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A note on optimal (s,S) and (R,nQ) policies under a stuttering Poisson demand process

Christian Larsen

Aarhus University, Department of Economics and Business, CORAL,
Fuglesangs Allé 4, DK-8210 Aarhus V, Denmark

Abstract

In this note, a new efficient algorithm is proposed to find an optimal (s,S) replenishment policy for inventory systems with continuous reviews and where the demand follows a stuttering Poisson process (the compound element is geometrically distributed). We also derive three upper bounds for the relative increase in cost if one uses the best (R,nQ) policy instead of the optimal (s,S) policy. One of these upper bounds (the most loose of those) can be expressed as the fraction of the variance-to-mean ratio of the geometric distribution and the economic order quantity. We explore numerically when these upper bounds are tight.

Keywords: Inventory control policies, stuttering Poisson process, upper bounds.

1. Introduction

We consider a continuous review inventory system where any unfilled demand is backordered. All replenishment orders issued from the system have a constant lead-time L . We assume there is a fixed cost K associated with any replenishment. This is denoted an FCO (Fixed Cost per Order) cost convention, in contrast to a Fixed Cost per Batch (FCB) cost convention where a fixed cost is charged for each batch of size Q , see Gallego (1998). Inventory on-hand is charged a unit cost h per time unit and all backlogged units are charged a unit cost b_l per time unit. We assume both cost components h and b_l to be positive, while the cost component K is non-negative. Under this cost structure and assuming that the demand process is a compound Poisson process, the optimal policy is of type (s,S) , Zipkin (2000, p. 411). The (s,S) policy specifies that each time the inventory position is at or below the re-order level s then a replenishment order is issued that brings the inventory position at level S . The inventory position is the net-inventory plus any outstanding replenishment orders not yet received, and the net-inventory is the on-hand inventory minus the amount on the backlog list. The algorithm of Zheng and Federgruen (1991), which improves previous works by Veinott and Wagner (1965) and Archibald and Silver (1978), can be used to find an optimal (s,S) policy. Another policy is the (R,nQ) policy. It specifies that each time the inventory position is at or below the re-order point R then the least possible replenishment order, in multiples of Q , is made that brings the inventory position above level R . In order to find the best (R,nQ) policy one has to use the algorithm of Federgruen and Zheng (1992). The algorithm is though designed for the FCB cost convention. In Zheng and Chen (1992) the algorithm is extended to the case of an FCO cost convention.

We assume in this note that the demand process is a stuttering Poisson process. This means that customers arrive according to a Poisson process with intensity λ arrivals per time unit and the demand of a customer is specified by a geometric random variable J with probability distribution:

$$P(J = j) = (1-\beta)\beta^{j-1}, \quad j=1,2,\dots \quad (1)$$

where $0 < \beta < 1$. Note that J has a cumulative distribution function

$$P(J \leq j) = 1 - \beta^j \quad j=1,2,\dots \quad (2)$$

For some empirical evidence of the relevance of using a geometric distribution to describe customer orders, see Johnston et al (2003).

In Section 2 we derive that the probability distribution of the inventory position is “almost uniform”. We explain how this finding enables us to use a variant of the algorithm of Federgruen and Zheng (1992) to compute the optimal (s,S) policy. In Section 3 we show that if we take the optimal (s,S) policy and implement it as an (R,nQ) policy then it is possible to derive three upper bounds on how this (R,nQ) policy performs compared to the optimal (s,S) policy. In Section 4 we do a numerical study to explore how tight these bounds are.

2. Analysis of the (s,S) policy

We first specify the probability distribution for the inventory position, IP , of the inventory system when it is governed by an (s,S) policy. From Axsäter (2006, p. 108) it holds for any compound Poisson demand process that

$$P(IP = S - j) = \frac{m(j)}{\sum_{k=0}^{S-s-1} m(k)}, \quad j=0,\dots,S-s-1 \quad (3)$$

where

$$m(0) = 1 \quad (4)$$

and

$$m(j) = \sum_{k=0}^{j-1} m(k)P(J = j - k) \quad j=1,\dots,S-s-1. \quad (5)$$

For the special case of a stuttering Poisson demand process Archibald (1976, p. 50) derived the following result:

Lemma 1

When J is specified by (1), it holds that

$$m(j) = \begin{cases} 1 & j = 0 \\ 1 - \beta & j = 1, \dots, S - s - 1 \end{cases} \quad (6)$$

Proof

The proof is based on induction. From (4) and (5) it holds that $m(1) = P(J=1) = 1 - \beta$. Assume now we have shown that $m(j) = 1 - \beta$ for $j = 1, \dots, j'$. This means

$$m(j' + 1) = (1 - \beta)P(J \leq j') + P(J = j' + 1) = (1 - \beta)(1 - \beta^{j'} + \beta^{j'}) = 1 - \beta$$

□

We include the proof here in order to make our paper self-contained. From Lemma 1 we get that the probability distribution of the inventory position in the (s,S) inventory system is “almost” uniform.

