$= \frac{\sigma_x^2}{2\alpha}\left(1 - e^{-2\alpha(T-t)}\right)$$

$$\mathrm{Var}_t\left[r_T\right] = \int_t^T \mathrm{E}_t\left[\Sigma_{33,u}\right]\mathrm{d}u$$

$$= \frac{\sigma_r^2}{2\kappa}\left(1 - e^{-2\kappa(T-t)}\right).$$

Expressions for the full covariance matrix of the future values of the state variables have now been provided.