This is interesting as uniformity is a characteristic which is normally attributed to the (R,nQ) inventory system, see Proposition 5.1 in Axsäter (2006).

According to Axsäter (2006, p. 132), the average cost per time unit for the continuous review (s,S) inventory system is given by

$$C(s, S) = \frac{K\lambda + \sum_{j=0}^{S-s-1} m(j)g(S-j)}{\sum_{j=0}^{S-s-1} m(j)} \quad (7)$$

The function $g(k)$ is the expected inventory and backorder costs if the inventory position contains k units and is according to Axsäter (2006, p. 130) given as

$$g(k) = -b_1 \left(k - \frac{\lambda L}{1-\beta} \right) + (h + b_1) \sum_{j=1}^k jP(D(L) = k - j), \quad k=S-s+1, \dots, S \quad (8)$$

$D(L)$ is the random variable for the total demand during the lead-time L . From Section 5.9.1 in Axsäter (2006) it follows that $g(k)$ is a convex function in k because the function $\Delta g(k) = g(k+1) - g(k)$ is increasing in k . Define $k^* = \operatorname{argmin}_k \{g(k)\}$.

Before we continue, we rewrite R as s and Q as $S - s$ to see the similarity with Federgruen and Zheng (1992), such that (7), using Lemma 1, can be rewritten as

$$C(R, Q) = \frac{K\lambda + (1-\beta) \sum_{j=R+1}^{R+Q-1} g(j) + g(R+Q)}{Q(1-\beta) + \beta} \quad (9)$$

Furthermore, let $R(Q) = \operatorname{argmin}_R \{C(R, Q)\}$ denote the re-order level that minimizes the average cost for a given Q and $C(Q) = C(R(Q), Q)$ as the corresponding costs. As in Archibald (1976) (combine relation (31) on p. 32 to relation (62) on p. 50), we can prove the following lemma.

Lemma 2

It holds that $0 \leq R(Q) - R(Q+1) \leq 1$.

Proof

Define the function $T(R, Q) = (1 - \beta)g(R) + \beta g(R+Q-1) - g(R+Q)$. Because $\Delta g(k)$ is an increasing function and $0 < \beta < 1$, we get that $T(R, Q)$ is decreasing in R for any value of Q . Therefore we get from (9) that

$$R(Q) = \operatorname{Max}\{R: T(R, Q) \geq 0\}. \quad (10)$$

Using the arguments as above it also follows that $T(R, Q)$ is decreasing in Q for any value of R . Therefore $T(R, Q+1) < T(R, Q) < 0$ for any value of $R > R(Q)$. This implies $R(Q+1) \leq R(Q)$. From (9) it follows that for any value of Q , $R(Q) \leq k^*$. Therefore we get $g(R(Q)-1) > g(R(Q))$ and we can conclude $T(R(Q)-1, Q+1) > T(R(Q), Q) \geq 0$ which implies $R(Q+1) \geq R(Q) - 1$.

□

The algorithm of Federgruen and Zheng (1992) concerns the minimization of a cost function of the form $\frac{A + \sum_{j=R+1}^{R+Q} g(j)}{Q}$, where A is some non-negative constant. The algorithm starts with $Q = 1$ and gradually increments Q by one until the optimal solution is reached. We modify this algorithm for the cost structure in (9). We start with $Q = 1$, $R(1) = k^* - 1$ and $C(1) = K\lambda + g(k^*)$ and we need to decide in every iteration whether to increment Q by one unit or to stop the search procedure. When Q is incremented by one, we either decrease the re-order level by one unit or let it be unchanged due to Lemma 2.

If the re-order point is decreased by one, then

$$\begin{aligned} C(R(Q) - 1, Q + 1) &= \frac{K\lambda + (1-\beta) \sum_{j=R(Q)+1}^{R(Q)+Q-1} g(j) + g(R(Q)+Q) + (1-\beta)g(R(Q))}{(Q+1)(1-\beta) + \beta} \\ &= \frac{C(Q)(Q(1-\beta) + \beta) + (1-\beta)g(R(Q))}{(Q+1)(1-\beta) + \beta} \end{aligned} \quad (11)$$

This expression is less than $C(Q)$ if $g(R(Q)) < C(Q)$.

If the re-order point is kept unchanged, then

$$\begin{aligned} C(R(Q), Q+1) &= \frac{K\lambda + (1-\beta) \sum_{j=R(Q)+1}^{R(Q)+Q-1} g(j) + g(R(Q)+Q) - \beta g(R(Q)+Q) + g(R(Q)+Q+1)}{(Q+1)(1-\beta) + \beta} \\ &= \frac{C(Q)(Q(1-\beta) + \beta) - \beta g(R(Q)+Q) + g(R(Q)+Q+1)}{(Q+1)(1-\beta) + \beta} \end{aligned} \quad (12)$$

This expression is less than $C(Q)$ if $g(R(Q)+Q+1) - \beta g(R(Q)+Q) < C(Q)(1-\beta)$. By comparing (11) and (12), we also see that $C(R(Q)-1, Q+1) < C(R(Q), Q+1)$ if and only if $(1-\beta)g(R(Q)) < g(R(Q)+Q+1) - \beta g(R(Q)+Q)$.

In summary: For a given Q , $R(Q)$ and $C(Q)$, compute $\min\{(1-\beta)g(R(Q)), g(R(Q)+Q+1) - \beta g(R(Q)+Q)\}$. If this minimum value is larger than $(1-\beta)C(Q)$ then stop the search procedure, because the optimal Q has been found. Otherwise, if the first argument in the minimization expression is the least, let $R(Q+1) = R(Q)-1$ and else let $R(Q+1) = R(Q)$. Increment Q by 1 and repeat the procedure. When the algorithm has terminated with a pair $(\bar{R}, \bar{Q})$, the optimal (s, S) policy has values s^* and S^* which are $s^* = \bar{R}$ and $S^* = \bar{R} + \bar{Q}$.

3. Relationships to the (R, nQ) policy

Assume we have computed the optimal (s, S) policy by the procedure developed in the previous section but implement it as an (R, nQ) policy by using the computed $\bar{R}$ and $\bar{Q}$. We refer to this policy the adapted (R, nQ) policy. From Zheng and Chen (1992) and Gallego (1998) the average cost per time unit for this policy is:

$$C_{(R,nQ)}(\bar{R}, \bar{Q}) = \frac{K\lambda \sum_{j=1}^{\bar{Q}} \beta^{j-1} + \sum_{j=\bar{R}+1}^{\bar{R}+\bar{Q}} g(j)}{\bar{Q}} = \frac{K\mu(1-\beta\bar{Q}) + \sum_{j=\bar{R}+1}^{\bar{R}+\bar{Q}} g(j)}{\bar{Q}} \quad (13)$$

Here $\mu = \lambda/(1 - \beta)$ which is the expected demand per time unit. Note that this expression is valid when there is a fixed cost per order rather than a fixed cost per batch. The latter convention does not allow for a comparison to the (s, S) policy, since there is only a one-time fixed order cost in such a policy. Next we compare the average cost of (13) to the average cost of the optimal (s, S) policy.

$$\begin{aligned} C_{(s,S)}(s^*, S^*) &= \frac{K\mu + \sum_{j=s^*}^{S^*-1} g(j) + \frac{1}{1-\beta}g(S^*)}{S^* - s^* + \frac{\beta}{1-\beta}} \\ &= \frac{\bar{Q}C_{(R,nQ)}(\bar{R}, \bar{Q}) + \frac{\beta}{1-\beta}g(\bar{R} + \bar{Q}) + K\mu\beta\bar{Q}}{\bar{Q} + \frac{\beta}{1-\beta}} \end{aligned} \quad (14)$$

As a result, the relative increase in average costs for using the adapted (R, nQ) policy rather than the optimal (s, S) policy equals

$$\frac{C_{(R,nQ)}(\bar{R}, \bar{Q}) - C_{(s,S)}(s^*, S^*)}{C_{(s,S)}(s^*, S^*)} = \frac{\frac{\beta}{1-\beta}(C_{(R,nQ)}(\bar{R}, \bar{Q}) - g(\bar{R} + \bar{Q})) - K\mu\beta\bar{Q}}{\bar{Q}C_{(R,nQ)}(\bar{R}, \bar{Q}) + \frac{\beta}{1-\beta}g(\bar{R} + \bar{Q}) + K\mu\beta\bar{Q}} \quad (15)$$

The following bounds can be derived for using (R, nQ) policies:

$$\begin{aligned} \frac{C_{(R,nQ)}(R^*, Q^*) - C_{(s,S)}(s^*, S^*)}{C_{(s,S)}(s^*, S^*)} &\leq \frac{C_{(R,nQ)}(\bar{R}, \bar{Q}) - C_{(s,S)}(s^*, S^*)}{C_{(s,S)}(s^*, S^*)} \\ &\leq \frac{\beta}{(1-\beta)\bar{Q}} = \frac{V[J]}{E[J]} \frac{1}{\bar{Q}} \leq \frac{V[J]}{E[J]} \sqrt{\frac{h}{2\mu K}} \end{aligned} \quad (16)$$

where $E[J] = 1/(1 - \beta)$ and $V[J] = \beta/(1 - \beta)^2$ represent the mean and the variance of the random variable J , respectively. The first inequality is trivial, since the best (R, nQ) policy (denoted (R^*, Q^*)) always perform better than a particular (R, nQ) policy (in this case $(\bar{R}, \bar{Q})$). The second inequality comes from examining right-hand side of (15). Because here, the term $\frac{\beta}{1-\beta}g(\bar{R} + \bar{Q}) + K\mu\beta\bar{Q}$ is subtracted in the numerator while it is added in the denominator. Therefore the this right-hand side is dominated by $\frac{\beta}{(1-\beta)\bar{Q}}$. The third inequality stems from a general finding that the optimal order size in any stochastic inventory system is larger than the EOQ in order to cope with random effects such that $\bar{Q} = S^* - s^* \geq EOQ = \sqrt{2\mu K/h}$. This has not be proven mathematically anywhere (except for a continuous approximation, albeit using the FCB cost convention, in Zheng (1992)), but for justification of the argument, see for instance the discussion at p. 186 in Zheng and Chen (1992) (actually they relate their empirical findings about the optimal Q

to the *EOQ* with backorders which is larger than *EOQ* thus the bound stated in our note is relaxed compared to this reference).

4. Numerical experiments

Throughout this section we set $h = 1$ and $L = 1$. We vary β to the following values: $0.1, 0.3, 0.5, 0.7, 0.9$ and 0.95 and the mean lead time demand (denoted by $E[D(L)] = \lambda L / (1 - \beta)$) is set to $5, 10, 15, 20, 25$ and 30 such that λ can be computed. To set the cost parameter b_1 , we use the expression from inventory systems with base stock policies to measure service as $FR = b_1 / (b_1 + h)$ where FR attains the following values $0.9, 0.95$ and 0.98 . Furthermore the ratio between the expected lead-time demand and the economic order quantity is an indicator of the order frequency (denoted by γ). Let γ have the values $0.5, 2$ and 4 such that $K = \gamma^2 E[D(L)] / 2$.

For each combination $(\beta, FR, E[D(L)], \gamma)$ we report the three upper bounds of (16) in Tables 1 – 3. All the three bounds are stated in increasing order (as in (16)) in each entry.

<Tables 1 – 3 about here>

In general the first bound in (16) is very tight. It reveals that the penalty of using the (R, nQ) policy, often considered more practical, instead of the (s, S) policy is small (less than 5% in all entries). When all the bounds are at or below 5% of the cost associated with the optimal (s, S) policy, the cell is marked in bold. From this we see that the bound $\frac{V[J]}{E[J]} \sqrt{\frac{h}{2\mu K}}$ is only of practical relevance if the parameter β is moderate or low in combination with a high frequent demand (control variable $E[D(L)]$), and some degree of economics of scale (the control parameter γ). However, if we had increased these two control parameters further, we would eventually have seen instances with a high value of β , say, $\beta = 0.9$ or even $\beta = 0.95$, where all bounds are at or below 5%. It also of interest to note that the tightness of the bounds seem unaffected by the level of the control variable FR .

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		360%		180%		120%		90.0%		72.0%		60.0%
	0.9 8	0.202% 225% 360%	0.9 8	0.411% 150% 180%	0.9 8	0.783% 90.0% 120%	0.9 8	0.837% 64.3% 90.0%	0.9 8	1.09% 52.9% 72.0%	0.9 8	1.03% 42.9% 60.0%
0.95	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds
	0.9	0.125% 633% 760%	0.9	0.371% 317% 380%	0.9	0.515% 190% 253%	0.9	0.831% 146% 190%	0.9	0.996% 119% 152%	0.9	1.14% 95.0% 127%
	0.9 5	0.0835% 475% 760%	0.9 5	0.279% 317% 380%	0.9 5	0.396% 211% 253%	0.9 5	0.635% 146% 190%	0.9 5	0.806% 119% 152%	0.9 5	0.944% 95.0% 127%
	0.9 8	0.0310% 633% 760%	0.9 8	0.134% 317% 380%	0.9 8	0.301% 211% 253%	0.9 8	0.436% 146% 190%	0.9 8	0.622% 119% 152%	0.9 8	0.719% 100% 127%

Table 1: $EOQ = 0.5E[D(L)]$.

$\beta \backslash E[D(L)]$	5		10		15		20		25		30	
0.1	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds
	0.9	0.0426 % 0.926% 1.11%	0.9	0.00678 % 0.483% 0.556%	0.9	0.0102 % 0.337% 0.370%	0.9	0.00325 % 0.252% 0.278%	0.9	0.00435 % 0.202% 0.222%	0.9	0.00170 % 0.168% 0.185%
	0.9 5	0.0568 % 0.926% 1.11%	0.9 5	0.0227 % 0.505% 0.560%	0.9 5	0.00147 % 0.337% 0.370%	0.9 5	0.000693 % 0.253% 0.278%	0.9 5	0.00426 % 0.210% 0.222%	0.9 5	0.00264 % 0.174% 0.185%
	0.9 8	0.00971 % 0.926% 1.11%	0.9 8	0.0208 % 0.505% 0.556%	0.9 8	0.00571 % 0.347% 0.370%	0.9 8	0.00567 % 0.265% 0.278%	0.9 8	0.00108 % 0.206% 0.222%	0.9 8	0.00140 % 0.174% 0.185%
	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds
	0.9	0.116% 3.30% 4.29%	0.9	0.111% 1.86% 2.14%	0.9	0.0408 % 1.26% 1.43%	0.9	0.0154% 0.952% 1.07%	0.9	0.0194 % 0.779% 0.857%	0.9	0.00906 % 0.649% 0.714%
	0.9 5	0.246% 3.57% 4.29%	0.9 5	0.0370 % 1.86% 2.14%	0.9 5	0.0263 % 1.26% 1.43%	0.9 5	0.0175% 0.974% 1.07%	0.9 5	0.0120 % 0.779% 0.857%	0.9 5	0.00827 % 0.659% 0.714%
	0.9 8	0.192% 3.57% 4.29%	0.9 8	0.0726 % 1.95% 2.14%	0.9 8	0.0260 % 1.30% 1.43%	0.9 8	0.0274% 1.00% 1.07%	0.9 8	0.0106 % 0.794% 0.857%	0.9 8	0.0113 % 0.670% 0.714%
0.3	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds
	0.9	0.982% 8.33% 10.0%	0.9	0.172% 4.17% 5.00%	0.9	0.162% 2.94% 3.33%	0.9	0.0821% 2.22% 2.50%	0.9	0.0424 % 1.75% 2.00%	0.9	0.0456 % 1.49% 1.67%
	0.9 5	0.924% 8.33% 10.0%	0.9 5	0.291% 4.35% 5.00%	0.9 5	0.0995 % 2.94% 3.33%	0.9 5	0.0721% 2.22% 2.50%	0.9 5	0.0537 % 1.82% 2.00%	0.9 5	0.0406 % 1.52% 1.67%
	0.9 8	0.639% 8.33% 10.0%	0.9 8	0.169% 4.35% 5.00%	0.9 8	0.0904 % 2.94% 3.33%	0.9 8	0.0479% 2.27% 2.50%	0.9 8	0.0521 % 1.85% 2.00%	0.9 8	0.0281 % 1.54% 1.67%
	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds
	0.9	1.66% 17.9% 23.3%	0.9	1.07% 9.72% 11.7%	0.9	0.432% 6.48% 7.78%	0.9	0.267% 4.96% 5.83%	0.9	0.172% 4.02% 4.67%	0.9	0.166% 3.43% 3.89%
	0.9 5	1.62% 17.9% 23.3%	0.9 5	0.695% 9.72% 11.7%	0.9 5	0.373% 6.67% 7.78%	0.9 5	0.293% 5.07% 5.83%	0.9 5	0.164% 4.09% 4.67%	0.9 5	0.137% 3.48% 3.89%
	0.9 8	1.50% 19.4% 23.3%	0.9 8	0.630% 9.72% 11.7%	0.9 8	0.461% 6.86% 7.78%	0.9 8	0.229% 5.19% 5.83%	0.9 8	0.162% 4.17% 4.67%	0.9 8	0.117% 3.53% 3.89%
0.5	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds
	0.9	2.56% 75.0% 90.0%	0.9	3.20% 37.5% 45.0%	0.9	2.76% 25.0% 30.0%	0.9	2.01% 18.4% 22.5%	0.9	1.56% 14.8% 18.0%	0.9	1.19% 12.3% 15.0%
	0.9 5	2.16% 75.0%	0.9 5	2.49% 37.5%	0.9 5	2.26% 25.0%	0.9 5	1.69% 18.8%	0.9 5	1.48% 15.3%	0.9 5	1.11% 12.7%

		90.0%		45.0%		30.0%		22.5%		18.0%		15.0%
	0.9	1.59%	0.9	2.13%	0.9	1.80%	0.9	1.57%	0.9	1.24%	0.9	0.925%
	8	75.0%	8	39.1%	8	25.0%	8	19.1%	8	15.5%	8	12.9%
		90.0%		45.0%		30.0%		22.5%		18.0%		15.0%
0.95	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds
	0.9	1.49%	0.9	2.62%	0.9	3.13%	0.9	3.15%	0.9	3.06%	0.9	2.77%
		173%		82.6%		52.8%		39.6%		31.7%		26.0%
		190%		95.0%		63.3%		47.5%		38.0%		31.7%
	0.9	1.00%	0.9	2.01%	0.9	2.57%	0.9	2.52%	0.9	2.52%	0.9	2.33%
	5	158%	5	82.6%	5	54.3%	5	39.6%	5	31.7%	5	26.4%
		190%		95.0%		63.3%		47.5%		38.0%		31.7%
	0.9	0.649%	0.9	1.58%	0.9	2.01%	0.9	2.09%	0.9	2.02%	0.9	1.93%
	8	173%	8	82.6%	8	54.3%	8	40.4%	8	32.2%	8	26.8%
		190%		95.0%		63.3%		47.5%		38.0%		31.2%

Table 2: $EOQ = 2E[D(L)]$.

$\beta \backslash E[D(L)]$	5		10		15		20		25		30	
0.1	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds
	0.9	0.00698% 0.483% 0.556%	0.9	0.00206% 0.253% 0.278%	0.9	0.000938% 0.171% 0.185%	0.9	0.000516% 0.129% 0.139%	0.9	0.000315% 0.104% 0.111%	0.9	0.000206% 0.0868% 0.0926%
	0.9 5	0.0147% 0.505% 0.556%	0.9 5	0.00244% 0.258% 0.278%	0.9 5	0.00298% 0.176% 0.185%	0.9 5	0.00102% 0.132% 0.139%	0.9 5	0.000192% 0.106% 0.111%	0.9 5	0.000539% 0.0889% 0.0926%
	0.9 8	0.00347% 0.505% 0.556%	0.9 8	0.00366% 0.265% 0.278%	0.9 8	0.00238% 0.179% 0.185%	0.9 8	0.00148% 0.134% 0.139%	0.9 8	0.00101% 0.108% 0.111%	0.9 8	0.000706% 0.0896% 0.0926%
	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds
	0.9	0.0842% 1.95% 2.14%	0.9	0.0243% 0.974% 1.07%	0.9	0.0114% 0.659% 0.714%	0.9	0.00654% 0.498% 0.536%	0.9	0.00420% 0.401% 0.429%	0.9	0.00290% 0.335% 0.357%
	0.9 5	0.0522% 1.95% 2.14%	0.9 5	0.0115% 0.997% 1.07%	0.9 5	0.0129% 0.670% 0.714%	0.9 5	0.00515% 0.504% 0.536%	0.9 5	0.00174% 0.404% 0.429%	0.9 5	0.00284% 0.340% 0.357%
	0.9 8	0.0463% 1.95% 2.14%	0.9 8	0.0278% 1.02% 1.07%	0.9 8	0.00633% 0.680% 0.714%	0.9 8	0.00490% 0.510% 0.536%	0.9 8	0.00374% 0.412% 0.429%	0.9 8	0.00292% 0.346% 0.357%
0.5	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds
	0.9	0.402% 4.55% 5.00%	0.9	0.0641% 2.27% 2.50%	0.9	0.0311% 1.52% 1.67%	0.9	0.0184% 1.15% 1.25%	0.9	0.0121% 0.926% 1.00%	0.9	0.00852% 0.775% 0.833%
	0.9 5	0.215% 4.35% 5.00%	0.9 5	0.0649% 2.27% 2.50%	0.9 5	0.0265% 1.54% 1.67%	0.9 5	0.0258% 1.18% 1.25%	0.9 5	0.0139% 0.943% 1.00%	0.9 5	0.00746% 0.787% 0.833%
	0.9 8	0.324% 4.55% 5.00%	0.9 8	0.0917% 2.33% 2.50%	0.9 8	0.0335% 1.56% 1.67%	0.9 8	0.0251% 1.19% 1.25%	0.9 8	0.0102% 0.952% 1.00%	0.9 8	0.00874% 0.800% 0.833%
	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds
	0.9	1.25% 10.1% 11.7%	0.9	0.313% 5.19% 5.83%	0.9	0.161% 3.54% 3.89%	0.9	0.0984% 2.68% 2.92%	0.9	0.0654% 2.14% 2.33%	0.9	0.0469% 1.79% 1.94%
	0.9 5	0.869% 10.1% 11.7%	0.9 5	0.332% 5.30% 5.83%	0.9 5	0.165% 3.59% 3.89%	0.9 5	0.0948% 2.71% 2.92%	0.9 5	0.0592% 2.18% 2.33%	0.9 5	0.0388% 1.82% 1.94%
	0.9 8	0.827% 10.1% 11.7%	0.9 8	0.337% 5.43% 5.83%	0.9 8	0.113% 3.59% 3.89%	0.9 8	0.0900% 2.75% 2.92%	0.9 8	0.0519% 2.20% 2.33%	0.9 8	0.0439% 1.85% 1.94%
0.9	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds
	0.9	4.27% 40.9% 45.0%	0.9	2.67% 19.6% 22.5%	0.9	1.64% 13.2% 15.0%	0.9	1.01% 10.0% 11.3%	0.9	0.654% 8.04% 9.00%	0.9	0.494% 6.77% 7.50%
	0.9 5	3.10% 39.1%	0.9 5	2.22% 20.0%	0.9 5	1.43% 13.4%	0.9 5	0.909% 10.1%	0.9 5	0.600% 8.18%	0.9 5	0.455% 6.87%

		45.0%		22.5%		15.0%		11.3%		9.00%		7.50%
	0.9	2.78%	0.9	1.96%	0.9	1.23%	0.9	0.804%	0.9	0.543%	0.9	0.417%
	8	40.9%	8	20.0%	8	13.6%	8	10.2%	8	8.26%	8	6.92%
		45.0%		22.5%		15.0%		11.3%		9.00%		7.50%
0.95	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds	FR	Bounds
	0.9	3.56%	0.9	4.16%	0.9	3.62%	0.9	2.86%	0.9	2.16%	0.9	1.68%
		86.4%		42.2%		27.9%		20.9%		16.8%		14.0%
		95.0%		47.5%		31.7%		23.8%		19.0%		15.8%
	0.9	2.72%	0.9	3.31%	0.9	3.10%	0.9	2.42%	0.9	1.88%	0.9	1.50%
	5	86.4%	5	42.2%	5	28.4%	5	21.1%	5	17.0%	5	14.2%
		95.0%		47.5%		31.7%		23.8%		19.0%		15.8%
	0.9	2.06%	0.9	2.66%	0.9	2.52%	0.9	2.08%	0.9	1.59%	0.9	1.32%
	8	86.4%	8	42.2%	8	28.4%	8	21.3%	8	17.1%	8	14.4%
		95.0%		47.5%		31.7%		23.8%		19.0%		15.8%

Table 3: $EOQ = 4E[D(L)]$.